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A Study of Positivity Analysis for Difference Operators in the Liouville–Caputo Setting

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Abstract: The class of symmetric function interacts extensively with other types of functions. One of these is the class of positivity of functions, which is closely related to the theory of symmetry. Here, we propose a positive analysis technique to analyse a class of Liouville–Caputo difference equations of fractional-order with extremal conditions. Our monotonicity results use difference conditions $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \geq (1-\mu)f(a+J_0)$ and $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \leq (1-\mu)f(a+J_0)$ to derive the corresponding relative minimum and maximum, respectively. We find alternative conditions corresponding to the main conditions in the main monotonicity results, which are simpler and stronger than the existing ones. Two numerical examples are solved by achieving the main conditions to verify the obtained monotonicity results.

Keywords: Liouville–Caputo fractional operators; positivity analysis; monotonicity analysis

MSC: 26A48; 33B10; 39A12; 39B62

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1. Introduction

In discrete fractional calculus and monotonicity analysis, the following implication holds.

$$f \text{ is increasing on } \{a, a+1, a+2, \dots\} \Longleftrightarrow (\Delta f)(x) \geq 0 \text{ for } x \in \{a, a+1, \dots\}, \quad (1)$$

where $a \in \mathbb{R}$, f is assumed to be a function from $\{a, a+1, a+2, \dots\}$ to $\mathbb{R}$, and

$$(\Delta f)(x) := f(x+1) - f(x), \, x \in \{a, a+1, \dots\}$$

is the delta first-order difference operator. Therefore, the implication (1) tell us that there exists a clear connection between the sign of the difference and the monotone behavior (decreasing or increasing) of the function on which the difference acts.

It is well-known that fractional operators play important roles in discrete fractional calculus theory. For example, Atici and Eloe [1–3] in 20107, as well as the subsequent work of Abdeljaward et al. in the article [4], Abdeljawad and Atici in the Ref. [5], Abdeljawad and Baleanu in the article [6], Abdeljawad and Madjidi in the Ref. [7], Chen et al. in the article [8], Ferreira and Torres in the Ref. [9], Lizama et al. in the article [10], and Wu and Baleanu in the Ref. [11] employed difference operators to develop the concept of discrete fractional calculus. In particular, there has been increasing interest in a nonlocal version of the difference calculus, that is, "discrete fractional calculus". For this reason and a wealth of additional information on a variety of nonlocal discrete operators and their properties, we refer to the great monograph in the Ref. [12] by Goodrich and Peterson.

Furthermore, a particularly curious and mathematically nontrivial aspect of this theory is that there is not a clean correlation between the sign of a discrete fractional operator and the monotone (or positive or convex) behavior of the function on which the operator acts. In fact, as has been shown time and time again, there is a highly complex and subtle relationship. This mathematically rich behavior was first documented in the monotonicity case by Dahal and Goodrich [13] in 2014. Since their initial work, numerous other studies have been published, including those by Atici and Uyanik in the Ref. [14], Baoguo et al. in the article [15], Bravo et al. in the article [16], Dahal and Goodrich in the Ref. [17], Du et al. in the article [18], Erbe et al. in the article [19], Goodrich in the Ref. [20], Goodrich et al. in the article [21], Goodrich and Lizama in the Ref. [22], Goodrich et al. in the article [23], Goodrich and Muellner in the Ref. [24], Mohammed et al. in the article [25], Liu et al. in the article [26], and Mohammed et al. in the article [27]. These papers investigate a variety of questions surrounding the qualitative properties inferred from the sign of a fractional difference acting on a function.

Discrete fractional calculus eventually developed into a suitable approach for describing the geometry of discrete operators with difference structures. Additionally, the ability of the difference (or derivative) to detect when a function is increasing or decreasing is of paramount importance in the application of both the continuous and the discrete calculus. Consequently, clarifying this aspect of the theory of fractional difference operators is important. This is particularly the case since there have been some initial attempts to apply discrete fractional calculus to biological modeling—see, for example, Atici, Atici, Nguyen, Zhoroev, and Koch in the Ref. [28], and Atici, Atici, Belcher, and Marshall in the Ref. [29].

Inspired by the above results and the results in the Ref. [30], we mainly consider analysing the discrete delta fractional difference operators of Liouville–Caputo type and to obtain the increasing and decreasing monotone as an outcome of the analyses. These allow us to establish the relative minimum and maximum of the functions at certain points. In addition, by finding a new stronger and simpler condition the main lemmas will be modified, and then the relative minimum and maximum results by considering the new lemmas will be rearranged. Lastly, we will discuss our main results with two examples via tables and figures. It is worth mentioning that this article is the Liouville–Caputo version of our recently published article [30].

We now give a brief outline of the study. We discuss and present in Section 2 the monotone increasing/decreasing and relative minimum/maximum of the discrete operators of Liouville-Caputo type. In Section 3 we demonstrate the effectiveness of the proposed method by means of numerical examples, and the conclusions of the article are collected in Section 4.

2. Discussions and Results

Throughout this study, we have the following identity:

$$x^{\underline{\mu}} := \frac{\Gamma(x+1)}{\Gamma(x+1-\mu)}.$$

In this section, the outcomes of our main results are discussed for the monotone decreasing and monotone increasing of the function f defined on $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$ by analysing the delta fractional differences of Liouville–Caputo type of order μ, which is expressed as follows (see [31] (Theorem 1 & Proposition 1)):

$$\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(x) = \frac{1}{\Gamma(-\mu)} \sum_{\mathrm{r}=a}^{x+\mu} (x - \mathrm{r} - 1)^{\underline{-\mu-1}} f(\mathrm{r}) - \frac{(x-a)^{\underline{-\mu}}}{\Gamma(1-\mu)} f(a), \tag{2}$$

for $x \in \mathbb{N}_{a-\mu+1}$ and $0 < \mu < 1$.

Lemma 1. *Let $1 > \mu > 0$ and f be nonnegative. If the following conditions hold*

(i) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) \leq 0,$

(ii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \leq \left(\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right) f(a), \quad x \in \mathbb{N}_2,$

then f is decreasing on $\mathbb{N}_a$.

Proof. Firstly, we see that $\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!}$ is negative. For $x \in \mathbb{N}_2$, one can have

$$\frac{1}{(i+1)!} \cdot \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)} = \frac{(-\mu)(1-\mu)\cdots(i-1-\mu)(i-\mu)}{(i+1)!} < 0, \tag{3}$$

because $-\mu < 0$ and $(1-\mu)$, $(i-1-\mu)$ and $(i-\mu)$ are all > 0. Computing the first condition to have

$$\begin{aligned}
\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) &= \frac{1}{\Gamma(-\mu)} \sum_{\mathrm{r}=a}^{a+1} (a-\mu-\mathrm{r})^{\underline{-\mu-1}} f(\mathrm{r}) - (1-\mu) f(a) \\
&= f(a+1) - f(a) \\
&\Downarrow \\
(\Delta f)(a) = \left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) &\underset{\text{condition (i)}}{\overset{by}{\leq}} 0.
\end{aligned} \tag{4}$$

Assume that $(\Delta f)(x) \leq 0$ for all $x \in \mathbb{N}_a^{J_0+a-1}$, when $J_0 \in \mathbb{N}_1$. Then we are planning to show that $(\Delta f)(a+J_0) \leq 0$. To do this, we consider the definition (2) to have

$$\begin{aligned}
&\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \\
&= \frac{1}{\Gamma(-\mu)} \sum_{\mathrm{r}=a}^{a+J_0+1} (a+J_0-\mu-\mathrm{r})^{\underline{-\mu-1}} f(\mathrm{r}) - \frac{(J_0+1-\mu)^{\underline{-\mu}}}{\Gamma(1-\mu)} f(a) \\
&= \frac{\Gamma(J_0+1-\mu)}{\Gamma(-\mu)(J_0+1)!} f(a) + \frac{\Gamma(J_0-\mu)}{\Gamma(-\mu)J_0!} f(a+1) + \cdots + \frac{(-\mu)(1-\mu)}{2} f(a+J_0-1) \\
&\quad + (-\mu) f(a+J_0) + f(a+J_0+1) - \frac{\Gamma(J_0+2-\mu)}{\Gamma(1-\mu)(J_0+1)!} f(a).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) - (\Delta f)(a+J_0) \\
&= \frac{\Gamma(J_0+1-\mu)}{\Gamma(-\mu)(J_0+1)!} f(a) + \frac{\Gamma(J_0-\mu)}{\Gamma(-\mu)J_0!} f(a+1) + \cdots \\
&+ \frac{(-\mu)(1-\mu)}{2} f(a+J_0-1) + (1-\mu) f(a+J_0) - \frac{\Gamma(J_0+2-\mu)}{\Gamma(1-\mu)(J_0+1)!} f(a),
\end{aligned} \tag{5}$$

and consequently,

$$\begin{aligned}
(\Delta f)(a+J_0) &\leq \left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \\
&\quad - \frac{\Gamma(J_0+1-\mu)}{\Gamma(-\mu)(J_0+1)!} f(a) - \frac{\Gamma(J_0-\mu)}{\Gamma(-\mu)J_0!} f(a) + \cdots \\
&\quad - \frac{(-\mu)(1-\mu)}{2} f(a) \underbrace{-(1-\mu) f(a)}_{\leq 0} + \frac{\Gamma(J_0+2-\mu)}{\Gamma(1-\mu)(J_0+1)!} f(a) \\
&\leq \left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \\
&\quad - \left(\sum_{i=1}^{(J_0+1)-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(J_0+2-\mu)}{\Gamma(1-\mu)(J_0+1)!} \right) f(a) \\
&\leq 0,
\end{aligned} \tag{6}$$

where we have used condition (ii), (3), $x = J_0 + 1 \in N_2$, and the hypothesis nonnegativity of f:

$$f(a+J_0-1) \leq f(a+J_0-2) \leq \cdots \leq f(a).$$

Hence, the inequality (4) combined with (6) gives us the required result. □

Lemma 2. *Let $1 > \mu > 0$ and f be nonpositive. If the following conditions hold*

(i) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) \geq 0,$

(ii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \geq \left(\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!} \right) f(a), \quad x \in \mathbb{N}_2,$

then f is increasing on $\mathbb{N}_a$.

Depending on the Lemma 1, we can obtain the following relativity (min) result.

Theorem 1. *Let $1 > \mu > 0$ and f be nonnegative. If the following conditions hold*

(i) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) \leq 0,$

(ii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \leq \left(\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!} \right) f(a), \quad x \in \mathbb{N}_2^{J_0},$

(iii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \geq (1-\mu) f(a+J_0),$

for a fixed $J_0 \in \mathbb{N}_3$, then f is a relative minimum at $a + J_0$.

Proof. Collecting the Conditions (i) and (ii) in Lemma 1, we have that $(\Delta f)(x) \leq 0$ for each $x \in \mathbb{N}_a^{a+J_0-1}$, when $J_0 \geq 1$, specifically, we have $(\Delta f)(a+J_0-1) \leq 0$. In order to show that f is a relative minimum at $a+J_0+1$, we shall claim that $\big(\Delta f\big)(a+J_0)\geq 0$. Rewrite (5) to get

$$\begin{aligned}&\big(\Delta f\big)(a+J_0)-\Big({}^{\mathrm{LC}}_{a}\Delta^{\mu}f\Big)(a+J_0+1-\mu)\\&=\frac{\Gamma(J_0+2-\mu)}{\Gamma(1-\mu)(J_0+1)!}f(a)-\frac{\Gamma(J_0+1-\mu)}{\Gamma(-\mu)(J_0+1)!}f(a)-\frac{\Gamma(J_0-\mu)}{\Gamma(-\mu)J_0!}f(a+1)\\&\qquad-\cdots-\frac{(-\mu)(1-\mu)}{2}f(a+J_0)-(1-\mu)f(a+J_0-1)\\&\geq-(1-\mu)f(a+J_0),\end{aligned}\tag{7}$$

where we have used $\frac{\Gamma(J_0+2-\mu)}{\Gamma(1-\mu)(J_0+1)!}f(a)\geq 0$ and

$$-\frac{\Gamma(J_0+1-\mu)}{\Gamma(-\mu)(J_0+1)!}f(a)-\frac{\Gamma(J_0-\mu)}{\Gamma(-\mu)J_0!}f(a+1)-\cdots-\frac{(-\mu)(1-\mu)}{2}f(a+J_0-1)\geq 0,$$

according to (3) and the nonnegativity of f. Rearrange (7) to have the required result as follows:

$$\begin{aligned}\big(\Delta f\big)(a+J_0)&\geq\Big({}^{\mathrm{LC}}_{a}\Delta^{\mu}f\Big)(a+J_0+1-\mu)-(1-\mu)f(a+J_0)\\&\underset{\text{Condition (iii)}}{\overset{by}{\geq}}0.\end{aligned}$$

This implies that f is a relative minimum at $a+J_0$. □

Theorem 2. *Let $1>\mu>0$ and f be nonpositive. If the following conditions hold*

(i) $\Big({}^{\mathrm{LC}}_{a}\Delta^{\mu}f\Big)(a+1-\mu)\geq 0,$

(ii) $\Big({}^{\mathrm{LC}}_{a}\Delta^{\mu}f\Big)(a+x-\mu)\geq\left(\sum_{i=1}^{x-1}\frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!}-\frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right)f(a),\quad x\in\mathbb{N}_2^{J_0},$

(iii) $\Big({}^{\mathrm{LC}}_{a}\Delta^{\mu}f\Big)(a+J_0+1-\mu)\leq(1-\mu)f(a+J_0),$

for a fixed $J_0\in\mathbb{N}_3$, then f is a relative maximum at $a+J_0$.

In the next couple of lemmas, we state a new condition stronger than the existing Condition (ii) in Lemmas 1 and 2.

Lemma 3. *If $1>\mu>0$ and f is a nonnegative function, then the* Condition (ii) *in Lemma 1 can be replaced with*

$$\Big({}^{\mathrm{LC}}_{a}\Delta^{\mu}f\Big)(a+x-\mu)\leq\frac{\{\mu(x-1)+2-\mu\}(\mu-1)}{2}f(a),$$

for $x\in\mathbb{N}_2$.

Proof. First, for $i=1$, we see that $\frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!}$ leads to

$$\frac{\Gamma(3-\mu)}{\Gamma(-\mu)(3!)}=\frac{(-\mu)(1-\mu)(2-\mu)}{6}>\frac{(-\mu)(1-\mu)}{2},$$

for $\mu>-1$. We proceed with it to show that

$$0>\frac{\Gamma(i+1-\mu)}{(i+1)!\Gamma(-\mu)}>\frac{(1-\mu)(-\mu)}{2},\tag{8}$$

for all $i \geq 2$. The first inequality is clear by (3). For the second one, we assume that

$$\frac{\Gamma(i_0+1-\mu)}{\Gamma(-\mu)(i_0+1)!} > \frac{(-\mu)(1-\mu)}{2},$$

for all $i_0 \geq 2$. Then we see that

$$\begin{aligned}\frac{\Gamma(i_0+2-\mu)}{\Gamma(-\mu)(i_0+2)!} &> \frac{i_0+1-\mu}{i_0+2}\cdot\frac{\Gamma(i_0+1-\mu)}{\Gamma(-\mu)(i_0+1)!}\\ &\geq \underbrace{\frac{i_0+1-\mu}{i_0+2}}_{0<\uparrow<1}\cdot\underbrace{\frac{(-\mu)(1-\mu)}{2}}_{<0}\\ &> \frac{(-\mu)(1-\mu)}{2}.\end{aligned}$$

Therefore, the inequalities (8) hold true for all $i \geq 2$ as we claimed. We know that $f(a+1) > 0$, and hence,

$$\begin{aligned}0 &> \left(\sum_{i=1}^{x-1}\frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right)f(a)\\ &> \left(\sum_{i=1}^{x-1}\frac{(-\mu)(1-\mu)}{2} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right)f(a)\\ &= \left(\frac{(-\mu)(1-\mu)}{2}(x-1) - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right)f(a)\\ &> \frac{\{\mu\,(x-1)+2-\mu\}(\mu-1)}{2}f(a),\end{aligned}$$

where we used

$$\frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!} < \frac{\Gamma(3-\mu)}{\Gamma(1-\mu)\,2!} = \frac{(1-\mu)(2-\mu)}{2}.$$

Hence, we conclude that if

$$\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \leq \frac{\{\mu\,(x-1)+2-\mu\}(\mu-1)}{2}f(a),$$

then Condition (ii) in Lemma 1 is satisfied. Thus, the result is obtained. □

Lemma 4. *If $1 > \mu > 0$ and f is a nonpositive function, then the* Condition (ii) *in Lemma 2 can be replaced with*

$$\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \geq \frac{\{\mu\,(x-1)+2-\mu\}(\mu-1)}{2}f(a),$$

for $x \in \mathbb{N}_2$.

Corollary 1. *Let for $1 > \mu > 0$ and f be nonnegative. If the following conditions hold*

(i) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) \leq 0,$

(ii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \leq \dfrac{\{\mu\,(x-1)+2-\mu\}(\mu-1)}{2}f(a), \quad x \in \mathbb{N}_2^{J_0},$

(iii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \geq (1-\mu)f(a+J_0+1),$

for a fixed $J_0 \in \mathbb{N}_3$, then f is a relative minimum at $a+J_0$.

Proof. The proof obviously follows from Theorem 1 and Lemma 3. □

Corollary 2. *Let for $1 > \mu > 0$ and f be nonpositive. If the following conditions hold*

(i) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+1-\mu) \geq 0,$

(ii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu) \geq \dfrac{\{\mu\,(x-1)+2-\mu\}(\mu-1)}{2} f(a), \quad x \in \mathbb{N}_2^{J_0},$

(iii) $\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+J_0+1-\mu) \leq (1-\mu) f(a+J_0+1),$

for a fixed $J_0 \in \mathbb{N}_3$, then f is a relative maximum at $a+J_0$.

Proof. The proof follows from Theorem 2 and Lemma 4 directly. □

3. Test Examples

Let us denote the main points in Section 2 by the following notations:

$$A_1(x) := \left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(a+x-\mu), \quad x \in \mathbb{N}_1,$$

and

$$A_2(x) := \begin{cases} 0, & \text{if} \quad x = 1, \\ \left(\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right) f(a), & \text{if} \quad x \in \mathbb{N}_2. \end{cases}$$

Then we present here some examples of application of the main results.

Example 1. *Let $\mu = 0.9$, $a = 0$, and the nonnegative function f be defined by*

$$f(x) = \left(\frac{4}{5}\right)^{x-a}, \quad \text{for } x \in \mathbb{N}_a.$$

Condition (i) is trivially valid due to

$$\begin{aligned} \left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(1-\mu) &= \frac{1}{\Gamma(-\mu)} \sum_{\mathrm{r}=0}^{1} (-\mu-\mathrm{r})^{\underline{-\mu-1}} f(\mathrm{r}) \\ &= f(1) - f(0) = -\frac{1}{5} \leq 0. \end{aligned}$$

On the other hand, according to Table 1 and Figure 1, we can observe that

$$\left({}^{\mathrm{LC}}_{0}\Delta^{\mu} f\right)(a+x-\mu) \leq \left(\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right) f(a),$$

for $x = 1,2,3,4$. Thus, all the conditions of the statement of Lemma 1 are verified, and hence the function will be decreasing on $\{1,2,3,4\}$.

Table 1. Comparison of $A_1(x)$ and $A_2(x)$ values.

	$x = a+1$	$x = a+2$	$x = a+3$	$x = a+4\cdots$
$A_1(x)$	−0.2000	−0.1800	−0.1550	$-0.1317\cdots$
$A_2(x)$	0	−0.0748	−0.0913	$-0.1000\cdots$

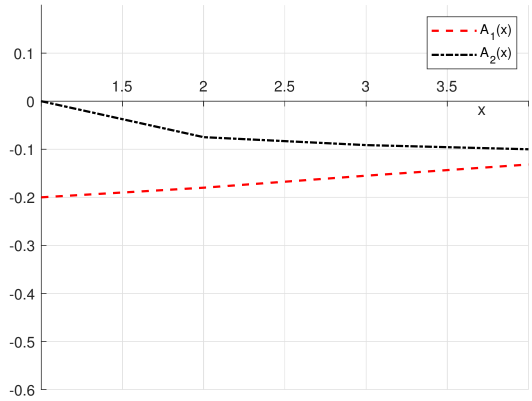

Figure 1. Plot of $A_1(x)$ and $A_2(x)$ in the interval $[1,4]$.

Example 2. *Assume that $\mu = 0.95$, $a = 0$, and the nonpositive function f is defined by*

$$f(x) = -\left(\frac{11}{20}\right)^{x-a}, \quad \text{for } x \in \mathbb{N}_a.$$

It is obvious that

$$\left({}^{\mathrm{LC}}_{a}\Delta^{\mu} f\right)(1-\mu) = \frac{1}{\Gamma(-\mu)} \sum_{\mathrm{r}=0}^{1} (-\mu - \mathrm{r})^{\underline{-\mu-1}} f(\mathrm{r})$$
$$= f(1) - f(0) = \frac{9}{20} \geq 0.$$

Moreover, the numerical results reported in Table 2 and Figure 2 tell us that

$$\left({}^{\mathrm{LC}}_{0}\Delta^{\mu} f\right)(a + x - \mu) \geq \left(\sum_{i=1}^{x-1} \frac{\Gamma(i+1-\mu)}{\Gamma(-\mu)(i+1)!} - \frac{\Gamma(x+1-\mu)}{\Gamma(1-\mu)(x)!}\right) f(a),$$

for $x = 1,2,3,4$. Thus, the entire conditions of the statement of Lemma 2 hold. Therefore, the given function is increasing on the set $\{1,2,3,4\}$.

Table 2. The values of $A_1(x)$ and $A_2(x)$.

	$x = a+1$	$x = a+2$	$x = a+3$	$x = a+4 \cdots$
$A_1(x)$	$\frac{9}{20}$	$\frac{27}{100}$	$\frac{513}{3200}$	$\frac{597}{6203} \cdots$
$A_2(x)$	0	$\frac{103}{2752}$	$\frac{634}{13861}$	$\frac{1}{20} \cdots$

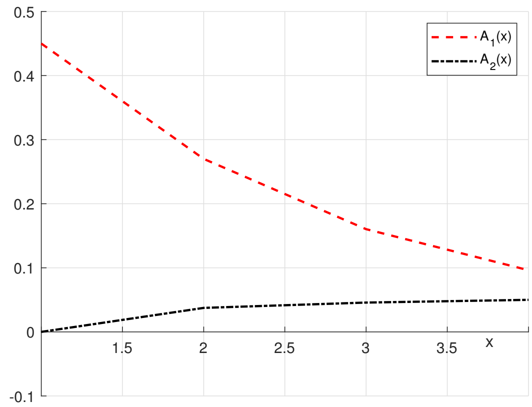

Figure 2. Plot of $A_1(x)$ and $A_2(x)$ in the interval $[1,4]$.

4. Concluding Remarks

In the present work, the positivity analysis technique was applied to investigate the monotonicity behavior of delta fractional differences of Liouville–Caputo type. Although the procedures are general, due to the limitation of singular and nonsingular kernels employed, for application purposes, we restricted the present work to a Liouville–Caputo difference operator of a nonsingular kernel. A condition corresponding to the original condition (ii) in both Lemmas 1 and 2 was found as an alternative condition. This condition was derived, and it was found that it is simpler and easier to use.

On the other hand, approximations of the solutions were computed by using the definition of a Liouville–Caputo difference operator. The numerical results were in agreement with the main Lemmas 1 and 2. The monotonicity and relativity results in this problem can be extended to study other, and more complicated types of fractional sums, and also to other types of difference operators, where singular and nonsingular kernels are both involved in the discrete operators—see the Refs. [32,33]. Furthermore, the importance of discrete fractional differences is increasing daily due to their high capabilities in modeling physical problems of the real world. Thus, one may develop our results in this paper and apply them in physical problems, as numerous generalizations of this model have recently been developed and explored in the published articles [34,35].

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