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ARTICLE

A Weighted Average Finite Difference Scheme for the Numerical Solution of Stochastic Parabolic Partial Differential Equations

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ABSTRACT

In the present paper, the numerical solution of Itô type stochastic parabolic equation with a time white noise process is imparted based on a stochastic finite difference scheme. At the beginning, an implicit stochastic finite difference scheme is presented for this equation. Some mathematical analyses of the scheme are then discussed. Lastly, to ascertain the efficacy and accuracy of the suggested technique, the numerical results are discussed and compared with the exact solution.

KEYWORDS

Itô equation; stochastic process; finite difference scheme; stability and convergence; consistency

1 Introduction

Stochastic partial differential equations (SPDEs) driven by white noise are one of the essential classes of partial differential equations (PDEs). This class of equations arises in many branches of applied sciences and engineering, such as nonlinear filtering [1], turbulent flows [2], population biology [3], microscopic particle dynamics [4], groundwater flow [5], etc. Few numbers of SPDEs can be solved by analytical techniques [6], most of which cannot be analyzed by well-known analytical schemes suitably. Due to this reason, various numerical methods have been discussed to solve such equations [7–9]. For instance, the authors in [10] proposed an explicit scheme to obtain the approximate solution of stochastic equations. In [11], a compact finite difference method for solving a stochastic advection-diffusion equation was proposed. In [12], two techniques on the basis of Saul'yev method and finite difference scheme were suggested for solving linear SPDEs. In [13], explicit and implicit finite difference methods were proposed to obtain the solution of general SPDEs. In [14], a stochastic compact finite difference scheme was suggested for solving a stochastic fractional advection-diffusion equation. In [15], high-resolution finite volume methods were used to solve SPDEs. In [16], the authors proposed a spectral collocation method for the numerical solution of SPDEs driven by infinite dimensional fractional Brownian motions. More than these, some authors used spectral methods for the discretization of spatial variables and applied a Crank-Nicolson scheme or a stochastic Runge-Kutta method for solving the resultant system of stochastic differential equations [17].

In [18,19], the authors investigated the convergence and stability of two stochastic finite difference schemes for a class of SPDEs. In more detail, the study [19] employed a Crank-Nicolson technique for the approximation of second-order derivatives. Although the reported results in [18,19] are interesting in some senses, the solution methods presented are only conditionally stable. To overcome this issue, here we extend a type of finite difference scheme to a stochastic version in order to approximate the solution of a stochastic advection-diffusion equation. To do so, instead of the Crank-Nicolson method used in [19], we consider a convex combination of discretized second-order derivatives in two consecutive time grid points. As a result, the proposed method in our case is unconditionally stable under a necessary condition, so there will be no limitation for the selection of space and time step sizes. This important feature makes the computational cost of our suggested technique less than the other methods available in the literature [18,19]. In the following, the main contributions of our study are summarized and highlighted as below:

- In this paper, a stochastic finite difference scheme is developed for the numerical solution of Itô type stochastic parabolic equation.
- As a theoretical investigation, some mathematical results for the proposed scheme are studied.
- In addition, the convergence of the suggested technique is discussed, and the necessary conditions for its conditional and unconditional stability are explored.
- Finally, the efficiency of the proposed method is shown by some numerical examples, and its key qualifications are examined as well.

The rest of this paper is structured as follows. An implicit finite difference scheme is proposed in Section 2, where some mathematical analyses are also investigated. Next, some numerical results are given in Section 3. Finally, the paper is closed by some concluding remarks in the last section.

2 Proposed Scheme

In this work, the following problem for the stochastic equation of Itô type is considered:

$$\begin{aligned}
&\frac{\partial v}{\partial t} = \rho \frac{\partial^2 v}{\partial x^2}(x,t) + \sigma\, v(x,t)\, \dot{\xi}(t),\\
&v(x,0) = v_0(x),\\
&v(0,t) = f_1(t),\\
&v(1,t) = f_2(t),
\end{aligned} \tag{1}$$

for $x \in (0,1)$ and $t \in (0,1]$. In this problem, the coefficients ρ and σ are constants, and $\xi(t)$ indicates a standard Wiener process. Also, the noise term $\dot{\xi}(t)$ is introduced to present a time white noise. Formally, $\dot{\xi}(t)$ is a Gaussian distribution with zero mean value [20].

Finite difference schemes are the most natural way of solving PDEs numerically. Furthermore, these methods are widely used in approximating the solution of SPDEs like (1). The idea behind these schemes is to discretize the continuous time and space into a finite number of discrete grid points. Then the values of state variables are calculated at any point of the grid. By considering a uniform space grid Δx and time grid Δt in the time-space lattice, the solution of the equation can be estimated at the lattice points. The value of the approximate solution at the point $(h\Delta x, m\Delta t)$ is indicated by the random variable v_h^m, where m and h are integer. The later stage is to approximate the problem (1) on the mentioned grid. For this purpose, the time and the space derivatives in the SPDE (1) are replaced by the following finite difference approximations:

$$v_t\left(h\Delta x, m\Delta t\right) \approx \frac{v_h^{m+1} - v_h^m}{\Delta t},$$

$$v_{xx}\left(h\Delta x, m\Delta t\right) \approx \lambda \frac{v_{h-1}^{m+1} - 2v_h^{m+1} + v_{h+1}^{m+1}}{\Delta x^2} + (1-\lambda)\frac{v_{h-1}^{m} - 2v_h^{m} + v_{h+1}^{m}}{\Delta x^2}, \tag{2}$$

where $0 \le \lambda \le 1$. Indeed, we use a forward finite difference scheme for the approximation of $v_t(h\Delta x, m\Delta t)$ as

$$v_t\left(h\Delta x, m\Delta t\right) = \frac{v(h\Delta x, (m+1)\Delta t) - v(h\Delta x, m\Delta t)}{\Delta t} + O\left(\Delta t\right), \tag{3}$$

and employ a convex combination of second-order derivatives in the time steps m and $m+1$ for the approximation of $v_{xx}(h\Delta x, m\Delta t)$ by

$$\begin{aligned} v_{xx}\left(h\Delta x, m\Delta t\right) = {} & \lambda \frac{v((h-1)\Delta x, (m+1)\Delta t) - 2v(h\Delta x, m\Delta t) + v((h+1)\Delta x, (m+1)\Delta t)}{\Delta x^2} \\ & + (1-\lambda)\frac{v((h-1)\Delta x, m\Delta t) - 2v(h\Delta x, m\Delta t) + v((h+1)\Delta x, m\Delta t)}{\Delta x^2} + O\left(\Delta x^2\right). \end{aligned} \tag{4}$$

For more details, the interested reader can refer to [21]. Substituting the approximations from (2) into (1), we can find

$$-\lambda r v_{h-1}^{m+1} + (1+2\lambda r)v_h^{m+1} - \lambda r v_{h+1}^{m+1} = (1-\lambda)r v_{h-1}^m + (1-2(1-\lambda)r)v_h^m + (1-\lambda)r v_{h+1}^m + \sigma v_h^m \Delta\xi_m, \tag{5}$$

where $r = \frac{\rho\Delta t}{\Delta x^2}$, and $\Delta\xi_m = \xi((m+1)\Delta t) - \xi(m\Delta t)$ is a Gaussian distribution with zero mean value and variance Δt, i.e., $\Delta\xi_m \sim N(0, \Delta t)$.

Remark 2.1. In the proposed scheme, the Wiener process increments are not dependent on the state v_h^m.

Substantially, the convergence of the stochastic difference scheme to the SPDE solution is very important. To achieve this, consider an SPDE in the form of $Lu = F$, wherein F is an inhomogeneity and L represents the differential operator. Suppose that the random variable v_h^m be a solution that is approximated by a stochastic finite difference scheme indicated by L_h^m. By applying the stochastic scheme to this SPDE, we obtain

$$L_h^m v_h^m = F_h^m, \tag{6}$$

where F_h^n is the approximation of inhomogeneity F. In favor of accessing the consistency, stability, and convergence results, a norm is needed. Because of this, for the sequence $v = \{\ldots, v_{-1}, v_0, v_1, \ldots\}$, we define the sup–norm as $\|v\|_\infty = \sqrt{\sup_h |v_h|^2}$. For additional details concerning the concepts of consistency, stability and convergence, see [10].

Definition 2.1. A stochastic finite difference scheme $L_h^m v_h^m = F_h^m$ is said to be point-wise consistent in mean square with PDE $Lu = F$ at point (x, t), if for any continuously differentiable solution $\Upsilon = \Upsilon(x, t)$ of this equation, we have

$$\mathbb{E}\| \left(L\Upsilon - F\right)|_h^m - \left[L_h^m \Upsilon(h\Delta x, m\Delta t) - F_h^m\right] \|^2 \to 0,$$

as $(\Delta x, \Delta t) \to (0, 0)$ and $(h\Delta x, (m+1)\Delta t) \to (x, t)$.

Theorem 2.1. The numerical scheme (5) is consistent in mean square in the sense of Definition 2.1.

Proof. For the smooth function $\Upsilon(x, t)$, we have

$$L(\Upsilon)|_h^m = \Upsilon(h\Delta x, (m+1)\Delta t) - \Upsilon(h\Delta x, m\Delta t) - \rho\int_{m\Delta t}^{(m+1)\Delta t} \Upsilon_{xx}(h\Delta x, w)\,\mathrm{d}w$$
$$- \sigma\int_{m\Delta t}^{(m+1)\Delta t} \Upsilon(h\Delta x, w)\,\mathrm{d}\xi(w), \tag{7}$$

and

$$L_h^m\Upsilon = \Upsilon(h\Delta x, (m+1)\Delta t) - \Upsilon(h\Delta x, m\Delta t) - \sigma\Upsilon(h\Delta x, m\Delta t)(\xi((m+1)\Delta t) - \xi(m\Delta t))$$
$$- \rho\lambda\Delta t\frac{\Upsilon((h-1)\Delta x, (m+1)\Delta t) - 2\Upsilon(h\Delta x, (m+1)\Delta t) + \Upsilon((h+1)\Delta x, (m+1)\Delta t)}{\Delta x^2}$$
$$- \rho\,(1-\lambda)\,\Delta t\frac{\Upsilon((h-1)\Delta x, m\Delta t) - 2\Upsilon(h\Delta x, m\Delta t) + \Upsilon((h+1)\Delta x, m\Delta t)}{\Delta x^2}. \tag{8}$$

Accordingly,

$$\mathbb{E}|L(\Upsilon)|_h^m - L_h^m\Upsilon|^2$$
$$\leq 2\rho^2\mathbb{E}|\int_{m\Delta t}^{(m+1)\Delta t} [\Upsilon_{xx}(h\Delta x, w) - \frac{1}{\Delta x^2}(\lambda[\Upsilon((h-1)\Delta x, (m+1)\Delta t) - 2\Upsilon(h\Delta x, (m+1)\Delta t)$$
$$+ \Upsilon((h+1)\Delta x, (m+1)\Delta t)] + (1-\lambda)[\Upsilon((h-1)\Delta x, m\Delta t) - 2\Upsilon(h\Delta x, m\Delta t) + \Upsilon((h+1)\Delta x, m\Delta t)])]\mathrm{d}w|^2$$
$$+ 2\sigma^2\int_{m\Delta t}^{(m+1)\Delta t} |\Upsilon(h\Delta x, w) - \Upsilon(h\Delta x, m\Delta t)|^2\,\mathrm{d}w. \tag{9}$$

In as much as $\Upsilon(x, t)$ is a deterministic function, $\mathbb{E}|L(\Upsilon)|_h^m - L_h^m\Upsilon|^2 \to 0$ as $m, h \to \infty$. Hence, the numerical scheme (5) is consistent with the SPDE (1).

By the assumption that $\hat{v}^{m+1}$ is the Fourier transform of v^{m+1}, the Fourier inversion formula results in

$$v_n^{m+1} = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{\Delta x}}^{\frac{\pi}{\Delta x}} e^{in\Delta x\eta}\hat{v}^{m+1}(\eta)\,\mathrm{d}\eta, \tag{10}$$

where

$$\hat{v}^{m+1} = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{-in\Delta x\eta}v_n^{m+1}\Delta x, \tag{11}$$

and η is a real variable. We utilize the Von Neumann method to investigate the stability of the stochastic difference scheme. By substituting Eq. (11) into the stochastic difference equation and using the equality of Fourier transformation, one achieves

$$\hat{v}^{m+1}(\eta) = \hat{v}^m(\eta)\,g(\Delta x\eta, \Delta t, \Delta x), \tag{12}$$

where $\hat{v}^m$ is the Fourier transform of v^m. Therefore,

$$\mathbb{E}|g(\Delta x\eta, \Delta t, \Delta x)|^2 \leq 1 + K\Delta t, \tag{13}$$

will be the necessary and sufficient condition for the stability [10].

Theorem 2.2. For the stochastic advection-diffusion Eq. (1), the stochastic scheme (5) is unconditionally stable for $\lambda \geq \frac{1}{2}$ based on the Fourier transformation analysis, and is conditionally stable for $\lambda \leq \frac{1}{2}$ under the condition $r \leq \frac{1}{2(1-2\lambda)}$.

Proof. Substituting (11) into (5), we get

$$\left(-\lambda re^{-i\Delta x\eta} + (1+2\lambda r) - \lambda re^{i\Delta x\eta}\right)\hat{v}^{m+1}(\eta) = \left((1-\lambda)\, re^{-i\Delta x\eta} + (1-2(1-\lambda)\, r) + (1-\lambda)\, re^{i\Delta x\eta} + \sigma\,\Delta\xi_m\right)\hat{v}^{m}(\eta). \tag{14}$$

Then we have

$$\hat{v}^{m+1}(\eta) = \frac{(1-\lambda)re^{-i\Delta x\eta} + (1-2(1-\lambda)r) + (1-\lambda)re^{i\Delta x\eta}}{-\lambda re^{-i\Delta x\eta} + (1+2\lambda r) - \lambda re^{i\Delta x\eta}} + \frac{\sigma\,\Delta\xi_m}{(1-\lambda)re^{-i\Delta x\eta} + (1-2(1-\lambda)r) + (1-\lambda)re^{i\Delta x\eta}}\hat{v}^{m}(\eta). \tag{15}$$

Hence, the stochastic difference scheme amplification factor is

$$\begin{aligned} g(\Delta x\eta, \Delta t, \Delta x) &= \frac{(1-\lambda)re^{-i\Delta x\eta} + (1-2(1-\lambda)r) + (1-\lambda)re^{i\Delta x\eta}}{-\lambda re^{-i\Delta x\eta} + (1+2\lambda r) - \lambda re^{i\Delta x\eta}} \\ &\quad + \frac{\sigma\,\Delta\xi_m}{(1-\lambda)re^{-i\Delta x\eta} + (1-2(1-\lambda)r) + (1-\lambda)re^{i\Delta x\eta}} \\ &= \frac{1-4(1-\lambda)\, r\sin^2\dfrac{\Delta x\eta}{2}}{1+4\lambda r\sin^2\dfrac{\Delta x\eta}{2}} + \frac{\sigma^2}{1+4\lambda r\sin^2\dfrac{\Delta x\eta}{2}}\Delta t. \end{aligned} \tag{16}$$

Set $\chi(\Delta x\eta) = \frac{1-4(1-\lambda)r\sin^2\frac{\Delta x\eta}{2}}{1+4\lambda r\sin^2\frac{\Delta x\eta}{2}}$, so $\chi(\theta) = \frac{1-4(1-\lambda)r\sin^2\frac{\theta}{2}}{1+4\lambda r\sin^2\frac{\theta}{2}}$. Now, by setting the derivative of $\chi(\theta)$ equal to zero, the critical points are obtained as $\theta = 0, \pm\pi$. Then one notes that $\chi(0) = 1$ and

$$\chi(\pm\pi) = \frac{1-4(1-\lambda)r}{1+4\lambda r}. \tag{17}$$

It is easy to see that $\frac{1-4(1-\lambda)r}{1+4\lambda r} \leq 1$. Since the equation

$$-1 \leq \frac{1-4(1-\lambda)r}{1+4\lambda r}, \tag{18}$$

is equivalent to $4r(1-2\lambda) \leq 2$, clearly if $\lambda \geq \frac{1}{2}$, then the inequality (18) is always satisfied, and if $\lambda < \frac{1}{2}$, then the inequality (18) is satisfied only if

$$r \leq \frac{1}{2(1-2\lambda)}. \tag{19}$$

And also

$$\left|\frac{1}{1+4\lambda r\sin^2\frac{\Delta x\eta}{2}}\right| \leq 1,$$

is always satisfied. Hence, we see that if $\lambda \geq \frac{1}{2}$, the scheme (5) is unconditionally stable, and if $\lambda < \frac{1}{2}$, the difference scheme (5) is conditionally stable with the condition for the stability given by (19).

Definition 2.2. The stochastic difference scheme $L_h^m v_h^m = F_h^m$, which approximates the SPDE $Lu = F$, is convergent in mean square at time t when $(m+1)\Delta t$ converges to t, $\mathbb{E}\|v^{m+1} - u^{m+1}\|^2 \to 0$ for $(m+1)\Delta t = t$, $\Delta x \to 0$, and $\Delta t \to 0$.

Theorem 2.3. The numerical scheme (5) for the Eq. (1) is convergent in mean square with respect to $\|\cdot\|_\infty = \sqrt{\sup_h |\cdot|^2}$ with $r \le \frac{1}{2(1-\lambda)}$ and $t = (m+1)\Delta t$.

Proof. The stochastic finite difference scheme is given by

$$v_h^{m+1} = v_h^m + \rho\Delta t\left((1-\lambda)\frac{v_{h+1}^m - 2v_h^m + v_{h-1}^m}{\Delta x^2} + \lambda\frac{v_{h+1}^{m+1} - 2v_h^{m+1} + v_{h-1}^{m+1}}{\Delta x^2}\right)$$
$$+ \sigma v_h^m(\xi((m+1)\Delta t) - \xi(m\Delta t)). \quad (20)$$

The solution u_h^{m+1} is represented by the Taylor's expansion $u_{xx}(x, w)$ with respect to the space variable as follows:

$$u_h^{m+1} = u_h^m + \rho\int_{m\Delta t}^{(m+1)\Delta t} u_{xx}(x_h, w)\mathrm{d}w$$
$$+ \sigma\int_{m\Delta t}^{(m+1)\Delta t} u(x_h, w)\mathrm{d}\xi(w)$$
$$= u_h^m + \rho\int_{m\Delta t}^{(m+1)\Delta t}\left((1-\lambda)\frac{u_{h+1}^m - 2u_h^m + u_{h-1}^m}{\Delta x^2} + \lambda\frac{u_{h+1}^{m+1} - 2u_h^{m+1} + u_{h-1}^{m+1}}{\Delta x^2}\right.$$
$$-\frac{\Delta x^2}{4!}((1-\lambda)[u_{xxxx}((h+\beta_1)\Delta x, w)$$
$$+u_{xxxx}((h+\beta_2)\Delta x, w)] + \lambda[u_{xxxx}((h+\beta_3)\Delta x, w+\Delta t)$$
$$+u_{xxxx}((h+\beta_4)\Delta x, w+\Delta t)]) - \lambda\Delta t\, u_{xxt}(h\Delta x, w+\delta\Delta t))\,\mathrm{d}w$$
$$+ \sigma\int_{m\Delta t}^{(m+1)\Delta t} u(x_h, w)\mathrm{d}\xi(w), \quad (21)$$

where $\beta_1, \beta_2, \beta_3, \beta_4, \Delta \in (0, 1)$. Then we have

$$u_h^{m+1} = u_h^m + \rho\int_{m\Delta t}^{\Delta t + m\Delta t}\left(\lambda\frac{u_{h+1}^{m+1} - 2u_h^{m+1} + u_{h-1}^{m+1}}{\Delta x^2} + (1-\lambda)\frac{u_{h+1}^m - 2u_h^m + u_{h-1}^m}{\Delta x^2}\right.$$
$$-\frac{\Delta x^2}{4!}((1-\lambda)[u_{xxxx}((h+\beta_1)\Delta x, w)$$
$$+u_{xxxx}((h+\beta_2)\Delta x, w)] + \lambda[u_{xxxx}((h+\beta_3)\Delta x, w+\Delta t)$$
$$+u_{xxxx}((h+\beta_4)\Delta x, w+\Delta t)]) - \lambda\rho\Delta t\, u_{xxxx}(h\Delta x, w+\delta\Delta t))\,\mathrm{d}w$$
$$-\Delta t\lambda\sigma\int_{m\Delta t}^{\Delta t + m\Delta t} u_{xx}(x_h, w)\mathrm{d}\xi(w)$$
$$+ \sigma\int_{m\Delta t}^{\Delta t + m\Delta t} u(x_h, w)\mathrm{d}\xi(w). \quad (22)$$

Let $z_h^m = u_h^m - v_h^m$, so we get

$$z_h^{m+1} = z_h^m + \rho\int_{m\Delta t}^{\Delta t+m\Delta t}\Bigg((1-\lambda)\,\frac{z_{h+1}^m - 2z_h^m + z_{h-1}^m}{\Delta x^2} + \lambda\frac{z_{h+1}^{m+1} - 2z_h^{m+1} + z_{h-1}^{m+1}}{\Delta x^2}$$
$$-\frac{\Delta x^2}{4!}((1-\lambda)\,[u_{xxxx}\,((h+\beta_1)\,\Delta x, w)$$
$$+u_{xxxx}((h+\beta_2)\Delta x, w)] + \lambda[u_{xxxx}((h+\beta_3)\Delta x, w+\Delta t)$$
$$+u_{xxxx}((h+\beta_4)\Delta x, w+\Delta t)]) - \lambda\rho\Delta t\,u_{xxxx}(h\Delta x, w+\delta\Delta t))\,\mathrm{d}w$$
$$-\lambda\sigma\Delta t\int_{m\Delta t}^{\Delta t+m\Delta t} u_{xx}(x,w)|_{x=x_h}\mathrm{d}\xi(w)$$
$$+\sigma\int_{m\Delta t}^{\Delta t+m\Delta t}(u(x,w)|_{x=x_h} - v_h^m)\mathrm{d}\xi(w). \tag{23}$$

It gives that

$$-\lambda r z_{h-1}^{m+1} + (1+2\lambda r)z_h^{m+1} - \lambda r z_{h+1}^{m+1}$$
$$= (1-\lambda)rz_{h-1}^m + (1-2(1-\lambda)r)z_h^m + (1-\lambda)rz_{h+1}^m$$
$$+\rho\int_{m\Delta t}^{\Delta t+m\Delta t}[-\frac{\Delta x^2}{4!}((1-\lambda)[u_{xxxx}((h+\beta_1)\Delta x, w)$$
$$+u_{xxxx}((h+\beta_2)\Delta x, w)] + \lambda[u_{xxxx}((h+\beta_3)\Delta x, w+\Delta t)$$
$$+u_{xxxx}((h+\beta_4)\Delta x, w+\Delta t)]) - \lambda\rho\Delta t\,u_{xxxx}(h\Delta x, w+\delta\Delta t)]\mathrm{d}w$$
$$-\lambda\sigma\Delta t\int_{m\Delta t}^{\Delta t+m\Delta t} u_{xx}(x,w)|_{x=x_h}\mathrm{d}\xi(w)$$
$$+\sigma\int_{m\Delta t}^{\Delta t+m\Delta t}(u(x,w)|_{x=x_h} - v_h^m)\mathrm{d}\xi(w), \tag{24}$$

where $r = \rho\frac{\Delta t}{\Delta x^2}$. Applying $\mathbb{E}|\cdot|^2$ to the above equation and using the following inequality:

$\mathbb{E}|X+Y+Z+R|^2 \le 8\mathbb{E}|Z|^2 + 8\mathbb{E}|Y|^2 + 2\mathbb{E}|R|^2 + 4\mathbb{E}|X|^2$,

we have

$$\mathbb{E}|\lambda r z_{h-1}^{m+1} + (1+2\lambda r)z_h^{m+1} - \lambda r z_{h+1}^{m+1}|^2$$
$$\le 4\mathbb{E}|(1-\lambda)rz_{h-1}^m + (1-2(1-\lambda)r)z_h^m + (1-\lambda)rz_{h+1}^m|^2$$
$$+8\mathbb{E}|\rho\int_{m\Delta t}^{\Delta t+m\Delta t}[-\frac{\Delta x^2}{4!}((1-\lambda)[u_{xxxx}((h+\beta_1)\Delta x, w)$$
$$+u_{xxxx}((h+\beta_2)\Delta x, w)] + \lambda[u_{xxxx}((h+\beta_3)\Delta x, w+\Delta t)$$
$$+u_{xxxx}((h+\beta_4)\Delta x, w+\Delta t)]) - \lambda\rho\Delta t\,u_{xxxx}(h\Delta x, w+\delta\Delta t)]\mathrm{d}w|^2$$
$$+8(\lambda\sigma\Delta t)^2\mathbb{E}|\int_{m\Delta t}^{\Delta t+m\Delta t} u_{xx}(x,w)|_{x=x_h}\mathrm{d}\xi(w)|^2$$

$$+ 4\sigma^2 \int_{m\Delta t}^{\Delta t + m\Delta t} \mathbb{E}|u(x,w)|_{x=x_h} - u_h^m|^2 \mathrm{d}w + 4\sigma^2 \underbrace{\int_{m\Delta t}^{(m+1)\Delta t} \mathbb{E}|u_h^m - v_h^m|^2 \mathrm{d}w}_{\mathbb{E}|z_h^m|^2 \Delta t}, \tag{25}$$

and so

$$\begin{aligned}
&\mathbb{E}|\lambda r z_{h-1}^{m+1} + (1+2\lambda r) z_h^{m+1} - \lambda r z_{h+1}^{m+1}|^2 \\
&\quad \le 4[\sigma^2 \Delta t + |(1-\lambda) r z_{h-1}^m + (1 - 2(1-\lambda) r) z_h^m + (1-\lambda) r z_{h+1}^m|^2] \sup_h \mathbb{E}|z_h^m|^2 \\
&\quad + 8 \sup_h \mathbb{E}|\rho \int_{m\Delta t}^{\Delta t + m\Delta t} [-\frac{\Delta x^2}{4!}((1-\lambda)[u_{xxxx}((h+\beta_1)\Delta x, w) \\
&\quad + u_{xxxx}((h+\beta_2)\Delta x, w)] + \lambda[u_{xxxx}((h+\beta_3)\Delta x, w + \Delta t) \\
&\quad + u_{xxxx}((h+\beta_4)\Delta x, w + \Delta t)]) - \lambda \rho \Delta t \, u_{xxxx}(h\Delta x, w + \delta \Delta t)] \mathrm{d}w|^2 \\
&\quad + 8(\lambda \sigma \Delta t)^2 \sup_h \int_{m\Delta t}^{\Delta t + m\Delta t} \mathbb{E}|u_{xx}(x,w)|_{x=x_h}|^2 \mathrm{d}w \\
&\quad + 4\sigma^2 \sup_h \int_{m\Delta t}^{\Delta t + m\Delta t} \mathbb{E}|u(x,w)|_{x=x_h} - u_h^m|^2 \mathrm{d}w.
\end{aligned} \tag{26}$$

By introducing the notation $\Theta_{1h} = u_{xxxx}((h+\beta_1)\Delta x, s) < \infty$, $\Theta_{2h} = u_{xxxx}((h+\beta_2)\Delta x, s) < \infty$, $\Theta_{3h} = u_{xxxx}((h+\beta_3)\Delta x, s + \Delta t) < \infty$, $\Theta_{4h} = u_{xxxx}((h+\beta_4)\Delta x, s + \Delta t) < \infty$, $\Theta_{5h} = u_{xxxx}(h\Delta x, s + \delta \Delta t) < \infty$, $\psi_{1h} = u_{xx}(x,s) < \infty$, taking into account

$$\begin{aligned}
\int_{m\Delta t}^{\Delta t + m\Delta t} \mathbb{E}|u(x_h, w) - u_h^m|^2 &= \mathbb{E}\int_{m\Delta t}^{\Delta t + m\Delta t} |u(x_h, w) - u_h^m|^2 \mathrm{d}w \\
&\le \sup_{w \in [m\Delta t, (m+1)\Delta t]} |u\left(h\Delta x, m\Delta t\right) - u\left(x_h, w\right)|^2 \Delta t \le \Delta t \psi',
\end{aligned} \tag{27}$$

and the usage of supposition $r \le \frac{1}{2(1-\lambda)}$, one concludes that

$$\begin{aligned}
&\sup_h \mathbb{E}|\lambda r z_{h-1}^{m+1} + (1+2\lambda r)\, z_h^{m+1} - \lambda r z_{h+1}^{m+1}|^2 \\
&\le 4\left(\sigma^2 \Delta t + 1\right) \sup_h \mathbb{E}|z_h^m|^2 \\
&\quad + 8 \sup_h \mathbb{E}|\rho \int_{m\Delta t}^{(m+1)\Delta t} \left[-\frac{\Delta x^2}{4!}\left((1-\lambda)\left(\Theta_{1h} + \Theta_{2h}\right) + \lambda\left(\Theta_{3h} + \Theta_{4h}\right)\right) - \lambda \rho \Delta t \, \Theta_{5h}\right] \mathrm{d}w|^2 \\
&\quad + 8\left(\lambda \sigma \Delta t\right)^2 \sup_h \int_{m\Delta t}^{(m+1)\Delta t} \mathbb{E}|\psi_{1h}|^2 \mathrm{d}w + 4\sigma^2 \psi' \Delta t.
\end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned}
&(-|\lambda r| + |1 + 2\lambda r| - |\lambda r|)^2 \sup_h \mathbb{E}|z_h^{m+1}|^2 \le 8 \sup_h \mathbb{E}|\psi_1|^2 \Delta t + 8 \sup_h \mathbb{E}|\psi_2|^2 \Delta t \\
&\quad + \psi_4 \Delta t + 4 \sup_h \mathbb{E}|z_h^m|^2 \left(1 + \sigma^2 \Delta t\right),
\end{aligned} \tag{29}$$

and

$$\sup_h \mathbb{E}|z_h^{m+1}|^2 \leq \psi\Delta t + 4\sup_h \mathbb{E}|z_h^m|^2\left(1+\sigma^2\Delta t\right). \tag{30}$$

It gives that

$$\begin{aligned}\mathbb{E}\|z^{m+1}\|_\infty^2 &\leq \psi\Delta t + (4+4\sigma^2\Delta t)\mathbb{E}\|z^m\|_\infty^2\\ &\leq \psi\Delta t + \left(\frac{\sigma^2 t}{m+1}+1\right)^{m+1}\sum_{k=1}^m (4\psi\Delta t)^k\\ &\leq e^{\sigma^2 t}\sum_{k=1}^m (4\psi\Delta t)^k + \psi\Delta t.\end{aligned} \tag{31}$$

When $\Delta t \to 0$, we have

$$\begin{aligned}\mathbb{E}\|z^{m+1}\|_\infty^2 &\leq 4e^{\sigma^2 t}\psi\Delta t + \psi\Delta t + (m-1)e^{\sigma^2 t}(4\psi\Delta t)^2\\ &\leq te^{\sigma^2 t}\Delta t(4\psi)^2 + (4e^{\sigma^2 t}+1)\psi\Delta t\\ &\leq (16te^{\sigma^2 t}\psi + 4e^{\sigma^2 t}+1)\psi\Delta t,\end{aligned} \tag{32}$$

and so $\mathbb{E}\|z^{m+1}\|_\infty^2 \to 0$.

Here, it is worth mentioning that according to the inequality (32), the error of the proposed scheme (5) is of first order with respect to the time.

3 Numerical Results and Discussion

In this part, we demonstrate the efficacy and accuracy of the suggested technique, developed in the previous section, by solving some numerical examples. Indeed, we investigate the theoretical consequences of previous section about the stability and convergence of the proposed scheme (5). In more detail, we discuss the convergence of the scheme (5) for each example and explore the necessary conditions for its conditional and unconditional stability. Numerical results in this section verify the previously presented theoretical analysis.

Example 3.1. Consider an SPDE in the following form:

$$v_t(x,t) = \rho v_{xx}(x,t) + \sigma v(x,t)\dot{\xi}(t), (x,t) \in (0,1)\times(0,1], \tag{33}$$

supplemented with the initial and boundary conditions

$$\begin{aligned} v(x,0) &= \exp\left(-\frac{(x-0.2)^2}{\rho}\right), \qquad x\in(0,1),\\ v(0,t) &= \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{0.04}{\rho(4t+1)}\right), \qquad t\in(0,1],\\ v(1,t) &= \frac{1}{\sqrt{4t+1}}\exp\left(-\frac{0.64}{\rho(4t+1)}\right), \qquad t\in(0,1].\end{aligned} \tag{34}$$

The exact solution is

$$v(x,t) = \frac{\exp\left(-\frac{(x-0.2)^2}{\rho(1+4t)}\right)}{\sqrt{1+4t}}, \tag{35}$$

if there is no noise term. Following the proposed idea developed in this paper, the stochastic finite difference scheme can be written as follows:

$$-\lambda r v_{h-1}^{m+1} + (1+2\lambda r) v_h^{m+1} - \lambda r v_{h+1}^{m+1} = (1-\lambda) r v_{h-1}^m + (1-2(1-\lambda)r) v_h^m + (1-\lambda) r v_{h+1}^m + \sigma v_h^m \Delta\xi_n, \quad (36)$$

where $r = \frac{\rho\Delta t}{\Delta x^2}$. To qualify the numerical results obtained in this example, the exact and numerical solutions are plotted in Fig. 1. Let M and N be the total numbers of grid points for the space and time discretization, respectively. If we set $\rho = 0.01$, $\sigma = 1$, and $M = 125$ ($\Delta x = 0.008$), then according to Theorem 2.2, the stochastic finite difference scheme (36) is unconditionally stable for all $\lambda \geq \frac{1}{2}$ (see Table 1), and it is conditionally stable for $\lambda < \frac{1}{2}$ with $\frac{1}{2(1-2\lambda)} \geq r$ where $r = \rho\frac{\Delta t}{\Delta x^2}$. To test the conditional stability as well, let $\lambda = 0.01$; then Theorem 2.2 implies that the numerical scheme (36) is stable for $N \geq 303$, a fact which is verified by the reported results in Table 2. Also, the absolute errors of the numerical scheme (36) with $\sigma = 1.5$, $\Delta x = 0.01$, $\rho = 0.001$, $\lambda = 0.25$, and $\Delta t = 0.01, 0.04, 0.05$ are reported in Table 3.

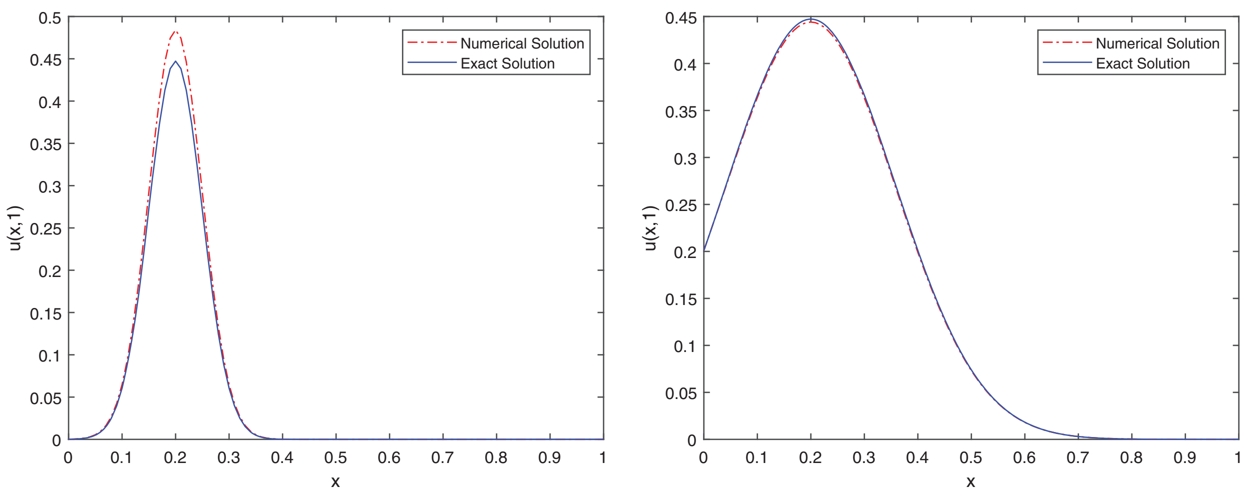

Figure 1: Comparison between the exact solution and the stochastic numerical solution of (33) with $\rho = 0.01$, $\Delta x = 0.008$, $\sigma = 1$, $\Delta t = 0.008$, $\lambda = 0.75$ (right figure), and $\sigma = 1.5$, $\Delta x = 0.01$, $\rho = 0.001$, $\Delta t = 0.01$, $\lambda = 0$:25 (left figure) (Example 3.1)

Table 1: Examination of unconditional stability for the stochastic scheme (36) (Example 3.1)

λ	N	$E(v(0.2, 1))$
0.5	20	0.4658
0.52	25	0.4481
0.6	80	0.4732
0.7	100	0.4552
0.8	125	0.4442

Table 2: Examination of conditional stability for the stochastic scheme (36) (Example 3.1)

N	$E(v(0.2, 1))$
50	6.2493×10^{37}
100	-1.1401×10^{59}
200	-8.2924×10^{52}
303	0.4181
400	0.4460
600	0.4191

Table 3: Absolute errors of the numerical scheme (36) for Example 3.1 with $\sigma = 1.5$, $\Delta x = 0.01$, $\rho = 0.001$, $\Delta t = 0.01, 0.04, 0.05$, and $\lambda = 0.25$

	Δt		
x	0.01	0.04	0.05
0.1	4.3000×10^{-3}	5.8578×10^{-5}	6.0000×10^{-3}
0.2	3.6800×10^{-2}	2.2000×10^{-3}	4.7400×10^{-2}
0.3	4.3000×10^{-3}	5.8578×10^{-5}	6.0000×10^{-3}
0.4	3.4287×10^{-5}	5.0582×10^{-6}	6.1739×10^{-6}
0.5	9.7327×10^{-9}	7.3834×10^{-10}	2.2026×10^{-9}
0.6	6.7303×10^{-14}	1.0652×10^{-15}	1.6669×10^{-15}
0.7	2.3412×10^{-20}	3.8851×10^{-22}	2.1245×10^{-22}
0.8	7.7082×10^{-28}	5.1278×10^{-30}	4.8151×10^{-30}
0.9	3.4080×10^{-36}	1.5900×10^{-37}	2.9264×10^{-38}
1	6.4864×10^{-70}	6.4864×10^{-70}	6.4864×10^{-70}

Example 3.2. Let us consider the following problem for the next example:

$$\begin{aligned} &v_t(x,t) = v(x,t)\dot{\xi}(t) + v_{xx}(x,t), (x,t) \in (0,1) \times (0,1], \\ &v(x,0) = \sin(\pi x), \qquad x \in (0,1), \\ &v(0,t) = v(1,t) = 0, t \in (0,1], \end{aligned} \tag{37}$$

with the exact solution

$$v(x,t) = \sin(\pi x)e^{-\pi^2 t}. \tag{38}$$

In Fig. 2, the exact solution and the stochastic numerical solution of (37) are compared for the two sets of $M = 100, N = 100, \lambda = 0.5$ (left plot) and $M = 120, N = 120, \lambda = 0.55$ (right plot). More comparisons between the exact and the numerical solutions are given in Fig. 3 for the values of $\rho = 1, \sigma = 1, \lambda = 0.55, \Delta x = \frac{1}{120}, \Delta t = 0.01$ (left plot) and $\rho = 1, \sigma = 1, \lambda = 0.5, \Delta x = 0.01, \Delta t = 0.02$ (right plot). From the numerical results in Figs. 2 and 3, one can see the high accuracy of the presented method for solving the SPDE (37). In Table 4, the unconditional stability of the proposed scheme (5) is shown for $\lambda \geq \frac{1}{2}$. For $\lambda = 0.4$ and $M = 100$, Table 5 portrays the conditional stability of the suggested technique (5) when $N \geq 4000$, a fact which is also shown in Fig. 4. In addition, the absolute errors of the numerical scheme (5) with $M = 100$, $N = 100$, $\lambda = 0.5$, $\Delta x = 0.01$, and $\Delta t = 0.01, 0.02, 0.2$ are reported in Table 6.

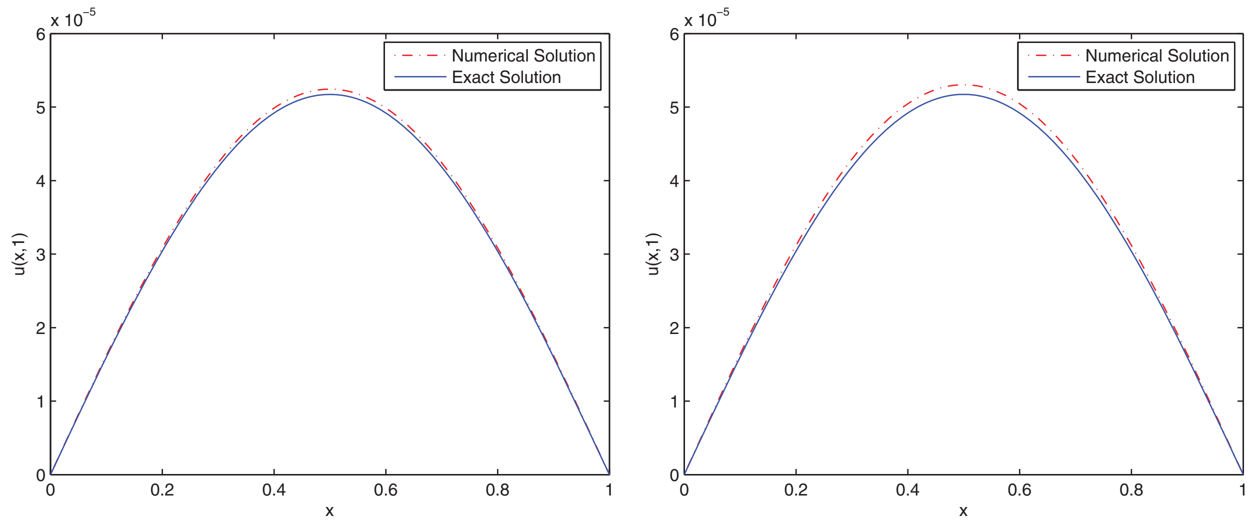

Figure 2: Comparison between the exact solution and the stochastic numerical solution of (37) with $M = 100$, $N = 100$, $\lambda = 0.5$ (left figure), and $M = 120$, $N = 120$, $\lambda = 0.55$ (right figure) (Example 3.2)

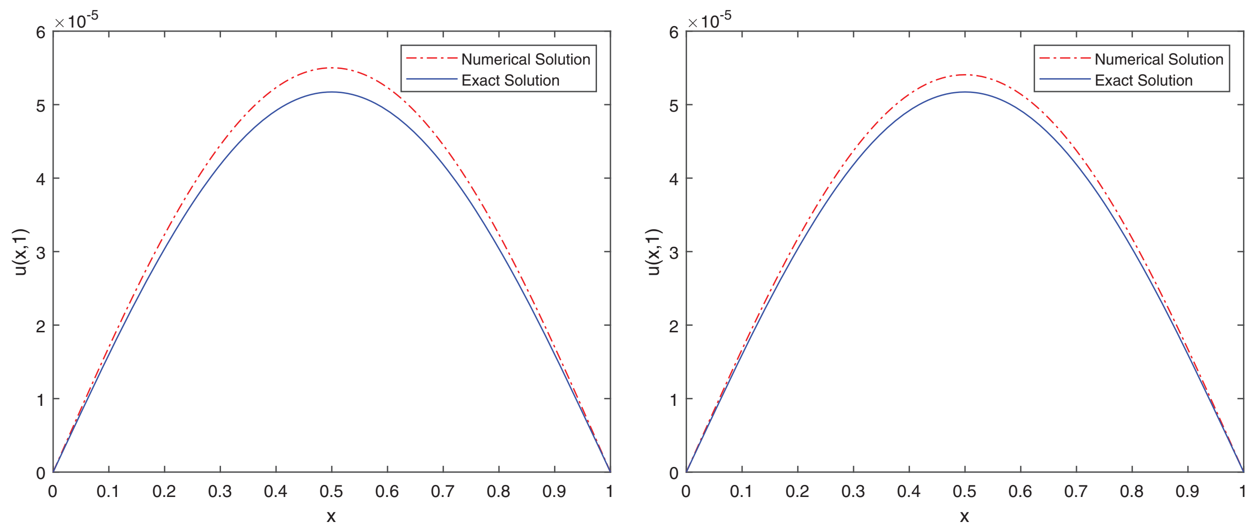

Figure 3: Comparison between the exact solution and the stochastic numerical solution of (37) with $\rho = 1$, $\sigma = 1$, $\lambda = 0.55$, $\Delta x = \frac{1}{120}$, $\Delta t = 0.01$ (left plot), and $\rho = 1$, $\sigma = 1$, $\lambda = 0.5$, $\Delta x = 0.01$, $\Delta t = 0.02$ (rightplot) (Example 3.2)

Table 4: Examination of unconditional stability for the stochastic scheme (5) (Example 3.2)

λ	N	$E(v(0.5,1))$
0.5	50	5.4067×10^{-05}
0.52	80	5.5825×10^{-05}
0.6	100	5.7715×10^{-05}
0.7	120	5.8729×10^{-05}
0.8	200	5.6157×10^{-05}

Table 5: Examination of conditional stability for the stochastic scheme (5) (Example 3.2)

N	$E(v(0.9,1))$
500	5.1365×10^{60}
1000	1.3167×10^{115}
2000	5.2079×10^{157}
4000	1.4083×10^{-05}
4100	1.3973×10^{-05}
4200	1.4054×10^{-05}

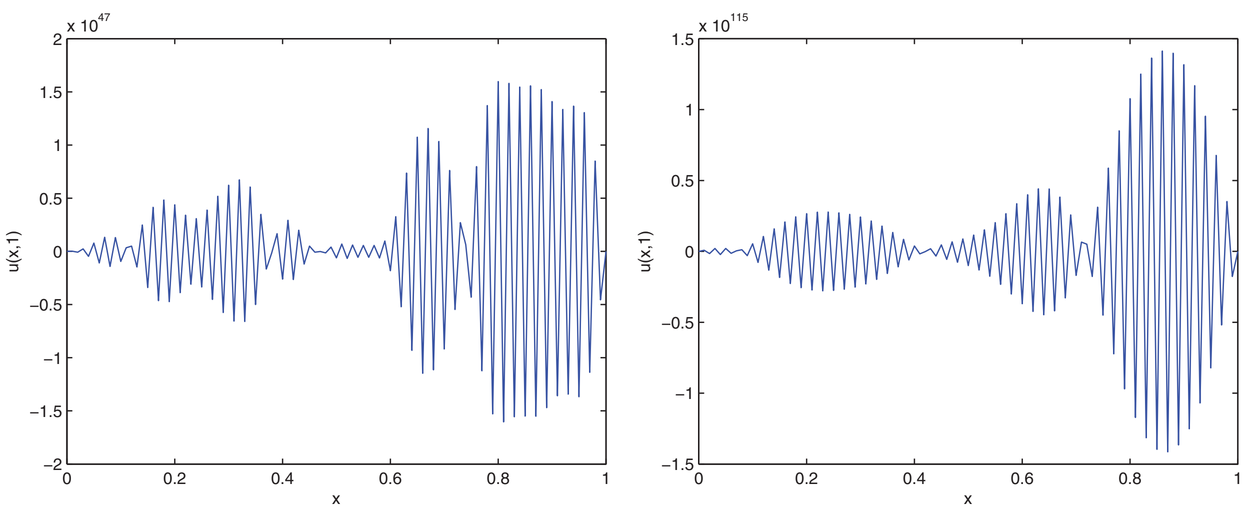

Figure 4: (Continued)

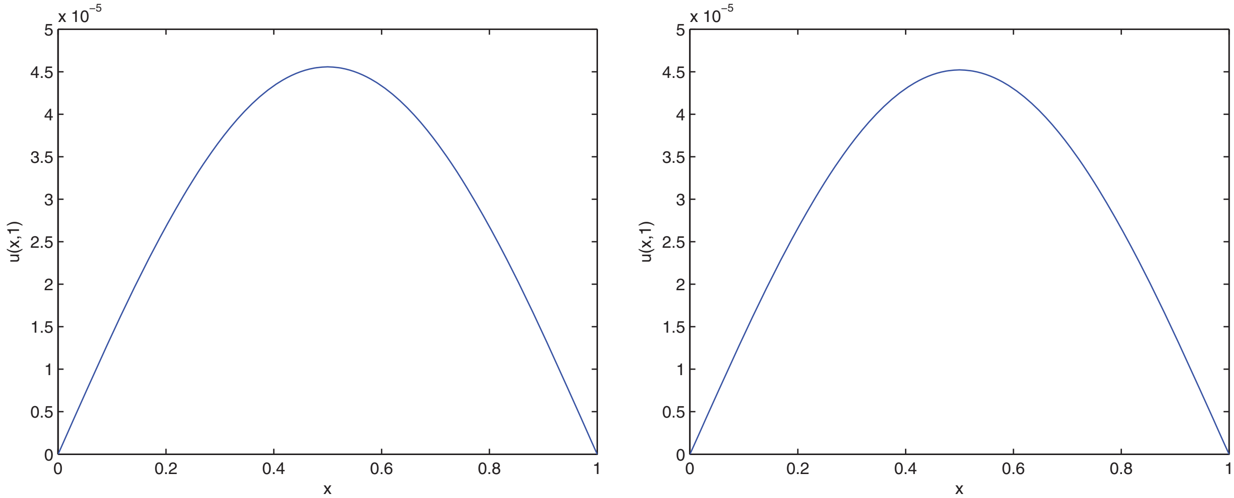

Figure 4: Display of conditional stability for various values of $N = 400, 1000, 4000, 4100$ (Example 3.2)

Table 6: Absolute errors of the numerical scheme (5) for Example 3.2 with $M = 100$, $N = 100$, $\lambda = 0.5$, $\Delta x = 0.01$, and $\Delta t = 0.01, 0.02, 0.2$

	Δt		
x	0.01	0.04	0.05
0.1	2.2049×10^{-7}	7.2438×10^{-7}	1.0274×10^{-6}
0.2	4.1940×10^{-7}	1.3779×10^{-6}	1.9542×10^{-6}
0.3	5.7726×10^{-7}	1.8965×10^{-6}	2.6897×10^{-6}
0.4	6.7861×10^{-7}	2.2294×10^{-6}	3.1619×10^{-6}
0.5	7.1353×10^{-7}	2.3441×10^{-6}	3.3247×10^{-6}
0.6	6.7861×10^{-7}	2.2294×10^{-6}	3.1619×10^{-6}
0.7	5.7726×10^{-7}	1.8965×10^{-6}	2.6897×10^{-6}
0.8	9.1940×10^{-7}	1.3779×10^{-6}	1.9542×10^{-6}
0.9	2.2049×10^{-7}	7.2438×10^{-7}	1.0274×10^{-6}
1	6.3343×10^{-21}	6.3343×10^{-21}	6.3343×10^{-21}

Example 3.3. As the third example, consider the following problem:

$$v_t(x,t) = 0.01 v_{xx}(x,t) + v(x,t)\dot{\xi}(t), (x,t) \in (0,1) \times (0,1],$$
$$v(x,0) = x - x^2, \quad x \in (0,1),$$
$$v(0,t) = v(1,t) = 0, t \in (0,1]. \tag{39}$$

Fig. 5 shows the approximation of SPDE (39) using the stochastic difference scheme (5) with the values $N = 50, 60, 80, 100$. In Table 7, the unconditional stability of the proposed method is depicted for $\lambda \geq \frac{1}{2}$. To test the conditional stability for $\rho = 0.01$, $M = 100$, and $\lambda = 0.3$, simulation results of (39) for various values of N are presented in Fig. 5 and Table 8. According to these results for

Example 3.3, it is apparent that the stochastic finite difference scheme (5) is stable when $N \geq 80$, a fact which coincides with the stability condition provided by Theorem 2.2.

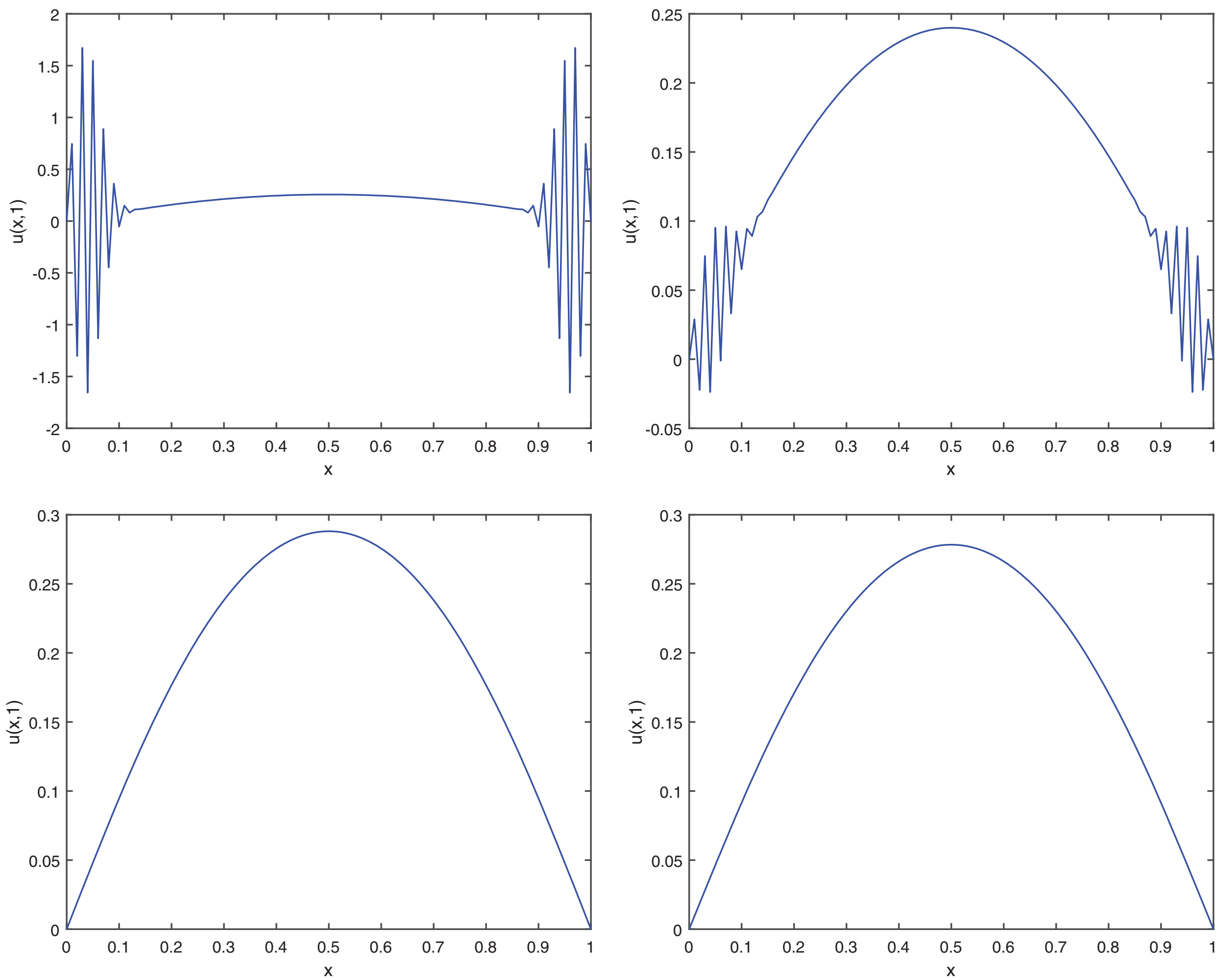

Figure 5: Display of conditional stability for various values of $N = 50$, 60, 80, 100 (Example 3.3)

Table 7: Examination of unconditional stability for the stochastic scheme (5) (Example 3.3)

λ	N	$E(v(0.5, 1))$
0.5	50	0.2566
0.52	80	0.2880
0.6	100	0.2783
0.7	110	0.2429
0.8	150	0.2263

Table 8: Examination of conditional stability for the stochastic scheme (5) (Example 3.3)

N	$E(v(0.03, 2))$
20	0.3828
50	1.6972
80	0.0245
110	0.0233
140	0.0234
170	0.0226

4 Conclusion

This study presented a numerical method based on the weighted average finite difference scheme for the solution of SPDEs. In this paper, we provided some mathematical analyses for the proposed numerical scheme. To ascertain the accuracy and efficacy of the proffered technique, we presented three numerical examples with different boundary conditions, and compared the associated numerical results with the exact solution. Additionally, we explored the necessary conditions for the conditional and unconditional stability of the presented method and verified the theoretical consequences in this regard by some figures and tables.

Future works can be focused on applying some new discrete schemes, such as those discussed in [22] with a second-order time convergence rate, for the numerical solution of stochastic problem studied in this paper.

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