



Research article

An efficient method for 3D Helmholtz equation with complex solution

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Abstract: The Helmholtz equation as an elliptic partial differential equation possesses many applications in the time-harmonic wave propagation phenomena, such as the acoustic cavity and radiation wave. In this paper, we establish a numerical method based on the orthonormal shifted discrete Chebyshev polynomials for finding complex solution of this equation. The presented method transforms the Helmholtz equation into an algebraic system of equations that can be easily solved. Four practical examples are examined to show the accuracy of the proposed technique.

Keywords: 3D Helmholtz equation; orthonormal shifted discrete Chebyshev polynomials; complex solution

Mathematics Subject Classification: 32W50, 65M70

1. Introduction

This paper is devoted to establishing a highly accurate method for solving the Helmholtz equation that has many applications in time-harmonic wave propagation phenomena, such as the acoustic cavity and radiation wave [1]. Some of the numerical methods that have been used to solve the Helmholtz equation in recent years are: Legendre wavelet method [2], least-squares finite element method [3], meshless Chebyshev collocation method [4], implicit finite difference method [5], staggered discontinuous Galerkin method [6], Fourier-Bessel method [7], radial basis function-generated finite difference scheme [8] and hybrid approach proposed in [9].

Nowadays, researchers use orthogonal polynomials instead of the Fourier functions to solve various classes of equations numerically, because these types of basis functions have spectral accuracy property and their computational volume is less than other basis functions. For instance, see [10–14]. Orthogonal polynomials are defined into two categories based on the definition of their internal

product [15], discrete and continuous. In numerical algorithms, the expansion coefficients for orthogonal continuous polynomials are achieved by integrating, but the expansion coefficients for orthogonal discrete polynomials are got using a finite summation. In recent years, researchers have been successfully used discrete polynomials to provide numerical methods for solving ordinary and fractional differential equations. Some differential equations that are solved using discrete polynomials are: variable-order fractional extended Fisher-Kolmogorov equation [16], time-delay fractional optimal control problems [17], fractional Volterra partial integro-differential equations [18], fractional viscoelastic model [19], fractional reaction-advection-diffusion equations [20], etc..

In this paper, we utilize the orthonormal shifted discrete Chebyshev polynomials (OSDCPs) to obtain the numerical solution of the 3D Helmholtz equation.

The main objectives of this study are briefly expressed in the following:

- Investigating the 1D and 2D OSDCPs and their derivative matrices.
- Establishing a computational method based on the 1D and 2D OSDCPs to obtain the numerical solution of the 3D Helmholtz equation.

To solve the intended equation, we first approximate the unknown solution in the main problem by the 1D and 2D OSDCPs. Next, using these polynomials, we expand the second-order derivatives. So, the derivative matrices of the second order are obtained. In the following, we replace approximations obtained in the previous steps into the main problem. By this replacing and the collocation technique, the primary problem is turned into an algebraic equations system. Some of the most important advantages of the established method are listed in the following:

- The proposed method has spectral accuracy and low computational volume.
- A small number of the OSDCPs are required to obtain highly accurate results.
- The developed method transforms the solution of the main problem into the solution of a linear system of algebraic equations, which is easily solvable.

In the sequel, we provide five sections as follows: the 1D and 2D OSDCPs and some of their properties are provided in Section 2. A numerical method is adopted by applying the 2D OSDCPs in Section 3. Section 4 provides some numerical examples for investigating the accuracy of the proposed method. Section 5 explains the conclusion of the article.

2. The Helmholtz equation and discrete Chebyshev polynomials

Here, we review the 3D Helmholtz equation and introduce the 1D and 2D OSDCPs together with some of their attributes.

2.1. The 3D Helmholtz equation

In this work, we focus on the 3D Helmholtz equation

$$\nabla^2 \vartheta(\xi, \gamma, z) + \eta^2 \vartheta(\xi, \gamma, z) = f(\xi, \gamma, z), \quad (\xi, \gamma, z) \in [0, 1]^3, \quad (2.1)$$

with the boundary conditions

$$\left\{ \begin{array}{l} \vartheta(0, \gamma, z) = h_1(\gamma, z), \quad (\gamma, z) \in [0, 1]^2, \\ \vartheta(1, \gamma, z) = h_2(\gamma, z), \quad (\gamma, z) \in [0, 1]^2, \\ \vartheta(\xi, 0, z) = h_3(\xi, z), \quad (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, 1, z) = h_4(\xi, z), \quad (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 0) = h_5(\xi, \gamma), \quad (\xi, \gamma) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 1) = h_6(\xi, \gamma), \quad (\xi, \gamma) \in [0, 1]^2, \end{array} \right. \quad (2.2)$$

or

$$\left\{ \begin{array}{l} \vartheta(0, \gamma, z) = h_1(\gamma, z), \quad (\gamma, z) \in [0, 1]^2, \\ \frac{\partial \vartheta(1, \gamma, z)}{\partial \xi} = iL_1 \vartheta(1, \gamma, z), \quad (\gamma, z) \in [0, 1]^2, \\ \vartheta(\xi, 0, z) = h_3(\xi, z), \quad (\xi, z) \in [0, 1]^2, \\ \frac{\partial \vartheta(\xi, 1, z)}{\partial \gamma} = iL_2 \vartheta(\xi, 1, z), \quad (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 0) = h_5(\xi, \gamma), \quad (\xi, \gamma) \in [0, 1]^2, \\ \frac{\partial \vartheta(\xi, \gamma, 1)}{\partial z} = iL_3 \vartheta(\xi, \gamma, 1), \quad (\xi, \gamma) \in [0, 1]^2, \end{array} \right. = \left\{ \begin{array}{l} \vartheta(0, \gamma, z) = h_1(\gamma, z), \quad (\gamma, z) \in [0, 1]^2, \\ \vartheta(1, \gamma, z) = \underbrace{\frac{-i \partial \vartheta(1, \gamma, z)}{L_1 \partial \xi}}_{h_2(\gamma, z)}, \quad (\gamma, z) \in [0, 1]^2, \\ \vartheta(\xi, 0, z) = h_3(\xi, z), \quad (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, 1, z) = \underbrace{\frac{-i \partial \vartheta(\xi, 1, z)}{L_2 \partial \gamma}}_{h_4(\xi, z)}, \quad (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 0) = h_5(\xi, \gamma), \quad (\xi, \gamma) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 1) = \underbrace{\frac{-i \partial \vartheta(\xi, \gamma, 1)}{L_3 \partial z}}_{h_6(\xi, \gamma)}, \quad (\xi, \gamma) \in [0, 1]^2, \end{array} \right. \quad (2.3)$$

where η is the wave number, f and h_ζ for $\zeta = 1, 2, \dots, 6$ are given complex functions, L_1 – L_3 are real constants. Also, ϑ is an unknown complex function.

2.2. The 1D basis polynomials

The 1D OSDCPs are defined over $[0, z_f]$ in the following form [21]:

$$\mathcal{CP}_{z_f, \ell}(z, \tilde{M}) = c(\ell, \tilde{M}) \sum_{\kappa=0}^{\ell} \sum_{r=0}^{\kappa} (-1)^\kappa \frac{\binom{\ell + \kappa}{\ell} \binom{\tilde{M} - \kappa}{\ell - \kappa}}{\kappa!} \left(\frac{\tilde{M}}{z_f}\right)^r S_\kappa^{(r)} z^r, \quad \ell = 0, 1, \dots, \tilde{M}, \quad (2.4)$$

where $S_\kappa^{(r)}$ is the first kind Stirling numbers and

$$c(\ell, \tilde{M}) = \sqrt{\frac{(2\ell + 1)(\tilde{M} - \ell)!(\ell!)^2}{(\tilde{M} + \ell + 1)!}}. \quad (2.5)$$

The set $\{\mathcal{CP}_{z_f, \ell}(z, \tilde{M})\}_{\ell=0}^{\tilde{M}}$ produces an orthogonal set over $[0, z_f]$ based on the following internal product definition:

$$\langle \varphi, \psi \rangle = \sum_{i=0}^{\tilde{M}} \varphi\left(\frac{z_f}{\tilde{M}}i\right) \psi\left(\frac{z_f}{\tilde{M}}i\right). \quad (2.6)$$

So, we have

$$\langle \mathcal{CP}_{z_f, \ell}(z, \tilde{M}), \mathcal{CP}_{z_f, \kappa}(z, \tilde{M}) \rangle = \begin{cases} 0, & \ell \neq \kappa, \\ 1, & \ell = \kappa. \end{cases} \quad (2.7)$$

Any function $\vartheta(z) \in [0, z_f]$ can be expanded by the 1D OSDCPs as follows:

$$w(z) \simeq \sum_{i=0}^{\tilde{M}} w_i \mathcal{CP}_{z_f, i}(z; \tilde{M}) \triangleq \mathbf{W}^T \mathbb{CP}_{z_f, \tilde{M}}(z; \tilde{M}), \quad (2.8)$$

where

$$\mathbf{W} = [w_0 \ w_1 \ \dots \ w_{\tilde{M}}]^T,$$

in which

$$w_i = \langle w(z), \mathcal{CP}_{z_f, i}(z; \tilde{M}) \rangle = \sum_{r=0}^{\tilde{M}} \omega\left(\frac{z_f}{\tilde{M}}r\right) \mathcal{CP}_{z_f, i}\left(\frac{z_f}{\tilde{M}}r; \tilde{M}\right), \quad (2.9)$$

and

$$\mathbb{CP}_{z_f, \tilde{M}}(z) = [\mathcal{CP}_{z_f, 0}(z; \tilde{M}) \ \mathcal{CP}_{z_f, 1}(z; \tilde{M}) \ \dots \ \mathcal{CP}_{z_f, \tilde{M}}(z; \tilde{M})]^T. \quad (2.10)$$

Theorem 2.1. [21] Let $\mathbb{CP}_{z_f, \tilde{M}}(z)$ is the vector introduced in (2.10). Then, the ρ th derivative of this vector can be gotten as

$$\frac{d^\rho \mathbb{CP}_{z_f, \tilde{M}}(z)}{dz^\rho} = D_z^{(\rho, z_f)} \mathbb{CP}_{z_f, \tilde{M}}(z), \quad (2.11)$$

where components of the square matrix $D_z^{(\rho, z_f)} = [d_{ij}^{(\rho, z_f)}]$ are calculated by the following formula:

$$d_{ij}^{(\rho, z_f)} = \begin{cases} c(i-1, \tilde{M}) \sum_{\ell=0}^{\tilde{M}} \left(\sum_{\kappa=\rho}^{i-1} \sum_{r=\rho}^{\kappa} (-1)^\kappa \frac{\binom{i+\kappa-1}{i-1} \binom{\tilde{M}-\kappa}{i-\kappa-1}}{\kappa!} \right) \\ \times \left(\frac{\tilde{M}}{z_f} \right)^\rho S_\kappa^{(r)} r(r-1) \dots (r-\rho+1) \ell^{r-\rho} \mathcal{CP}_{z_f, j-1}\left(\frac{z_f}{\tilde{M}}\ell; \tilde{M}\right), & \rho+1 \leq i \leq \tilde{M}+1, \\ & 1 \leq j \leq i-\rho, \\ 0, & \text{otherwise.} \end{cases} \quad (2.12)$$

The 2D OSDCPs are constructed over $[0, 1] \times [0, 1]$ by the 1D OSDCPs as

$$\mathcal{CP}_{\kappa\ell}(\xi, \gamma; \mathbb{N}_1, \mathbb{N}_2) = \mathcal{CP}_\kappa(\xi; \mathbb{N}_1) \mathcal{CP}_\ell(\gamma; \mathbb{N}_2), \quad \kappa = 0, 1, \dots, \mathbb{N}_1, \ell = 0, 1, \dots, \mathbb{N}_2. \quad (2.13)$$

Utilizing the 2D OSDCPs, any continuous function $\vartheta(\xi, \gamma)$ can be expanded over $[0, 1] \times [0, 1]$ in the following form:

$$\vartheta(\xi, \gamma) \simeq \sum_{\kappa=0}^{\mathbb{N}_1} \sum_{\ell=0}^{\mathbb{N}_2} \lambda_{\kappa\ell} \mathcal{CP}_{\kappa\ell}(\xi, \gamma; \mathbb{N}_1, \mathbb{N}_2) \triangleq \Lambda^T \mathbb{CP}_{\mathbb{N}_1, \mathbb{N}_2}(\xi, \gamma) = \mathbb{CP}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Lambda, \quad (2.14)$$

where

$$\Lambda = \left[\lambda_{00} \ \lambda_{01} \ \dots \ \lambda_{0\mathbb{N}_2} \mid \lambda_{10} \ \lambda_{11} \ \dots \ \lambda_{1\mathbb{N}_2} \mid \dots \mid \lambda_{\mathbb{N}_1, 0} \ \lambda_{\mathbb{N}_1, 1} \ \dots \ \lambda_{\mathbb{N}_1, \mathbb{N}_2} \right]^T, \quad (2.15)$$

with

$$\lambda_{(\kappa-1)(\ell-1)} = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \vartheta \left(\frac{i}{N_1}, \frac{j}{N_2} \right) \mathcal{CP}_{(\kappa-1)(\ell-1)} \left(\frac{i}{N_1}, \frac{j}{N_2}; N_1, N_2 \right), \quad (2.16)$$

$$\kappa = 1, 2, \dots, N_1 + 1, \ell = 1, 2, \dots, N_2 + 1,$$

and

$$\mathbb{CP}_{N_1, N_2}(\xi, \gamma) = \left[\mathcal{CP}_{00}(\xi, \gamma) \dots \mathcal{CP}_{0N_2}(\xi, \gamma) | \mathcal{CP}_{10}(\xi, \gamma) \dots \mathcal{CP}_{1N_2}(\xi, \gamma) | \dots | \right. \\ \left. \mathcal{CP}_{N_1,0}(\xi, \gamma) \dots \mathcal{CP}_{N_1, N_2}(\xi, \gamma) \right]^T. \quad (2.17)$$

Theorem 2.2. Suppose that $\mathbb{CP}_{N_1, N_2}(\xi, \gamma)$ is the 2D OSDCPs vector demonstrated in (2.17). Then, the ρ th derivative of this vector can be calculated as follow:

$$\frac{\partial^\rho \mathbb{CP}_{N_1, N_2}(\xi, \gamma)}{\partial \xi^\rho} = D_\xi^{(\rho)} \mathbb{CP}_{N_1, N_2}(\xi, \gamma), \quad (2.18)$$

where

$$D_\xi^{(\rho)} = \begin{pmatrix} \check{d}_{11}^{(\rho)} I & \check{d}_{12}^{(\rho)} I & \dots & \check{d}_{1(N_1+1)}^{(\rho)} I \\ \check{d}_{21}^{(\rho)} I & \check{d}_{22}^{(\rho)} I & \dots & \check{d}_{2(N_1+1)}^{(\rho)} I \\ \vdots & \vdots & \ddots & \vdots \\ \check{d}_{(N_1+1)1}^{(\rho)} I & \check{d}_{(N_1+1)2}^{(\rho)} I & \dots & \check{d}_{(N_1+1)(N_1+1)}^{(\rho)} I \end{pmatrix},$$

in which $I_{(N_2+1) \times (N_2+1)} = \text{diag}(1, 1, \dots, 1)$ and

$$\check{d}_{ij}^{(\rho)} = \begin{cases} c(i-1, N_1) \sum_{\ell=0}^{N_1} \left(\sum_{\kappa=\rho}^{i-1} \sum_{r=\rho}^{\kappa} (-1)^\kappa \frac{\binom{i+\kappa-1}{i-1} \binom{N_1-\kappa}{i-\kappa-1}}{\kappa!} \right) \\ N_1 S_\kappa^{(r)} r(r-1) \dots (r-\rho+1) \ell^{r-\rho} \mathcal{CP}_{j-1} \left(\frac{\ell}{N_1}; N_1 \right), & \rho+1 \leq i \leq N_1+1, \\ & 1 \leq j \leq i-\rho, \\ 0, & \text{otherwise.} \end{cases} \quad (2.19)$$

Proof. According to (2.13), the ρ th derivative of the vector $\mathcal{CP}_{ij}(\xi, \gamma; N_1, N_2)$ ($i = 1, 2, \dots, N_1$, $j=1, 2, \dots, N_2$) can be written in the following form:

$$\frac{\partial^\rho}{\partial \xi^\rho} \mathcal{CP}_{ij}(\xi, \gamma; N_1, N_2) = \frac{\partial^\rho}{\partial \xi^\rho} \mathcal{CP}_i(\xi; N_1) \mathcal{CP}_j(\gamma; N_2) = \mathcal{CP}_j(\gamma; N_2) \frac{d^\rho}{d \xi^\rho} \mathcal{CP}_i(\xi; N_1). \quad (2.20)$$

By computing $\frac{d^p}{d\xi^\rho} \mathcal{CP}_i(\xi; N_1)$ ($i = 1, 2, \dots, N_1$) and substituting in the above relation, we get

$$\begin{aligned} \frac{\partial^\rho \mathbb{CP}_{N_1, N_2}(\xi, \gamma)}{\partial \xi^\rho} = & \left[\mathcal{CP}_1(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_1(\xi; N_1) \quad \mathcal{CP}_2(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_1(\xi; N_1) \dots \right. \\ & \mathcal{CP}_{N_2+1}(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_1(\xi; N_1) | \mathcal{CP}_1(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_2(\xi; N_1) \quad \mathcal{CP}_2(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_2(\xi; N_1) \\ & \dots \mathcal{CP}_{N_2+1}(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_2(\xi; N_1) | \dots | \mathcal{CP}_1(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_{N_1+1}(\xi; N_1) \\ & \left. \mathcal{CP}_2(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_{N_1+1}(\xi; N_1) \dots \mathcal{CP}_{N_2+1}(\gamma; N_2) \frac{d^p}{d\xi^\rho} \mathcal{CP}_{N_1+1}(\xi; N_1) \right]^T. \end{aligned}$$

Then, by applying Theorem 2.1, we have

$$\frac{\partial^\rho \mathbb{CP}_{N_1, N_2}(\xi, \gamma)}{\partial \xi^\rho} = D_\xi^{(\rho)} \mathbb{CP}_{N_1, N_2}(\xi, \gamma),$$

where

$$D_\xi^{(\rho)} = \begin{pmatrix} \check{d}_{11}^{(\rho)} & 0 & \dots & 0 & \check{d}_{12}^{(\rho)} & 0 & \dots & 0 & \dots & \check{d}_{1(N_1+1)}^{(\rho)} & 0 & \dots & 0 \\ 0 & \check{d}_{11}^{(\rho)} & \dots & 0 & 0 & \check{d}_{12}^{(\rho)} & \dots & 0 & \dots & 0 & \check{d}_{1(N_1+1)}^{(\rho)} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \check{d}_{11}^{(\rho)} & 0 & 0 & \dots & \check{d}_{12}^{(\rho)} & \dots & 0 & 0 & \dots & \check{d}_{1(N_1+1)}^{(\rho)} \\ \check{d}_{21}^{(\rho)} & 0 & \dots & 0 & \check{d}_{22}^{(\rho)} & 0 & \dots & 0 & \dots & \check{d}_{2(N_1+1)}^{(\rho)} & 0 & \dots & 0 \\ 0 & \check{d}_{21}^{(\rho)} & \dots & 0 & 0 & \check{d}_{22}^{(\rho)} & \dots & 0 & \dots & 0 & \check{d}_{2(N_1+1)}^{(\rho)} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \check{d}_{21}^{(\rho)} & 0 & 0 & \dots & \check{d}_{22}^{(\rho)} & \dots & 0 & 0 & \dots & \check{d}_{2(N_1+1)}^{(\rho)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \check{d}_{(N_1+1)1}^{(\rho)} & 0 & \dots & 0 & \check{d}_{(N_1+1)2}^{(\rho)} & 0 & \dots & 0 & \dots & \check{d}_{(N_1+1)(N_1+1)}^{(\rho)} & 0 & \dots & 0 \\ 0 & \check{d}_{(N_1+1)1}^{(\rho)} & \dots & 0 & 0 & \check{d}_{(N_1+1)2}^{(\rho)} & \dots & 0 & \dots & 0 & \check{d}_{(N_1+1)(N_1+1)}^{(\rho)} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \check{d}_{(N_1+1)1}^{(\rho)} & 0 & 0 & \dots & \check{d}_{(N_1+1)2}^{(\rho)} & \dots & 0 & 0 & \dots & \check{d}_{(N_1+1)(N_1+1)}^{(\rho)} \end{pmatrix},$$

or equivalently

$$D_\xi^{(\rho)} = \begin{pmatrix} \check{d}_{11}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} & \check{d}_{12}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} & \dots & \check{d}_{1(N_1+1)}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} \\ \check{d}_{21}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} & \check{d}_{22}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} & \dots & \check{d}_{2(N_1+1)}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \check{d}_{(N_1+1)1}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} & \check{d}_{(N_1+1)2}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} & \dots & \check{d}_{(N_1+1)(N_1+1)}^{(\rho)} I_{\frac{1}{2}(N_2+1)(N_2+1)} \end{pmatrix}.$$

□

As a numerical example, we want to compute matrix $D_\xi^{(1)}$ expressed in the Theorem 2.2. By

assuming that $N_1 = N_2 = 3$, we have

$$\left[\hat{d}_{ij}^{(1)} \right] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 \\ 0 & \frac{-15}{\sqrt{5}} & 0 & 0 \\ \frac{-17\sqrt{5}}{5} & 0 & -6\sqrt{5} & 0 \end{pmatrix},$$

and

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we obtain

$$\mathbb{D}_\xi^{(1)} = \begin{pmatrix} 0 \times \mathbb{I} & 0 \times \mathbb{I} & 0 \times \mathbb{I} & 0 \times \mathbb{I} \\ \frac{-6\sqrt{5}}{5} \times \mathbb{I} & 0 \times \mathbb{I} & 0 \times \mathbb{I} & 0 \times \mathbb{I} \\ 0 \times \mathbb{I} & \frac{-15}{\sqrt{5}} \times \mathbb{I} & 0 \times \mathbb{I} & 0 \times \mathbb{I} \\ \frac{-17\sqrt{5}}{5} \times \mathbb{I} & 0 \times \mathbb{I} & -6\sqrt{5} \times \mathbb{I} & 0 \times \mathbb{I} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{-17\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-17\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-17\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-17\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, the matrix $D_{\xi}^{(2)}$ is computed as

$$D_{\xi}^{(2)} = \begin{pmatrix} 0 \times I & 0 \times I & 0 \times I & 0 \times I \\ 0 \times I & 0 \times I & 0 \times I & 0 \times I \\ 18 \times I & 0 \times I & 0 \times I & 0 \times I \\ 0 \times I & 90 \times I & 0 \times I & 0 \times I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 90 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.3. Suppose that $\mathbb{CP}_{N_1, N_2}(\xi, \gamma)$ is the 2D OSDCPs vector demonstrated in (2.17). Then, the ρ th derivative of this vector can be calculated as follow:

$$\frac{\partial^\rho \mathbb{CP}_{N_1, N_2}(\xi, \gamma)}{\partial \gamma^\rho} = D_{\gamma}^{(\rho)} \mathbb{CP}_{N_1, N_2}(\xi, \gamma), \quad (2.21)$$

where

$$D_{\gamma}^{(\rho)} = \begin{pmatrix} \mathbb{D}^{(\rho)} & Q & \dots & Q \\ Q & \mathbb{D}^{(\rho)} & \dots & Q \\ \vdots & \vdots & \ddots & \vdots \\ Q & Q & \dots & \mathbb{D}^{(\rho)} \end{pmatrix},$$

such that Q demonstrates the zero matrix of order $(N_2 + 1) \times (N_2 + 1)$ and

$$\dot{d}_{ij}^{(\rho)} = \begin{cases} c(i-1, N_2) \sum_{\ell=0}^{N_2} \left(\sum_{\kappa=\rho}^{i-1} \sum_{r=\rho}^{\kappa} (-1)^\kappa \frac{\binom{i+\kappa-1}{i-1} \binom{N_2-\kappa}{i-\kappa-1}}{\kappa!} \right) \\ \times N_2^\rho S_\kappa^{(r)} r(r-1)\dots(r-\rho+1) \ell^{r-\rho} \mathbb{CP}_{j-1} \left(\frac{\ell}{N_2}; N_2 \right), & \rho+1 \leq i \leq N_2+1, \\ & 1 \leq j \leq i-\rho, \\ 0, & \text{otherwise.} \end{cases} \quad (2.22)$$

Proof. According to (2.13), the ρ th derivative of the vector $\mathcal{CP}_{ij}(\xi, \gamma; \mathbb{N}_1, \mathbb{N}_2)$ ($i = 1, 2, \dots, \mathbb{N}_1, j=1, 2, \dots, \mathbb{N}_2$) can be written in the following form:

$$\frac{\partial^\rho}{\partial \gamma^\rho} \mathcal{CP}_{ij}(\xi, \gamma; \mathbb{N}_1, \mathbb{N}_2) = \mathcal{CP}_i(\xi; \mathbb{N}_1) \frac{\partial^\rho}{\partial \gamma^\rho} \mathcal{CP}_j(\gamma; \mathbb{N}_2) = \mathcal{CP}_i(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_j(\gamma; \mathbb{N}_2). \tag{2.23}$$

By computing $\frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_j(\gamma; \mathbb{N}_2)$ ($j = 1, 2, \dots, \mathbb{N}_2$) and substituting in the above relation, we get

$$\begin{aligned} \frac{\partial^\rho \mathcal{CP}_{\mathbb{N}_1, \mathbb{N}_2}(\xi, \gamma)}{\partial \gamma^\rho} &= \left[\mathcal{CP}_1(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_1(\gamma; \mathbb{N}_2) \quad \mathcal{CP}_1(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_2(\gamma; \mathbb{N}_2) \dots \right. \\ &\quad \mathcal{CP}_1(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_{\mathbb{N}_2+1}(\gamma; \mathbb{N}_2) | \mathcal{CP}_2(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_1(\gamma; \mathbb{N}_2) \quad \mathcal{CP}_2(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_2(\gamma; \mathbb{N}_2) \\ &\quad \dots | \mathcal{CP}_2(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_{\mathbb{N}_2+1}(\gamma; \mathbb{N}_2) | \dots | \mathcal{CP}_{\mathbb{N}_1+1}(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_1(\gamma; \mathbb{N}_2) \\ &\quad \left. \mathcal{CP}_{\mathbb{N}_1+1}(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_2(\gamma; \mathbb{N}_2) \dots \mathcal{CP}_{\mathbb{N}_1+1}(\xi; \mathbb{N}_1) \frac{d^\rho}{d\gamma^\rho} \mathcal{CP}_{\mathbb{N}_2+1}(\gamma; \mathbb{N}_2) \right]^T. \end{aligned}$$

Then, by applying Theorem 2.1, we have

$$\frac{\partial^\rho \mathcal{CP}_{\mathbb{N}_1, \mathbb{N}_2}(\xi, \gamma)}{\partial \gamma^\rho} = \mathbb{D}_\gamma^{(\rho)} \mathcal{CP}_{\mathbb{N}_1, \mathbb{N}_2}(\xi, \gamma),$$

where

$$\mathbb{D}_\gamma^{(\rho)} = \begin{pmatrix} \check{q}_{11}^{(\rho)} & \check{q}_{12}^{(\rho)} & \dots & \check{q}_{1(\mathbb{N}_2+1)}^{(\rho)} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \check{q}_{21}^{(\rho)} & \check{q}_{22}^{(\rho)} & \dots & \check{q}_{2(\mathbb{N}_2+1)}^{(\rho)} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \check{q}_{(\mathbb{N}_2+1)1}^{(\rho)} & \check{q}_{(\mathbb{N}_2+1)2}^{(\rho)} & \dots & \check{q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)}^{(\rho)} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \check{q}_{11}^{(\rho)} & d_{12} & \dots & \check{q}_{1(\mathbb{N}_2+1)}^{(\rho)} & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \check{q}_{21}^{(\rho)} & d_{22} & \dots & \check{q}_{2(\mathbb{N}_2+1)}^{(\rho)} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \check{q}_{(\mathbb{N}_2+1)1}^{(\rho)} & d_{(\mathbb{N}_2+1)2} & \dots & \check{q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)}^{(\rho)} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \check{q}_{11}^{(\rho)} & \check{q}_{12}^{(\rho)} & \dots & \check{q}_{1(\mathbb{N}_2+1)}^{(\rho)} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \check{q}_{21}^{(\rho)} & \check{q}_{22}^{(\rho)} & \dots & \check{q}_{2(\mathbb{N}_2+1)}^{(\rho)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \check{q}_{(\mathbb{N}_2+1)1}^{(\rho)} & d_{(\mathbb{N}_2+1)2} & \dots & \check{q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)}^{(\rho)} \end{pmatrix},$$

or equivalently

$$\mathbb{D}_\gamma^{(\rho)} = \begin{pmatrix} \mathbb{D}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} & \mathbb{Q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} & \dots & \mathbb{Q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} \\ \mathbb{Q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} & \mathbb{D}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} & \dots & \mathbb{Q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{Q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} & \mathbb{Q}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} & \dots & \mathbb{D}_{(\mathbb{N}_2+1)(\mathbb{N}_2+1)} \end{pmatrix}.$$

□

As a numerical example, for $(N_1 = N_2 = 3)$, we get

$$D_\gamma^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-17\sqrt{5}}{5} & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-17\sqrt{5}}{5} & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-17\sqrt{5}}{5} & 0 & -6\sqrt{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-6\sqrt{5}}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-15}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-17\sqrt{5}}{5} & 0 & -6\sqrt{5} & 0 \end{pmatrix}.$$

3. Numerical scheme

In this section, a numerical scheme is provided for the 3D Helmholtz Eq (2.1) by utilizing the OSDCPs. To this end, we represent the complex function $\vartheta(\xi, \gamma, z)$ as follow:

$$\vartheta(\xi, \gamma, z) = \omega(\xi, \gamma, z) + i\nu(\xi, \gamma, z), \quad (3.1)$$

where ω and ν are real functions. Moreover, we consider the complex functions $f(\xi, \gamma, z)$, $g(\xi, \gamma)$ and $h_\ell(.,.)$ ($\ell = 1, 2, 3, 4$) in the following forms:

$$\left\{ \begin{array}{l} f(\xi, \gamma, z) = f_1(\xi, \gamma, z) + if_2(\xi, \gamma, z), \\ h_1(\gamma, z) = h_{11}(\gamma, z) + ih_{12}(\gamma, z), \\ h_2(\gamma, z) = h_{21}(\gamma, z) + ih_{22}(\gamma, z), \\ h_3(\xi, z) = h_{31}(\xi, z) + ih_{32}(\xi, z), \\ h_4(\xi, z) = h_{41}(\xi, z) + ih_{42}(\xi, z), \\ h_5(\xi, \gamma) = h_{51}(\xi, \gamma) + ih_{52}(\xi, \gamma), \\ h_6(\xi, \gamma) = h_{61}(\xi, \gamma) + ih_{62}(\xi, \gamma), \end{array} \right. \quad (3.2)$$

where $f_k(.,.,.)$, and $h_{lk}(.,.)$ for $l = 1, 2, 3, 4, 5, 6$, $k = 1, 2$ are given real functions. By putting (3.1) into (2.1), we obtain

$$\begin{cases} \Delta\omega(\xi, \gamma, z) + \eta^2\omega(\xi, \gamma, z) = f_1(\xi, \gamma, z), & (\xi, \gamma, z) \in [0, 1]^3, \\ \Delta\nu(\xi, \gamma, z) + \eta^2\nu(\xi, \gamma, z) = f_2(\xi, \gamma, z), & (\xi, \gamma, z) \in [0, 1]^3. \end{cases} \quad (3.3)$$

Also, by replacing (3.2) into (2.2) or (2.3), the boundary conditions are obtained as follows:

$$\begin{cases} \omega(0, \gamma, z) = h_{11}(\gamma, z), & \nu(0, \gamma, z) = h_{12}(\gamma, z), \\ \omega(1, \gamma, z) = h_{21}(\gamma, z), & \nu(1, \gamma, z) = h_{22}(\gamma, z), \\ \omega(\xi, 0, z) = h_{31}(\xi, z), & \nu(\xi, 0, z) = h_{32}(\xi, z), \\ \omega(\xi, 1, z) = h_{41}(\xi, z), & \nu(\xi, 1, z) = h_{42}(\xi, z), \\ \omega(\xi, \gamma, 0) = h_{51}(\xi, \gamma), & \nu(\xi, \gamma, 0) = h_{52}(\xi, \gamma), \\ \omega(\xi, \gamma, 1) = h_{61}(\xi, \gamma), & \nu(\xi, \gamma, 1) = h_{62}(\xi, \gamma), \end{cases} \quad (3.4)$$

or

$$\begin{cases} \omega(0, \gamma, z) = h_{11}(\gamma, z), & \nu(0, \gamma, z) = h_{12}(\gamma, z), \\ \omega(1, \gamma, z) = \underbrace{\frac{1}{L_1} \frac{\partial \nu(1, \gamma, z)}{\partial \xi}}_{h_{21}(\gamma, z)}, & \nu(1, \gamma, z) = \underbrace{\frac{-1}{L_1} \frac{\partial \omega(1, \gamma, z)}{\partial \xi}}_{h_{22}(\gamma, z)}, \\ \omega(\xi, 0, z) = h_{31}(\xi, z), & \nu(\xi, 0, z) = h_{32}(\xi, z), \\ \omega(\xi, 1, z) = \underbrace{\frac{1}{L_2} \frac{\partial \nu(\xi, 1, z)}{\partial \gamma}}_{h_{41}(\xi, z)}, & \nu(\xi, 1, z) = \underbrace{\frac{-1}{L_2} \frac{\partial \omega(\xi, 1, z)}{\partial \gamma}}_{h_{42}(\xi, z)}, \\ \omega(\xi, \gamma, 0) = h_{51}(\xi, \gamma), & \nu(\xi, \gamma, 0) = h_{52}(\xi, \gamma), \\ \omega(\xi, \gamma, 1) = \underbrace{\frac{1}{L_3} \frac{\partial \nu(\xi, \gamma, 1)}{\partial z}}_{h_{61}(\xi, \gamma)}, & \nu(\xi, \gamma, 1) = \underbrace{\frac{-1}{L_3} \frac{\partial \omega(\xi, \gamma, 1)}{\partial z}}_{h_{62}(\xi, \gamma)}. \end{cases} \quad (3.5)$$

In the following, using (2.8) and (2.14), we expand the unknown functions $\omega(\xi, \gamma, z)$ and $\nu(\xi, \gamma, z)$ as follows:

$$\begin{cases} \omega(\xi, \gamma, z) \simeq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Upsilon \mathbb{C}\mathbb{P}_{z_f, \bar{M}}(z), \\ \nu(\xi, \gamma, z) \simeq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Lambda \mathbb{C}\mathbb{P}_{z_f, \bar{M}}(z), \end{cases} \quad (3.6)$$

where Υ and Λ are matrices with $(\mathbb{N}_1 + 1)(\mathbb{N}_2 + 1) \times (\bar{M} + 1)$ unknown elements. Regarding Theorems 2.1–2.3, we have

$$\begin{cases} \frac{\partial^2 \omega(\xi, \gamma, z)}{\partial \xi^2} \simeq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) (\mathbb{D}_\xi^{(2)})^T \Upsilon \mathbb{C}\mathbb{P}_{z_f, \bar{M}}(z), \\ \frac{\partial^2 \omega(\xi, \gamma, z)}{\partial \gamma^2} \simeq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) (\mathbb{D}_\gamma^{(2)})^T \Upsilon \mathbb{C}\mathbb{P}_{z_f, \bar{M}}(z), \\ \frac{\partial^2 \omega(\xi, \gamma, z)}{\partial z^2} \simeq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Upsilon \mathbb{D}_z^{(1, z_f)} \mathbb{C}\mathbb{P}_{z_f, \bar{M}}(z), \end{cases} \quad (3.7)$$

and

$$\begin{cases} \frac{\partial^2 v(\xi, \gamma, z)}{\partial \xi^2} \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \left(\mathbb{D}_{\xi}^{(2)} \right)^T \Lambda \mathbb{CP}_{z_f, \bar{M}}(z), \\ \frac{\partial^2 v(\xi, \gamma, z)}{\partial \gamma^2} \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \left(\mathbb{D}_{\gamma}^{(2)} \right)^T \Lambda \mathbb{CP}_{z_f, \bar{M}}(z), \\ \frac{\partial^2 v(\xi, \gamma, z)}{\partial z^2} \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \Lambda \mathbb{D}_z^{(1, z_f)} \mathbb{CP}_{z_f, \bar{M}}(z). \end{cases} \quad (3.8)$$

The functions expressed in (3.4) or (3.5) can be expanded by the OSDCPs as

$$\begin{cases} h_{11}(\gamma, z) \simeq \mathcal{H}_{11}(\gamma) \mathbb{CP}_{z_f, \bar{M}}(z), & h_{12}(\gamma, z) \simeq \mathcal{H}_{12}(\gamma) \mathbb{CP}_{z_f, \bar{M}}(z), \\ h_{21}(\gamma, z) \simeq \mathcal{H}_{21}(\gamma) \mathbb{CP}_{z_f, \bar{M}}(z), & h_{22}(\gamma, z) \simeq \mathcal{H}_{22}(\gamma) \mathbb{CP}_{z_f, \bar{M}}(z), \\ h_{31}(\xi, z) \simeq \mathcal{H}_{31}(\xi) \mathbb{CP}_{z_f, \bar{M}}(z), & h_{32}(\xi, z) \simeq \mathcal{H}_{32}(\xi) \mathbb{CP}_{z_f, \bar{M}}(z), \\ h_{41}(\xi, z) \simeq \mathcal{H}_{41}(\xi) \mathbb{CP}_{z_f, \bar{M}}(z), & h_{42}(\xi, z) \simeq \mathcal{H}_{42}(\xi) \mathbb{CP}_{z_f, \bar{M}}(z), \\ h_{51}(\xi, \gamma) \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \mathbb{G}_{11}, & h_{52}(\xi, \gamma) \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \mathbb{G}_{12}, \\ h_{61}(\xi, \gamma) \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \mathbb{G}_{21}, & h_{62}(\xi, \gamma) \simeq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \mathbb{G}_{22}, \end{cases} \quad (3.9)$$

where \mathcal{G}_{kj} and $\mathcal{H}_{ij}(\cdot)$ for $k = 1, 2$, $i = 1, 2, 3, 4$, $j = 1, 2$ are given vectors. Using (2.3), (3.6) and (3.9), we obtain the following relations:

$$\begin{aligned} & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(0, \gamma) \Upsilon - \mathcal{H}_{11}(\gamma) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{11}(\gamma, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(0, \gamma) \Lambda - \mathcal{H}_{12}(\gamma) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{12}(\gamma, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(1, \gamma) \Upsilon - \mathcal{H}_{21}(\gamma) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{21}(\gamma, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(1, \gamma) \Lambda - \mathcal{H}_{22}(\gamma) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{22}(\gamma, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, 0) \Upsilon - \mathcal{H}_{31}(\xi) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{31}(\xi, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, 0) \Lambda - \mathcal{H}_{32}(\xi) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{32}(\xi, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, 1) \Upsilon - \mathcal{H}_{41}(\xi) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{41}(\xi, z) \simeq 0, \\ & \left[\mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, 1) \Lambda - \mathcal{H}_{42}(\xi) \right] \mathbb{CP}_{z_f, \bar{M}}(z) \triangleq \tilde{\mathcal{H}}_{42}(\xi, z) \simeq 0, \\ & \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \left[\Upsilon \mathbb{CP}_{z_f, \bar{M}}(0) - \mathbb{G}_{11}^T \right] \triangleq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \tilde{\mathbb{G}}_{11} \simeq 0, \\ & \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \left[\Upsilon \mathbb{CP}_{z_f, \bar{M}}(1) - \mathbb{G}_{21}^T \right] \triangleq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \tilde{\mathbb{G}}_{12} \simeq 0, \\ & \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \left[\Lambda \mathbb{CP}_{z_f, \bar{M}}(0) - \mathbb{G}_{21}^T \right] \triangleq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \tilde{\mathbb{G}}_{21} \simeq 0, \\ & \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \left[\Lambda \mathbb{CP}_{z_f, \bar{M}}(1) - \mathbb{G}_{22}^T \right] \triangleq \mathbb{CP}_{\mathbb{N}_1 \mathbb{N}_2}^T(\xi, \gamma) \tilde{\mathbb{G}}_{22} \simeq 0. \end{aligned} \quad (3.10)$$

The residual functions can be defined via (2.1) and (3.6)–(3.8) as follows:

$$\left\{ \begin{array}{l} \text{Res}_1(\xi, \gamma, z) \triangleq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \left(\mathbb{D}_\xi^{(2)} \right)^T \Upsilon \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) + \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \left(\mathbb{D}_\gamma^{(2)} \right)^T \Upsilon \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) \\ \quad + \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Upsilon \mathbb{D}_z^{(2)} \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) + \eta^2 \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Upsilon \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) - f_1(\xi, \gamma, z) \simeq 0, \\ \text{Res}_2(\xi, \gamma, z) \triangleq \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \left(\mathbb{D}_\xi^{(2)} \right)^T \Lambda \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) + \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \left(\mathbb{D}_\gamma^{(2)} \right)^T \Lambda \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) \\ \quad + \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Lambda \mathbb{D}_z^{(2)} \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) + \eta^2 \mathbb{C}\mathbb{P}_{\mathbb{N}_1, \mathbb{N}_2}^T(\xi, \gamma) \Lambda \mathbb{C}\mathbb{P}_{z_f, \widetilde{M}}(z) - f_2(\xi, \gamma, z) \simeq 0. \end{array} \right. \quad (3.11)$$

Finally, we get a system containing $2(\mathbb{N}_1 + 1)(\mathbb{N}_2 + 1) \times (\mathcal{M} + 1)$ equations by (3.10) and (3.11) as

$$\left\{ \begin{array}{l} \text{Res}_1(\xi_\kappa, \gamma_\ell, z_m) = 0, \quad \kappa = 2, 3, 4, \dots, \mathbb{N}_1, \quad \ell = 2, 3, \dots, \mathbb{N}_2, \quad m = 2, 3, \dots, \widetilde{M}, \\ \text{Res}_2(\xi_\kappa, \gamma_\ell, z_m) = 0, \quad \kappa = 2, 3, 4, \dots, \mathbb{N}_1, \quad \ell = 2, 3, \dots, \mathbb{N}_2, \quad m = 2, 3, \dots, \widetilde{M}, \\ [\widetilde{\mathbb{G}}_{ij}]_{\kappa\ell} = 0, \quad i, j = 1, 2, \quad \kappa = 1, 2, \dots, \mathbb{N}_1 + 1, \quad \ell = 1, 2, \dots, \mathbb{N}_2 + 1, \\ \widetilde{\mathcal{H}}_{ij}(\gamma_\ell, z_m) = 0, \quad i, j = 1, 2, \quad \ell = 1, 2, \dots, \mathbb{N}_2 + 1, \quad m = 2, 3, \dots, \widetilde{M}, \\ \widetilde{\mathcal{H}}_{ij}(\xi_\kappa, z_m) = 0, \quad i = 3, 4, \quad j = 1, 2, \quad \kappa = 2, 3, \dots, \mathbb{N}_1, \quad m = 2, 3, \dots, \widetilde{M}, \end{array} \right. \quad (3.12)$$

where $(\xi_\kappa, \gamma_\ell, z_m)$ are defined as follows:

$$\begin{aligned} \xi_\kappa &= \frac{\kappa - 1}{\mathbb{N}_1 + 1}, & \kappa &= 1, 2, \dots, \mathbb{N}_1 + 1, \\ \gamma_\ell &= \frac{\ell - 1}{\mathbb{N}_2 + 1}, & \ell &= 1, 2, \dots, \mathbb{N}_2 + 1, \\ z_m &= \frac{m - 1}{\widetilde{M} + 1}, & m &= 1, 2, \dots, \widetilde{M} + 1. \end{aligned}$$

By solving (3.12), we determine the elements of the matrices Υ and Λ , and subsequently using (3.6), we derive a solution for primary problem (2.1). In this work, we have used the “linsolve” command of Matlab R2020b to solve this system.

The step-by-step algorithm of the proposed method is given in the following:

Algorithm

Input. The positive integer numbers \widetilde{M} , N_1 and N_2 .

Step 1. Generate the points $\xi_\kappa = \frac{\kappa - 1}{N_1 + 1}$ for $\kappa = 1, 2, \dots, N_1 + 1$, $\gamma_\ell = \frac{\ell - 1}{N_2 + 1}$ for

$$\ell = 1, 2, \dots, N_2 + 1 \text{ and } z_m = \frac{m - 1}{\widetilde{M} + 1} \text{ for } m = 1, 2, \dots, \widetilde{M} + 1.$$

Step 2. Define the first kind Stirling numbers $S_\kappa^{(r)}$ using Eq (2.5).

Step 3. Define the 1D polynomials $\mathcal{CP}_{z_f, \ell}(z, \widetilde{M})$ for $1 \leq \ell \leq \widetilde{M}$ by Eq (2.4) and 2D polynomials $\mathcal{CP}_{\kappa \ell}(\xi, \gamma; N_1, N_2)$ for $1 \leq \kappa \leq N_1$, $1 \leq \ell \leq N_2$ by Eq (2.13).

Step 4. Compute the vectors $\mathbb{CP}_{z_f, \widetilde{M}}(z)$ and $\mathbb{CP}_{N_1, N_2}^T(\xi, \gamma)$ by Eqs (2.10) and (2.17).

Step 5. Compute the matrices $\mathbb{D}_z^{(2, z_f)}$, $\mathbb{D}_\xi^{(2)}$ and $\mathbb{D}_\gamma^{(2)}$ using Theorems 2.1–2.3.

Step 6. Introduce the vectors Υ and Λ with unknown elements.

Step 7. Compute the residual functions $\text{Res}_1(\xi, \gamma, z)$ and $\text{Res}_2(\xi, \gamma, z)$ in Eq (3.11).

Step 8. Extract a linear system of algebraic equations using Eqs (3.10) and (3.11).

Step 9. Solve the system expressed in Step 8 and calculate vectors Υ and Λ .

Output. The approximate solutions: $\omega(\xi, \gamma, z) \simeq \mathbb{CP}_{N_1, N_2}^T(\xi, \gamma) \Upsilon \mathbb{CP}_{z_f, \widetilde{M}}(z)$ and

$$v(\xi, \gamma, z) \simeq \mathbb{CP}_{N_1, N_2}^T(\xi, \gamma) \Lambda \mathbb{CP}_{z_f, \widetilde{M}}(z).$$

4. Test problem

This section presents four test problems to demonstrate the validity of the suggested method. We evaluate the precision of our algorithm using L_∞ norm as follows:

$$\begin{cases} L_\infty^{\text{real}} = \max_{1 \leq i \leq N_1 + 1} \max_{1 \leq j \leq N_2 + 1} |\omega(\xi_i, \gamma_j, 1) - \tilde{\omega}(\xi_i, \gamma_j, 1)|, \\ L_\infty^{\text{image}} = \max_{1 \leq i \leq N_1 + 1} \max_{1 \leq j \leq N_2 + 1} |v(\xi_i, \gamma_j, 1) - \tilde{v}(\xi_i, \gamma_j, 1)|, \\ |L_\infty| = \sqrt{(L_\infty^{\text{real}})^2 + (L_\infty^{\text{image}})^2}, \end{cases}$$

where $\tilde{\omega}$ and \tilde{v} are the approximations of ω and v , respectively. In addition, we compute the convergence order (CO) as

$$\text{CO} = \log_{\frac{N_1}{N_2}} \frac{L_\infty(N_2)}{L_\infty(N_1)},$$

where N_1 and N_2 are the number of the 2D OSDCPs applied in the first and second implementations, respectively. It should be noted that in all examples we assume that $N_1 = N_2$.

Example 4.1. Consider the 3D Helmholtz equation

$$\Delta \vartheta(\xi, \gamma, z) + \eta^2 \vartheta(\xi, \gamma, z) = 0, \quad (\xi, \gamma, z) \in [0, 1]^3,$$

with the boundary conditions

$$\left\{ \begin{array}{ll} \vartheta(0, \gamma, z) = \exp(i(L_2\gamma + L_3z)), & (\gamma, z) \in [0, 1]^2, \\ \vartheta(1, \gamma, z) = \exp(i(L_1 + L_2\gamma + L_3z)), & (\gamma, z) \in [0, 1]^2, \\ \vartheta(\xi, 0, z) = \exp(i(L_1\xi + L_3z)), & (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, 1, z) = \exp(i(L_1\xi + L_2 + L_3z)), & (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 0) = \exp(i(L_1\xi + L_2\gamma)), & (\xi, \gamma) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 1) = \exp(i(L_1\xi + L_2\gamma + L_3)), & (\xi, \gamma) \in [0, 1]^2, \end{array} \right.$$

where $(L_1, L_2, L_3) = (\eta \cos \mu \cos \varpi, \eta \sin \mu \cos \varpi, \eta \sin \varpi)$. The true solution of this example is

$$\vartheta(\xi, \gamma, z) = e^{i(L_1\xi + L_2\gamma + L_3z)}.$$

We apply the method proposed in Section 3 for solving this example with some values (μ, ϖ) , (N, \tilde{M}) and η . The obtained results are presented in Tables 1 and 2 and Figures 1–3. These results indicate that the proposed method is sufficiently accurate.

Table 1. The values of L_∞ at $z = 0.5$ with $\eta = \sqrt{3}$, $\tilde{M} = 9$ and some values of (μ, ϖ) and N for Example 4.1.

N	$(\mu = \frac{\pi}{32}, \varpi = \frac{\pi}{16})$			$(\mu = \frac{3\pi}{32}, \varpi = \frac{3\pi}{16})$			$(\mu = \frac{5\pi}{32}, \varpi = \frac{5\pi}{16})$		
	L_∞^{real}	L_∞^{image}	$ L_\infty $	L_∞^{real}	L_∞^{image}	$ L_\infty $	L_∞^{real}	L_∞^{image}	$ L_\infty $
4	5.5570E-4	4.6427E-4	7.2412E-4	4.7169E-4	3.3362E-4	5.7775E-4	3.1603E-4	1.9379E-4	3.7071E-4
6	6.8299E-6	5.3973E-6	8.7051E-6	5.4661E-6	3.5644E-6	6.5256E-6	3.1504E-6	1.8635E-6	3.6603E-6
8	4.2722E-8	3.2964E-8	5.3961E-8	3.1874E-8	2.0441E-8	3.7865E-8	1.5717E-8	9.0795E-9	1.8151E-8
10	1.6709E-10	1.2795E-10	2.1045E-10	1.1668E-10	7.3901E-11	7.4816E-10	4.9133E-11	2.8366E-11	5.6733E-11
12	4.5254E-13	3.4379E-13	5.6832E-13	2.9349E-13	1.8428E-13	3.4655E-13	1.0530E-13	6.1587E-14	6.2481E-13

Table 2. The values of L_∞ and CO at $z = 0.5$ with $\eta = 2\sqrt{3}, 3\sqrt{3}$, $(\mu = \frac{5\pi}{32}, \varpi = \frac{5\pi}{16})$ and some values of N for Example 4.1.

N	$\eta = 2\sqrt{3}$					$\eta = 3\sqrt{3}$				
	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $
4	1.5005E-2	-	1.4972E-2	-	2.1197E-2	1.6803E-1	-	1.6758E-1	-	2.3731E-1
6	4.7165E-4	8.5332	4.6907E-4	8.5413	6.6519E-4	8.1850E-3	7.4528	1.1423E-2	6.6241	1.4053E-2
8	8.2620E-6	14.0590	8.7443E-6	1.8430	1.2030E-5	3.0222E-4	11.4670	4.7133E-4	11.0810	5.5990E-4
10	1.0487E-7	19.5690	1.0403E-7	19.8590	1.4772E-7	7.7584E-6	16.4130	1.2296E-5	16.3410	1.4539E-5
12	8.8228E-10	26.2060	8.7339E-10	26.2180	8.8659E-9	1.5564E-7	21.4400	2.2843E-7	21.8610	2.7641E-7

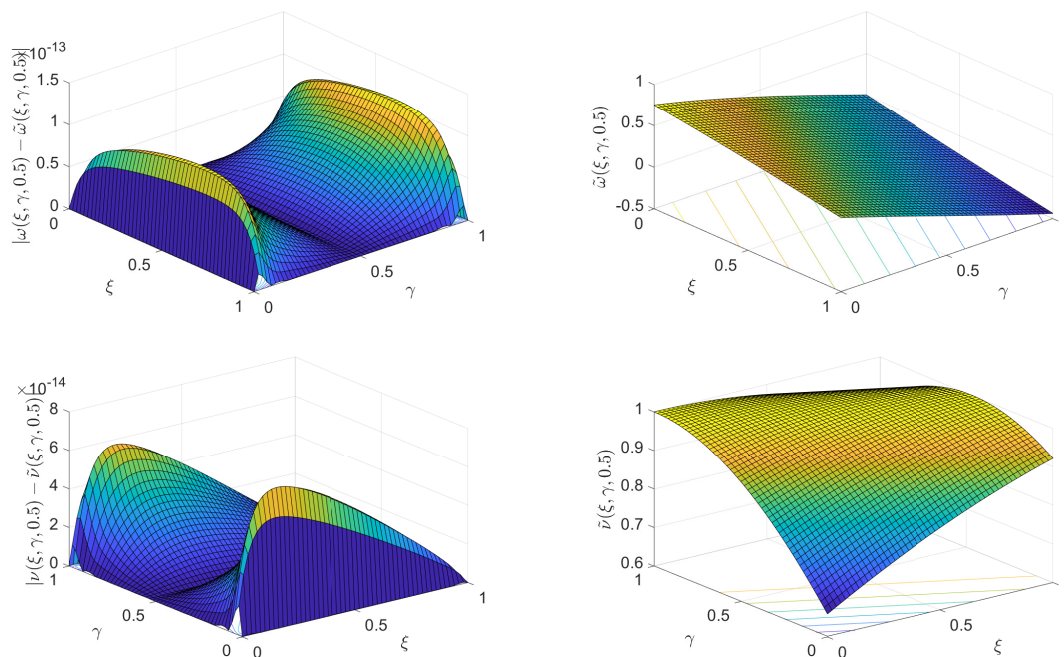


Figure 1. Results achieved at $z = 0.5$ with $\eta = \sqrt{3}$, $N = 12$ and $(\mu = \frac{5\pi}{32}, \varpi = \frac{5\pi}{16})$ for Example 4.1.

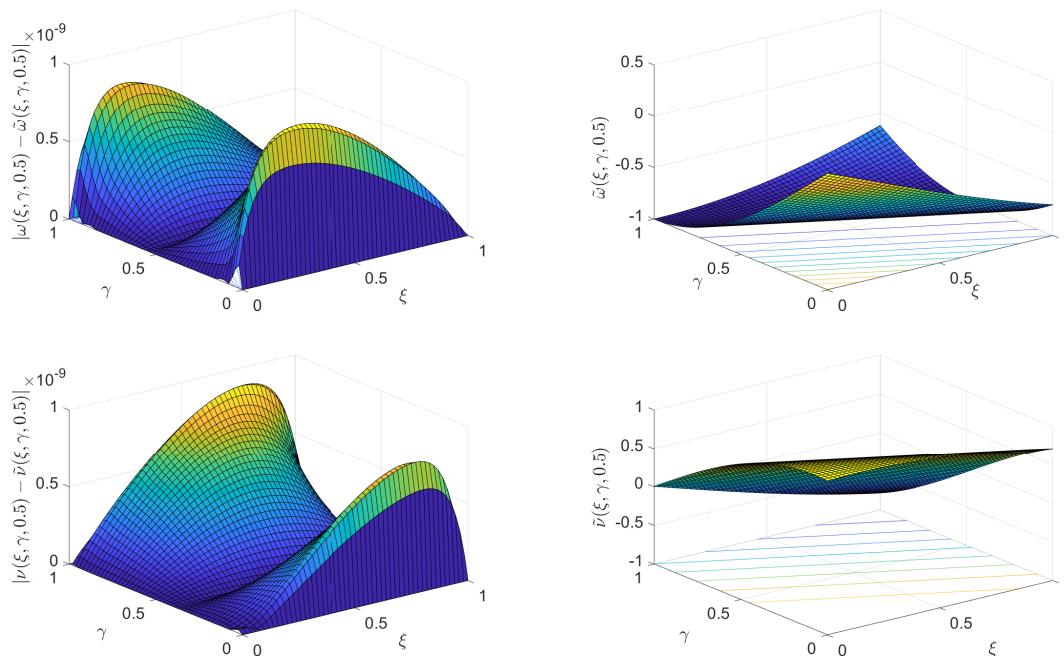


Figure 2. Results achieved at $z = 0.5$ with $\eta = 2\sqrt{3}$, $N = 12$ and $(\mu = \frac{5\pi}{32}, \varpi = \frac{5\pi}{16})$ for Example 4.1.

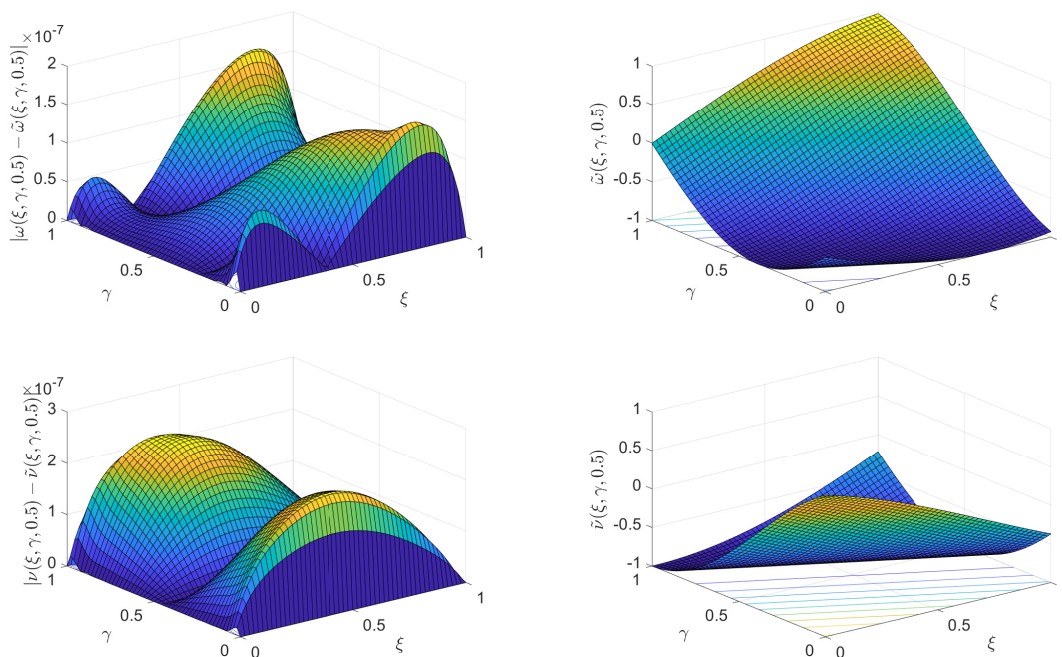


Figure 3. Results achieved at $z = 0.5$ with $\eta = 3\sqrt{3}$, $N = 12$ and $(\mu = \frac{5\pi}{32}, \varpi = \frac{5\pi}{16})$ for Example 4.1.

Example 4.2. Consider the 3D Helmholtz equation

$$\Delta\vartheta(\xi, \gamma, z) + \eta^2\vartheta(\xi, \gamma, z) = 0, \quad (\xi, \gamma, z) \in [0, 1]^3,$$

with the boundary conditions

$$\left\{ \begin{array}{ll} \vartheta(0, \gamma, z) = h_1(\gamma, z), & (\gamma, z) \in [0, 1]^2, \\ \frac{\partial\vartheta(1, \gamma, z)}{\partial\xi} = iL_1\vartheta(1, \gamma, z), & (\gamma, z) \in [0, 1]^2, \\ \vartheta(\xi, 0, z) = h_3(\xi, z), & (\xi, z) \in [0, 1]^2, \\ \frac{\partial\vartheta(\xi, 1, z)}{\partial\gamma} = iL_2\vartheta(\xi, 1, z), & (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 0) = h_5(\xi, \gamma), & (\xi, \gamma) \in [0, 1]^2, \\ \frac{\partial\vartheta(\xi, \gamma, 1)}{\partial z} = iL_3\vartheta(\xi, \gamma, 1), & (\xi, \gamma) \in [0, 1]^2, \end{array} \right.$$

where $(L_1, L_2, L_3) = (\eta \cos \mu \cos \varpi, \eta \sin \mu \cos \varpi, \eta \sin \varpi)$. The analytic solution of this example is

$$\vartheta(\xi, \gamma, z) = e^{i(L_1\xi + L_2\gamma + L_3z)},$$

that is a plane wave. Using the above analytic solution, we can determine functions $h_\ell(\cdot, \cdot)$, $\ell = 1, 3, 5$. We apply the method proposed in Section 3 for solving this example with $(\mu, \varpi) = (\frac{\pi}{4}, 0)$. Results

obtained for $\eta = \sqrt{2}, 5\sqrt{2}, 10\sqrt{2}, 15\sqrt{2}$ are presented in Tables 3 and 4, Figures 4–7. The results listed in Tables 3 and 4 indicate that the proposed method is sufficiently accurate even for large wave numbers. Figure 8 shows the logarithm of errors for various values of wave number that decrease with increasing N .

Table 3. The values of L_∞ and the CO with $\eta = \sqrt{2}, \eta = 5\sqrt{2}$ and some values of N for Example 4.2.

N	$\eta = \sqrt{2}$					$\eta = 5\sqrt{2}$				
	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $
3	2.4080E-03	-	3.6760E-03	-	4.3945E-04	1.5280	-	6.2590E-01	-	1.6526
5	3.2660E-06	12.9502	4.4055E-06	13.1683	5.4604E-06	9.6405E-01	0.9016	4.8246E-01	0.5208	1.0780
7	3.3233E-09	20.4416	5.1757E-09	20.0511	6.1026E-09	2.2086E-02	11.2230	6.4915E-03	12.8046	2.3020E-02
9	2.2094E-12	29.1109	3.0378E-12	29.6067	3.7563E-12	3.8518E-04	16.1113	1.2590E-04	15.6886	4.0523E-04
11	1.0184E-15	38.2828	1.4821E-15	37.9997	1.7983E-15	5.2881E-06	21.3696	1.6748E-06	21.5268	4.2002E-06
13	3.3151E-19	48.0688	4.5715E-19	48.3913	5.6470E-19	4.6430E-08	28.3457	1.4938E-08	28.2516	4.8774E-08
15	2.3274E-20	15.9009	3.7306E-20	15.0003	4.3971E-20	3.4047E-10	34.3490	1.1216E-10	34.1839	3.5847E-10

Table 4. The values of L_∞ and CO with $\eta = 10\sqrt{2}, \eta = 15\sqrt{2}$ and some values of N for Example 4.2.

N	$\eta = 10\sqrt{2}$					$\eta = 15\sqrt{2}$				
	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $
3	3.9050	-	5.3276	-	6.6055	5.9586	-	5.9515	-	8.4217
5	1.7718	1.5470	2.3754	1.5812	2.9634	3.3894	1.04453	3.2673	1.1739	4.7078
7	5.6250E-1	3.4100	6.2821E-1	3.9530	8.4324E-1	2.1202	1.3943	2.3651	0.9604	3.1763
9	2.7280E-2	12.0417	2.8424E-2	12.3183	3.9397E-2	1.6842	0.9161	1.7351	1.2325	2.4181
11	1.1852E-3	15.6288	1.5255E-3	14.5750	1.9318E-3	1.3744	1.0130	1.2816	1.5097	1.8792
13	5.2001E-5	18.7149	6.6344E-5	18.7677	8.4295E-5	1.1920E-2	28.4193	1.3800E-2	27.1241	1.8235E-2
15	1.2452E-6	26.0793	1.6798E-6	25.6894	2.0910E-6	8.9276E-4	18.1107	1.0043E-3	18.1314	1.3437E-3

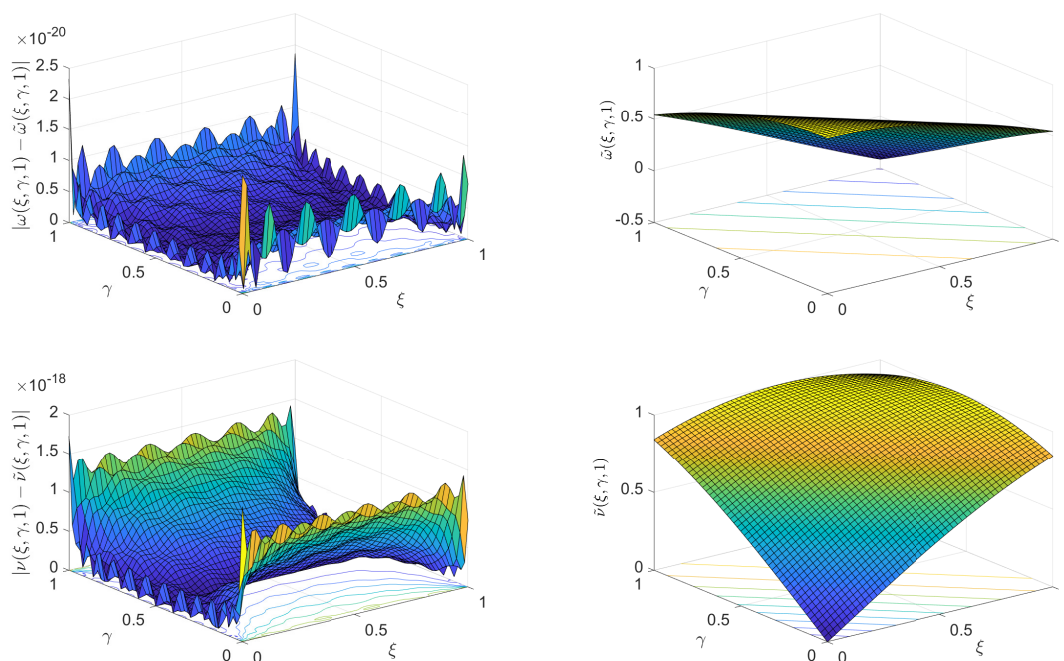


Figure 4. Results achieved with $\eta = \sqrt{2}$, $N = 15$ and $(\mu = \frac{\pi}{4}, \varpi = 0)$ for Example 4.2.

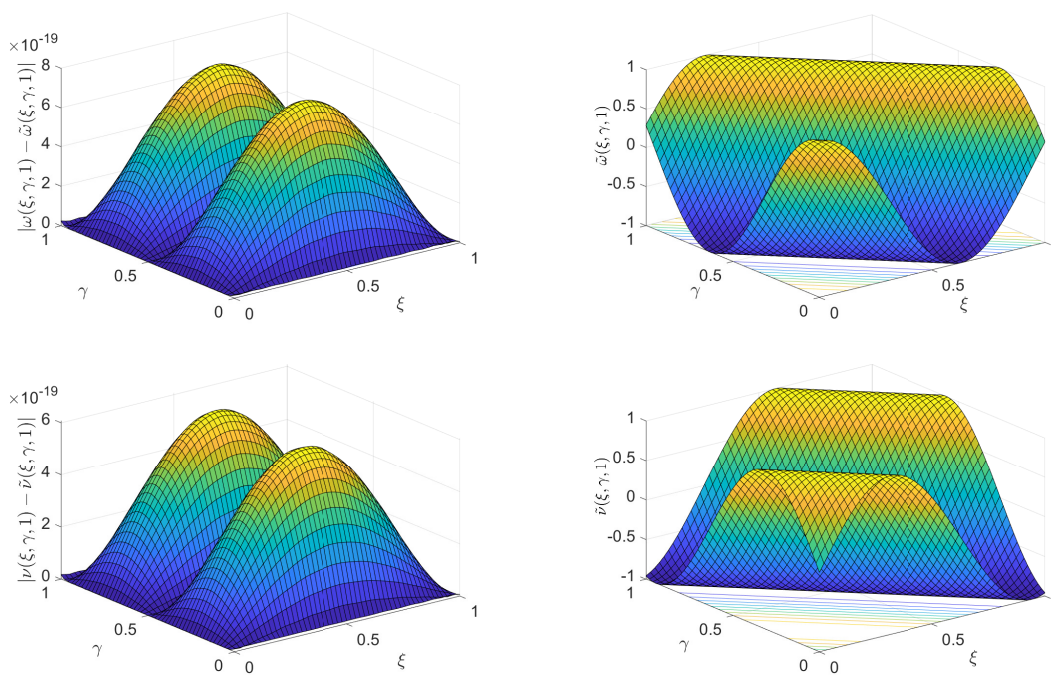


Figure 5. Results achieved with $\eta = 5\sqrt{2}$, $N = 23$ and $(\mu = \frac{\pi}{4}, \varpi = 0)$ for Example 4.2.

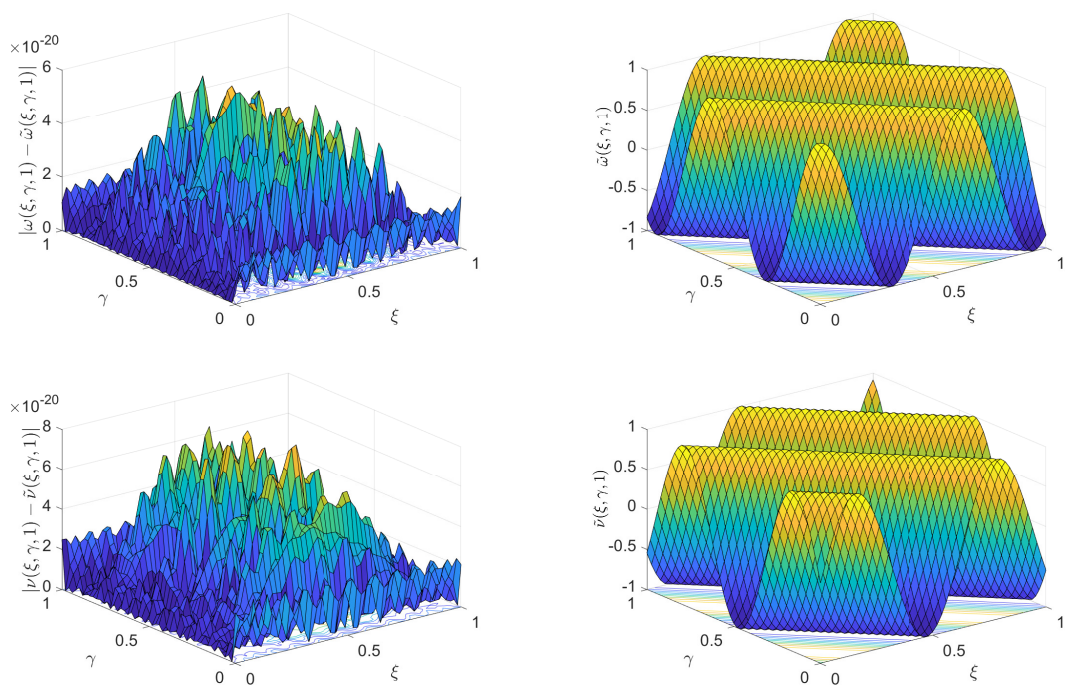


Figure 6. Results achieved with $\eta = 10\sqrt{2}$, $N = 29$ and $(\mu = \frac{\pi}{4}, \varpi = 0)$ for Example 4.2.

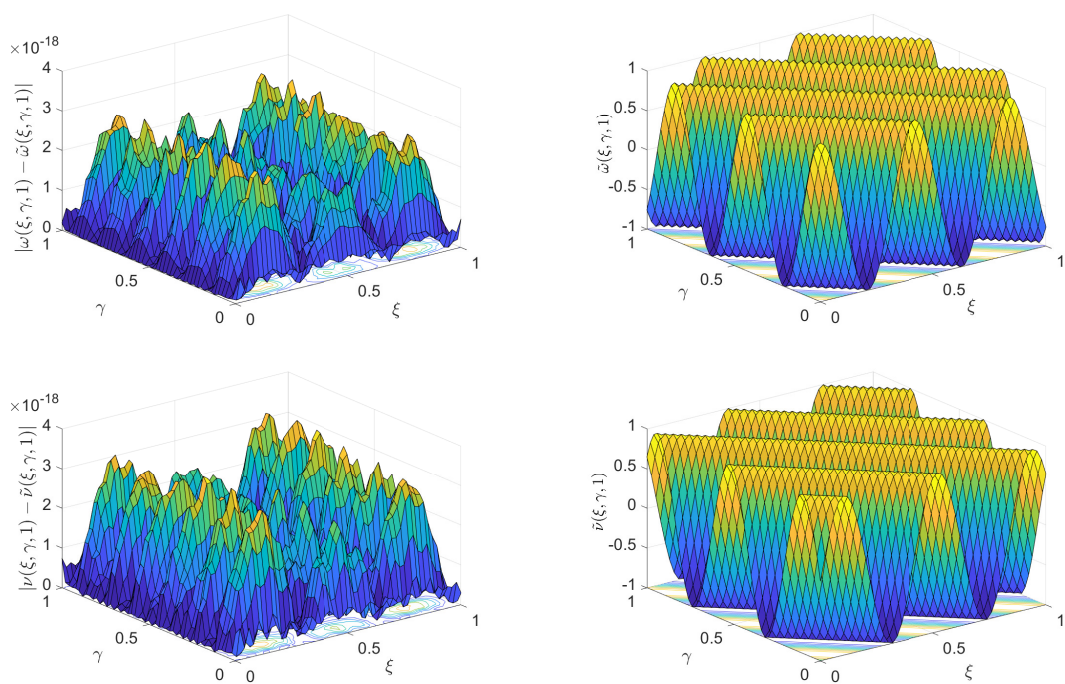


Figure 7. Results achieved with $\eta = 15\sqrt{2}$, $N = 33$ and $(\mu = \frac{\pi}{4}, \varpi = 0)$ for Example 4.2.

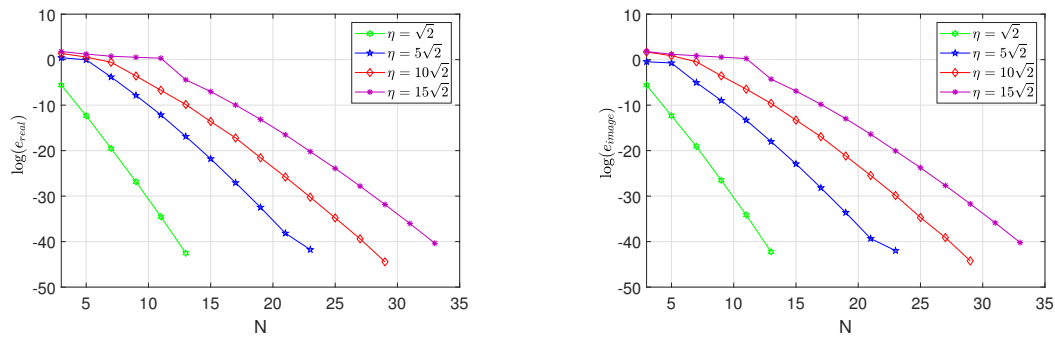


Figure 8. Logarithm of the absolute errors for different values of η and N in Example 4.2.

Example 4.3. Consider the 3D Helmholtz equation

$$\Delta\vartheta(\xi, \gamma, z) + \eta^2\vartheta(\xi, \gamma, z) = 0, \quad (\xi, \gamma, z) \in [0, 1]^3,$$

with the boundary conditions

$$\left\{ \begin{array}{ll} \vartheta(0, \gamma, z) = \exp(i(L_2\gamma + L_3z)), & (\gamma, z) \in [0, 1]^2, \\ \vartheta(1, \gamma, z) = \exp(i(L_1 + L_2\gamma + L_3z)), & (\gamma, z) \in [0, 1]^2, \\ \vartheta(\xi, 0, z) = \exp(i(L_1\xi + L_3z)), & (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, 1, z) = \exp(i(L_1\xi + L_2 + L_3z)), & (\xi, z) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 0) = \exp(i(L_1\xi + L_2\gamma)), & (\xi, \gamma) \in [0, 1]^2, \\ \vartheta(\xi, \gamma, 1) = \exp(i(L_1\xi + L_2\gamma + L_3)), & (\xi, \gamma) \in [0, 1]^2, \end{array} \right.$$

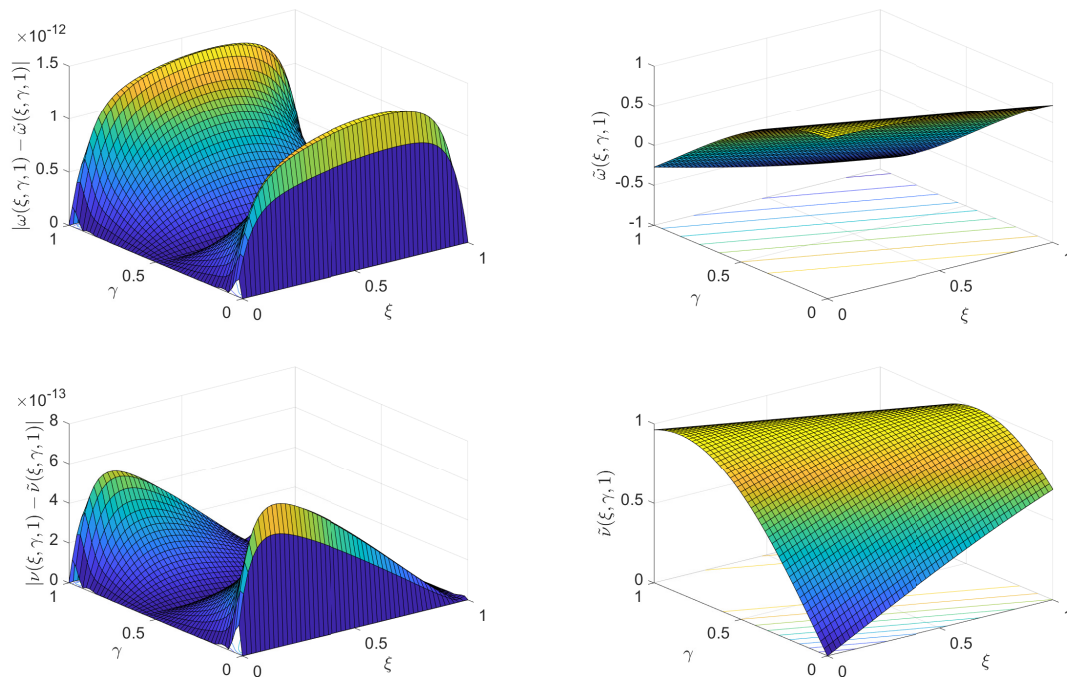
where $(L_1, L_2, L_3) = (\eta \cos \mu \cos \varpi, \eta \sin \mu \cos \varpi, \eta \sin \varpi)$. The analytic solution of this example is $\vartheta(\xi, \gamma, z) = e^{i(L_1\xi + L_2\gamma + L_3z)}$. We apply the method proposed in Section 3 for solving this example with $(\mu = \frac{\pi}{8}, \varpi = 0)$ and some values of (N, \tilde{M}) and η . The obtained results are presented in Tables 5, 6 and Figures 9–12. These results indicate that the proposed method is sufficiently accurate.

Table 5. The values of L_∞ and CO with $\eta = 2, 3$ and some values of N for Example 4.3.

N	$\eta = 2$					$\eta = 3$				
	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $
3	3.3098E-3	-	7.4487E-3	-	8.1509E-3	3.1269E-2	-	5.3151E-2	-	6.1667E-2
4	8.5322E-4	4.7122	4.6445E-4	9.6459	9.7144E-4	7.7571E-3	4.8457	6.3093E-3	7.4079	9.9990E-3
5	6.2661E-5	11.7020	1.4818E-4	5.1197	1.6088E-4	1.2314E-3	8.2479	2.1290E-3	4.8685	2.4595E-3
6	1.2525E-5	8.8306	6.0859E-6	17.5100	1.3925E-5	2.3630E-4	9.0544	1.6522E-4	14.0199	2.8833E-4
7	5.7241E-7	20.0169	1.3519E-6	9.7252	1.4747E-6	2.3448E-5	14.9874	4.1441E-5	8.9717	4.7615E-5
8	9.1367E-8	13.7419	4.2548E-8	25.9410	1.0079E-7	3.7416E-6	13.7441	2.4626E-6	21.1415	4.4793E-6
9	3.1008E-9	28.7242	7.3886E-9	14.8640	8.0129E-9	2.8188E-7	21.9538	4.8832E-7	13.7371	5.6384E-7
10	4.1402E-10	19.1106	1.8960E-10	34.7642	4.5537E-10	3.7671E-8	19.1019	2.4596E-8	28.3634	4.4990E-8
11	1.1572E-11	37.5335	2.6875E-11	20.4985	2.9260E-11	2.2964E-9	29.3520	3.8934E-9	19.3400	4.5202E-9
12	1.2974E-12	25.1487	6.0059E-13	43.6844	1.4297E-12	2.6247E-10	24.9273	1.7160E-10	35.8789	3.1359E-10

Table 6. The values of L_∞ and CO with $\eta = 4, 5$ and some values of N for Example 4.3.

N	$\eta = 4$					$\eta = 5$				
	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $	L_∞^{real}	CO	L_∞^{image}	CO	$ L_\infty $
3	6.0417E-1	-	3.8960E-1	-	7.1889E-1	4.5872E-1	-	3.7373E-1	-	5.9169E-1
4	6.6147E-2	7.6889	5.7464E-2	6.6530	8.7622E-2	2.5479E-1	7.6809	1.5532E-1	3.0521	2.9840E-1
5	3.0496E-2	3.4698	1.8676E-2	5.0368	3.5760E-2	1.1164E-1	3.4698	2.9700E-2	7.4138	1.1552E-1
6	2.2562E-3	13.6620	2.4021E-3	11.2488	3.2955E-3	1.0991E-2	13.6620	1.1495E-2	5.2064	1.5904E-2
7	1.1182E-3	4.5537	6.7140E-4	8.2694	1.3043E-3	6.1771E-3	4.5537	1.4321E-3	13.5113	6.3409E-3
8	5.3904E-5	22.7083	6.0476E-5	18.0277	8.1006E-5	3.5283E-4	22.7083	4.9562E-4	7.9464	6.0838E-4
9	2.4291E-5	6.7675	1.4451E-5	12.1523	2.8265E-5	2.1461E-5	6.7675	4.3630E-5	20.6317	4.8623E-5
10	8.6966E-7	31.6035	1.0273E-6	25.0932	1.3460E-6	7.8944E-6	31.6035	1.3407E-5	11.1993	1.5559E-5
11	3.5421E-7	9.4241	2.0970E-7	16.6720	4.1163E-7	4.9461E-6	9.4241	9.6286E-7	27.6321	5.0389E-6
12	1.0167E-8	40.8078	1.2275E-8	32.6177	1.5939E-8	1.3022E-7	40.8078	2.5241E-7	15.3871	2.8402E-7

**Figure 9.** Results achieved with $\eta = 2$, $N = 12$ and $(\mu = \frac{\pi}{8}, \varpi = 0)$ for Example 4.3.

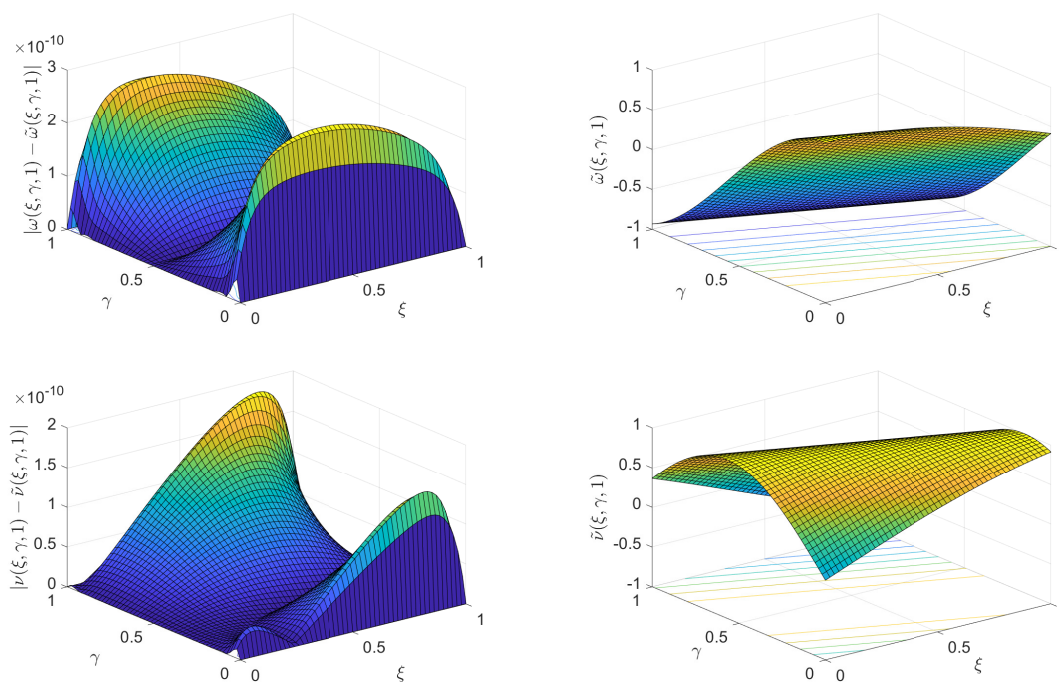


Figure 10. Results achieved with $\eta = 3$, $N = 12$ and $(\mu = \frac{\pi}{8}, \varpi = 0)$ for Example 4.3.

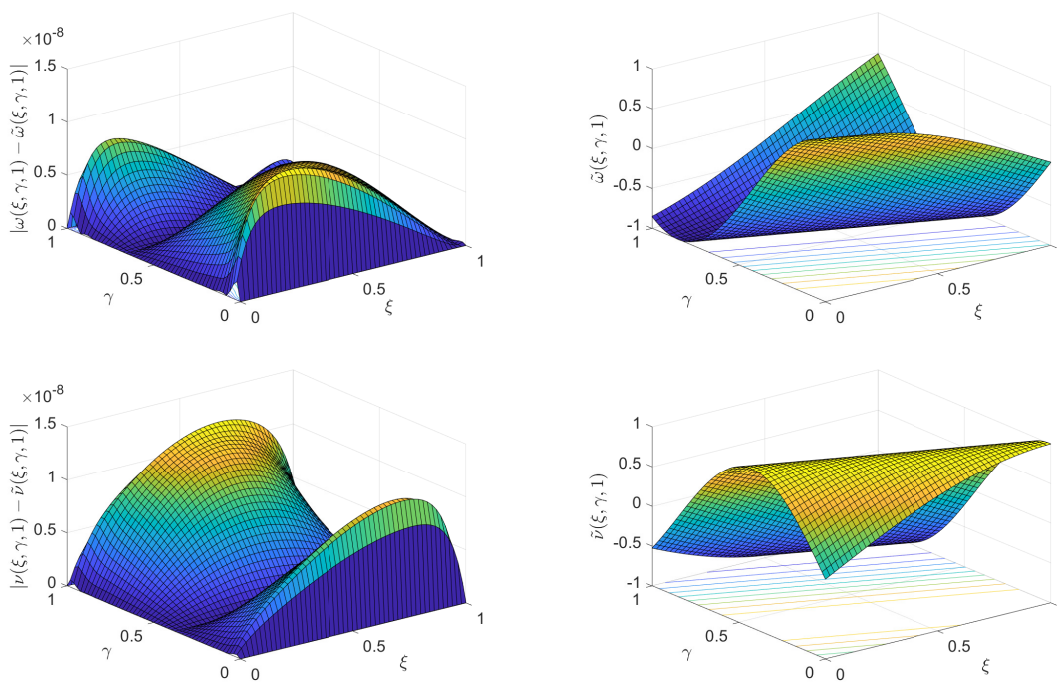


Figure 11. Results achieved with $\eta = 4$, $N = 12$ and $(\mu = \frac{\pi}{8}, \varpi = 0)$ for Example 4.3.

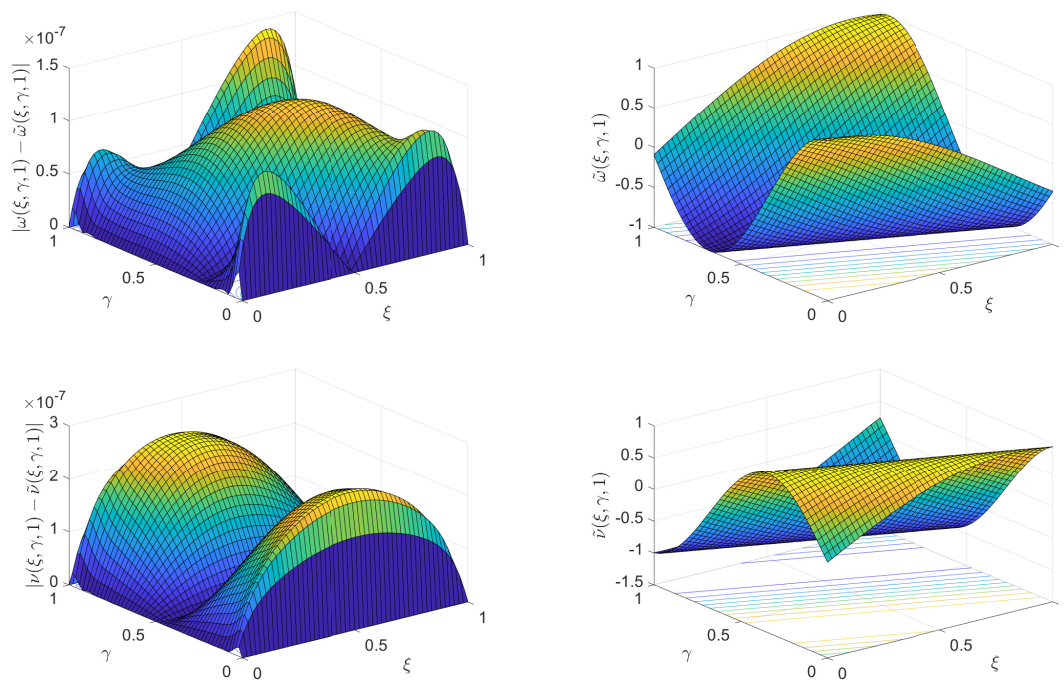


Figure 12. Results achieved with $\eta = 5$, $N = 12$ and $(\mu = \frac{\pi}{8}, \varpi = 0)$ for Example 4.3.

Example 4.4. Consider the 3D Helmholtz equation

$$\Delta\vartheta(\xi, \gamma, z) + \eta^2\vartheta(\xi, \gamma, z) = f(\xi, \gamma, z), \quad (\xi, \gamma, z) \in [0, 1]^3,$$

with

$$f(\xi, \gamma, z) = -2 \sin(\eta\xi) \sin(\eta\gamma) \sin(\eta z),$$

where the exact solution is $\vartheta(\xi, \gamma, z) = \frac{\sin(\eta\xi) \sin(\eta\gamma) \sin(\eta z)}{\eta^2}$. Other required information can be derived from the expressed exact solution. We apply the method proposed in Section 3 for solving this example with some values of (N, \bar{M}) and η . The obtained results are presented in Table 7, Figures 13 and 14. These results indicate that the proposed method is sufficiently accurate.

Table 7. The values of L_∞ and CO with $\eta = \frac{\pi}{2}, \pi, 2\pi, 3\pi$ and some values of N for Example 4.4.

N	$\eta = \frac{\pi}{2}$		$\eta = \pi$		$\eta = 2\pi$		$\eta = 3\pi$	
	L_∞^{real}	CO	L_∞^{real}	CO	L_∞^{real}	CO	L_∞^{real}	CO
4	1.0805E-4	-	1.4439E-3	-	1.5450E-2	-	1.8589E-1	-
6	1.2378E-6	11.0225	3.6391E-5	9.0779	2.0078E-3	5.0327	4.5350E-3	9.1582
8	6.5440E-8	18.2234	4.9688E-7	14.9253	1.6308E-4	8.7268	3.8311E-3	0.5863
10	2.1336E-11	25.6602	5.0409E-9	20.5731	8.5400E-6	13.2179	2.0592E-4	13.1011
12	4.7472E-14	33.5013	3.7234E-11	26.9201	3.1128E-7	18.1647	1.5229E-5	14.2845
14	1.1144E-15	24.1664	2.0899E-13	33.6209	8.2027E-9	50.9515	7.9471E-7	19.1564

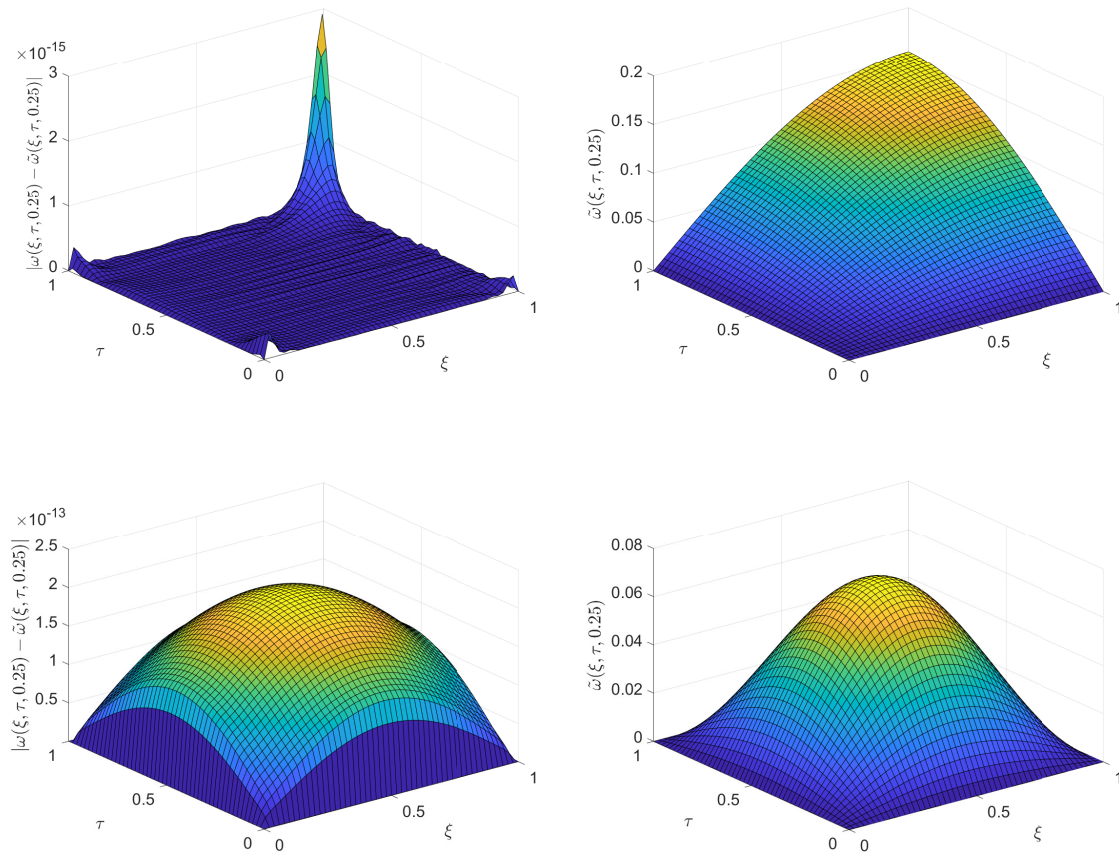


Figure 13. Results achieved with $\eta = \frac{\pi}{2}, \pi$ and $N = 14$ for Example 4.4.

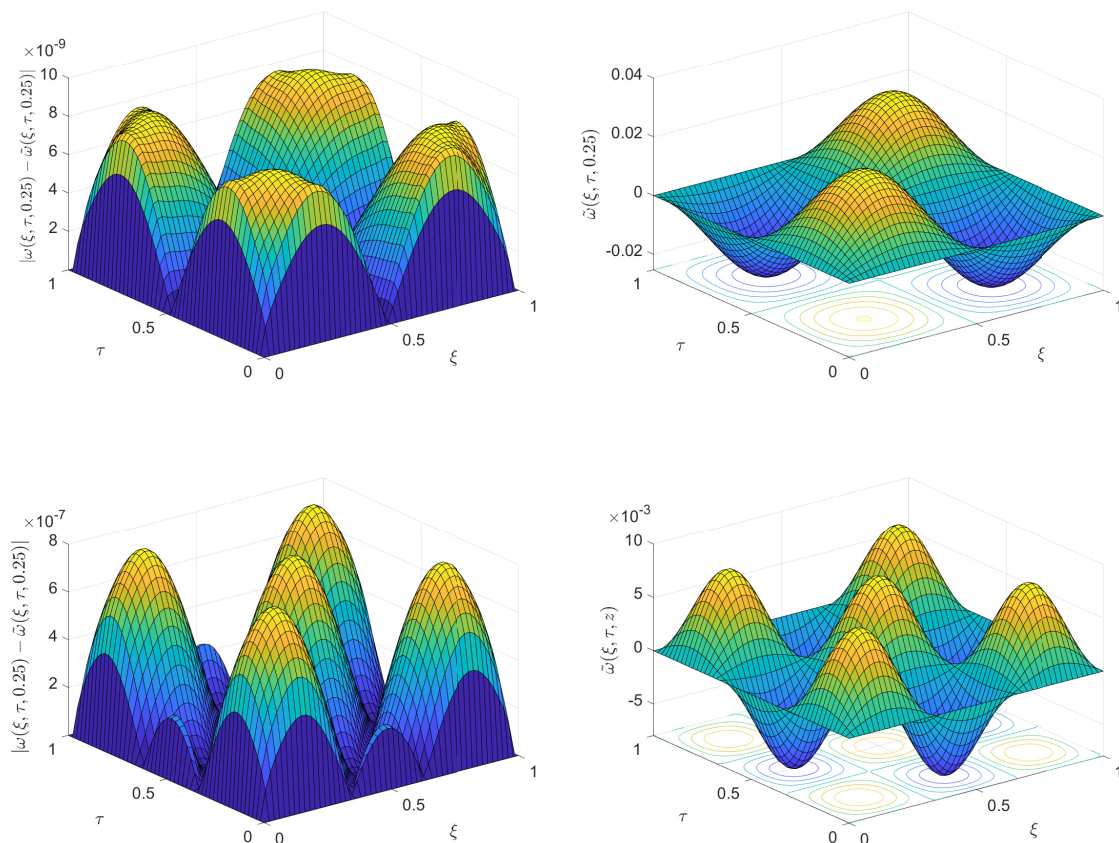


Figure 14. Results achieved with $\eta = \frac{\pi}{2}, \pi, 2\pi, 3\pi$ and $N = 14$ for Example 4.4.

5. Conclusions

In this paper, we provided a numerical scheme using the orthonormal shifted discrete Chebyshev polynomials. Using the proposed method, we convert the 3D Helmholtz equation into a system of algebraic equations which can be easily solved. The results obtained of solving some numerical examples confirmed the high accuracy of the presented algorithm. Note that the proposed scheme can be developed for other types of partial differential equations, such as the Kdv-Burgers-Kuramoto equation, the Schrödinger equation, and the Benjamin-Bona-Mahony equation.

Conflict of interest

The authors declare that they have no competing interests.

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