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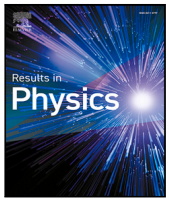

Analysis of a fractional order Bovine Brucellosis disease model with discrete generalized Mittag–Leffler kernels

Muhammad Farman [a,b,d,1], Aamir Shehzad [a,1], Ali Akgül [b,c,d,1], Dumitru Baleanu [e,1], Nourhane Attia [f,*,1], Ahmed M. Hassan [g,1]

[a] Institute of Mathematics, Khawaja Fareed University of Engineering and Information Technology, Rahim Yar Khan 64200, Pakistan
[b] Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon
[c] Siirt University, Art and Science Faculty, Department of Mathematics, 56100 Siirt, Turkey
[d] Near East University, Mathematics Research Center, Department of Mathematics, Near East Boulevard, PC: 99138, Nicosia/Mersin 10, Turkey
[e] Department of Mathematics and Computer Science, Cankaya University, Angara, Turkey
[f] National High School for Marine Sciences and Coastal (ENSSMAL), Dely Ibrahim University Campus, Bois des Cars, B.P. 19, 16320, Algiers, Algeria
[g] Faculty of engineering, Future University in Egypt, Egypt

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ABSTRACT

Bovine Brucellosis, a zoonotic disease, can infect cattle in tropical and subtropical areas. It remains a critical issue for both human and animal health in many parts of the world, especially those where livestock is an important source of food and income. An efficient method for monitoring the illness's increasing prevalence and developing low-cost prevention strategies for both its effects and recurrence is brucellosis disease modeling. We create a fractional-order model of Bovine Brucellosis using a discrete modified Atangana–Baleanu fractional difference operator of the Liouville–Caputo type. An analysis of the suggested system's well-posedness and a qualitative investigation are both conducted. The examination of the Volterra-type Lyapunov function for global stability is supported by the first and derivative tests. The Lipschitz condition is also used for the model in order to meet the criterion of the uniqueness of the exact solution. We created an endemic and disease-free equilibrium. Solutions are built in the discrete generalized form of the Mittag-Leffler kernel in order to analyze the effect of the fractional operator with numerical simulations and emphasize the effects of the sickness due to the many factors involved. The capacity of the suggested model to forecast an infectious disease like brucellosis can help researchers and decision-makers take preventive actions.

Introduction

A bacterium of the genus Brucella causes the infectious and contagious zoonotic disease known as bovine brucellosis, which affects both animals and humans. In decreasing order, the four Brucella species that cause the illness are: B. melitensis in small ruminants like goats and sheep; B. abortus commonly found in cattle; B. suis in swine; and B. canis in dogs. Direct contact with diseased animal tissues, urine, or blood or with the environment that has been contaminated by discharges from sick cattle are the two main ways that bovine brucellosis is transmitted to vulnerable livestock. Moreover, it may be vertically transmitted from sick mothers to their newborns [1]. Typically, biovars of B. abortus induce brucellosis in cattle. The bacteria B. melitensis can also infect cattle housed in close proximity to sheep or goats in various regions, particularly in southern Europe and western Asia. Both B. abortus and B. melitensis can infect humans. The two main sources of human brucellosis are environmental Brucella and contaminated cattle. Contact with diseased animals or animal products in meals can spread brucellosis to people. By consuming infected, unpasteurized dairy products, the illness can also be spread indirectly [2]. In countries all over the world, Brucella infections produce substantial financial losses as well as societal health concerns. Brucellosis impedes the trade of animals, animal products, and animal migration. In cattle, the condition is known to cause abortion in the final stages of pregnancy, followed by foetal membrane retention and infertility in subsequent pregnancies [3]. The Brucella organism spreads through contact with aborted cattle and aborted materials, as well as contaminated fomites. Numerous risk variables connected to production methods, the biology

* Corresponding author.
E-mail address: nourhane.attia@enssmal.edu.dz (N. Attia).
[1] All authors have equal contributions.

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of the particular host, and environmental factors affect the likelihood of contracting brucellosis. Age, herd size and composition, farm hygiene, the frequency of interaction between infected and vulnerable animals, farm bio-security, and climate are a few of these factors [4]. Cattle that are sexually mature and pregnant are more likely to contract Brucella than cattle that are not sexually mature. This is due to the concentration of erythritol sugar produced inside the foetal tissues of cattle, which causes the Brucella organism to confer a reaction in the reproductive tract and increases the growth of Brucella organisms [5]. The isolation and identification of the brucellosis-causing bacteria is the gold standard for diagnosis, but this method is risky, requires a high-security lab and highly qualified staff, and takes a while to complete. Therefore, the mainstay of diagnosis relies on the identification of antibodies in serum using serological tests like the Rapid Plate Agglutination Test (RPAT), Rose Bengal Test (RBT), Standard Agglutination Test (SAT), 2-Mercapto Ethanol Test (2ME), and the Complement Fixation Test (CFT). They offer a helpful benefit in detecting the prevalence of Brucella infection and are quite easy to carry out [6].

According to international data, approximately 500,000 instances of bovine brucellosis are reported worldwide each year. According to research, the illness causes an annual loss in cattle production of US 3.4 billion in economic terms. Consequently, it follows that the disease's toll in underdeveloped nations cannot be understated [7]. Therefore, if we are to control or eradicate the disease, we must evaluate the current control measures and their financial viability. We can better understand how infectious diseases like bovine brucellosis propagate by using mathematical modeling. In [8], a mathematical model was developed to explain how bovine brucellosis spreads among cattle. According to numerical simulation, control strategies should focus on lowering the parameters for infectious cattle's contact rate with susceptible and recovered animals and raising the treatment rate for infected cattle. Incorporating the impacts of seasonality, researchers have provided a mathematical model for the dynamics of brucellosis transmission [9]. Their findings demonstrate that seasonality has a significant impact on the long-term dynamics of brucellosis, which in turn affects how its best control techniques are developed. A mathematical model for the effects of various control options on the dynamics of Brucellosis transmission was developed and examined by researchers [10]. They concentrated on environmental hygiene and sanitation, personal protection in humans, progressive culling by killing seropositive cattle and small ruminants, and livestock vaccination. An individual-based model was presented by Nepomuceno et al. [11] to look at the dynamics and management of bovine brucellosis. The model concentrated on geographical elements like herd motion and varied populations. Additionally, it emphasized pulse interventions. In order to describe both within- and between-herd transmission of Brucella abortus in dairy cattle herds, a stochastic, age-structured model was constructed. This project gave valuable information about brucellosis control in India, which has the world's biggest population of cattle, as well as a basic framework for assessing control measures in endemic areas [12]. Researchers from all across the globe are paying close focus on mathematical modeling of novel infectious diseases that are emerging. An analysis of the effects of varying contact rates and the usage of face masks on the fluctuations of coronavirus infection in KSA was done mathematically in [13]. A mathematical model was created by researchers to examine the monkeypox infection in light of the epidemic's reported instances in the USA. They created a model and thoroughly explored its fundamental findings [14].

Nearly all branches of science use fractional calculus. Fractional differential equations have generated a lot of attention recently due to their wide range of applications in the physics and engineering areas [15–19]. Different mathematical models' genetic and memory qualities can be distinguished using fractional differential equations, which is their most notable characteristic. Hence, fractional order models seem more factual and empirical when compared to normal integer order models [20]. The Riemann–Liouville, Grünwald–Letnikov, and Caputo definitions of fractional-order derivatives are the most well-known and often applicable definitions. It is the most widely used fractional-order derivative in mathematical modeling because the initial conditions in the definition owed to Caputo can be represented in a way similar to the integer-order differentiation. To analyze the local and global stability of the equilibrium point(s) of a fractional-order system, the linearization, Lyapunov technique, and Lyapunov direct method have emerged recently [21]. The existence and controllability of nonlocal mixed Volterra–Fredholm type fractional delay integro-differential equations have been proven in the same line. Several derivatives of fractional differential operators, such as the Hilfer, nonsingular kernel type, and Caputo–Fabrizio operator, have recently been employed by a number of writers to study diverse scenarios. Each operator has unique qualities and flaws [22]. The Caputo–Fabrizio non-integer order Brucellosis model was studied in [23]. Using a set of values for the model parameters that were taken from the literature as their sources, the simulations were carried out using the iterative Laplace transform technique. The simulations' findings support the reliability and effectiveness of the Caputo–Fabrizio derivative for estimating the dynamics of the brucellosis disease. In [24], a new fractional-order model for the dynamics of brucellosis transmission with a focus on sheep-to-sheep transmission was investigated. Culling and vaccination rates were two control strategies that were included in the model. They specifically noticed that the disease disappears from the population if the rate of culling and vaccination is higher than 40% and 50%, respectively. It demonstrated the consistency, originality, and applicability of the model for the regulation of blood glucose concentration in healthy individuals and individuals with type 1 diabetes. The population of cattle and different animals, particularly buffalo and cows, are mostly affected by lumpy skin disease (LSD). By considering all potential paths for a disease to propagate within a population, [25] created a mathematical model to comprehend LSD. A fresh method for solving fractional order systems has been presented and used with the suggested model.

In the fractional order models, chaos makes a number of biological and physical elements extremely sensitive to the beginning conditions. Chaos was studied in fractional-order bio-mathematical models of the Ebola virus, diabetes, HIV, and dengue [26]. The existence of chaos in these models accelerates the evolution of the disease, making it undesirable. A modified Atangana–Baleanu Caputo derivative (MABC) fractional order model of the HIV/AIDS pandemic was discussed in [27]. Using the Laplace Adomian decomposition approach, the MABC model of HIV/AIDS was numerically solved. The reported numerical method took advantage of the recently constructed Newton polynomial. To study and monitor the dynamical transmission of the disease under the influence of vaccination, researchers introduced a novel fractional-order measles model employing a constant proportional (CP) Caputo operator [28]. Due to its numerous significant applications in resolving the complex dynamics of various complicated systems originating from various branches of science and engineering, discrete fractional calculus began to be a fascinating area of fractional calculus [29–31]. In addition, readers will be particularly interested in this area of fractional calculus due to the potential for using discrete fractional calculus to enhance certain artificial intelligence (AI) methodologies and procedures. Finding the discrete equivalent of a modified or generalized fractional operator is also a fascinating topic, largely because the continuous non-local operators' discrete counterpart has fundamentally distinct characteristics. Particularly interesting are the fractional continuous and discrete operators involving the Mittag-Leffler kernels [32,33]. A novel fractional discrete COVID-19 model was introduced and studied by Abbes et al. [34]. It was demonstrated that the dynamic behaviors of the model shift from stable to chaotic behavior by modifying fractional orders using maximum Lyapunov exponents, phase attractors, bifurcation diagrams, the 0–1 test, and approximation entropy. The fractional discrete model fits the true data of the epidemic, according to the results. Atangana

Baleanu–Caputo derivative with a nonsingular Mittag-Leffler function as its kernel served as the derivative in a study by Narayanan and colleagues [35] on the stability of a new fractional Nabla difference biological model of the glucose-insulin regulatory system in diabetes mellitus. It was assumed that the model for diabetes mellitus was a deterministic fractional Nabla difference model, which offers a better control strategy at fractional values for the creation of an artificial pancreas.

The discussion above served as the impetus for this study, which proposes a fractional order model with discrete generalized Mittag-Leffler kernels for the dynamics of the bovine brucellosis illness in cattle. For system dynamics models, researchers have recently concentrated on fractional order operators. Currently, both singular and nonsingular kernels have received much scholarly study. Though it can be challenging to select the best operator, researchers are always examining various operators to uncover new findings. This work generalizes the majority of the outcomes examined for the ABC operator however such a system has not been investigated for the operator indicated. Additionally, a lot of dynamical problems will have this work as a foundation for existence, uniqueness, and numerical simulations.

The manuscript is organized as follows: Section "Introduction" offers an evaluation of the literature and an introduction. We explain the basics of the fractional operator utilized in the suggested model in Section "Preliminaries". In Section "Proposed model", we propose a fractional order Bovine Brucellosis disease model with discrete generalized Mittag-Leffler kernels. The analysis of equilibrium states, the fundamental reproductive number $\mathfrak{R}_0$ and sensitivity analysis, the model's global asymptotic stability, potential region and well-posedness of responses, and positiveness and boundedness of solutions are all covered in Section "Analysis of proposed model". Additional analysis of the proposed operator is provided in Section "Further analysis on discrete Atangana–Baleanu, and modified Atangana–Baleanu operators". To analyze the impact of the proposed fractional operator, solutions are developed in the discrete generalized version of the Mittag-Leffler kernel in Section "Solution of proposed system of fractional differential equations". The results' discussion and significant conclusions of our analysis are covered in Sections "Results and discussion", "Conclusion".

Preliminaries

Here, we give a few key definitions that might be useful to analyze the system.

Definition 1 ([32]). For $\upsilon, \chi, \omega, t \in \mathbb{C}$ with $Re(\upsilon) > 0$, the generalized discrete Mittag-Leffler function is defined as:

$$\mathbf{E}_{\frac{\omega}{\upsilon,\chi}}(\Lambda, t) = \sum_{k=0}^{\infty} \Lambda^k \frac{t^{\overline{k\upsilon+\chi-1}}(\omega)_k}{\Gamma(k\upsilon+\chi)} \quad , \quad \{\forall \Lambda \in \mathbb{R} \ : \ |\Lambda| < 1\}, \tag{1}$$

where

$$(\omega)_k = \frac{\Gamma(\omega+k)}{\Gamma(k)}. \tag{2}$$

The discrete Mittag-Leffler function of two parameters can be found as follows when $\omega = 1$:

$$\mathbf{E}_{\frac{1}{\upsilon,\chi}}(\Lambda, t) = \sum_{k=0}^{\infty} \Lambda^k \frac{t^{\overline{k\upsilon+\chi-1}}}{\Gamma(k\upsilon+\chi)} \quad , \quad \{\forall \Lambda \in \mathbb{R} \ : \ |\Lambda| < 1\}. \tag{3}$$

The discrete Mittag-Leffler function of one parameter can be determined as follows when $\omega = \chi = 1$:

$$\mathbf{E}_{\frac{1}{\upsilon,1}}(\Lambda, t) = \mathbf{E}_{\overline{\upsilon}}(\Lambda, t) = \sum_{k=0}^{\infty} \Lambda^k \frac{t^{\overline{k\upsilon}}}{\Gamma(k\upsilon+1)} \quad , \quad \{\forall \Lambda \in \mathbb{R} \ : \ |\Lambda| < 1\}. \tag{4}$$

Lemma 2 ([32]). *Let* H *and* G *be functions defined on* $\mathbb{N}_a = \{a, a+1, a+2, \dots\}, a \in \mathbb{R}$. *Then, the discrete Laplace transform can be defined by*

$$\Big\{\mathfrak{L}_a \mathsf{H}(t)\Big\}\{\mathcal{S}\} = \sum_{t=a+1}^{\infty} \frac{\mathsf{H}(t)}{(1-\mathcal{S})^{a+1-t}}. \tag{5}$$

Also, the discrete Laplace transform of convolution of H *and* G *can be defined by*

$$\Big\{\mathfrak{L}_a(\mathsf{H} \circ \mathsf{G})(t)\Big\}\{\mathcal{S}\} = \big(\mathfrak{L}_a \mathsf{H}\big)\{\mathcal{S}\} * \big(\mathfrak{L}_a \mathsf{G}\big)\{\mathcal{S}\}. \tag{6}$$

Lemma 3 ([32,36]). *Let* H *be a function defined on* $\mathbb{N}_a$. *Then, the following result holds.*

$$\Big\{\mathfrak{L}_a\big[\nabla \mathsf{H}(t)\big]\Big\}\{\mathcal{S}\} = \mathcal{S}\big\{\mathfrak{L}_a \mathsf{H}(t)\big\}\{\mathcal{S}\} - \mathsf{H}(a). \tag{7}$$

More generally,

$$\Big\{\mathfrak{L}_a\big[\nabla^n \mathsf{H}(t)\big]\Big\}\{\mathcal{S}\} = \mathcal{S}^n\big\{\mathfrak{L}_a \mathsf{H}(t)\big\}\{\mathcal{S}\} - \sum_{i=0}^{n-1} \mathcal{S}^{(n-1-i)}\, \nabla^i \mathsf{H}(a+1). \tag{8}$$

Lemma 4 ([32,36]). *Let* υ *be any real number. Then, we find*

$$\Big\{\mathfrak{L}_a\big[{}^{RL}_{a}\nabla^{-\upsilon} \mathsf{H}(t)\big]\Big\}\{\mathcal{S}\} = \frac{1}{\mathcal{S}^{\upsilon}}\big\{\mathfrak{L}_a \mathsf{H}(t)\big\}\{\mathcal{S}\}. \tag{9}$$

Lemma 5 ([37]). *Let* $\upsilon, \chi, \Lambda, \mathcal{S} \in \mathbb{C}$ *with* $Re(\chi) > 0$. *If* $|\Lambda \mathcal{S}^{-\upsilon}| < 1$ *with* $Re(\mathcal{S}) > 0$, *then we have*

$$\Big\{\mathfrak{L}_{\upsilon} \mathbf{E}_{\overline{\upsilon,\chi}}(\Lambda, t-a)\Big\}\{\mathcal{S}\} = \frac{1}{\mathcal{S}^{\chi}(1-\Lambda \mathcal{S}^{-\upsilon})}. \tag{10}$$

In particular,

$$\Big\{\mathfrak{L}_{\upsilon} \mathbf{E}_{\overline{\upsilon,\upsilon}}(\Lambda, t-a)\Big\}\{\mathcal{S}\} = \frac{1}{\mathcal{S}^{\upsilon} - \Lambda}. \tag{11}$$

Definition 6 ([32,36]). Let H be defined on $\mathbb{N}_a \bigcap {}_b\mathbb{N}$, $a < b$, where ${}_b\mathbb{N} = \{b, b-1, \dots\}$ and $b \in \mathbb{R}$. Then, for $\Lambda = -\frac{\upsilon}{1-\upsilon}$ and $0 < \upsilon < \frac{1}{2}$, the left discrete generalized Atangana–Baleanu of the Liouville–Caputo type fractional difference is defined by

$${}^{AB}_{a}\nabla^{\upsilon} \mathsf{H}(t) = \frac{AB(\upsilon)}{1-\upsilon} \sum_{s=a+1}^{t} \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-s+1) \nabla \mathsf{H}(s) \quad , \quad \forall t \in \mathbb{N}_a, \tag{12}$$

where $AB(\upsilon) > 0$ such that $AB(0) = AB(1) = 1$. Also right discrete generalized Atangana–Baleanu of the Liouville–Caputo type fractional difference is defined by

$${}^{AB}\nabla^{\upsilon}_{b} \mathsf{H}(t) = \frac{-AB(\upsilon)}{1-\upsilon} \sum_{s=t}^{b-1} \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-s+1) \Delta \mathsf{H}(s) \quad , \quad \forall t \in {}_b\mathbb{N}. \tag{13}$$

The associated discrete Atangana–Baleanu fractional sum is defined by

$${}^{AB}_{a}\nabla^{-\upsilon} \mathsf{H}(t) = \frac{1-\upsilon}{AB(\upsilon)} \mathsf{H}(t) + \frac{\upsilon}{AB(\upsilon)} \big[{}_a\nabla^{-\upsilon} \mathsf{H}(t)\big] \quad , \quad \forall t \in \mathbb{N}_a. \tag{14}$$

Where ${}_a\nabla^{-\upsilon} \mathsf{H}(t)$ is the nabla left-sided Riemann–Liouville fractional sum of order υ, given by

$${}_a\nabla^{-\upsilon} \mathsf{H}(t) = \frac{1}{\Gamma(\upsilon)} \sum_{\phi=a+1}^{t} (t+1-\phi)^{\overline{\upsilon-1}} \mathsf{H}(\phi) \quad , \quad \forall t \in \mathbb{N}_{a+1}. \tag{15}$$

Definition 7 ([36]). Let H be defined on $\mathbb{N}_a \bigcap {}_b\mathbb{N}$, $a < b$. Then for $\Lambda = -\frac{\upsilon}{1-\upsilon}$ and $0 < \upsilon < \frac{1}{2}$, the left discrete modified Atangana–Baleanu of the Liouville–Caputo type fractional difference is defined by

$$\begin{aligned} {}^{MABC}_{a}\nabla^{\upsilon} \mathsf{H}(t) = \frac{AB(\upsilon)}{1-\upsilon} \Big\{ & \mathsf{H}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathsf{H}(a) \\ & + \Lambda \sum_{s=a+1}^{t} \mathbf{E}_{\overline{\upsilon,\upsilon}}(\Lambda, t-s+1)\mathsf{H}(s) \Big\}, \forall t \in \mathbb{N}_a. \end{aligned} \tag{16}$$

Also right discrete modified Atangana–Baleanu of the Liouville–Caputo type fractional difference is defined by

$$\begin{aligned} {}^{MABC}\nabla^{\upsilon}_{b} \mathsf{H}(t) = \frac{AB(\upsilon)}{1-\upsilon} \Big\{ & \mathsf{H}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, b-t)\mathsf{H}(b) \\ & + \Lambda \sum_{s=t}^{b-1} \mathbf{E}_{\overline{\upsilon,\upsilon}}(\Lambda, s-t+1)\mathsf{H}(s) \Big\}, \forall t \in {}_b\mathbb{N}. \end{aligned} \tag{17}$$

By generalizing using the same method as for $0 < \upsilon < 1$, we can get the following conclusions:

Definition 8 ([36]). Let H be defined on ${}_a\mathbb{N} \bigcap {}_b\mathbb{N}$, $a < b$. Let $j \in \mathbb{N}_0$ then for $\Lambda_j = -\frac{\upsilon - j}{j+1-\upsilon}$ and $j < \upsilon < j + \frac{1}{2}$, the left discrete modified Atangana–Baleanu of the Liouville–Caputo type fractional difference of a higher order is defined by

$$\begin{aligned} {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{H}(t) &= {}^{MABC}_{a}\nabla^{\upsilon-j}\nabla^{j}f(t) \\ &= \frac{AB(\upsilon-j)}{j+1-\upsilon}\left\{\nabla^{j}f(t) - \mathbf{E}_{\overline{\upsilon-j}}(\Lambda_j, t-a)\nabla^{j}f(a) \right. \\ &\quad \left. + \Lambda_j \sum_{s=a+1}^{t} \mathbf{E}_{\overline{\upsilon-j,\upsilon-j}}(\Lambda_j, t-s+1)\nabla^{j}f(s)\right\} \end{aligned} \tag{18}$$

$\forall t \in \mathbb{N}_a$.

Moreover, $\forall t \in \mathbb{N}_a$ right discrete modified Atangana–Baleanu of the Liouville–Caputo type fractional difference of a higher order is defined by

$$\begin{aligned} {}^{MABC}\nabla^{\upsilon}_{b}\mathrm{H}(t) &= {}^{MABC}\nabla^{\upsilon-j}_{b}\nabla^{j}f(t) \\ &= \frac{AB(\upsilon-j)}{j+1-\upsilon}\left\{\nabla^{j}f(t) - \mathbf{E}_{\overline{\upsilon-j}}(\Lambda_j, b-t)\nabla^{j}f(a) \right. \\ &\quad \left. + \Lambda_j \sum_{s=t}^{b-1} \mathbf{E}_{\overline{\upsilon-j,\upsilon-j}}(\Lambda_j, s-t+1)\nabla^{j}f(s)\right\} \end{aligned} \tag{19}$$

$\forall t \in {}_b\mathbb{N}$.

Proposed model

To analyze the impact of various treatments on the spread of bovine brucellosis, a deterministic model [7] is suggested. The cattle population is separated into five classes, including the susceptible (S), infected (I), recovered (R), biosecured (B), and vaccinated (V), Cattle that are not immune and are prone to getting brucellosis are classified as vulnerable. Cattle that have caught the disease and may infect other animals are classified as belonging to the infected class. Cattle that have recovered from infections with bovine brucellosis belong to the recovered class. Cattle of the biosecured class are incapable of contracting bovine brucellosis due to their strict adherence to biosecurity guidelines. Cattle who have received a temporary immunity vaccination are represented by the vaccinated class. $\mathrm{S}(t)$, $\mathrm{I}(t)$, $\mathrm{R}(t)$, $\mathrm{B}(t)$, and $\mathrm{V}(t)$ are the abbreviations for the proportions of cattle in compartments S, I, R, B, and V at time "t", respectively. We assume that

- the rate of cattle recruitment into the susceptible class is $\}\}\vartheta\varepsilon$, while the rate of livestock vaccination is $\}\}\eta\mathrm{S}\varepsilon$.
- the immune-boosted animals return to the susceptible compartment at a rate of $\}\}\sigma\mathrm{V}\varepsilon$ and lose their protection. While thorough biosecurity precautions are followed, cattle move from the vulnerable compartment to the biosecured compartment at a rate of $\}\}\rho\mathrm{S}\varepsilon$.
- at a rate of $\}\}\beta\varepsilon$, infected cattle spread bovine brucellosis to the susceptibles.
- the diseased cattle infected with the infection recover and then move into the compartment that has recovered at a rate of $\}\}\delta\mathrm{I}\varepsilon$. Moreover, the diseased people are killed off at a rate of $\}\}\zeta\mathrm{I}\varepsilon$.
- the restored animals gradually lose their innate resistance before rejoining the susceptible class at a rate of $\}\}\lambda\mathrm{R}\varepsilon$.
- the rates of natural death and disease-induced death are seen as being $\}\}\varphi\varepsilon$ and $\}\}\xi\varepsilon$, respectively.

The following non-linear fractional differential equations are used to formulate the new fractional-order model under left discrete modified Atangana–Baleanu of the Liouville–Caputo type fractional difference:

$$\begin{cases} {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t) = \vartheta - \beta\mathrm{SI} - (\varphi+\eta+\rho)\mathrm{S} + \lambda\mathrm{R} + \sigma\mathrm{V}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t) = \beta\mathrm{SI} - (\varphi+\xi+\delta+\zeta)\mathrm{I}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t) = \delta\mathrm{I} - (\varphi+\lambda)\mathrm{R}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t) = \rho\mathrm{S} - \varphi\mathrm{B}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t) = \eta\mathrm{S} - (\varphi+\sigma)\mathrm{V}. \end{cases} \tag{20}$$

with non-negative initial constraints,

$$\mathrm{S}(0) \geq 0, \quad \mathrm{I}(0) \geq 0, \quad \mathrm{R}(0) \geq 0, \quad \mathrm{B}(0) \geq 0, \quad \mathrm{V}(0) \geq 0. \tag{21}$$

Where $0 < \upsilon < \frac{1}{2}$.

Analysis of proposed model

Positive-ness and bounded-ness of solutions

We investigate the conditions that ensure the positivity of the proposed model's solutions to show that they are fine and restricted, assuming that they include real-world conditions with relevant values. For this, we have

$$\mathrm{I}(t) \geq \mathrm{I}(0)e^{-(\varphi+\xi+\delta+\zeta)t} \quad , \quad \forall t \geq 0. \tag{22}$$

$$\mathrm{R}(t) \geq \mathrm{R}(0)e^{-(\varphi+\lambda)t} \quad , \quad \forall t \geq 0. \tag{23}$$

$$\mathrm{B}(t) \geq \mathrm{B}(0)e^{-(\varphi)t} \quad , \quad \forall t \geq 0. \tag{24}$$

$$\mathrm{V}(t) \geq \mathrm{V}(0)e^{-(\varphi+\sigma)t} \quad , \quad \forall t \geq 0. \tag{25}$$

Define the norm

$$\| \mathcal{U} \|_{\infty} = \sup_{t \in D_{\mathcal{U}}} |\mathcal{U}(t)|, \tag{26}$$

such that $D_{\mathcal{U}}$ is the domain of $\mathcal{U}$. Using this norm, we have for the function $\mathrm{S}(t)$;

$$\begin{aligned} {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t) &= \vartheta - \beta\mathrm{SI} - (\varphi+\eta+\rho)\mathrm{S} + \lambda\mathrm{R} + \sigma\mathrm{V} \geq -\beta\mathrm{SI} - (\varphi+\eta+\rho)\mathrm{S} \\ &\geq -\Big[\beta|\mathrm{I}| + (\varphi+\eta+\rho)\Big]\mathrm{S} \ \geq -\Big[\beta\sup_{t\in D_1}|\mathrm{I}| + (\varphi+\eta+\rho)\Big]\mathrm{S} \\ &= -\Big[\beta|\mathrm{I}|_{\infty} + (\varphi+\eta+\rho)\Big]\mathrm{S}. \end{aligned} \tag{27}$$

One can find,

$$\mathrm{S}(t) \geq \mathrm{S}(0)\ e^{-\left(\beta|\mathrm{I}|_{\infty}+(\varphi+\eta+\rho)\right)t} \quad , \quad \forall t \geq 0. \tag{28}$$

Positively invariant region

Lemma 9. *If*

$$\mathfrak{M} = \left\{(\mathrm{S},\mathrm{I},\mathrm{R},\mathrm{B},\mathrm{V}) \in \mathbb{R}^5_+ : 0 \leq \mathbf{N}(t) \leq \frac{\vartheta}{\varphi}\right\}, \tag{29}$$

then the region $\mathfrak{M}$ attracts all solutions of the proposed system in $\mathbb{R}^5_+$ and is positively invariant when applied to non-negative starting conditions if $\mathbf{N}(0) \leq \frac{\vartheta}{\varphi}$.

Proof. We shall demonstrate the system (20)'s positive solution, and the outcomes are given as follows:

$$\begin{cases} {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t)\Big|_{\mathrm{S}=0} = \vartheta + \lambda\mathrm{R} + \sigma\mathrm{V} \geq 0, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t)\Big|_{\mathrm{I}=0} = 0, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t)\Big|_{\mathrm{R}=0} = \delta\mathrm{I} \geq 0, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t)\Big|_{\mathrm{B}=0} = \rho\mathrm{S} \geq 0, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t)\Big|_{\mathrm{V}=0} = \eta\mathrm{S} \geq 0. \end{cases} \tag{30}$$

The system (30) states that the vector field is located on each hyperplane containing the non-negative orthant with $t \geq 0$ in the area $\mathbb{R}^5_+$.

After adding the component portions of the human population in model (20), we get the following total population:

$$\begin{aligned} {}_{a}^{MABC}\nabla^{\upsilon}\mathbf{N}(t) &= {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t) + {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t) \\ &\quad + {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t) + {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t) + {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t) \\ &= \vartheta - \varphi(\mathrm{S}+\mathrm{I}+\mathrm{R}+\mathrm{B}+\mathrm{V}) - (\xi+\zeta)\mathrm{I} = \vartheta - \varphi\mathbf{N} - (\xi+\zeta)\mathrm{I} \\ &\leq \vartheta - \varphi\mathbf{N}. \end{aligned} \tag{31}$$

Suppose that $\mathbf{N}(0) \leq \frac{\vartheta}{\varphi}$

$$\Rightarrow \quad \mathbf{N}(t) \leq \frac{\vartheta}{\varphi}. \tag{32}$$

Hence, for every $t > 0$, a solution of the fractional model (20) exists in $\mathfrak{M}$. As a result, the closed set $\mathfrak{M}$ is positively invariant with regard to the fractional model. Therefore, we can test our model (20) in the feasible region:

$$\mathfrak{M} = \left\{(\mathrm{S},\mathrm{I},\mathrm{R},\mathrm{B},\mathrm{V}) \in \mathbb{R}_{+}^{5} : \mathbf{N}(t) \leq \frac{\vartheta}{\varphi}\right\}. \quad \square \tag{33}$$

Equilibrium points

Constant functions are used to represent the equilibrium states of a system of differential equations with constant parameters. An equilibrium in our case is represented by

$$\begin{aligned} {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t) &= {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t) = {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t) = {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t) \\ &= {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t) = 0. \end{aligned} \tag{34}$$

We have the disease-free equilibrium states ($\mathcal{E}^0$) as:

$$\begin{aligned} \mathcal{E}^0 &= \{\mathrm{S}^0,\ \mathrm{I}^0,\ \mathrm{R}^0,\ \mathrm{B}^0,\ \mathrm{V}^0\} \\ &= \left\{ \frac{(\varphi+\sigma)\vartheta}{\varphi(\varphi+\sigma+\eta)+\rho(\varphi+\sigma)},\ 0,\ 0,\ \frac{\rho(\varphi+\sigma)\vartheta}{\varphi\left[\,\varphi(\varphi+\sigma+\eta)+\rho(\varphi+\sigma)\,\right]}, \right. \\ &\quad \left. \frac{\eta\vartheta}{\varphi(\varphi+\sigma+\eta)+\rho(\varphi+\sigma)} \right\}. \end{aligned} \tag{35}$$

When there is an infection, endemic equilibrium happens. To reach the endemic equilibrium values ($\mathcal{E}^*$), the right side of the system's Eqs. (20) must be set to zero. We have

$$\mathcal{E}^* = \begin{cases} \mathrm{S}^* = \frac{\Xi_1}{\beta}, \\ \mathrm{I}^* = -\frac{(\sigma+\varphi)\left[(\varphi^2+\rho\lambda)\Xi_1+\varphi(\lambda+\rho)\Xi_4\right]+\varphi^2\eta\Xi_1+\zeta(\lambda+\rho)(\varphi^2+\sigma)}{\beta(\sigma+\varphi)(\lambda\Xi_3+\varphi\Xi_2)}, \\ \mathrm{R}^* = -\frac{\delta\left[\varphi\eta\Xi_2+(\sigma+\varphi)(\rho+\varphi)\Xi_1-\beta\vartheta(\sigma+\varphi)\right]}{\beta(\sigma+\varphi)(\lambda\Xi_3+\varphi\Xi_2)}, \\ \mathrm{B}^* = \frac{\Xi_1\rho}{\beta\varphi}, \\ \mathrm{V}^* = \frac{\Xi_1\eta}{\beta(\sigma+\varphi)}. \end{cases} \tag{36}$$

Where,

$$\begin{aligned} \Xi_1 &= \zeta+\xi+\varphi+\delta \quad , \quad \Xi_3 = \zeta+\xi+\varphi\ , \\ \Xi_2 &= \zeta+\xi+\lambda+\delta \quad , \quad \Xi_4 = \xi+\varphi+\delta. \end{aligned} \tag{37}$$

Reproductive number

We utilize the next generation matrix technique [38] on system (20) to obtain the reproductive number ($\mathfrak{R}$) as given below. next generation matrices F and V are;

$$F = \begin{pmatrix} -\beta\mathrm{I} & -\beta\mathrm{S} & 0 & 0 & 0 \\ \beta\mathrm{I} & \beta\mathrm{S} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \varphi+\eta+\rho & 0 & -\lambda & 0 & -\sigma \\ 0 & \varphi+\xi+\delta+\zeta & 0 & 0 & 0 \\ 0 & -\delta & \varphi+\lambda & 0 & 0 \\ -\rho & 0 & 0 & \varphi & 0 \\ -\eta & 0 & 0 & 0 & \sigma+\varphi \end{pmatrix},$$

Table 1
$\mathfrak{R}$ Parameters sensitivity indices.

Parameter	Index	Parameter	Index	Parameter	Index
β	+1.0000	δ	−0.0293	φ	−0.5218
ϑ	+1.0000	ξ	−0.5574	ζ	−0.4107
σ	+0.4904	ρ	−0.2425	η	−0.7355

and

$$F(\mathcal{E}^0) = \begin{pmatrix} 0 & -\beta\mathrm{S}^0 & 0 & 0 & 0 \\ 0 & \beta\mathrm{S}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $det\left|F(\mathcal{E}^0)V^{-1} - \lambda I\right| = 0$ yields the reproductive number ($\mathfrak{R}$):

$$\mathfrak{R} = \frac{\beta\mathrm{S}^0}{\zeta+\xi+\varphi+\delta}, \qquad where \ \ \mathrm{S}^0 = \frac{(\varphi+\sigma)\vartheta}{\varphi(\varphi+\sigma+\eta)+\rho(\varphi+\sigma)}. \tag{38}$$

Strength number

These days, the "Strength Number", an extension of the reproduction number, is currently undergoing numerous evaluations to see if it can be utilized to detect spread complexity, or at the very least, if it can detect waves in a diffusion. We determine the strength value "$\mathfrak{S}^0$" by computing the second derivative of the infected compartments using the next-generation matrix.

$$\frac{\partial^2}{\partial \mathrm{I}^2}\left(\frac{\beta\mathrm{SI}}{\mathbf{N}}\right) = \beta\mathrm{S}\frac{\partial^2}{\partial \mathrm{I}^2}\left(\frac{(\mathbf{N}-\mathrm{I})}{\mathbf{N}^3}\right) = -\frac{2\beta\mathrm{S}\left|\mathbf{N}-\mathrm{I}\right|}{\mathbf{N}^3} = -\frac{2\beta\mathrm{S}(\mathrm{S}+\mathrm{R}+\mathrm{B}+\mathrm{V})}{(\mathrm{S}+\mathrm{I}+\mathrm{R}+\mathrm{B}+\mathrm{V})^3}. \tag{39}$$

At disease-free equilibrium, we have

$$\left(-\frac{2\beta\mathrm{S}(\mathrm{S}+\mathrm{R}+\mathrm{B}+\mathrm{V})}{(\mathrm{S}+\mathrm{I}+\mathrm{R}+\mathrm{B}+\mathrm{V})^3}\right) = -\frac{2\beta\mathrm{S}^0}{(\mathrm{S}^0+\mathrm{B}^0+\mathrm{V}^0)^2}. \tag{40}$$

Hence, $\det\left|F(\mathcal{E}^0)\mathcal{V}^{-1} - \lambda I\right| = 0$ produces strength number $\mathcal{S}^0$:

$$\mathfrak{S}^0 = -\frac{2\beta\mathrm{S}^0}{(\zeta+\xi+\varphi+\delta)(\mathrm{S}^0+\mathrm{B}^0+\mathrm{V}^0)^2}. \tag{41}$$

An infection that declines quickly from the disease-free equilibrium and then rises after a minimal point before stabilizing or ceasing later is suggested by a negative strength value, as is an infection that has two infection factors at the maximum indicating a single wave. This will be confirmed by looking at the sign of the second derivative of infected compartments.

Sensitivity indices of reproductive ($\mathfrak{R}$)′s parameters

If the major impact of each element in the dynamics of the disease is taken into consideration, the incidence and prevalence of bovine brucellosis can be decreased or managed. It explores how each parameter affects the basic reproduction number $\mathfrak{R}$. The following table includes all parameter's sensitivity indices (see Table 1).

Existence and uniqueness of the proposed model

In this part, we demonstrate that the proposed system's coefficients meet the Lipschitz and linear growth conditions, demonstrating the existence and uniqueness of the solution. We now recall our model

$$\begin{cases} {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t) = \vartheta - \beta\mathrm{SI} - (\varphi+\eta+\rho)\mathrm{S} + \lambda\mathrm{R} + \sigma\mathrm{V}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t) = \beta\mathrm{SI} - (\varphi+\xi+\delta+\zeta)\mathrm{I}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t) = \delta\mathrm{I} - (\varphi+\lambda)\mathrm{R}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t) = \rho\mathrm{S} - \varphi\mathrm{B}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t) = \eta\mathrm{S} - (\varphi+\sigma)\mathrm{V}. \end{cases} \tag{42}$$

Let $\mathcal{G} = (\mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}) \in \mathcal{H}$, where $\mathcal{H} = [\mathrm{C}([0, \mathbb{T}], \mathbb{R}_+)]^5$ is a Banach space built with the norm:

$$\|\mathcal{G}\| = \sup_{0\le t\le \mathbb{T}} \Big[|\mathrm{S}(t)| + |\mathrm{I}(t)| + |\mathrm{R}(t)| + |\mathrm{B}(t)| + |\mathrm{V}(t)|\Big],$$

also let $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5)$ such that

$$\begin{cases} \mathcal{M}_1(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}) = \vartheta - \beta \mathrm{SI} - (\varphi + \eta + \rho)\mathrm{S} + \lambda \mathrm{R} + \sigma \mathrm{V} \\ \mathcal{M}_2(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}) = \beta \mathrm{SI} - (\varphi + \xi + \delta + \zeta)\mathrm{I}, \\ \mathcal{M}_3(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}) = \delta \mathrm{I} - (\varphi + \lambda)\mathrm{R}, \\ \mathcal{M}_4(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}) = \rho \mathrm{S} - \varphi \mathrm{B}, \\ \mathcal{M}_5(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}) = \eta \mathrm{S} - (\varphi + \sigma)\mathrm{V}, \end{cases} \tag{43}$$

We can observe that the function $\mathcal{M} \in [0, \mathbb{T}] \times \mathcal{H}^5$ is continuous. And

$$\mathcal{G}(t) = \begin{cases} \mathrm{S}(t) \\ \mathrm{I}(t) \\ \mathrm{R}(t) \\ \mathrm{B}(t) \\ \mathrm{V}(t) \end{cases} \quad \mathcal{G}_0(t) = \begin{cases} \mathrm{S}_0 \\ \mathrm{I}_0 \\ \mathrm{R}_0 \\ \mathrm{B}_0 \\ \mathrm{V}_0 \end{cases} = \begin{cases} \mathcal{M}_1(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}), \\ \mathcal{M}_2(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}), \\ \mathcal{M}_3(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}), \\ \mathcal{M}_4(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}), \\ \mathcal{M}_5(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}). \end{cases}$$

We must validate the following theorem in order to demonstrate the existence and uniqueness of the solution.

Theorem 10. *Assume the existence of positive constants, ϖ_i, ψ_i, such that*

$$(i)\quad |\mathcal{M}_i(\mathcal{G}, t)|^2 \le \varpi_i(1 + |\mathcal{G}|^2) \quad , \quad \forall (\mathcal{G}, t) \in \mathbb{R}^5 \times [0, \mathbb{T}]. \tag{44}$$

$$(ii)\quad |\mathcal{M}_i(\mathcal{G}_1, t) - \mathcal{M}_i(\mathcal{G}_2, t)|^2 \le \psi_i |\mathcal{G}_1 - \mathcal{G}_2|^2 \quad , \quad \forall i \in \{1, 2, \ldots, 5\}. \tag{45}$$

Proof. We begin with the linear growth.

$$\begin{cases} |\mathcal{M}_1(\mathrm{S}, t)|^2 = |\vartheta - \beta \mathrm{SI} - (\varphi + \eta + \rho)\mathrm{S} + \lambda \mathrm{R} + \sigma \mathrm{V}|^2 \\ \qquad\qquad = |\vartheta + \lambda \mathrm{R} + \sigma \mathrm{V} - (\beta \mathrm{I} + \varphi + \eta + \rho)\mathrm{S}|^2, \\ |\mathcal{M}_1(\mathrm{S}, t)|^2 \le 2\{\vartheta + \lambda|\mathrm{R}| + \sigma|\mathrm{V}|\}^2 + 2\{\beta|\mathrm{I}| + \varphi + \eta + \rho\}^2|\mathrm{S}|^2, \\ |\mathcal{M}_1(\mathrm{S}, t)|^2 \le 2\{\vartheta + \lambda \sup_{0\le t\le T}|\mathrm{R}| + \sigma \sup_{0\le t\le T}|\mathrm{V}|\}^2 \\ \qquad\qquad +2\{\beta \sup_{0\le t\le T}|\mathrm{I}| + \varphi + \eta + \rho\}^2|\mathrm{S}|^2, \\ |\mathcal{M}_1(\mathrm{S}, t)|^2 \le 2\{\vartheta + \lambda|\mathrm{R}|_\infty + \sigma|\mathrm{V}|_\infty\}^2 + 2\{\beta|\mathrm{I}|_\infty + \varphi + \eta + \rho\}^2|\mathrm{S}|^2, \\ |\mathcal{M}_1(\mathrm{S}, t)|^2 \le 2\{\vartheta + \lambda|\mathrm{R}|_\infty + \sigma|\mathrm{V}|_\infty\}^2\Big[1 + \frac{\{\beta|\mathrm{I}|_\infty + \varphi + \eta + \rho\}^2}{\{\vartheta + \lambda|\mathrm{R}|_\infty + \sigma|\mathrm{V}|_\infty\}^2}|\mathrm{S}|^2\Big] \\ \qquad\qquad \le \varpi_1(1 + |\mathrm{S}|^2), \end{cases} \tag{46}$$

under the condition $\left\{\frac{\{\beta|\mathrm{I}|_\infty + \varphi + \eta + \rho\}^2}{\{\vartheta + \lambda|\mathrm{R}|_\infty + \sigma|\mathrm{V}|_\infty\}^2}\right\} < 1$,

$$\begin{cases} |\mathcal{M}_2(\mathrm{I}, t)|^2 = |\beta \mathrm{SI} - (\varphi + \xi + \delta + \zeta)\mathrm{I}|^2 |\mathcal{M}_2(\mathrm{I}, t)|^2 \le 2 \\ \qquad\qquad +2\{\beta|\mathrm{S}| + \varphi + \xi + \delta + \zeta\}^2|\mathrm{I}|^2, \\ |\mathcal{M}_2(\mathrm{I}, t)|^2 \le 2 + 2\{\beta \sup_{0\le t\le T}|\mathrm{S}| + \varphi + \xi + \delta + \zeta\}^2|\mathrm{I}|^2, \\ |\mathcal{M}_2(\mathrm{I}, t)|^2 \le 2 + 2\{\beta|\mathrm{S}|_\infty + \varphi + \xi + \delta + \zeta\}^2|\mathrm{I}|^2, \\ |\mathcal{M}_2(\mathrm{I}, t)|^2 \le 2\Big[1 + \frac{\{\beta|\mathrm{S}|_\infty + \varphi + \xi + \delta + \zeta\}^2}{2}|\mathrm{I}|^2\Big] \le \varpi_2(1 + |\mathrm{I}|^2), \end{cases} \tag{47}$$

under the condition $\left\{\frac{\{\beta|\mathrm{S}|_\infty + \varphi + \xi + \delta + \zeta\}^2}{2}\right\} < 1$,

$$\begin{cases} |\mathcal{M}_3(\mathrm{R}, t)|^2 = |\delta \mathrm{I} - (\varphi + \lambda)\mathrm{R}|^2 \le 2\{\delta|\mathrm{I}|\}^2 + 2\{\varphi + \lambda\}^2|\mathrm{R}|^2, \\ |\mathcal{M}_3(\mathrm{R}, t)|^2 \le 2\{\delta \sup_{0\le t\le T}|\mathrm{I}|\}^2 + 2\{\varphi + \lambda\}^2|\mathrm{R}|^2, \\ |\mathcal{M}_3(\mathrm{R}, t)|^2 \le 2\{\delta|\mathrm{I}|_\infty\}^2 + 2\{\varphi + \lambda\}^2|\mathrm{R}|^2, \\ |\mathcal{M}_3(\mathrm{R}, t)|^2 \le 2\{\delta|\mathrm{I}|_\infty\}^2\Big[1 + \frac{\{\varphi + \lambda\}^2}{\{\delta|\mathrm{I}|_\infty\}^2}|\mathrm{R}|^2\Big] \le \varpi_3(1 + |\mathrm{R}|^2), \end{cases} \tag{48}$$

under the condition $\left\{\frac{\{\varphi + \lambda\}^2}{\{\delta|\mathrm{I}|_\infty\}^2}\right\} < 1$,

$$\begin{cases} |\mathcal{M}_4(\mathrm{B}, t)|^2 = |\rho \mathrm{S} - \varphi \mathrm{B}|^2 \le 2\{\rho|\mathrm{S}|\}^2 + 2\{\varphi\}^2|\mathrm{B}|^2, \\ |\mathcal{M}_4(\mathrm{B}, t)|^2 \le 2\{\rho \sup_{0\le t\le T}|\mathrm{S}|\}^2 + 2\{\varphi\}^2|\mathrm{B}|^2 \le 2\{\rho|\mathrm{S}|_\infty\}^2 + 2\{\varphi\}^2|\mathrm{B}|^2, \\ |\mathcal{M}_4(\mathrm{B}, t)|^2 \le 2\{\rho|\mathrm{S}|_\infty\}^2\Big[1 + \frac{\{\varphi\}^2}{\{\rho|\mathrm{S}|_\infty\}^2}|\mathrm{B}|^2\Big] \le \varpi_4(1 + |\mathrm{B}|^2), \end{cases} \tag{49}$$

under the condition $\left\{\frac{\{\varphi\}^2}{\{\rho|\mathrm{S}|_\infty\}^2}\right\} < 1$,

$$\begin{cases} |\mathcal{M}_5(\mathrm{V}, t)|^2 = |\eta \mathrm{S} - (\varphi + \sigma)\mathrm{V}|^2 \le 2\{\eta|\mathrm{S}|\}^2 + 2\{\varphi + \sigma\}^2|\mathrm{V}|^2, \\ |\mathcal{M}_5(\mathrm{V}, t)|^2 \le 2\{\eta \sup_{0\le t\le T}|\mathrm{S}|\}^2 + 2\{\varphi + \sigma\}^2|\mathrm{V}|^2, \\ |\mathcal{M}_5(\mathrm{V}, t)|^2 \le 2\{\eta|\mathrm{S}|_\infty\}^2 + 2\{\varphi + \sigma\}^2|\mathrm{V}|^2, \\ |\mathcal{M}_5(\mathrm{V}, t)|^2 \le 2\{\eta|\mathrm{S}|_\infty\}^2\Big[1 + \frac{\{\varphi + \sigma\}^2}{\{\eta|\mathrm{S}|_\infty\}^2}|\mathrm{V}|^2\Big] \le \varpi_5(1 + |\mathrm{V}|^2), \end{cases} \tag{50}$$

under the condition $\left\{\frac{\{\varphi + \sigma\}^2}{\{\eta|\mathrm{S}|_\infty\}^2}\right\} < 1$.

The function thus meets the requirement for growth. We will check the Lipschitz condition now. We start with the function $\mathcal{M}_1(\mathcal{G}, t)$. Then, we will show that

$$|\mathcal{M}_1(\mathrm{S}_1, t) - \mathcal{M}_1(\mathrm{S}_2, t)|^2 \le \psi_1 |\mathrm{S}_1 - \mathrm{S}_2|^2. \tag{51}$$

Then, we write

$$\begin{aligned} |\mathcal{M}_1(\mathrm{S}_1, t) - \mathcal{M}_1(\mathrm{S}_2, t)|^2 &= |-(\beta \mathrm{I})(\mathrm{S}_1 - \mathrm{S}_2) - (\varphi + \eta + \rho)(\mathrm{S}_1 - \mathrm{S}_2)|^2 \\ &= |-(\beta \mathrm{I} + \varphi + \eta + \rho)(\mathrm{S}_1 - \mathrm{S}_2)|^2 \\ &= |(\beta \mathrm{I} + \varphi + \eta + \rho)(\mathrm{S}_1 - \mathrm{S}_2)|^2 \\ |\mathcal{M}_1(\mathrm{S}_1, t) - \mathcal{M}_1(\mathrm{S}_2, t)|^2 &\le \Big\{(2\beta^2|\mathrm{I}|^2 + 2\varphi^2 + 2\eta^2 + 2\rho^2)\Big\}(\mathrm{S}_1 - \mathrm{S}_2)|^2 \\ &\le \Big\{(2\beta^2 \sup_{0\le t\le T}|\mathrm{I}|^2 + 2\varphi^2 + 2\eta^2 + 2\rho^2)\Big\} \\ &\quad \times (\mathrm{S}_1 - \mathrm{S}_2)|^2 \\ &= \Big\{(2\beta^2|\mathrm{I}|_\infty^2 + 2\varphi^2 + 2\eta^2 + 2\rho^2)\Big\}(\mathrm{S}_1 - \mathrm{S}_2)|^2 \end{aligned}$$

$$\Rightarrow\ |\mathcal{M}_1(\mathrm{S}_1, t) - \mathcal{M}_1(\mathrm{S}_2, t)|^2 \le \psi_1 |\mathrm{S}_1 - \mathrm{S}_2|^2. \tag{52}$$

Where

$$\psi_1 = \Big\{(2\beta^2|\mathrm{I}|_\infty^2 + 2\varphi^2 + 2\eta^2 + 2\rho^2)\Big\}. \tag{53}$$

And

$$\begin{aligned} |\mathcal{M}_2(\mathrm{I}_1, t) - \mathcal{M}_2(\mathrm{I}_2, t)|^2 &= |(\beta \mathrm{S})(\mathrm{I}_1 - \mathrm{I}_2) - (\varphi + \xi + \delta + \zeta)(\mathrm{I}_1 - \mathrm{I}_2)|^2 \\ &= |(\beta \mathrm{S} - \varphi - \xi - \delta - \zeta)(\mathrm{I}_1 - \mathrm{I}_2)|^2 \\ |\mathcal{M}_2(\mathrm{I}_1, t) - \mathcal{M}_2(\mathrm{I}_2, t)|^2 &\le \{2\beta^2|I|^2 + 2\varphi^2 + 2\xi^2 + 2\delta^2 + 2\zeta^2\}(\mathrm{I}_1 - \mathrm{I}_2)|^2 \\ &\le \{2\beta^2 \sup_{0\le t\le T}|I|^2 + 2\varphi^2 + 2\xi^2 + 2\delta^2 + 2\zeta^2\} \\ &\quad \times (\mathrm{I}_1 - \mathrm{I}_2)|^2 \\ &= \{2\beta^2|I|_\infty^2 + 2\varphi^2 + 2\xi^2 + 2\delta^2 + 2\zeta^2\}(\mathrm{I}_1 - \mathrm{I}_2)|^2 \end{aligned}$$

$$\Rightarrow\ |\mathcal{M}_2(\mathrm{I}_1, t) - \mathcal{M}_2(\mathrm{I}_2, t)|^2 \le \psi_2 |(\mathrm{I}_1 - \mathrm{I}_2)|^2. \tag{54}$$

where

$$\psi_2 = \{2\varphi^2 + 2\xi^2 + 2\delta^2 + 2\zeta^2\}. \tag{55}$$

$$\begin{aligned} |\mathcal{M}_3(\mathrm{R}_1, t) - \mathcal{M}_3(\mathrm{R}_2, t)|^2 &= |-(\varphi + \lambda)(\mathrm{R}_1 - \mathrm{R}_2)|^2 \\ &= |(\varphi + \lambda)(\mathrm{R}_1 - \mathrm{R}_2)|^2 \\ |\mathcal{M}_3(\mathrm{R}_1, t) - \mathcal{M}_3(\mathrm{R}_2, t)|^2 &\le (2\varphi^2 + 2\lambda^2)|(\mathrm{R}_1 - \mathrm{R}_2)|^2 \end{aligned}$$

$$\Rightarrow\ |\mathcal{M}_3(\mathrm{R}_1, t) - \mathcal{M}_3(\mathrm{R}_2, t)|^2 \le \psi_3 |(\mathrm{R}_1 - \mathrm{R}_2)|^2, \tag{56}$$

where

$$\psi_3 = \{2\varphi^2 + 2\lambda^2\}. \tag{57}$$

$$\begin{aligned} |\mathcal{M}_4(\mathrm{B}_1, t) - \mathcal{M}_4(\mathrm{B}_2, t)|^2 &= |-(\varphi)(\mathrm{B}_1 - \mathrm{B}_2)|^2 \\ &= |(\varphi)(\mathrm{B}_1 - \mathrm{B}_2)|^2 \end{aligned}$$

$$|\mathcal{M}_4(B_1,t)-\mathcal{M}_4(B_2,t)|^2 \le (2\varphi^2)|(B_1-B_2)|^2$$
$$\Rightarrow\ |\mathcal{M}_4(B_1,t)-\mathcal{M}_4(B_2,t)|^2 \le \psi_4|(B_1-B_2)|^2, \tag{58}$$

where

$$\psi_4 = \{2\varphi^2\}. \tag{59}$$

$$|\mathcal{M}_5(V_1,t)-\mathcal{M}_5(V_2,t)|^2 = |-(\varphi+\sigma)(V_1-V_2)|^2$$
$$= |(\varphi+\sigma)(V_1-V_2)|^2$$
$$|\mathcal{M}_5(V_1,t)-\mathcal{M}_5(V_2,t)|^2 \le (2\varphi^2+2\sigma^2)|(V_1-V_2)|^2$$
$$\Rightarrow\ |\mathcal{M}_5(V_1,t)-\mathcal{M}_5(V_2,t)|^2 \le \psi_5|(V_1-V_2)|^2, \tag{60}$$

where

$$\psi_5 = \{2\varphi^2+2\sigma^2\}. \tag{61}$$

Then, given the circumstances, the answer to our system exists and is unique.

$$Max\begin{cases}\left\{\frac{\{\beta|I|_\infty+\varphi+\eta+\rho\}^2}{\{\vartheta+\lambda|R|_\infty+\sigma|V|_\infty\}^2}\right\},\\ \left\{\frac{\{\beta|S|_\infty+\varphi+\xi+\delta+\zeta\}^2}{2}\right\},\\ \left\{\frac{\{\varphi+\lambda\}^2}{\{\delta|I|_\infty\}^2}\right\}, \quad < 1. \ \square\\ \left\{\frac{\{\varphi\}^2}{\{\rho|S|_\infty\}^2}\right\},\\ \left\{\frac{\{\varphi+\sigma\}^2}{\{\eta|S|_\infty\}^2}\right\}\end{cases} \tag{62}$$

Stability analysis

The global stability analysis of epidemiological algorithms plays a crucial part in determining the stage of infection and offering suggestions for disease management strategies. The Volterra–Lyapunov matrix theory phenomena has recently drawn significant attention in the field of the prevention of diseases. The examination of disease-free equilibrium's global stability is first presented here. The risk endemic equilibrium's global stability is subsequently researched. First, we go through a crucial Lemma, [39], which helps us analyze the proposed system's global stability.

Lemma 11. *Let $\mathbf{G} \in \mathbb{R}^+$ be a continuous function such that for every $t \ge t_0$;*

$${}_a^{ABC}\nabla^{\upsilon}\left(\mathbf{G}-\mathbf{G}^*-\mathbf{G}^*\log\frac{\mathbf{G}}{\mathbf{G}^*}\right) \le \left(1-\frac{\mathbf{G}^*}{\mathbf{G}}\right){}_a^{ABC}\nabla^{\upsilon}\mathbf{G}, \tag{63}$$

$\mathbf{G}^* \in \mathbb{R}^+, \forall\ \upsilon \in (0,\frac{1}{2})$.

Theorem 12. *If the reproductive number $\Re < 1$, then the disease-free equilibrium states $(\mathcal{E}^0)$ are globally asymptotically stable.*

Proof. We establish a Volterra-type Lyapunov function as;

$$\mathbf{G} = \left[S-S^0-S^0\log\frac{S}{S^0}\right]+I+R+\left[B-B^0-B^0\log\frac{B}{B^0}\right]+\left[V-V^0-V^0\log\frac{V}{V^0}\right]. \tag{64}$$

From Lemma 11, we have

$${}_a^{ABC}\nabla^{\upsilon}\mathbf{G} \le \left(1-\frac{S^0}{S}\right){}_a^{ABC}\nabla^{\upsilon}S + {}_a^{ABC}\nabla^{\upsilon}I + {}_a^{ABC}\nabla^{\upsilon}R + \left(1-\frac{B^0}{B}\right){}_a^{ABC}\nabla^{\upsilon}B + \left(1-\frac{V^0}{V}\right){}_a^{ABC}\nabla^{\upsilon}V. \tag{65}$$

Substituting the values of ${}_a^{ABC}\nabla^{\upsilon}S(t)$, ${}_a^{ABC}\nabla^{\upsilon}I(t)$, ${}_a^{ABC}\nabla^{\upsilon}R(t)$, ${}_a^{ABC}\nabla^{\upsilon}B(t)$, and ${}_a^{ABC}\nabla^{\upsilon}V(t)$ from (20), we find

$${}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t) \le \left(1-\frac{S^0}{S}\right)\left(\vartheta-\beta SI-(\varphi+\eta+\rho)S+\lambda R+\sigma V\right) + \left(\beta SI-(\varphi+\xi+\delta+\zeta)I\right) + \left(\delta I-(\varphi+\lambda)R\right)+\left(1-\frac{B^0}{B}\right)\left(\rho S-\varphi B\right) + \left(1-\frac{V^0}{V}\right)\left(\eta S-(\varphi+\sigma)V\right), \tag{66}$$

putting $S = S-S^0$, $I = I-I^0$, $R = R-R^0$, $B = B-B^0$, $V = V-V^0$, we have

$$\begin{aligned}{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t) \le{}& \left(1-\frac{S^0}{S}\right)\left[\vartheta-\beta(S-S^0)(I-I^0)-(\varphi+\eta+\rho)(S-S^0)\right.\\ &\left.+\lambda(R-R^0)+\sigma(V-V^0)\right]\\ &+\left[\beta(S-S^0)(I-I^0)-(\varphi+\xi+\delta+\zeta)(I-I^0)\right]\\ &+\left[\delta(I-I^0)-(\varphi+\lambda)(R-R^0)\right]\\ &+\left(1-\frac{B^0}{B}\right)\left[\rho(S-S^0)-\varphi(B-B^0)\right]\\ &+\left(1-\frac{V^0}{V}\right)\left[\eta(S-S^0)-(\varphi+\sigma)(V-V^0)\right].\end{aligned} \tag{67}$$

We observe that

- ${}_a^{ABC}\nabla^{\upsilon}\mathbf{G} \le 0$ for $\Re < 1$.
- ${}_a^{ABC}\nabla^{\upsilon}\mathbf{G} = 0$ only when $S = S^0$, $I = I^0$, $R = R^0$, $B = B^0$, and $V = V^0$.

Hence we conclude that disease-free equilibrium states $\mathcal{E}^0$ are globally asymptotically stable. $\square$

For the endemic Lyapunov function, we set all independent variables in suggested model, in our case, $\{S,I,R,B,V\}$, $\mathbf{G} < 0$ is the endemic equilibrium $(\mathcal{E}^*)$.

Theorem 13. *If the reproductive number $\Re > 1$, the endemic equilibrium points of harmful impact equilibrium points $\mathcal{E}^*$ of the survival of fractional order system are globally asymptotically stable.*

Proof. We can write Volterra-type Lyapunov function as;

$$\begin{aligned}\mathbf{G} ={}& \mu_1\left(S-S^*-S^*\log\frac{S}{S^*}\right)+\mu_2\left(I-I^*-I^*\log\frac{I}{I^*}\right)\\ &+\mu_3\left(R-R^*-R^*\log\frac{R}{R^*}\right)\\ &+\mu_4\left(B-B^*-B^*\log\frac{B}{B^*}\right)+\mu_5\left(V-V^*-V^*\log\frac{V}{V^*}\right).\end{aligned} \tag{68}$$

Where μ_i, $i = 1,2,3,4,5$ are positive constants that we can choose later. Substituting Eq. (68) into system (20) and utilizing Lemma 11, we find

$$\begin{aligned}{}_a^{ABC}\nabla^{\upsilon}\mathbf{G} \le{}& \mu_1\left(\frac{S-S^*}{S}\right){}_a^{ABC}\nabla^{\upsilon}S+\mu_2\left(\frac{I-I^*}{I}\right){}_a^{ABC}\nabla^{\upsilon}I\\ &+\mu_3\left(\frac{R-R^*}{R}\right){}_a^{ABC}\nabla^{\upsilon}R\\ &+\mu_4\left(\frac{B-B^*}{B}\right){}_a^{ABC}\nabla^{\upsilon}B+\mu_5\left(\frac{V-V^*}{V}\right){}_a^{ABC}\nabla^{\upsilon}V,\end{aligned} \tag{69}$$

writing their expressions for derivatives (20) as follows;

$$\begin{aligned}{}_a^{ABC}\nabla^{\upsilon}\mathbf{G} \le{}& \mu_1\left(\frac{S-S^*}{S}\right)\left[\vartheta-\beta SI-(\varphi+\eta+\rho)S+\lambda R+\sigma V\right]\\ &+\mu_2\left(\frac{I-I^*}{I}\right)\left[\beta SI-(\varphi+\xi+\delta+\zeta)I\right]\\ &+\mu_3\left(\frac{R-R^*}{R}\right)\left[\delta I-(\varphi+\lambda)R\right]+\mu_4\left(\frac{B-B^*}{B}\right)\left[\rho S-\varphi B\right]\\ &+\mu_5\left(\frac{V-V^*}{V}\right)\left[\eta S-(\varphi+\sigma)V\right],\end{aligned} \tag{70}$$

putting $S = S-S^*$, $I = I-I^*$, $R = R-R^*$, $B = B-B^*$, $V = V-V^*$, we have

$$\begin{aligned}{}_a^{ABC}\nabla^{\upsilon}\mathbf{G} \le{}& \mu_1\left(\frac{S-S^*}{S}\right)\left[\vartheta-\beta(S-S^*)(I-I^*)-(\varphi+\eta+\rho)(S-S^*)\right.\\ &\left.+\lambda(R-R^*)+\sigma(V-V^*)\right]\\ &+\mu_2\left(\frac{I-I^*}{I}\right)\left[\beta(S-S^*)(I-I^*)-(\varphi+\xi+\delta+\zeta)(I-I^*)\right]\\ &+\mu_3\left(\frac{R-R^*}{R}\right)\left[\delta(I-I^*)-(\varphi+\lambda)(R-R^*)\right]\end{aligned}$$

$$+\ \mu_4\Big(\frac{B-B^*}{B}\Big)\Big[\rho(S-S^*)-\varphi(B-B^*)\Big] + \mu_5\Big(\frac{V-V^*}{V}\Big)\Big[\eta(S-S^*)-(\varphi+\sigma)(V-V^*)\Big], \tag{71}$$

we have

$$\begin{aligned}
{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t) \le\ & \mu_1\vartheta-\mu_1\Big(\frac{S^*}{S}\Big)\vartheta-\mu_1\frac{\beta\big(S-S^*\big)^2(I-I^*)}{S}\\
&-\mu_1\frac{(\varphi+\eta+\rho)(S-S^*)^2}{S}+\mu_1\lambda R\\
&-\mu_1\lambda R^*-\mu_1\lambda\Big(\frac{S^*}{S}\Big)R+\mu_1\lambda\Big(\frac{S^*}{S}\Big)R^*+\mu_1\sigma V-\mu_1\sigma V^*\\
&-\mu_1\sigma\Big(\frac{S^*}{S}\Big)V+\mu_1\sigma\Big(\frac{S^*}{S}\Big)V^*\\
&+\mu_2\frac{\beta\big(I-I^*\big)^2(S-S^*)}{I}-\mu_2\frac{(\varphi+\xi+\delta+\zeta)(I-I^*)^2}{I}\\
&+\mu_3\delta I-\mu_3\delta I^*-\mu_3\delta\Big(\frac{R^*}{R}\Big)I\\
&+\mu_3\delta\Big(\frac{R^*}{R}\Big)I^*-\mu_3\frac{(\varphi+\lambda)(R-R^*)^2}{R}+\mu_4\rho S\\
&-\mu_4\rho S^*-\mu_4\rho\Big(\frac{B^*}{B}\Big)S+\mu_4\rho\Big(\frac{B^*}{B}\Big)S^*\\
&-\mu_4\frac{\varphi(B-B^*)^2}{B}+\mu_5\eta S-\mu_5\eta S^*-\mu_5\eta\Big(\frac{V^*}{V}\Big)S\\
&+\mu_5\eta\Big(\frac{V^*}{V}\Big)S^*-\mu_5\frac{(\varphi+\sigma)(V-V^*)^2}{V},
\end{aligned}\tag{72}$$

Selecting $\mu_1=\mu_2=\mu_3=\mu_4=\mu_5=1$ and simplifying above, we can write

$${}_0^{FFM}D_t^{\alpha,\beta}\mathbf{G}(t)\le\mathfrak{X}-\mathfrak{Z}, \tag{73}$$

where

$$\begin{aligned}
\mathfrak{X}=\ & \vartheta+\lambda R+\lambda\Big(\frac{S^*}{S}\Big)R^*+\sigma V+\sigma\Big(\frac{S^*}{S}\Big)V^*+\frac{\beta\big(I-I^*\big)^2(S-S^*)}{I}+\delta I\\
&+\delta\Big(\frac{R^*}{R}\Big)I^*+\rho S+\rho\Big(\frac{B^*}{B}\Big)S^*+\eta S+\eta\Big(\frac{V^*}{V}\Big)S^*.
\end{aligned}\tag{74}$$

$$\begin{aligned}
\mathfrak{Z}=\ & \Big(\frac{S^*}{S}\Big)\vartheta+\frac{\beta\big(S-S^*\big)^2(I-I^*)}{S}+\frac{(\varphi+\eta+\rho)(S-S^*)^2}{S}\\
&+\lambda R^*-\lambda\Big(\frac{S^*}{S}\Big)R+\sigma V^*+\sigma\Big(\frac{S^*}{S}\Big)V\\
&+\frac{(\varphi+\xi+\delta+\zeta)(I-I^*)^2}{I}+\delta I^*+\delta\Big(\frac{R^*}{R}\Big)I+\frac{(\varphi+\lambda)(R-R^*)^2}{R}\\
&+\rho S^*+\rho\Big(\frac{B^*}{B}\Big)S+\frac{\varphi(B-B^*)^2}{B}+\eta S^*+\eta\Big(\frac{V^*}{V}\Big)S+\frac{(\varphi+\sigma)(V-V^*)^2}{V}.
\end{aligned}\tag{75}$$

We observe that

- if $\mathfrak{X}<\mathfrak{Z}\Rightarrow {}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)<0$.
- if $S(t)=S^*,\ I(t)=I^*,\ R(t)=R^*,\ B(t)=B^*,\ V(t)=V^*$, then

$$\mathfrak{X}-\mathfrak{Z}=0\quad\Longrightarrow\quad {}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)=0. \tag{76}$$

We find the proposed model's largest compact invariant set in

$$\Big\{(S^*,I^*,R^*,B^*,V^*)\in\Pi:\ {}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)=0\Big\}, \tag{77}$$

is the point $\mathcal{E}^*$, the endemic equilibrium of the proposed model. Therefore, we can conclude that $\mathcal{E}^*$ is globally asymptotically stable in Π if $\mathfrak{X}<\mathfrak{Z}$. □

Second derivative of Lyapunov

Because the first derivative evaluation of an arbitrary function is not a completely effective resource for illustrating its variations, extra observation on the specifications of each variation is required. As a result, we investigate the second derivative of the system's correspondingLyapunov function.

$$\begin{aligned}
&{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}\big]\\
&\le\ {}_a^{ABC}\nabla^{\upsilon}\Big\{\mathbf{q}_1\Big(\frac{S-S^*}{S}\Big){}_a^{ABC}\nabla^{\upsilon}S+\mathbf{q}_2\Big(\frac{I-I^*}{I}\Big){}_a^{ABC}\nabla^{\upsilon}I\\
&\quad+\mathbf{q}_3\Big(\frac{R-R^*}{R}\Big){}_a^{ABC}\nabla^{\upsilon}R\\
&\quad+\mathbf{q}_4\Big(\frac{B-B^*}{B}\Big){}_a^{ABC}\nabla^{\upsilon}B+\mathbf{q}_5\Big(\frac{V-V^*}{V}\Big){}_a^{ABC}\nabla^{\upsilon}V\Big\}.
\end{aligned}\tag{78}$$

And we have

$$\begin{aligned}
&{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)\big]\\
&\le\mathbf{q}_1\Big({}_a^{ABC}\nabla^{\upsilon}S\Big)^2S^*+\mathbf{q}_2\Big({}_a^{ABC}\nabla^{\upsilon}I\Big)^2I^*+\mathbf{q}_3\Big({}_a^{ABC}\nabla^{\upsilon}R\Big)^2R^*\\
&\quad+\mathbf{q}_4\Big({}_a^{ABC}\nabla^{\upsilon}B\Big)^2B^*+\mathbf{q}_5\Big({}_a^{ABC}\nabla^{\upsilon}V\Big)^2V^*\\
&\quad+\mathbf{q}_1\Big(1-\frac{S^*}{S}\Big){}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}S\big]\\
&\quad+\mathbf{q}_2\Big(1-\frac{I^*}{I}\Big){}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}I\big]\\
&\quad+\mathbf{q}_3\Big(1-\frac{R^*}{R}\Big){}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}R\big]\\
&\quad+\mathbf{q}_4\Big(1-\frac{B^*}{B}\Big){}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}B\big]\\
&\quad+\mathbf{q}_5\Big(1-\frac{V^*}{V}\Big){}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}V\big],
\end{aligned}\tag{79}$$

where

$$\begin{aligned}
{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}S(t)\big]=\ & \vartheta-\beta\Big(\big[{}_a^{ABC}\nabla^{\upsilon}S\big]I+\big[{}_a^{ABC}\nabla^{\upsilon}I\big]S\Big)\\
&-(\varphi+\eta+\rho)\big[{}_a^{ABC}\nabla^{\upsilon}S\big]\\
&+\lambda\big[{}_a^{ABC}\nabla^{\upsilon}R\big]+\sigma\big[{}_a^{ABC}\nabla^{\upsilon}V\big],\\
{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}I(t)\big]=\ & \beta\Big(\big[{}_a^{ABC}\nabla^{\upsilon}S\big]I+\big[{}_a^{ABC}\nabla^{\upsilon}I\big]S\Big)\\
&-(\varphi+\xi+\delta+\zeta)\big[{}_a^{ABC}\nabla^{\upsilon}I\big],\\
{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}R(t)\big]=\ & \delta\big[{}_a^{ABC}\nabla^{\upsilon}I\big]-(\varphi+\lambda)\big[{}_a^{ABC}\nabla^{\upsilon}R\big],\\
{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}B(t)\big]=\ & \rho\big[{}_a^{ABC}\nabla^{\upsilon}S\big]-\varphi\big[{}_a^{ABC}\nabla^{\upsilon}B\big],\\
{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}V(t)\big]=\ & \eta\big[{}_a^{ABC}\nabla^{\upsilon}S\big]-(\varphi+\sigma)\big[{}_a^{ABC}\nabla^{\upsilon}V\big],
\end{aligned}\tag{80}$$

then we have

$$\begin{cases}
{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)\big]\le\ {}_a^{ABC}\nabla^{\upsilon}\Theta(S,I,R,B,V)\\
\quad+\mathbf{q}_1\Big(1-\frac{S^*}{S}\Big)\Big\{\vartheta-\beta\Big(\big[{}_a^{ABC}\nabla^{\upsilon}S\big]I+\big[{}_a^{ABC}\nabla^{\upsilon}I\big]S\Big)\\
-(\varphi+\eta+\rho)\big[{}_a^{ABC}\nabla^{\upsilon}S\big]\Big\}+\mathbf{q}_2\Big(1-\frac{I^*}{I}\Big)\Big\{\beta\Big(\big[{}_a^{ABC}\nabla^{\upsilon}S\big]I\\
\quad+\big[{}_a^{ABC}\nabla^{\upsilon}I\big]S\Big)\\
-(\varphi+\xi+\delta+\zeta)\big[{}_a^{ABC}\nabla^{\upsilon}I\big]\Big\}+\mathbf{q}_3\Big(1-\frac{R^*}{R}\Big)\Big\{\delta\big[{}_a^{ABC}\nabla^{\upsilon}I\big]\\
\quad-(\varphi+\lambda)\big[{}_a^{ABC}\nabla^{\upsilon}R\big]\Big\}\\
+\mathbf{q}_4\Big(1-\frac{B^*}{B}\Big)\Big\{\rho\big[{}_a^{ABC}\nabla^{\upsilon}S\big]-\varphi\big[{}_a^{ABC}\nabla^{\upsilon}B\big]\Big\}\\
\quad+\mathbf{q}_5\Big(1-\frac{V^*}{V}\Big)\Big\{\eta\big[{}_a^{ABC}\nabla^{\upsilon}S\big]-(\varphi+\sigma)\big[{}_a^{ABC}\nabla^{\upsilon}V\big]\Big\}.
\end{cases}$$

Where

$$\begin{aligned}
&{}_a^{ABC}\nabla^{\upsilon}\Theta(S,I,R,B,V)\\
&=\mathbf{q}_1\Big({}_a^{ABC}\nabla^{\upsilon}S\Big)^2S^*+\mathbf{q}_2\Big({}_a^{ABC}\nabla^{\upsilon}I\Big)^2I^*+\mathbf{q}_3\Big({}_a^{ABC}\nabla^{\upsilon}R\Big)^2R^*\\
&\quad+\mathbf{q}_4\Big({}_a^{ABC}\nabla^{\upsilon}B\Big)^2B^*+\mathbf{q}_5\Big({}_a^{ABC}\nabla^{\upsilon}V\Big)^2V^*,
\end{aligned}\tag{81}$$

now substituting ${}_a^{ABC}\nabla^{\upsilon}S(t)$, ${}_a^{ABC}\nabla^{\upsilon}I(t)$, ${}_a^{ABC}\nabla^{\upsilon}R(t)$, ${}_a^{ABC}\nabla^{\upsilon}B(t)$, and ${}_a^{ABC}\nabla^{\upsilon}V(t)$ with their relative formula from the proposed system (20) and after arranging the above equation, we can write

$${}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)\big]\le\vartheta_1-\vartheta_2. \tag{82}$$

We observe that

$$\begin{aligned}
&{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)\big]>0\quad if\quad \vartheta_1>\vartheta_2,\\
&{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)\big]<0\quad if\quad \vartheta_1<\vartheta_2,\\
&{}_a^{ABC}\nabla^{\upsilon}\big[{}_a^{ABC}\nabla^{\upsilon}\mathbf{G}(t)\big]=0\quad if\quad \vartheta_1=\vartheta_2.
\end{aligned}\tag{83}$$

Further analysis on discrete Atangana–Baleanu, and modified Atangana–Baleanu operators

Theorem 14. *For $\upsilon \in (0, \frac{1}{2})$, the following results hold true:*

$$\begin{cases} {}_{a}^{AB}\nabla^{-\upsilon}[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t)] = \mathrm{S}(t) - \mathrm{S}(a), \\ {}_{a}^{AB}\nabla^{-\upsilon}[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t)] = \mathrm{I}(t) - \mathrm{I}(a), \\ {}_{a}^{AB}\nabla^{-\upsilon}[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t)] = \mathrm{R}(t) - \mathrm{R}(a), \\ {}_{a}^{AB}\nabla^{-\upsilon}[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t)] = \mathrm{B}(t) - \mathrm{B}(a), \\ {}_{a}^{AB}\nabla^{-\upsilon}[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t)] = \mathrm{V}(t) - \mathrm{V}(a). \end{cases} \tag{84}$$

And

$$\begin{cases} {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{S}(t)] = \mathrm{S}(t) - \mathrm{S}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{I}(t)] = \mathrm{I}(t) - \mathrm{I}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{R}(t)] = \mathrm{R}(t) - \mathrm{R}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{B}(t)] = \mathrm{B}(t) - \mathrm{B}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{V}(t)] = \mathrm{V}(t) - \mathrm{V}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a), \end{cases} \tag{85}$$

for $t \in \mathbb{N}_{a+1}$.

Proof. Define the following:

$$\begin{cases} {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{S}(t) = \mathbf{Q}_1(t), & \forall t \in \mathbb{N}_{a+1}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{I}(t) = \mathbf{Q}_2(t), & \forall t \in \mathbb{N}_{a+1}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{R}(t) = \mathbf{Q}_3(t), & \forall t \in \mathbb{N}_{a+1}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{B}(t) = \mathbf{Q}_4(t), & \forall t \in \mathbb{N}_{a+1}, \\ {}_{a}^{MABC}\nabla^{\upsilon}\mathrm{V}(t) = \mathbf{Q}_5(t), & \forall t \in \mathbb{N}_{a+1}. \end{cases} \tag{86}$$

Taking discrete Laplace transform $\mathfrak{L}_a$ on both sides of (86) and utilizing Lemma 5, we have

$$\begin{cases} \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{S}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{S}(a) + \frac{\Lambda}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{S}(\mathcal{S})\right\} = \mathbf{Q}_1(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{I}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{I}(a) + \frac{\Lambda}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{I}(\mathcal{S})\right\} = \mathbf{Q}_2(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{R}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{R}(a) + \frac{\Lambda}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{R}(\mathcal{S})\right\} = \mathbf{Q}_3(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{B}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{B}(a) + \frac{\Lambda}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{B}(\mathcal{S})\right\} = \mathbf{Q}_4(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{V}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{V}(a) + \frac{\Lambda}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{V}(\mathcal{S})\right\} = \mathbf{Q}_5(\mathcal{S}). \end{cases} \tag{87}$$

$$\Rightarrow \begin{cases} \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{S}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{S}(a)\right\} = \mathbf{Q}_1(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{I}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{I}(a)\right\} = \mathbf{Q}_2(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{R}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{R}(a)\right\} = \mathbf{Q}_3(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{B}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{B}(a)\right\} = \mathbf{Q}_4(\mathcal{S}), \\ \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{V}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{V}(a)\right\} = \mathbf{Q}_5(\mathcal{S}). \end{cases} \tag{88}$$

Where $\mathfrak{L}_a[\mathrm{S}(t)] = \mathbf{S}(\mathcal{S})$, $\mathfrak{L}_a[\mathrm{I}(t)] = \mathbf{I}(\mathcal{S})$, $\mathfrak{L}_a[\mathrm{R}(t)] = \mathbf{R}(\mathcal{S})$, $\mathfrak{L}_a[\mathrm{B}(t)] = \mathbf{B}(\mathcal{S})$, $\mathfrak{L}_a[\mathrm{V}(t)] = \mathbf{V}(\mathcal{S})$, and $\mathfrak{L}_a[\mathbf{Q}_i(t)] = \mathbf{Q}_i(\mathcal{S})$ for $i = 1, 2, 3, 4, 5$.

Solving for $\mathbf{S}(\mathcal{S})$, $\mathbf{I}(\mathcal{S})$, $\mathbf{R}(\mathcal{S})$, $\mathbf{B}(\mathcal{S})$, and $\mathbf{V}(\mathcal{S})$, we have

$$\begin{cases} \mathbf{S}(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{Q}_1(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{Q}_1(\mathcal{S}) + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{S}(a), \\ \mathbf{I}(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{Q}_2(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{Q}_2(\mathcal{S}) + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{I}(a), \\ \mathbf{R}(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{Q}_3(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{Q}_3(\mathcal{S}) + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{R}(a), \\ \mathbf{B}(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{Q}_4(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{Q}_4(\mathcal{S}) + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{B}(a), \\ \mathbf{V}(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{Q}_5(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{Q}_5(\mathcal{S}) + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{V}(a). \end{cases} \tag{89}$$

It follows from [32]

$$\begin{cases} \mathbf{S}(\mathcal{S}) = \left\{\mathfrak{L}_a[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathbf{Q}_1(t)]\right\}\{\mathcal{S}\} + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{S}(a), \\ \mathbf{I}(\mathcal{S}) = \left\{\mathfrak{L}_a[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathbf{Q}_2(t)]\right\}\{\mathcal{S}\} + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{I}(a), \\ \mathbf{R}(\mathcal{S}) = \left\{\mathfrak{L}_a[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathbf{Q}_3(t)]\right\}\{\mathcal{S}\} + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{R}(a), \\ \mathbf{B}(\mathcal{S}) = \left\{\mathfrak{L}_a[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathbf{Q}_4(t)]\right\}\{\mathcal{S}\} + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{B}(a), \\ \mathbf{V}(\mathcal{S}) = \left\{\mathfrak{L}_a[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathbf{Q}_5(t)]\right\}\{\mathcal{S}\} + \frac{(1-\mathcal{S})^{\upsilon}}{\mathcal{S}}\mathrm{V}(a). \end{cases} \tag{90}$$

Taking discrete inverse Laplace transform on both sides of (90), we have

$$\begin{cases} \mathrm{S}(t) = {}_{a}^{AB}\nabla^{-\upsilon}[\mathbf{Q}_1(t)] + \mathrm{S}(a), \\ \mathrm{I}(t) = {}_{a}^{AB}\nabla^{-\upsilon}[\mathbf{Q}_2(t)] + \mathrm{I}(a), \\ \mathrm{R}(t) = {}_{a}^{AB}\nabla^{-\upsilon}[\mathbf{Q}_3(t)] + \mathrm{R}(a), \\ \mathrm{B}(t) = {}_{a}^{AB}\nabla^{-\upsilon}[\mathbf{Q}_4(t)] + \mathrm{B}(a), \\ \mathrm{V}(t) = {}_{a}^{AB}\nabla^{-\upsilon}[\mathbf{Q}_5(t)] + \mathrm{V}(a). \end{cases} \tag{91}$$

The first portion of the theorem is satisfied by using Eq. (86) in Eq. (91).

Now let

$$\begin{cases} {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{S}(t) = \mathbf{W}_1(t), & \forall t \in \mathbb{N}_{a+1} \\ {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{I}(t) = \mathbf{W}_2(t), & \forall t \in \mathbb{N}_{a+1} \\ {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{R}(t) = \mathbf{W}_3(t), & \forall t \in \mathbb{N}_{a+1} \\ {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{B}(t) = \mathbf{W}_4(t), & \forall t \in \mathbb{N}_{a+1} \\ {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{V}(t) = \mathbf{W}_5(t), & \forall t \in \mathbb{N}_{a+1}. \end{cases} \tag{92}$$

$$\Rightarrow \begin{cases} {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{S}(t)] = {}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_1(t), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{I}(t)] = {}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_2(t), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{R}(t)] = {}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_3(t), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{B}(t)] = {}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_4(t), \\ {}_{a}^{MABC}\nabla^{\upsilon}[\,{}_{a}^{AB}\nabla^{-\upsilon}\mathrm{V}(t)] = {}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_5(t). \end{cases} \tag{93}$$

Taking discrete Laplace transform $\mathfrak{L}_a$ on ${}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_i(t)$, $i = 1, 2, 3, 4, 5$.

$$\begin{cases} \left\{\mathfrak{L}_a[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_1(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_1(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_1(a)\right\}. \\ \left\{\mathfrak{L}_a[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_2(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_2(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_2(a)\right\}. \\ \left\{\mathfrak{L}_a[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_3(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_3(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_3(a)\right\}. \\ \left\{\mathfrak{L}_a[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_4(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_4(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_4(a)\right\}. \\ \left\{\mathfrak{L}_a[\,{}_{a}^{MABC}\nabla^{\upsilon}\mathbf{W}_5(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_5(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathbf{W}_5(a)\right\}. \end{cases} \tag{94}$$

From (89) and (90), we have

$$\begin{cases} \mathbf{W}_1(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{S}(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{S}(\mathcal{S}), \\ \mathbf{W}_2(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{I}(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{I}(\mathcal{S}), \\ \mathbf{W}_3(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{R}(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{R}(\mathcal{S}), \\ \mathbf{W}_4(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{B}(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{B}(\mathcal{S}), \\ \mathbf{W}_5(\mathcal{S}) = \frac{1-\upsilon}{AB(\upsilon)}\mathbf{V}(\mathcal{S}) + \frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{V}(\mathcal{S}). \end{cases} \tag{95}$$

Also, from (14), we find

$$\begin{cases} \mathbf{W}_1(a) = {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{S}(a) = \frac{1-\upsilon}{AB(\upsilon)}\mathrm{S}(a), \\ \mathbf{W}_2(a) = {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{I}(a) = \frac{1-\upsilon}{AB(\upsilon)}\mathrm{I}(a), \\ \mathbf{W}_3(a) = {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{R}(a) = \frac{1-\upsilon}{AB(\upsilon)}\mathrm{R}(a), \\ \mathbf{W}_4(a) = {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{B}(a) = \frac{1-\upsilon}{AB(\upsilon)}\mathrm{B}(a), \\ \mathbf{W}_5(a) = {}_{a}^{AB}\nabla^{-\upsilon}\mathrm{V}(a) = \frac{1-\upsilon}{AB(\upsilon)}\mathrm{V}(a). \end{cases} \tag{96}$$

Eq. (94) becomes:

$$\begin{cases}\left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_1(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathbf{S}(\mathcal{S})+\frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{S}(\mathcal{S})\right]\right.\\ \left.-\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathrm{S}(a)\right]\right\},\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_2(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathbf{I}(\mathcal{S})+\frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{I}(\mathcal{S})\right]\right.\\ \left.-\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathrm{I}(a)\right]\right\},\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_3(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathbf{R}(\mathcal{S})+\frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{R}(\mathcal{S})\right]\right.\\ \left.-\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathrm{R}(a)\right]\right\},\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_4(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathbf{B}(\mathcal{S})+\frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{B}(\mathcal{S})\right]\right.\\ \left.-\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathrm{B}(a)\right]\right\},\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_5(t)]\right\}\{\mathcal{S}\} = \frac{AB(\upsilon)}{1-\upsilon}\left\{\frac{\mathcal{S}^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathbf{V}(\mathcal{S})+\frac{\upsilon}{AB(\upsilon)\mathcal{S}^{\upsilon}}\mathbf{V}(\mathcal{S})\right]\right.\\ \left.-\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\left[\frac{1-\upsilon}{AB(\upsilon)}\mathrm{V}(a)\right]\right\}.\end{cases} \tag{97}$$

$$\Rightarrow\begin{cases}\left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_1(t)]\right\}\{\mathcal{S}\} = \mathbf{S}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{S}(a),\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_2(t)]\right\}\{\mathcal{S}\} = \mathbf{I}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{I}(a),\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_3(t)]\right\}\{\mathcal{S}\} = \mathbf{R}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{R}(a),\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_4(t)]\right\}\{\mathcal{S}\} = \mathbf{B}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{B}(a),\\ \left\{\mathfrak{L}_a[\,{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_5(t)]\right\}\{\mathcal{S}\} = \mathbf{V}(\mathcal{S}) - \frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\mathrm{V}(a).\end{cases} \tag{98}$$

Taking inverse Laplace transform on both sides of (98), we find

$$\begin{cases}{}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_1(t) = \mathrm{S}(t) - \mathrm{S}(a)\mathfrak{L}_a^{-1}\left\{\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\right\} = \mathrm{S}(t) - \mathrm{S}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_2(t) = \mathrm{I}(t) - \mathrm{I}(a)\mathfrak{L}_a^{-1}\left\{\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\right\} = \mathrm{I}(t) - \mathrm{I}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_3(t) = \mathrm{R}(t) - \mathrm{R}(a)\mathfrak{L}_a^{-1}\left\{\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\right\} = \mathrm{R}(t) - \mathrm{R}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_4(t) = \mathrm{B}(t) - \mathrm{B}(a)\mathfrak{L}_a^{-1}\left\{\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\right\} = \mathrm{B}(t) - \mathrm{B}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathbf{W}_5(t) = \mathrm{V}(t) - \mathrm{V}(a)\mathfrak{L}_a^{-1}\left\{\frac{\mathcal{S}^{\upsilon-1}(1-\mathcal{S})^{\upsilon}}{\mathcal{S}^{\upsilon}-\Lambda}\right\} = \mathrm{V}(t) - \mathrm{V}(a)\mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a).\end{cases} \tag{99}$$

In view of Eq. (92), the second portion of Theorem is satisfied. □

Theorem 15. *For $\upsilon \in (0, \frac{1}{2})$, the following result gives an alternative series representation of the discrete modified Atangana–Baleanu of the Liouville–Caputo type fractional difference:*

$$\begin{cases}{}^{MABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{S}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{S}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{S}(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{I}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{I}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{I}(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{R}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{R}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{R}(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{B}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{B}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{B}(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{V}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{V}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{V}(t)\right]\right\},\end{cases} \tag{100}$$

for $t \in \mathbb{N}_{a+1}$.

Proof. From definitions 1 and 7, we have

$$\begin{cases}{}^{MABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{S}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{S}(a) + \Lambda\sum_{s=a+1}^{t}\mathbf{E}_{\overline{\upsilon},\upsilon}(\Lambda, t-s+1)\mathrm{S}(s)\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{I}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{I}(a) + \Lambda\sum_{s=a+1}^{t}\mathbf{E}_{\overline{\upsilon},\upsilon}(\Lambda, t-s+1)\mathrm{I}(s)\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{R}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{R}(a) + \Lambda\sum_{s=a+1}^{t}\mathbf{E}_{\overline{\upsilon},\upsilon}(\Lambda, t-s+1)\mathrm{R}(s)\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{B}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{B}(a) + \Lambda\sum_{s=a+1}^{t}\mathbf{E}_{\overline{\upsilon},\upsilon}(\Lambda, t-s+1)\mathrm{B}(s)\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{V}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{V}(a) + \Lambda\sum_{s=a+1}^{t}\mathbf{E}_{\overline{\upsilon},\upsilon}(\Lambda, t-s+1)\mathrm{V}(s)\right\},\end{cases} \tag{101}$$

$$\begin{cases}= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{S}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{S}(a) + \Lambda\sum_{s=a+1}^{t}\sum_{k=0}^{\infty}\Lambda^{k}\frac{(t-s+1)^{\overline{\upsilon k+\upsilon+1}}}{\Gamma(\upsilon k+\upsilon)}\mathrm{S}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{I}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{I}(a) + \Lambda\sum_{s=a+1}^{t}\sum_{k=0}^{\infty}\Lambda^{k}\frac{(t-s+1)^{\overline{\upsilon k+\upsilon+1}}}{\Gamma(\upsilon k+\upsilon)}\mathrm{I}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{R}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{R}(a) + \Lambda\sum_{s=a+1}^{t}\sum_{k=0}^{\infty}\Lambda^{k}\frac{(t-s+1)^{\overline{\upsilon k+\upsilon+1}}}{\Gamma(\upsilon k+\upsilon)}\mathrm{R}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{B}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{B}(a) + \Lambda\sum_{s=a+1}^{t}\sum_{k=0}^{\infty}\Lambda^{k}\frac{(t-s+1)^{\overline{\upsilon k+\upsilon+1}}}{\Gamma(\upsilon k+\upsilon)}\mathrm{B}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{V}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{V}(a) + \Lambda\sum_{s=a+1}^{t}\sum_{k=0}^{\infty}\Lambda^{k}\frac{(t-s+1)^{\overline{\upsilon k+\upsilon+1}}}{\Gamma(\upsilon k+\upsilon)}\mathrm{V}(s)\right\},\end{cases} \tag{102}$$

$$\begin{cases}= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{S}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{S}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\frac{1}{\Gamma(\upsilon k+\upsilon)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\upsilon k+\upsilon+1}}\mathrm{S}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{I}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{I}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\frac{1}{\Gamma(\upsilon k+\upsilon)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\upsilon k+\upsilon+1}}\mathrm{I}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{R}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{R}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\frac{1}{\Gamma(\upsilon k+\upsilon)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\upsilon k+\upsilon+1}}\mathrm{R}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{B}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{B}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\frac{1}{\Gamma(\upsilon k+\upsilon)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\upsilon k+\upsilon+1}}\mathrm{B}(s)\right\},\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{V}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{V}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\frac{1}{\Gamma(\upsilon k+\upsilon)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\upsilon k+\upsilon+1}}\mathrm{V}(s)\right\},\end{cases} \tag{103}$$

$$\begin{cases}= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{S}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{S}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{S}(t)\right]\right\}.\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{I}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{I}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{I}(t)\right]\right\}.\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{R}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{R}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{R}(t)\right]\right\}.\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{B}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{B}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{B}(t)\right]\right\}.\\ = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathrm{V}(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathrm{V}(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathrm{V}(t)\right]\right\}.\end{cases} \tag{104}$$

Solution of proposed system of fractional differential equations

Our proposed system is:

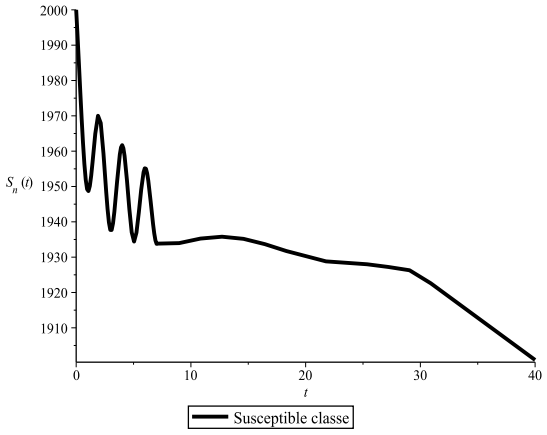
$$\begin{aligned}{}^{MABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) &= \vartheta - \beta\mathrm{SI} - (\varphi+\eta+\rho)\mathrm{S} + \lambda\mathrm{R} + \sigma\mathrm{V},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) &= \beta\mathrm{SI} - (\varphi+\xi+\delta+\zeta)\mathrm{I},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) &= \delta\mathrm{I} - (\varphi+\lambda)\mathrm{R},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) &= \rho\mathrm{S} - \varphi\mathrm{B},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) &= \eta\mathrm{S} - (\varphi+\sigma)\mathrm{V}.\end{aligned} \tag{105}$$

We can write above as:

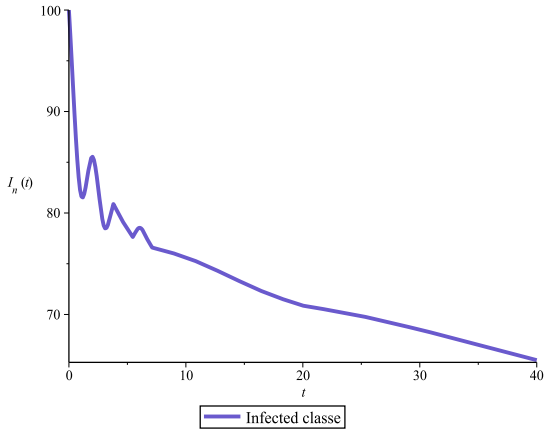
$$\begin{aligned}{}^{MABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) &= \mathbf{M}_1(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) &= \mathbf{M}_2(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) &= \mathbf{M}_3(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) &= \mathbf{M}_4(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}),\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) &= \mathbf{M}_5(t, \mathrm{S}, \mathrm{I}, \mathrm{R}, \mathrm{B}, \mathrm{V}).\end{aligned} \tag{106}$$

From Theorem 15, we obtain

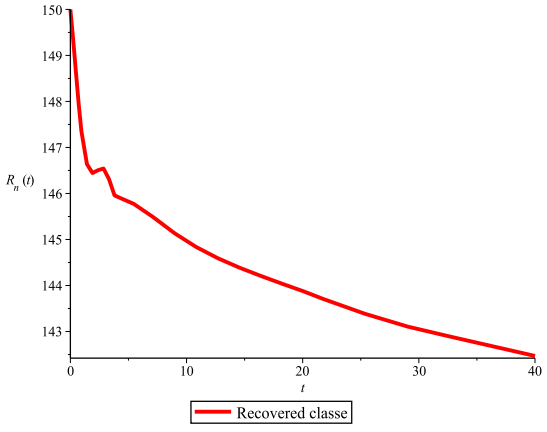
$$\begin{cases}{}^{MABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_1(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathbf{M}_1(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathbf{M}_1(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_2(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathbf{M}_2(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathbf{M}_2(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_3(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathbf{M}_3(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathbf{M}_3(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_4(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathbf{M}_4(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathbf{M}_4(t)\right]\right\},\\ {}^{MABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) = \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_5(t) - \mathbf{E}_{\overline{\upsilon}}(\Lambda, t-a)\mathbf{M}_5(a) + \sum_{k=0}^{\infty}\Lambda^{k+1}\left[{}_{a}\nabla^{-(\upsilon k+\upsilon)}\mathbf{M}_5(t)\right]\right\}.\end{cases} \tag{107}$$



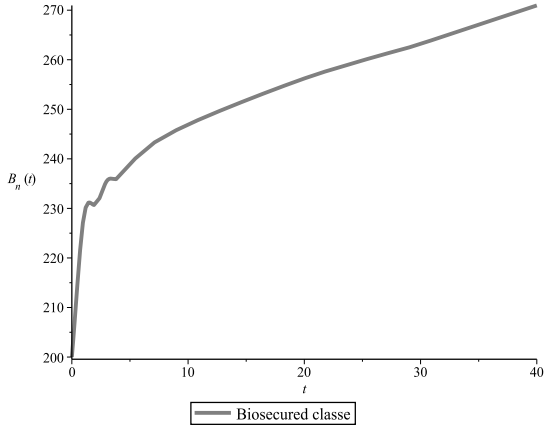
(a) S



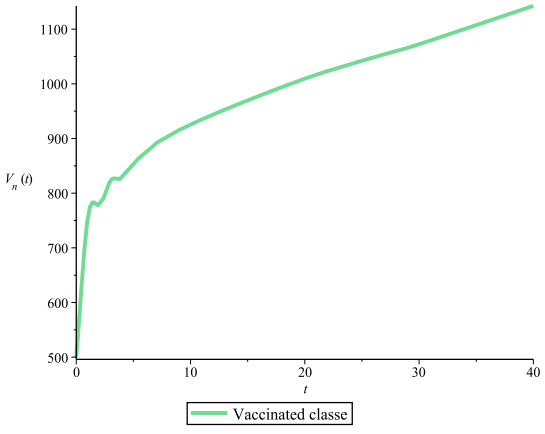
(b) I



(c) R

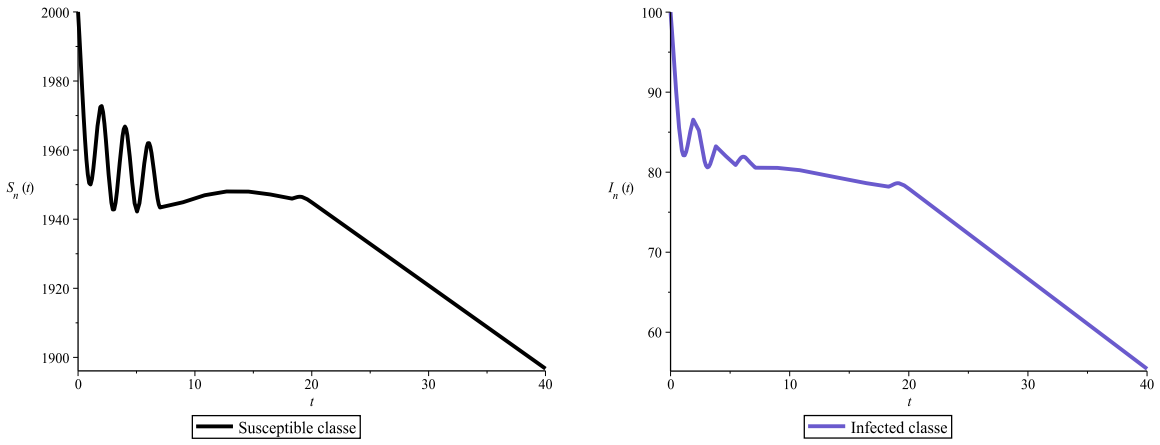


(d) B



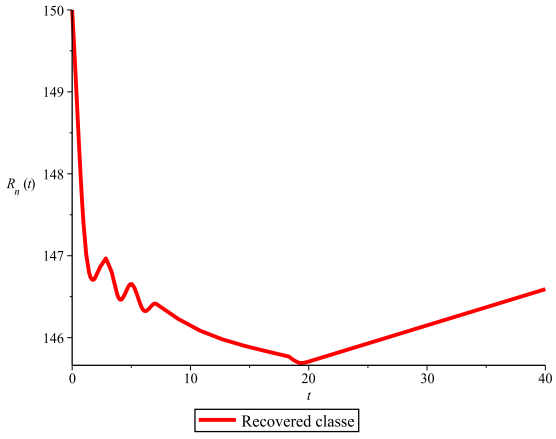
(e) V

Fig. 1. The solutions' trajectories of bovine brucellosis illness in cattle using reproducing kernel Hilbert space method with $\upsilon = 0.4$.

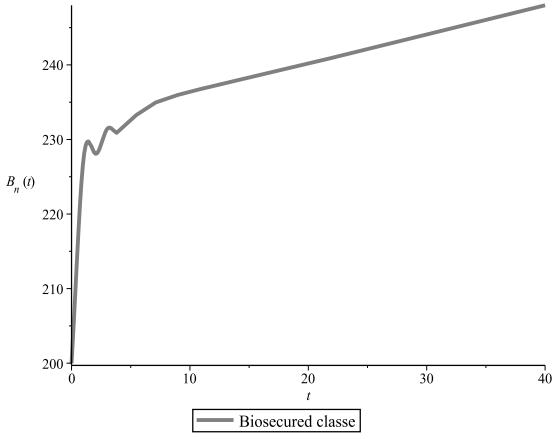


(a) S

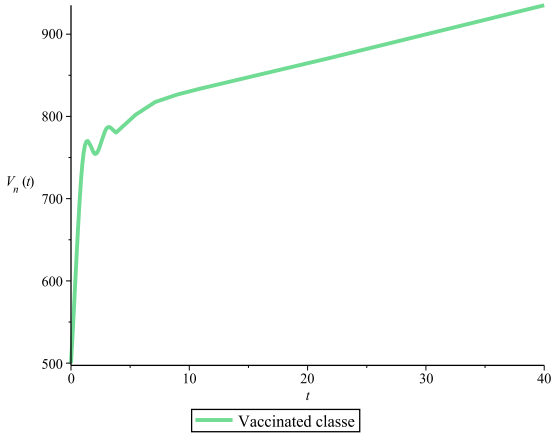
(b) I



(c) R



(d) B



(e) V

Fig. 2. The solutions' trajectories of bovine brucellosis illness in cattle using reproducing kernel Hilbert space method with $\upsilon = 0.3$.

$$
\begin{aligned}
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_1(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_1(a)+\sum_{k=0}^{\infty}\Lambda^{k+1}\ \frac{(t-a)^{\overline{\upsilon k+\upsilon}}}{\Gamma(\upsilon k+\upsilon+1)}\mathbf{M}_1(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_2(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_2(a)+\sum_{k=0}^{\infty}\Lambda^{k+1}\ \frac{(t-a)^{\overline{\upsilon k+\upsilon}}}{\Gamma(\upsilon k+\upsilon+1)}\mathbf{M}_2(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_3(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_3(a)+\sum_{k=0}^{\infty}\Lambda^{k+1}\ \frac{(t-a)^{\overline{\upsilon k+\upsilon}}}{\Gamma(\upsilon k+\upsilon+1)}\mathbf{M}_3(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_4(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_4(a)+\sum_{k=0}^{\infty}\Lambda^{k+1}\ \frac{(t-a)^{\overline{\upsilon k+\upsilon}}}{\Gamma(\upsilon k+\upsilon+1)}\mathbf{M}_4(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_5(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_5(a)+\sum_{k=0}^{\infty}\Lambda^{k+1}\ \frac{(t-a)^{\overline{\upsilon k+\upsilon}}}{\Gamma(\upsilon k+\upsilon+1)}\mathbf{M}_5(t)\right\}.
\end{aligned}
\tag{108}
$$

$$
\Rightarrow
\begin{aligned}
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{S}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_1(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_1(a)+\sum_{k=1}^{\infty}\Lambda^{k}\ \frac{(t-a)^{\overline{\upsilon k}}}{\Gamma(\upsilon k+1)}\mathbf{M}_1(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{I}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_2(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_2(a)+\sum_{k=1}^{\infty}\Lambda^{k}\ \frac{(t-a)^{\overline{\upsilon k}}}{\Gamma(\upsilon k+1)}\mathbf{M}_2(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{R}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_3(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_3(a)+\sum_{k=1}^{\infty}\Lambda^{k}\ \frac{(t-a)^{\overline{\upsilon k}}}{\Gamma(\upsilon k+1)}\mathbf{M}_3(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{B}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_4(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_4(a)+\sum_{k=1}^{\infty}\Lambda^{k}\ \frac{(t-a)^{\overline{\upsilon k}}}{\Gamma(\upsilon k+1)}\mathbf{M}_4(t)\right\},\\
{}^{ABC}_{a}\nabla^{\upsilon}\mathrm{V}(t) &= \frac{AB(\upsilon)}{1-\upsilon}\left\{\mathbf{M}_5(t)-\mathbf{E}_{\overline{\upsilon}}(\Lambda,t-a)\mathbf{M}_5(a)+\sum_{k=1}^{\infty}\Lambda^{k}\ \frac{(t-a)^{\overline{\upsilon k}}}{\Gamma(\upsilon k+1)}\mathbf{M}_5(t)\right\}.
\end{aligned}
\tag{109}
$$

Results and discussion

In this section, we simulate the outcomes for the specified initial circumstances and the parameter values from [7]. For the simulations, we use the fractional orders $\upsilon = 0.3$ and $\upsilon = 0.4$ to examine the dynamics of the suggested model. To demonstrate how the discovered solution works, we use graphs. The behavior of the suggested approach solutions at fractional orders $\upsilon = 0.4$ and $\upsilon = 0.3$ is depicted in Figs. 1 and 2, respectively.

- The susceptible (S) compartment is examined in Figs. 1(a) and 2(a) for the fractional orders $\upsilon = 0.4, 0.3$, respectively, which demonstrate a significant reduction in the susceptible population after a short period of time. We see a rapid decline in the susceptible class as we raise the fractional order.
- The infected (I) compartment exhibits similar behavior, as seen in Figs. 1(b) and 2(b). As we increase the fractional order, we observe a sharp reduction in the infected class.
- For the fractional orders $\upsilon = 0.4, 0.3$, the recovered (R) compartment is analyzed in Figs. 1(c) and 2(c), which show a rapid decline in the recovered population at higher fractional orders. In contrast, at lower fractional orders, we initially see a quick fall and, after a little period, an increase in the population that has recovered.
- Additionally, the bio-secured (B) and vaccinated (V) compartments are studied in Fig. 1(d), Figs. 1(e) and 2(d), Fig. 2(e), respectively. Both the bio-secured and immunized groups show a large rise. Both of these groups rapidly rise as fractional order is increased.

All classes in the model exhibit the effects of fractional order. The fractional order affects the progression. The model under consideration extends the classical model since graphs offer a wide range of compartments' geometrical information. It demonstrates that global behavior, rather than integer order, is provided by the fractional model of the bovine brucellosis illness.

Conclusion

In this study, a fractional-order model of the bovine brucellosis disease in cattle was constructed using discrete generalized Mittag-Leffler kernels. The fundamental reproduction number, the analysis of equilibrium states, the solutions' positivity and boundedness, the positively invariant region, and existence and uniqueness were all topics we covered. We also looked at the fractional-order model's global asymptotic stability. A sensitivity analysis has been performed on the variables that are most responsive to the fundamental reproduction number. The recently updated fractional differences were carefully reviewed. We argue that numerical simulations of the proposed method could be used to track the dynamics of the bovine brucellosis disease in cattle. The results of this investigation may be useful in delivering information to public health experts and policymakers in order to stop the spread of bovine brucellosis. By using the best control strategy and an adequate variable or parameter to reduce the affected population, we can stop the spread of disease.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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All authors have read and agreed to the last version of the manuscript.

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