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DISCRETE LEFT-DEFINITE HAMILTONIAN SYSTEMS

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Abstract In this paper we consider an even-dimensional discrete Hamiltonian system on the set of nonnegative integers in the left-definite form. Using the inertia indices of the hermitian form related with the solutions of the equation we construct some maximal subspaces of the solution space. After constructing some ellipsoids preserving nesting properties we introduce a lower bound for the number of Dirichlet-summable solutions of the equation. Moreover we introduce a limit-point criterion.

Keywords Discrete Hamiltonian system, Weyl theory, left-definite equation, Sylvester's inertia indices, subspace theory.

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1. Introduction

This paper aims to introduce a way to handle and a lower bound for the number of linearly independent *summable-square* solutions of the following $2m-$dimensional discrete left-definite Hamiltonian system

$$J(n+1)y(n+1) - J(n)y(n) = \lambda A(n)\widetilde{y}(n) + B(n)\widetilde{y}(n),\ \ n \in \mathbb{N}, \tag{1.1}$$

where $\mathbb{N} = \{0, 1, 2, ...\}$, $y(n)$ is a $2m \times 1$ vector-function such that $y(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix}$ with $m \times 1$ vector-functions y_1 and y_2, $\widetilde{y}(n) = \begin{bmatrix} y_1(n+1) \\ y_2(n) \end{bmatrix}$, J, A, B are $2m \times 2m$ matrix-functions such that

$$J(n) = \begin{bmatrix} 0 & -E^*(n) \\ E(n) & 0 \end{bmatrix},\ A(n) = \begin{bmatrix} P(n) & 0 \\ 0 & 0 \end{bmatrix},\ B(n) = \begin{bmatrix} -K(n) & L(n) \\ M(n) & N(n) \end{bmatrix}.$$

Here E, P, K, L, M, N are $m \times m$ matrix-functions, and the basic assumptions are as follows:

(i) $E(n)$ is nonsingular on $\mathbb{N}$, and $P^*(n) = P(n)$, $n \in \mathbb{N}$,

(ii) $E(n+1) - M(n)$ is invertible for each $n \in \mathbb{N}$,

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(*iii*) $K^*(n) = K(n) \geq 0,\ N^*(n) = N(n) \geq 0,\ n \in \mathbb{N}$, but not both zero at any $n \in \mathbb{N}$,

(*iv*) $E(n+1) - E(n) = M(n) - L^*(n),\ n \in \mathbb{N}$.

(*ii*) and (*iv*) also imply that $E(n) - L^*(n)$ is invertible on $\mathbb{N}$.

If $J(n)$ is chosen as a constant matrix for each $n \in \mathbb{N}$ then (1.1) takes the form

$$Jy(n+1) - Jy(n) = \lambda A(n)\widetilde{y}(n) + B(n)\widetilde{y}(n),\ n \in \mathbb{N}, \tag{1.2}$$

and (*iv*) requires that $M(n) = L^*(n)$, $n \in \mathbb{N}$. Eq. (1.2) with

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}, A(n) = \begin{bmatrix} P(n) & 0 \\ 0 & Q(n) \end{bmatrix} \geq 0, B^*(n) = B(n) = \begin{bmatrix} K(n) & L^*(n) \\ L(n) & N(n) \end{bmatrix}$$

such that I_m is the $m \times m$ identity matrix and $I_m - L(n)$ is invertible for each $n \in \mathbb{N}$ has been studied by Shi [26] (also see [16, 24, 25, 28, 31]). (1.2) is a right-definite Hamiltonian system because the *weight* matrix $A(x)$ is a nonnegative matrix. Using the nested-circles approach introduced by Hinton and Shaw [10, 11, 15] for the following continuous Hamiltonian system

$$Jy' = \lambda A(x)y + B(x)y,\ x \in [a,b), \tag{1.3}$$

where J is a $2m \times 2m$ constant matrix with $J^* = -J$, $A(x), B(x)$ are $2m \times 2m$ locally integrable functions on $[a,b)$ with $A^*(x) = A(x) \geq 0$, $B^*(x) = B(x)$ and y is a $2m \times 1$ locally absolutely continuous vector-function, Shi introduced a lower bound for the number of linearly independent summable-square solutions of (1.2).

At this stage we shall note that Hinton and Shaws' approach is very similar with the approach of Weyl [33] who introduced a lower bound for the number of linearly independent integrable-square solutions of the following scalar right-definite second-order equation

$$-(py')' + qy = \lambda wy,\ x \in [0,\infty),$$

where p, q, w $(w > 0)$ are real-valued locally integrable functions. In the paper [34] Weyl also shared an idea for the Dirichlet-integrable functions on $[0,\infty)$ as

$$\int_0^\infty p\left|y'\right|^2 + q\left|y\right|^2.$$

If p and q are chosen as nonnegative functions on $[0,\infty)$ then one obtains a left-definite equation and now the *weight* function w may have positive and negative signs on the subintervals of $[0,\infty)$. Weyl's results have been generalized by Kodaira [14], Kimura and Takahasi [13], Everitt [6, 7], Pleijel [20, 21], and the others to the right and left-definite higher-order formally symmetric differential equations. However, it should be noted that some of the ways introduced in these papers differ from the others.

We shall note that Eq. (1.2) contains the following right-definite Hamiltonian system introduced by Clark and Gesztasy [4]

$$\begin{bmatrix} 0 & \rho(k)S^+ \\ \rho^-(k)S^- & 0 \end{bmatrix} y(k) = \lambda A(k)y(k) + B(k)y(k),\ k \in \mathbb{N} \cup \{-\mathbb{N}\},$$

where $y(k)$ is a $2m \times r$ matrix, $1 \le r \le 2m$, $S^{\pm}y(k) = y(k \pm 1)$, $\rho^- S^-$ is the formal adjoint of ρS^+, ρ is a $m \times m$ nonsingular hermitian matrix, $A(k) \ge 0$, $B^*(k) = B(k)$, such that

$$A(k) = \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix}, \; B(k) = \begin{bmatrix} B_{11}(k) & B_{12}(k) \\ B_{12}^*(k) & B_{22}(k) \end{bmatrix} = B^*(k),$$

$\lambda A_{12}(k) + B_{12}(k)$ is invertible for $k \in \mathbb{N} \cup \{-\mathbb{N}\}$.

Although there are many results on right-definite equations (the readers may also see the papers [1–3, 5, 8, 9, 22, 23, 27, 30, 32]) there are only a few papers on left-definite discrete (scalar) equations. Indeed, Shi and Yan [29] considered the following scalar left-definite equation

$$-p(n)y(n+1) + (p(n-1) + p(n) + q(n))y(n) - p(n-1)y(n-1) = \lambda w(n)y(n),$$

on a finite discrete set together with some boundary conditions, where $p(n-1) \ge 0$, $p(n-1) + q(n) > 0$ on this set. This approach has been followed by Ma, Gao and their colleagues [17–19].

It seems that there is not any work on left-definite discrete Hamiltonian system even in the case that J is a constant matrix in (1.1). In this paper we will introduce a lower bound for the number of linearly independent summable-square solutions in the Dirichlet sense of (1.1). For this purpose we will use the hermitian forms and inertia indices of that forms that will help us to construct some maximal subspaces of the solution space and then we will be able to introduce a theorem on the number of summable-square solutions of the Eq. (1.1). We shall note that this way has been introduced by Pleijel [20,21] for scalar continuous formally symmetric right and left-definite differential equations. Finally we will introduce a limit-point criterion.

2. Hermitian forms

In this section we will introduce some basic results on the left-definite equation and hermitian form together with its inertia indices.

Eq. (1.1) can be handled as

$$\begin{aligned} &-E^*(n+1)y_2(n+1) + E^*(n)y_2(n) + K(n)y_1(n+1) - L(n)y_2(n) = \lambda P(n)y_1(n+1), \\ &E(n+1)y_1(n+1) - E(n)y_1(n) - M(n)y_1(n+1) - N(n)y_2(n) = 0. \end{aligned} \tag{2.1}$$

Existence and uniqueness of the solutions of (1.1) follow from the assumptions $(i) - (iv)$ and (2.1). Indeed, (2.1) can be handled as

$$y(n+1, \lambda) = S(n, \lambda)y(n, \lambda),$$

where

$$S(n, \lambda) = \begin{bmatrix} S_{11}(n, \lambda) & S_{12}(n, \lambda) \\ S_{21}(n, \lambda) & S_{22}(n, \lambda) \end{bmatrix},$$

$S_{11}(n, \lambda) = (E(n+1) - M(n))^{-1}E(n)$, $S_{12}(n, \lambda) = (E(n+1) - M(n))^{-1}N(n)$, $S_{21}(n, \lambda) = -E^{*-1}(n+1)(\lambda P(n) - K(n))(E(n+1) - M(n))^{-1}E(n)$, $S_{22}(n, \lambda) =$

$-E^{*-1}(n+1)(\lambda P(n)-K(n))(E(n+1)-M(n))^{-1}N(n)-E^{*-1}(n+1)(L(n)-E^*(n))$. Using $(i)-(iv)$ one gets easily the equation

$$S^*(n,\overline{\lambda})J(n+1)S(n,\lambda)=J(n).$$

Let $y=y(n,\lambda)$ and $z=z(n,\mu)$ be the solutions of (1.1) corresponding to the parameters λ and μ, respectively. Using (2.1) and the assumptions $(i)-(iv)$ one gets that

$$\begin{aligned}
&\sum_{n=s}^{r}\overline{\mu}\widetilde{z}^*(n)A(n)\widetilde{y}(n)\\
&=\sum_{n=s}^{r}\{-z_2^*(n+1)E(n+1)y_1(n+1)\\
&\quad+z_2^*(n)\left(E(n)-L^*(n)\right)y_1(n+1)+z_1^*(n+1)K(n)y_1(n+1)\}\\
&=\sum_{n=s}^{r}\{-z_2^*(n+1)E(n+1)y_1(n+1)+z_2^*(n)\left(E(n+1)-M(n)\right)y_1(n+1)\\
&\quad+z_1^*(n+1)K(n)y_1(n+1)\}=\sum_{n=s}^{r}\{-z_2^*(n+1)E(n+1)y_1(n+1)\\
&\quad+z_2^*(n)\left(E(n)+N(n)\right)y_1(n)+z_1^*(n+1)K(n)y_1(n+1)\}\\
&=\sum_{n=s}^{r}\{-z_2^*(n+1)E(n+1)y_1(n+1)+z_2^*(n)E(n)y_1(n)+z_2^*(n)N(n)y_2(n)\\
&\quad+z_1^*(n+1)K(n)y_1(n+1)\}\,,
\end{aligned}\tag{2.2}$$

and, similarly

$$\begin{aligned}
&\sum_{n=s}^{r}\lambda\widetilde{z}^*(n)A(n)\widetilde{y}(n)\\
&=\sum_{n=s}^{r}\{-z_1^*(n+1)E^*(n+1)y_2(n+1)\\
&\quad+z_1^*(n+1)\left(E^*(n)-L(n)\right)y_2(n)+z_1^*(n+1)K(n)y_1(n+1)\}\\
&=\sum_{n=s}^{r}\{-z_1^*(n+1)E^*(n+1)y_2(n+1)+z_1^*(n+1)\left(E^*(n+1)-M^*(n)\right)y_2(n)\\
&\quad+z_1^*(n+1)K(n)y_1(n+1)\}=\sum_{n=s}^{r}\{-z_1^*(n+1)E^*(n+1)y_2(n+1)\\
&\quad+z_1^*(n)E^*(n)y_2(n)+z_2^*(n)N(n)y_2(n)+z_1^*(n+1)K(n)y_1(n+1)\}\,.
\end{aligned}\tag{2.3}$$

From (2.2) and (2.3) we obtain that

$$\begin{aligned}
&\lambda\sum_{n=s}^{r}\widetilde{z}^*(n)A(n)\widetilde{y}(n)\\
&=-z_1^*(r+1)E^*(r+1)y_2(r+1)+z_1^*(s)E^*(s)y_2(s)\\
&\quad+\sum_{n=s}^{r}\begin{bmatrix}z_1^*(n+1) & z_2^*(n)\end{bmatrix}\begin{bmatrix}K(n) & \\ & N(n)\end{bmatrix}\begin{bmatrix}y_1(n+1)\\ y_2(n)\end{bmatrix}
\end{aligned}\tag{2.4}$$

and

$$
\begin{aligned}
&\overline{\mu}\sum_{n=s}^{r}\widetilde{z}^*(n)A(n)\widetilde{y}(n)\\
&= -z_2^*(r+1)E(r+1)y_1(r+1)+z_2^*(s)E(s)y_1(s)\\
&\quad+\sum_{n=s}^{r}\begin{bmatrix} z_1^*(n+1) & z_2^*(n)\end{bmatrix}\begin{bmatrix} K(n) & \\ & N(n)\end{bmatrix}\begin{bmatrix} y_1(n+1)\\ y_2(n)\end{bmatrix}.
\end{aligned}
\tag{2.5}
$$

For $f=f(n)=\begin{bmatrix} f_1(n)\\ f_2(n)\end{bmatrix}$, $g=g(n)=\begin{bmatrix} g_1(n)\\ g_2(n)\end{bmatrix}$ we shall adopt the notation

$$
\langle f,g\rangle\mid_{n=s}^{r}=\sum_{n=s}^{r}\begin{bmatrix} g_1^*(n+1) & g_2^*(n)\end{bmatrix}\begin{bmatrix} K(n) & \\ & N(n)\end{bmatrix}\begin{bmatrix} f_1(n+1)\\ f_2(n)\end{bmatrix}.
$$

(2.4) and (2.5) imply that

$$
\frac{\lambda-\overline{\mu}}{i}\langle y,z\rangle\mid_{n=s}^{r}=[y_\lambda,z_\mu]\mid_s^{r+1}, \tag{2.6}
$$

where $[y_\lambda,z_\mu]\mid_s^{r+1}:=[y_\lambda,z_\mu](r+1)-[y_\lambda,z_\mu](s)$ and

$$
[y_\lambda,z_\mu](k):=\begin{bmatrix}\overline{\mu}z_1^*(k) & z_2^*(k)\end{bmatrix}(J(k)/i)\begin{bmatrix}\lambda y_1(k)\\ y_2(k)\end{bmatrix}. \tag{2.7}
$$

Let D be a set of all functions $y=y(n,\lambda)$ satisfying (1.1). It is clear that (2.7) represents a hermitian form. Hermitian forms can be used to construct some maximal subspaces of D. However, before constructing these subspaces we shall write the following form with the aid of (2.7)

$$
\begin{aligned}
2[y_\lambda,y_\lambda](n) &= (\lambda y_1(n)+iE^*(n)y_2(n))^*(\lambda y_1(n)+iE^*(n)y_2(n))\\
&\quad-(\lambda y_1(n)-iE^*(n)y_2(n))^*(\lambda y_1(n)-iE^*(n)y_2(n)).
\end{aligned}
\tag{2.8}
$$

(2.8) implies that the hermitian form $[.,.](n):=[h,h](n)$ at a point $n\in\mathbb{N}$ can be introduced as a sum of $\mathbf{i}_+(n)$ squares of absolute values minus $\mathbf{i}_-(n)$ squares of absolute values from $\mathbf{i}_+(n)+\mathbf{i}_-(n)$ linearly independent linear forms, where

$$
\mathbf{i}_+(n)\le m,\ \mathbf{i}_-(n)\le m. \tag{2.9}
$$

Indeed, the inequalities in (2.9) come from Sylvester's positive and negative inertia indices of a hermitian form and they, in general, depent on the point $n\in\mathbb{N}$. However, following lemma shows that these indices are independent from $n\in\mathbb{N}$.

Lemma 2.1. *Let $\mathbf{i}_+(n)$ and $\mathbf{i}_-(n)$ be the positive and negative inertia indices, respectively, of the hermitian form $[.,.](n)$ defined on D at $n\in\mathbb{N}$. Then*
$\mathbf{i}_+(n)=\mathbf{i}_-(n)=m$.

Proof. From (2.6) we have

$$2\,\mathrm{Im}\,\lambda\,\langle y,y\rangle\,|_{n=s}^{r}= [y_\lambda,y_\lambda](r+1)-[y_\lambda,y_\lambda](s). \tag{2.10}$$

Let the positive and negative inertia indices of $[y_\lambda,y_\lambda](r+1)$ and $[y_\lambda,y_\lambda](s)$ be $(\mathbf{i}_+(r+1),\mathbf{i}_-(r+1))$ and $(\mathbf{i}_+(s),\mathbf{i}_-(s))$, respectively. Then the right-hand side of (2.10) is the subtraction of $\mathbf{i}_+(s)+\mathbf{i}_-(r+1)$ squares from $\mathbf{i}_+(r+1)+\mathbf{i}_-(s)$ squares. Clearly

$$\mathbf{i}_+(s)+\mathbf{i}_-(r+1)\le 2m \text{ and } \mathbf{i}_+(r+1)+\mathbf{i}_-(s)\le 2m. \tag{2.11}$$

In the case, for instance, $\mathbf{i}_+(r+1)+\mathbf{i}_-(s)<2m$ that we may assume that there is a positive dimensional subspace on which $\mathbf{i}_+(r+1)+\mathbf{i}_-(s)$ squares equals zero. But then the left-hand side of (2.10) is nonzero for $\mathrm{Im}\,\lambda\neq 0$ while the right-hand side of (2.10) is zero. Hence $y\equiv 0$. Consequently, (2.9) and (2.11) completes the proof. □

3. Maximal subspaces

In this section we will construct some subspaces of D on which the sign of the hermitian form $[.,.](n)$ will be fixed for each $n\in\mathbb{N}$.

Since the inertia indices $\mathbf{i}_+$ and $\mathbf{i}_-$ of the hermitian form $[.,.](n)$ on D are equal at any point $n\in\mathbb{N}$ there should exist an $m-$dimensional subspace $\mathbb{D}_0$ of D at $n=0$ such that $[y_\lambda,z_\lambda](0)=0$ for every $y=y(n,\lambda)$, $z=z(n,\lambda)$ belonging to this subspace. This subspace is maximal with respect to the hermitian form $[.,.](0)$ at $n=0$. Hence

$$[f_\lambda,f_\lambda](0)=0,\ \ f\in\mathbb{D}_0.$$

Now we shall consider $m-$dimensional subspaces D_r^- and D_r^+ of D on which the hermitian form $[.,.]$ satisfies, respectively, $[.,.](r+1)\le 0$ and $[.,.](r+1)\ge 0$ at $n=r+1$.

Let $y(n,\lambda)\in\mathbb{D}_0\cap D_r^-$. Using (2.6) one gets for $\mathrm{Im}\,\lambda>0$ that $y=0$. Hence

$$D=\mathbb{D}_0\oplus D_r^-,\ \ \mathrm{Im}\,\lambda>0. \tag{3.1}$$

Similarly, for $y(n,\lambda)\in\mathbb{D}_0\cap D_r^+$ we find for $\mathrm{Im}\,\lambda<0$ that $y=0$ and the following representation holds

$$D=\mathbb{D}_0\oplus D_r^+,\ \ \mathrm{Im}\,\lambda<0. \tag{3.2}$$

Let $\eta_1,...,\eta_m$ be a base of $\mathbb{D}_0$. Hence a function η of $\mathbb{D}_0$ has the representation

$$\eta=\sum_{k=1}^{m}\widetilde{c}_k\eta_k,\ \ \widetilde{c}_k \text{ constant}. \tag{3.3}$$

Now consider a function $y(n,\lambda)\in D$ for $\mathrm{Im}\,\lambda>0$. Then according to (3.1) y has the representation

$$y=\eta+\delta,$$

where $\eta\in\mathbb{D}_0$, $\delta\in D_r^-$. Using (2.6) we get the following inequality

$$[y-\eta,y-\eta](0)+2\,\mathrm{Im}\,\lambda\,\langle y-\eta,y-\eta\rangle\,|_0^r\le 0, \tag{3.4}$$

or with the aid of (3.3) we get that

$$\left[y-\sum_{k=1}^{m}\widetilde{c}_k\eta_k,\, y-\sum_{k=1}^{m}\widetilde{c}_k\eta_k\right](0)+2\operatorname{Im}\lambda\left\langle y-\sum_{k=1}^{m}\widetilde{c}_k\eta_k,\, y-\sum_{k=1}^{m}\widetilde{c}_k\eta_k\right\rangle\Big|_0^r\leq 0. \tag{3.5}$$

Let us denote by $\mathcal{E}(r)$ the set of all $m-$tuples $\widetilde{c}=(\widetilde{c}_1,...,\widetilde{c}_m)\in\mathbb{C}^m$ satisfying (3.5) and it is not empty by (3.3) and (3.4).

Lemma 3.1. *The sets $\mathcal{E}(r)$ are nested as r grows.*

Proof. The proof follows from (3.4) or (3.5) by considering two positive integers r and r_1 with $r<r_1$. □

Corollary 3.1. *$\mathcal{E}(\infty):=\lim_{r\to\infty}\mathcal{E}(r)$ contains at least an $m-$tuple $c=(c_1,...,c_m)\in\mathbb{C}^m$.*

Now using this $c=(c_1,...,c_m)\in\mathbb{C}^m$ we obtain from (3.4) and (3.5) that

$$[y-\eta,y-\eta](0)+2\operatorname{Im}\lambda\,\langle y-\eta,y-\eta\rangle\,|_0^\infty\leq 0,\ \ \operatorname{Im}\lambda>0. \tag{3.6}$$

For $\operatorname{Im}\lambda<0$ using (3.2) and similar steps given above we can introduce the following.

Theorem 3.1. *Let $y=y(n,\lambda)$ be a solution of (1.1) for $\operatorname{Im}\lambda\neq 0$. Then one may find a solution $\eta=\eta(n,\lambda)$ of (1.1) belonging to $\mathbb{D}_0$ such that the inequality*

$$\langle y-\eta,y-\eta\rangle\,|_0^\infty<\infty$$

holds.

Let $y_1,...,y_m$ be a completion of $\eta_1,...,\eta_m\in\mathbb{D}_0$ to a base of D. Then each $y_k-\eta_k$, $1\leq k\leq m$, should be summable-square and hence we obtain the following.

Theorem 3.2. *Eq. (1.1) has at least $m-$linearly independent solutions $f=f(n,\lambda)$ for $\operatorname{Im}\lambda\neq 0$ satisfying*

$$\langle f,f\rangle\,|_0^\infty<0. \tag{3.7}$$

Hinton and Shaw [12] characterized the limit

$$\lim_{x\to\infty}y_1^*(x,\lambda)Jy_2(x,\overline{\lambda})=0$$

as the limit-point case for (1.3), where $y_1(x,\lambda)$ and $y_2(x,\overline{\lambda})$ are the solutions of (1.3) corresponding to λ and $\overline{\lambda}$, respectively.

For the discrete Hamiltonian system (1.1) we shall define the case

$$\lim_{n\to\infty}[y_\lambda,z_\mu](n)=0 \tag{3.8}$$

as the limit-point case, where $y=y(n,\lambda), z=z(n,\mu)\in D[\mathbb{N}]$, where $\operatorname{Im}\lambda\neq 0$, $\operatorname{Im}\mu\neq 0$, $D[\mathbb{N}]$ is a subset of D consisting of all functions f satisfying (3.7). Then we may introduce the following.

Theorem 3.3. *Suppose that $J(n)$ is bounded on $\mathbb{N}$ and*

$$\begin{bmatrix}K(n) & \\ & N(n)\end{bmatrix}\geq\gamma I_{2m},\ \ n\in\mathbb{N}, \tag{3.9}$$

where I_{2m} is the identity matrix of dimension $2m$ and $\gamma > 0$. Then (3.8) holds for $y(n,\lambda), z(n,\mu) \in D[\mathbb{N}]$.

Proof. Since $y, z \in D[\mathbb{N}]$, $[y_\lambda, z_\mu](\infty) := \lim_{n\to\infty}[y_\lambda, z_\mu](n)$ is finite by (2.6).
We shall assume that

$$[y_\lambda, z_\mu](\infty) \neq 0. \tag{3.10}$$

Using (3.9) we obtain that

$$\sum_{n=0}^{\infty} \widetilde{y}^*(n) \begin{bmatrix} K(n) & \\ & N(n) \end{bmatrix} \widetilde{y}(n) \geq \gamma \sum_{n=0}^{\infty} \left(|y_1(n+1)|^2 + |y_2(n)|^2 \right). \tag{3.11}$$

Moreover we have

$$\begin{aligned} &\left| \sum_{n=0}^{\infty} \begin{bmatrix} \overline{\mu} z_1^*(n) \; z_2^*(n) \end{bmatrix} J(n) \begin{bmatrix} \lambda y_1(n) \\ y_2(n) \end{bmatrix} \right| \\ &\leq const. \sum_{t=1}^{2} \left(\sum_{n=0}^{\infty} |z_t(n)|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |y_{3-t}(n)|^2 \right)^{1/2}. \end{aligned} \tag{3.12}$$

(3.11) implies that (3.12) is finite. However, this contradicts to (3.10) and this completes the proof. □

4. Titchmarsh-Weyl function

We shall consider the following $2m \times 2m$ matrix function whose columns are linearly independent solutions of (1.1)

$$\mathcal{Y} = \begin{bmatrix} U \; V \end{bmatrix} = \begin{bmatrix} U_1 \; V_1 \\ U_2 \; V_2 \end{bmatrix} \tag{4.1}$$

satisfying $\mathcal{Y}(0) = I_{2m}$, where $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ are $2m \times m$ matrices, U_1, U_2, V_1, V_2 are $m \times m$ matrices and

$$U = \begin{bmatrix} u_1 \cdots u_m \end{bmatrix}, V = \begin{bmatrix} v_1 \cdots v_m \end{bmatrix}.$$

Here $u_1, ..., u_m, v_1, ..., v_m$ are $2m \times 1$ vector-functions.
A direct calculation shows that

$$[V_\lambda, V_\lambda](0) = \mathbb{O},$$

where $\mathbb{O}$ is the $m \times m$ zero matrix. This implies that $v_1, ..., v_m$ belong to $\mathbb{D}_0$. Hence from Theorem 3.1 we get that

$$\sum_{n=0}^{\infty} \widetilde{\Psi}^*(n) \begin{bmatrix} K(n) & \\ & N(n) \end{bmatrix} \widetilde{\Psi}(n) < \infty, \;\; \operatorname{Im} \lambda \neq 0,$$

where $\Psi(n)=U(n)-V(n)H$, H is a $m\times m$ matrix as

$$H=\begin{bmatrix} c_{11} & \cdots & c_{m1} \\ \vdots & & \vdots \\ c_{1m} & \cdots & c_{mm} \end{bmatrix},$$

$\Psi(n)=\begin{bmatrix} \Psi_1(n) \\ \Psi_2(n) \end{bmatrix}$ and $\widetilde{\Psi}(n)=\begin{bmatrix} \Psi_1(n+1) \\ \Psi_2(n) \end{bmatrix}$.

We shall note that H is the Titchmarsh-Weyl matrix.

For $\operatorname{Im}\lambda\neq 0$ we shall define the following

$$\mathcal{Y}^*\begin{bmatrix} \Lambda^* & \\ & I_m \end{bmatrix}(J/i)\begin{bmatrix} \Lambda & \\ & I_m \end{bmatrix}\mathcal{Y}=\varepsilon\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^* \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix}, \tag{4.2}$$

where Λ is the $m\times m$ matrix defined by $\Lambda = diag\{\lambda,...,\lambda\}$, $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ are $m\times m$ matrices, $\varepsilon=1$ when $\operatorname{Im}\lambda>0$ and $\varepsilon=-1$ when $\operatorname{Im}\lambda<0$.

Theorem 4.1. *For sufficiently large r , $\boldsymbol{C}(r)>0$.*

Proof. Using (4.1) and (4.2) one may see that

$$C(n)=V^*(n)\begin{bmatrix} \Lambda^* & \\ & I_m \end{bmatrix}(J(n)/i)\begin{bmatrix} \Lambda & \\ & I_m \end{bmatrix}V(n). \tag{4.3}$$

On the other side we have

$$V^*(r)\begin{bmatrix} \Lambda^* & \\ & I_m \end{bmatrix}(J(r)/i)\begin{bmatrix} \Lambda & \\ & I_m \end{bmatrix}V(r)=\begin{cases} 2\operatorname{Im}\lambda\langle V,V\rangle\,|_0^{r-1}, & \operatorname{Im}\lambda>0, \\ -2\operatorname{Im}\lambda\langle V,V\rangle\,|_0^{r-1}, & \operatorname{Im}\lambda<0. \end{cases} \tag{4.4}$$

Now (4.3) and (4.4) complete the proof. □

Corollary 4.1. *As r increases $\boldsymbol{C}(r)$ nondecreases.*

Theorem 4.2. *Let there exist $s-$linearly independent solutions of (1.1), $m\leq s\leq 2m$, satisfying (3.7) and let $\mu_1(r)\leq ...\leq \mu_m(r)$ be the eigenvalues of $\boldsymbol{C}(r)$. Then as $r\to\infty$, $\mu_1(\infty)\leq ...\leq \mu_{(s-m)}(\infty)$ remain finite and the others go to infinity, where $\mu_k(\infty):=\lim_{r\to\infty}\mu_k(r)$.*

Proof. Let e_r be an unit eigenvector of $\boldsymbol{C}(r)$ and we shall set $\Psi(n)=V(n)e_r$. Then one gets that

$$\begin{aligned} 2\operatorname{Im}\lambda\langle\Psi,\Psi\rangle\,|_0^{r-1} &= e_r^*V^*(r)\begin{bmatrix} \Lambda^* & \\ & I_m \end{bmatrix}(J(r)/i)\begin{bmatrix} \Lambda & \\ & I_m \end{bmatrix}V(r)e_r \\ &= \begin{cases} \mu(r), & \operatorname{Im}\lambda>0, \\ -\mu(r), & \operatorname{Im}\lambda<0, \end{cases} \end{aligned}$$

where $\mu(r) < \infty$. Hence we obtain that

$$\langle \Psi, \Psi \rangle \mid_0^{r-1} \leq \frac{\mu(r)}{|2 \operatorname{Im} \lambda|} < \infty.$$

Since $m-$linearly independent summable-square solutions come from $\Psi(n) = U(n) - V(n)H$ we complete the proof. □

Using (3.6) we may also introduce the following.

Theorem 4.3. *Following inequality holds*

$$2 \operatorname{Im} \lambda \langle \Psi(n), \Psi(n) \rangle \mid_0^{\infty} \leq i\lambda H^* E(0) - i\overline{\lambda} E^*(0) H.$$

5. Conclusion and remarks

In this paper we have introduced an approach to handle an even-dimensional discrete Hamiltonian system (1.1) and using Pleijel's idea [20], [21] we have shared a lower bound for the number of Dirichlet-summable solutions of (1.1). For this purpose we have constructed a nullspace $\mathbb{D}_0$. We should note that we could also use some non-nullspaces. Indeed, let us denote by D_0^- and D_0^+ $m-$dimensional subspaces of D on which the hermitian form $[.,.]$ has a certain sign as $[.,.](0) \leq 0$ and $[.,.](0) \geq 0$, respectively. For $y(n, \lambda) \in D_0^+ \cap D_r^-$ with $\operatorname{Im} \lambda > 0$ one gets that $y = 0$ and hence

$$D = D_0^+ \oplus D_r^-, \quad \operatorname{Im} \lambda > 0. \tag{5.1}$$

Similarly one obtains the representation

$$D = D_0^- \oplus D_r^+, \operatorname{Im} \lambda < 0. \tag{5.2}$$

Using similar steps introduced in section 3 we can construct nested-ellipsoids and hence a lower bound for the number of Dirichlet-summable solutions of (1.1) with the aid of the functions belonging to (5.1) and (5.2).

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