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Research Article

Discussions on Proinov-$\mathscr{C}_b$-Contraction Mapping on b-Metric Space

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In the present paper, we introduce the notion of Proinov-$\mathscr{C}_b$-contraction mapping and we discuss it within the most interesting abstract structure, namely, b-metric spaces. We further take into consideration the necessary conditions to guarantee the existence and uniqueness of fixed points for such mappings, as well as indicate the validity of the main results by providing illustrative examples.

1. Introduction and Preliminaries

The fixed point theory focuses on investigating the necessary and sufficient conditions on the operator as well as the abstract structure within which the operator is defined. Many research papers, on fixed point theory, aim to bring forth a new condition on the operator (contraction criteria) or suggest a new abstract structure, or both. The present paper highlights a new contraction condition, namely, a Proinov-$\mathscr{C}_b$-contraction, on the most interesting abstract structure of b-metric spaces.

The notion of b-metric has been approached by several researchers such as Bakhtin [1] and Czerwik [2, 3]. For instance, Berinde [4, 5] named this structure as "quasi-metric." To be more precise, by b-metric, we understand the natural successful extension of metric by weakening "the triangle inequality" with "the extended triangle inequality." In other words, the condition of metric $d(r, q) \leq d(r, p) + d(p, q)$ turns into the new condition $d(r, q) \leq s[d(r, p) + d(p, q)]$ for all p, q, r and for a real number $s \geq 1$. Evidently, in case of $s = 1$, these two notions coincide. Despite the high similarities of the definitions of the notion of metric and b-metric, there topological properties may differ. For instance, it is known that metric is a continuous map, but, as a mapping, b-metric is not necessarily continuous. Moreover, an open ball is not open and a closed ball is not a closed set. These differences make this structure very interesting to investigate. In particular, in [6], the authors characterized the weak ϕ-contractions in setting of b-metric spaces. In [7], the existence of the fixed point of certain set-valued mappings was discussed in the context of b-metric spaces. Additionally, Ulam Stability of the fixed point, in the framework of b-metric spaces, has been considered in [8]. On the other hand, in [9–12], the authors focused on the existence of distinct multivalued operators in the context of b-metric spaces. In [13], Pacurar dealt with a fixed point for ϕ-contractions in the same structures. Another fact worth mentioning is that Shukla [14] defined partial b-metric spaces while considering the fixed point theorem.

The notion of Proinov-$\mathscr{C}_b$-contraction mapping is based on two aspects: "Proinov-type mappings" [15] and "simulation functions" [16, 17]. Proinov [15] proved that several existing results are consequences of Skof's result [18] reported in 1977. On the other hand, the simulation function also helps to get a very general contraction condition whose consequences involve several existing fixed point theorems, including Banach's.

Throughout the paper, we presume that $\mathfrak{X}$ is a nonempty set.

The notion of simulation function, introduced by Joonaghany et al. [16], combine several existing results.

Definition 1 (see [16]). A function $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is called a simulation function if

(ζ_1) $\zeta(0, 0) = 0$

(ζ_2) $\zeta(r, p) < p - r$ for all $r, p > 0$

(ζ_3) $\{r_n\}, \{p_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\longrightarrow\infty} r_n = \lim_{n\longrightarrow\infty} p_n > 0$, then

$$\limsup_{n\longrightarrow\infty} \zeta(r_n, p_n) < 0. \tag{1}$$

The set of all simulation functions will be denoted by $\mathscr{Z}$. On account of (ζ_2), we observe that

$$\zeta(t, t) < 0 \text{ for all } t > 0, \zeta \in \mathscr{Z}. \tag{2}$$

We also notice that in [17], it was shown that (ζ_1) is superfluous.

Definition 2 (see [16]). Let $(\mathfrak{X}, d)$ be a metric space and $\zeta \in \mathscr{Z}$. We say that a self-mapping T on $\mathfrak{X}$ is a $\mathscr{Z}$-contraction with respect to ζ, if

$$\zeta(d(T(x), T(y)), d(x, y)) \geq 0, \text{for all } x, y \in \mathfrak{X}. \tag{3}$$

Considering $\zeta(r, p) = \kappa p - r$ with $\kappa \in [0, 1)$ and $r, p \in [0,\infty)$, it follows that the Banach contraction forms a $\mathscr{Z}$ -contraction with respect to ζ.

Theorem 3. *On a complete metric space, every $\mathscr{Z}$ -contraction has a unique fixed point.*

Definition 4. On a nonempty set X, let $b : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0,\infty)$ be a function such that the following conditions hold:

(b_1) $b(x, y) = 0$ if and only if $x = y$

(b_2) $b(x, y) = b(y, x)$ for all $x, y \in X$

(b_3) $b(x, y) \leq s[b(x, u) + b(u, y)]$ for all $x, y, u \in \mathfrak{X}$, with $s \geq 1$

Then, we say that function b is a b -metric. In this case, the tripled $(\mathfrak{X}, b, s)$ forms a b-metric space.

Of course, for s = 1, the above function b defines a distance (or metric) on $\mathfrak{X}$.

An illustrative example of b-metric would be the following:

Example 1. Let the space

$$l_{1/2} = \left\{ x = (x_1, x_2, \cdots, x_m, \cdots) : \sum_{j=1}^{\infty} |x_j| < \infty \right\}. \tag{4}$$

Then, the function $b : l_{1/2} \times l_{1/2} \longrightarrow [0,\infty)$, where

$$b(x, y) = \left(\sum_{j=1}^{\infty} \sqrt{|x_j - y_j|} \right)^2 \tag{5}$$

is a b-metric, with $s = 2$.

The concepts of convergent and Cauchy sequences on b-metric spaces can be defined in a similar way to the case of ordinary metric spaces.

Definition 5. Let $\{x_m\}_{m\geq 0}$ be a sequence in the b-metric space $(\mathfrak{X}, b, s)$. We say that the sequence $\{x_m\}_{m\geq 0}$ is

(c) *convergent* $\Longleftrightarrow$ there exists $u \in \mathfrak{X}$ such that for any $e > 0$, there exists $N(e) \in \mathbb{N}$ such that $b(x_m, u) < e$, for all $m \geq N(e)$

This means, $\lim_{m\longrightarrow \text{infty}} b(x_m, u) = 0$; we write $x_m \longrightarrow u$, or $\lim_{m\longrightarrow\infty} x_m = u$.

(C) *Cauchy* $\Longleftrightarrow$ for any $e > 0$, there exists $N(e) \in \mathbb{N}$ such that $b(x_m, x_p) < e$, for all $m, p \geq N(e)$

In case every Cauchy sequence in $\mathfrak{X}$ is convergent, we say that the b-metric space $(\mathfrak{X}, \mathrm{b}, \mathrm{s})$ is *complete*.

Lemma 6 (see [19]). *Let $(\mathfrak{X}, b)$ be a b-metric space and $\{x_n\}$ be a sequence of elements in $\mathfrak{X}$ such that there exists $\kappa \in [0, 1)$ such that $b(x_{n+1}, x_{n+2}) \leq \kappa(x_n, x_{n+1})$ for every $n \in \mathbb{N}$. Then, $\{x_n\}$ is a Cauchy sequence.*

Definition 7. Let $(\mathfrak{X}, b)$, $s \geq 1$, be a b-metric space and a function $\zeta_b : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ satisfying the following:

(ζ_{b1}) $\zeta_b(r, t) < t - r$ for all $r, t \in \mathbb{R}^+$

(ζ_{b2}) If $\{r_n\}, \{t_n\}$ are two sequences in $[0, +\infty)$, such that for $p > 0$

$$\limsup_{n\longrightarrow\infty} t_n = s^p \lim_{n\longrightarrow\infty} r_n > 0, \tag{6}$$

then

$$\limsup_{n\longrightarrow\infty} \zeta_b(s^p r_n, t_n) < 0. \tag{7}$$

Thus, ζ_b is said to be a b-ψ-simulation function. We shall denote by $\mathscr{C}_b$ the family of all b-simulation functions.

(See, e.g., [16, 20, 21], for more details and examples.)

In [22], the authors considered several fixed point theorems, in the setting of b-metric spaces, for a family of contractions (called multiparametric contractions) depending on two functions (that are not defined in $t = 0$) and some parameters.

Definition 8 (see [22]). Let $(\mathfrak{X}, b)$ be a b-metric space and $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ be a mapping. Let $\varkappa = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ be a set of five nonnegative real numbers, and we denote by

$$A_T : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0,\infty) \tag{8}$$

the function defined, for all $x, y \in \mathfrak{X}$, by

$$A_T(x, y) = \kappa_1 b(x, y) + \kappa_2 b(x, Tx) + \kappa_3 b(y, Ty) + \kappa_4 b(x, Ty) + \kappa_5 b(y, Tx). \tag{9}$$

We say that T is a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on $(\mathfrak{X}, b, s)$ if

$$\psi(s^q b(Tx, Ty)) \le \phi(A_T(x, y)) \quad \text{for all } x, y \in \mathfrak{X} \text{ such that } b(Tx, Ty) > 0, \tag{10}$$

where $\psi, \phi : (0,\infty) \longrightarrow \mathbb{R}$ are two auxiliary functions and $q \in [1,\infty)$.

Inspired by some results in [15], we will consider a pair of two functions $\psi, \phi : (0,\infty) \longrightarrow \mathbb{R}$ that satisfy the following:

(p_1) $\phi(u) < \psi(u)$ for any $u > 0$

(p_2) ψ is nondecreasing

Let $\mathscr{P}$ be the set of such pair of functions; that is,

$$\mathscr{P} = \{(\psi, \phi) | \psi, \phi: (0,\infty) \longrightarrow \mathbb{R}, \quad (p_1), (p_2) \text{ hold}\}. \tag{11}$$

2. Main Results

Definition 9. Let $(\mathfrak{X}, b, s)$ be a b-metric space. A mapping $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ is a Proinov-$\mathscr{C}_b$-contraction mapping of type R_i if there exist $(\psi, \phi) \in \mathscr{P}$, $\zeta_b \in \mathscr{C}_b$, a number $\beta \ge 1$, and nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, with $\alpha_1 + \alpha_2 + \alpha_3 > 0$, such that for all $x, y \in \mathfrak{X}$ with $b(Tx, Ty) > 0$, we have

$$\frac{1}{2s} \min \{b(x, Tx), b(y, Ty)\} \le b(x, y) \text{ implies } \zeta_b\left(\psi\left(s^\beta b(Tx, Ty)\right), \phi(R_i(x, y))\right) \ge 0, \tag{12}$$

where

$$\begin{aligned} R_1(x, y) &= \alpha_1 b(x, y) + \alpha_2 b(x, Tx) + \alpha_3 b(y, Ty) + \alpha_4 \frac{b(x, Tx) b(y, Ty)}{b(x, y)}, \quad \text{for any } x \ne y \\ R_2(x, y) &= \alpha_1 b(x, y) + \alpha_2 b(x, Tx) + \alpha_3 b(y, Ty) + \alpha_4 \frac{b(y, Ty))(1 + b(x, Tx))}{1 + b(x, y)}, \\ R_3(x, y) &= \alpha_1 b(x, y) + \alpha_2 b(x, Tx) + \alpha_3 b(y, Ty) + \alpha_4 \frac{b(x, Tx) b(x, Ty) + b(y, Ty) b(y, Tx)}{1 + \max\{b(x, Ty), b(y, Tx)\}} \\ &\quad + +\alpha_5 \frac{b(x, Tx) b(x, Ty) + b(y, Ty) b(y, Tx)}{1 + s \max\{b(x, Tx), b(y, Ty)\}}. \end{aligned} \tag{13}$$

Remark 10. We mention that following Corollary 11 in [22], we have that, for $\alpha_1 + \alpha_2 + \alpha_3 > 0$, either T admits at least one fixed point or $R_i(x, y) > 0$, $i = \overline{1, 3}$, for all distinct $x, y \in \mathfrak{X}$.

Theorem 11. *On a complete b-metric space $(\mathfrak{X}, b, s)$, any continuous Proinov-$\mathscr{C}_b$-contraction mapping of type R_1T has a unique fixed point provided that $\sum_{k=1}^4 \alpha_k < s^\beta$.*

Proof. Starting with a point $x_0 \in \mathfrak{X}$, we can consider the sequence $\{x_n\}$ in $\mathfrak{X}$, build as follows:

$$x_1 = Tx_0, \cdots x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}_0. \tag{14}$$

We observe that if there is some $m_0 \in \mathbb{N}$ such that $x_{m_0} = x_{m_0+1}$, it follows that $x_{m_0} = Tx_{m_0}$, so x_{m_0} is a fixed point of the mapping T. With this in mind, we will presume that $x_n \ne x_{n+1}$ for all n. Thus, since

$$\frac{1}{2s} \min \{b(x_n, Tx_n), b(x_{n+1}, Tx_{n+1})\} = \frac{1}{2s} \min \{b(x_n, x_{n+1}), b(x_{n+1}, x_{n+2})\} \le b(x_n, x_{n+1}), \tag{15}$$

by (12),

$$\zeta_b\left(\psi\left(s^\beta b(x_n, x_{n+1}), \phi(R_1(x_n, x_{n+1}))\right)\right) \ge 0, \tag{16}$$

which is equivalent, taking (ζ_{b1}) into account, with

$$\phi(R_1(x_n, x_{n+1})) - \psi\left(s^\beta b(Tx_n, Tx_{n+1})\right) > 0. \tag{17}$$

Moreover, since

$$\begin{aligned} R_1(x_n, x_{n+1}) &= \alpha_1 b(x_n, x_{n+1}) + \alpha_2 b(x_n, Tx_n) + \alpha_3 b(x_{n+1}, Tx_{n+1}) + \alpha_4 \frac{b(x_n, Tx_n) b(x_{n+1}, Tx_{n+1})}{b(x_n, x_{n+1})} \\ &= (\alpha_1 + \alpha_2) b(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) b(x_{n+1}, x_{n+2}), \end{aligned} \tag{18}$$

the above inequality becomes

$$\psi\left(s^\beta \mathrm{b}(x_{n+1}, x_{n+2})\right) < \phi((\alpha_1 + \alpha_2) \mathrm{b}(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) d(x_{n+1}, x_{n+2})). \tag{19}$$

Since the pair $(\psi, \phi) \in \mathscr{P}$, it follows

$$\begin{aligned} \psi\left(s^\beta b(x_{n+1}, x_{n+2}\right) &< \phi((\alpha_1 + \alpha_2) b(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) b(x_{n+1}, x_{n+2})) \\ &< \psi((\alpha_1 + \alpha_2) d(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) d(x_{n+1}, x_{n+2})). \end{aligned} \tag{20}$$

Consequently,

$$s^{\beta}b(x_{n+1},x_{n+2})<(\alpha_1+\alpha_2)b(x_n,x_{n+1})+(\alpha_3+\alpha_4)b(x_{n+1},x_{n+2}),$$
$$b(x_{n+1},x_{n+2})<\frac{\alpha_1+\alpha_2}{s^{\beta}-\alpha_3-\alpha_4}b(x_n,x_{n+1}). \tag{21}$$

Let $\kappa=(\alpha_1+\alpha_2)/(s^{\beta}-\alpha_3-\alpha_4)<1$. Consequently,

$$b(x_{n+1},x_{n+2})<\kappa b(x_n,x_{n+1})<\kappa^{n+1}b(x_0,x_1)\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{22}$$

Moreover, by Lemma 6, it follows that the sequence $\{x_n\}$ is Cauchy, and taking into account the completeness of the b-metric space $\mathfrak{X}$, we find that there exists $\omega\in\mathfrak{X}$ such that

$$\lim_{n\longrightarrow\infty}x_n=\omega. \tag{23}$$

But, the mapping T was supposed to be continuous, so that

$$T\omega=T\left(\lim_{n\longrightarrow\infty}x_n\right)=\lim_{n\longrightarrow\infty}T(x_n)=\lim_{n\longrightarrow\infty}x_{n+1}=\omega. \tag{24}$$

Thereupon, $T\omega=\omega$; that is, ω is a fixed point of the mapping T.

Supposing that there exists another point $v\in\mathfrak{X}$, such that $Tv=v\neq\omega=T\omega$, we have

$$\frac{1}{2s}\min\{b(\omega,T\omega),b(v,T(v)\}$$
$$=0<b(\omega,v)\Longrightarrow\zeta_b\left(\psi\left(s^{\beta}b(T\omega,Tv)\right),\phi(R_1(\omega,v))\right)\geq 0. \tag{25}$$

Thus,

$$\phi(R_1(\omega,v))-\psi\left(s^{\beta}b(T\omega,Tv)\right)>0$$
$$\Longleftrightarrow\psi\left(s^{\beta}b(T\omega,Tv)\right)<\phi(R_1(\omega,v)), \tag{26}$$

where

$$R_1(\omega,v)=\alpha_1b(\omega,v)+\alpha_2b(\omega,T\omega)+\alpha_3b(v,Tv)$$
$$+\alpha_4\frac{b(\omega,T\omega)b(v,Tv)}{b(\omega,v)}$$
$$=\alpha_1b(\omega,v). \tag{27}$$

We have in this case

$$\psi\left(s^{\beta}b(\omega,v)\right)=\psi\left(s^{\beta}b(T\omega,Tv)\right)<\phi(\alpha_1b(\omega,v))<\psi(\alpha_1b(\omega,v)), \tag{28}$$

or, since ψ is nondecreasing,

$$0<s^{\beta}b(\omega,v)<\alpha_1b(\omega,v), \tag{29}$$

which is a contradiction. Therefore, the mapping T admits a unique fixed point. □

Example 2. Let $\mathfrak{X}=[-1,1]$, the function $b:\mathfrak{X}\longrightarrow\mathfrak{X}\longrightarrow[0,\infty)$, and $b(x,y)=|x-y|^2$ be a b -metric with $s=2$, and let $T:\mathfrak{X}\longrightarrow\mathfrak{X}$ be a continuous mapping, where

$$Tx=\begin{cases}-1, & \text{for } x\in[-1,0),\\ \dfrac{x}{4}-1, & \text{for } x\in[0,1].\end{cases} \tag{30}$$

Let the pair $(\psi,\phi)\in\mathscr{P}$, with $\psi(u)=u$, $\phi(u)=u/2$, for any $u>0$, and $\zeta_b\in\mathscr{C}_b$, $\zeta_b(r,t)=(10/11)t-r$, for $r,t\geq 0$. Thus, choosing $\beta=1$, $\alpha_1=1$, $\alpha_2=\alpha_4=1/16$, and $\alpha_3=3/4$, we have

$$\zeta_b\left(\psi\left(s^{\beta}b(Tx,Ty)\right),\phi(R_1(x,y))\right)$$
$$=\frac{10}{11}\phi(R_1(x,y))-\psi(2b(Tx,Ty))$$
$$=\frac{5}{11}\left(b(x,y)+\frac{1}{16}b(x,Tx)+\frac{3}{4}b(y,Ty)\right.$$
$$\left.+\frac{1}{16}\cdot\frac{b(x,Tx)b(y,Ty)}{b(x,y)}\right)-2b(Tx,Ty). \tag{31}$$

For $x,y\in[0,1]$ such that $1/4\min\{b(x,Tx),b(y,Ty)\}=1/4\min\{(3x/4+1)^2,(3y/4+1)^2\}\leq|x-y|^2=b(x,y)$, we have $b(Tx,Ty)=|(x/4)-1-(y/4)+1|^2=(|x-y|^2)/16$ and

$$\zeta_b\left(\psi\left(s^{\beta}b(Tx,Ty)\right),\phi(R_1(x,y))\right)$$
$$=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}\left(\frac{3x}{4}+1\right)^2+\frac{3}{4}\left(\frac{3y}{4}+1\right)^2+\frac{1}{16}\right.$$
$$\left.\cdot\frac{((3x/4)+1)^2\cdot(3/4)((3y/4)+1)^2}{b(x,y)}\right)-2\frac{|x-y|^2}{16}$$
$$=\frac{5}{11}\left(\frac{29}{40}|x-y|^2+\frac{1}{16}\left(\frac{3x}{4}+1\right)^2+\frac{3}{4}\left(\frac{3y}{4}+1\right)^2+\frac{1}{16}\right.$$
$$\left.\cdot\frac{((3x/4)+1)^2\cdot(3/4)((3y/4)+1)^2}{b(x,y)}\right)\geq 0. \tag{32}$$

For $x\in[-1,0)$, $y\in[0,1]$ such that $1/4\min\{b(x,Tx),b(y,Ty)\}=1/4\min\{(x+1)^2,((3y/4)+1)^2\}\leq|x-y|^2=b(x,y)$, we have $b(Tx,Ty)=|-1-(y/4)+1|^2=y^2/16$ and

$$\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right),\phi(R_1(x,y))\right)$$
$$=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}(x+1)^2+\frac{3}{4}\left(\frac{3y}{4}+1\right)^2+\frac{1}{16}\cdot\frac{(x+1)^2((3y/4)+1)^2}{b(x,y)}\right)-2\frac{y^2}{16}$$
$$=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}(x+1)^2+\frac{3}{4}\left(\frac{9y^2}{16}+\frac{3y}{2}+1\right)+\frac{1}{16}\cdot\frac{(x+1)^2((3y/4)+1)^2}{b(x,y)}\right)-\frac{y^2}{8}$$
$$=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}(x+1)^2+\frac{3y}{2}+1\right)+\frac{1}{16}\cdot\frac{(x+1)^2((3y/4)+1)^2}{b(x,y)}\Bigg)+\left(\frac{5}{11}\cdot\frac{3}{4}\cdot\frac{9}{16}-\frac{1}{8}\right)y^2\geq 0. \tag{33}$$

Therefore, T is a continuous Proinov-$\mathscr{C}_b$-contraction mapping of type R_1, and from Theorem 11, it follows that T has a unique fixed point.

Corollary 12. *Let $(\mathfrak{X},b,s)$ be a complete b-metric space and $T:\mathfrak{X}\longrightarrow\mathfrak{X}$ be a continuous mapping such that there exist $(\psi,\phi)\in\mathscr{P}$, $\zeta\in\mathscr{C}_b$, a number $\beta\geq 1$, and nonnegative real numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that for all $x,y\in\mathfrak{X}$ with $b(Tx,Ty)>0$, we have*

$$\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right),\phi(R_1(x,y))\right)\geq 0, \tag{34}$$

where

$$R_1(x,y)=\alpha_1 b(x,y)+\alpha_2 b(x,Tx)+\alpha_3 b(y,Ty)+\alpha_4\frac{b(x,Tx)b(y,Ty)}{b(x,y)},\text{for any } x\neq y. \tag{35}$$

Then, T has a unique fixed point provided that $\sum_{k=1}^4\alpha_k<s^\beta$.

Theorem 13. *On a complete b-metric space $(\mathfrak{X},b,s)$ any T Proinov-$\mathscr{C}_b$-contraction mapping of type R_2 has a unique fixed point provided that $\sum_{k=1}^4\alpha_k<s^\beta$.*

Proof. Let $\{x_n\}$ be the sequence in $\mathfrak{X}$ defined by (14), with $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}$. Thus, by (12),

$$\begin{aligned}&\frac{1}{2s}\min\{b(x_n,Tx_n),b(x_{n+1},Tx_{n+1}\}\\&=\frac{1}{2s}\min\{b(x_n,x_{n+1}),b(x_{n+1},x_{n+2}\}\\&\Longrightarrow\zeta_b\left(\psi\left(s^\beta b(x_n,x_{n+1})\right),\phi(R_2(x_n,x_{n+1}))\right)\geq 0.\end{aligned} \tag{36}$$

Thus, using (ζ_{b1}), it follows

$$\phi(R_2(x_n,x_{n+1}))-\psi\left(s^\beta b(x_n,x_{n+1})\right)>0, \tag{37}$$

where

$$\begin{aligned}R_2(x_n,x_{n+1})&=\alpha_1 b(x_n,x_{n+1})+\alpha_2 b(x_n,Tx_n)+\alpha_3 b(x_{n+1},Tx_{n+1})+\alpha_4\frac{b(x_{n+1},Tx_{n+1})(1+b(x_n,Tx_n))}{1+b(x_n,x_{n+1})}\\&=\alpha_1 b(x_n,x_{n+1})+\alpha_2 b(x_n,x_{n+1})+\alpha_3 b(x_{n+1},x_{n+2})+\alpha_4\frac{b(x_{n+1},x_{n+2})(1+b(x_n,x_{n+1}))}{1+b(x_n,x_{n+1})}\\&=(\alpha_1+\alpha_2)b(x_n,x_{n+1})+(\alpha_3+\alpha_4)b(x_{n+1},x_{n+2}).\end{aligned} \tag{38}$$

Since $R_2(x_n,x_{n+1})=R_1(x_n,x_{n+1})$, proceeding in the previous proof, it follows that $\{x_n\}$ is a convergent sequence in $\mathfrak{X}$. Thus, there exists $\omega\in\mathfrak{X}$, such that $\lim_{n\longrightarrow\infty}x_n=\omega$.

We shall show that $T\omega=\omega$. First of all, we claim that

$$\frac{1}{2s}b(x_n,x_{n+1})\leq b(x_n,\omega) \tag{39}$$

or

$$\frac{1}{2s}b(x_{n+1},x_{n+2})\leq b(x_{n+1},\omega). \tag{40}$$

By contradiction, if we suppose that there exists $p_0\in\mathbb{N}$ such that neither (39) nor (40) hold, we have

$$\begin{aligned}b\left(x_{p_0},x_{p_0+1}\right)&\leq s\cdot\left[b\left(x_{p_0},\omega\right)+b\left(\omega,x_{p_0+1}\right)\right]\\&<s\cdot\left[\frac{1}{2s}b\left(x_{p_0},x_{p_0+1}\right)+\frac{1}{2s}b\left(x_{p_0+1},x_{p_0+2}\right)\right]\\&=\frac{b\left(x_{p_0},x_{p_0+1}\right)+b\left(x_{p_0+1},x_{p_0+2}\right)}{2}\\&<b\left(x_{p_0},x_{p_0+1}\right),\end{aligned} \tag{41}$$

which is a contradiction. Consequently, at least one of (39) or (40) holds, so that we can find a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$, such that

$$\begin{aligned}&\frac{1}{2s}\min\left\{b\left(x_{n(i)},Tx_{n(i)}\right),b(\omega,T\omega)\right\}\\&=\frac{1}{2s}b\left(x_{n(i)},x_{n(i)+1}\right)\leq b\left(x_{n(i)},\omega\right).\end{aligned} \tag{42}$$

Therefore, keeping (12) in mind,

$$\zeta_b\left(\psi\left(s^\beta b(Tx_n(i),T\omega)\right),\phi\left(R_2\left(x_{n(i)},\omega\right)\right)\right)\geq 0, \tag{43}$$

which is equivalent with

$$\psi\left(s^\beta b\left(Tx_{n(i)}, T\omega\right)\right) < \phi\left(R_2\left(x_{n(i)}, \omega\right)\right). \tag{44}$$

Moreover, since $(\psi, \phi) \in \mathscr{P}$,

$$\psi\left(s^\beta b\left(Tx_{n(i)}, T\omega\right)\right) < \phi\left(R_2\left(x_{n(i)}, \omega\right)\right) < \psi\left(R_2\left(x_{n(i)}, \omega\right)\right), \tag{45}$$

and then,

$$s^\beta b\left(Tx_{n(i)}, T\omega\right) < R_2\left(x_{n(i)}, \omega\right). \tag{46}$$

But,

$$\begin{aligned} R_2\left(x_{n(i)}, \omega\right) &= \alpha_1 b\left(x_{n(i)}, \omega\right) + \alpha_2 b\left(x_{n(i)}, Tx_{n(i)}\right) + \alpha_3 b(\omega, T\omega) \\ &\quad + \alpha_4 \frac{b(\omega, T\omega)\left(1 + b\left(x_{n(i)}, Tx_{n(i)}\right)\right)}{1 + b\left(x_{n(i)}, \omega\right)} \\ &= \alpha_1 b\left(x_{n(i)}, \omega\right) + \alpha_2 b\left(x_{n(i)}, x_{n(i)+1}\right) + \alpha_3 b(\omega, T\omega) \\ &\quad + \alpha_4 \frac{b(\omega, T\omega)\left(1 + b\left(x_{n(i)}, x_{n(i)+1}\right)\right)}{1 + b\left(x_{n(i)}, \omega\right)}. \end{aligned} \tag{47}$$

Consequently, there exists $\lim_{n\longrightarrow\infty} R_2(x_{n(i)}, \omega)$, and we have

$$\lim_{i\longrightarrow\infty} R_2\left(x_{n(i)}, \omega\right) = (\alpha_3 + \alpha_4) b(\omega, T\omega). \tag{48}$$

On the other hand,

$$\begin{aligned} 0 < b(\omega, T\omega) &\le s[b(\omega, Tx_n) + b(Tx_n, T\omega)] \\ &\le sb(\omega, x_{n+1}) + s^\beta b(Tx_n, T\omega) \\ &< sb(\omega, x_{n+1}) + R_2(x_n, \omega). \end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned} 0 < b(\omega, T\omega) &< \limsup_{n\longrightarrow\infty} R_2(x_n, \omega) \\ &= (\alpha_3 + \alpha_4) b(\omega, T\omega) \\ &\le b(\omega, T\omega), \end{aligned} \tag{50}$$

which is a contradiction. Thus, $T\omega = \omega$. Supposing that this point is not unique, we can find another point $v \in \mathfrak{X}$, such that $T\omega = \omega \neq v = Tv$. In this case,

$$\begin{aligned} 0 &= \frac{1}{2s} \min\{b(\omega, T\omega), b(v, Tv)\} < b(\omega, v) \\ &\Longrightarrow \zeta_b\left(\psi\left(s^\beta b(T\omega, Tv)\right), \phi(R_2(\omega, v))\right) \ge 0. \end{aligned} \tag{51}$$

We have,

$$\begin{aligned} \psi\left(s^\beta b(\omega, v)\right) &= \psi\left(s^\beta b(T\omega, Tv)\right) \le \phi(R_2(\omega, v)) \\ &= \phi(\alpha_1 b(\omega, v)) < \psi(\alpha_1 b(\omega, v)), \end{aligned} \tag{52}$$

and, taking (p_1) into account,

$$0 < s^\beta b(\omega, v) < \alpha_1 b(\omega, v), \tag{53}$$

which is a contradiction, because $\alpha_1 < s^\beta$. So, the mapping T possesses a unique fixed point. □

Corollary 14. *Let $(\mathfrak{X}, b, s)$ be a complete b-metric space and $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ be a continuous mapping such that there exist $(\psi, \phi) \in \mathscr{P}$, $\zeta \in \mathscr{C}_b$, a number $\beta \ge 1$, and nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that for all $x, y \in \mathfrak{X}$ with $b(Tx, Ty) > 0$, we have*

$$\zeta_b\left(\psi\left(s^\beta b(Tx, Ty)\right), \phi(R_2(x, y))\right) \ge 0, \tag{54}$$

where

$$\begin{aligned} R_2(x, y) &= \alpha_1 b(x, y) + \alpha_2 b(x, Tx) + \alpha_3 b(y, Ty) \\ &\quad + \alpha_4 \frac{b(x, Tx) b(y, Ty)}{b(x, y)}, \text{for any } x \neq y. \end{aligned} \tag{55}$$

Then, T has a unique fixed point provided that $\sum_{k=1}^4 \alpha_k < s^\beta$.

Theorem 15. *On a complete b-metric space $(\mathfrak{X}, b, s)$, any Proinov- $\mathscr{C}_b$-contraction mapping of type R_3T has a unique fixed point provided that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 < s^\beta$ and $\alpha_3 < 1$.*

Proof. Let $\{x_n\}$ be the sequence in $\mathfrak{X}$ defined by (14), with $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Thus, by (12),

$$\begin{aligned} &\frac{1}{2s} \min\{b(x_n, Tx_n), b(x_{n+1}, Tx_{n+1}\} \\ &\quad = \frac{1}{2s} \min\{b(x_n, x_{n+1}), b(x_{n+1}, x_{n+2}\} \\ &\quad \Longrightarrow \zeta_b\left(\psi\left(s^\beta b(Tx_n, Tx_{n+1})\right), \phi(R_2(x_n, x_{n+1}))\right) \ge 0. \end{aligned} \tag{56}$$

Thus, using (ζ_{b1}), it follows

$$\phi(R_3(x_n, x_{n+1})) - \psi\left(s^\beta b(Tx_n, Tx_{n+1})\right) > 0, \tag{57}$$

or, equivalent (keeping in mind (ζ_{b1}) and (p_1))

$$\psi\left(s^\beta b(Tx_n, Tx_{n+1})\right) < \phi(R_3(x_n, x_{n+1})) < \psi(R_3(x_n, x_{n+1})), \tag{58}$$

where

$$\begin{aligned}R_3(x_n, x_{n+1}) &= \alpha_1 b(x_n, x_{n+1}) + \alpha_2 b(x_n, Tx_n) + \alpha_3 b(x_{n+1}, Tx_{n+1})\\&\quad ++\alpha_4 \frac{b(x_n, Tx_n)b(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})b(x_{n+1}, Tx_n)}{1 + \max\{b(x_n, Tx_{n+1}), b(x_{n+1}, Tx_n)\}}\\&\quad ++\alpha_5 \frac{(b(x_n, Tx_n)b(x_n, Tx_{n+1}) + b(x_{n+1}, Tx_{n+1})b(x_{n+1}, Tx_n)}{1 + s\max\{b(x_n, Tx_n), b(x_{n+1}, Tx_{n+1})\}}\\&= \alpha_1 b(x_n, x_{n+1}) + \alpha_2 b(x_n, x_{n+1}) + \alpha_3 b(x_{n+1}, x_{n+2})\\&\quad ++\alpha_4 \frac{b(x_n, x_{n+1})b(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2})b(x_{n+1}, x_{n+1})}{1 + \max\{b(x_n, x_{n+2}), b(x_{n+1}, x_{n+1})\}}\\&\quad ++\alpha_5 \frac{(b(x_n, x_{n+1})b(x_n, x_{n+2}) + b(x_{n+1}, x_{n+2})b(x_{n+1}, x_{n+1})}{1 + s\max\{b(x_n, x_{n+1}), b(x_{n+1}, x_{n+2})\}}\\&= \alpha_1 b(x_n, x_{n+1}) + \alpha_2 b(x_n, x_{n+1}) + \alpha_3 b(x_{n+1}, x_{n+2})\\&\quad ++\alpha_4 \frac{b(x_n, x_{n+1})b(x_n, x_{n+2})}{1 + b(x_n, x_{n+2})}\\&\quad + \alpha_5 \frac{b(x_n, x_{n+1})b(x_n, x_{n+2})}{1 + s\max\{b(x_n, x_{n+1}), b(x_{n+1}, x_{n+2})\}}\\&\leq (\alpha_1 + \alpha_2 + \alpha_3)b(x_n, x_{n+1})\\&\quad + \alpha_5 \frac{s\cdot b(x_n, x_{n+1})(b(x_n, x_{n+1}) + b(x_{n+1}, x_{n+2}))}{1 + s\max\{b(x_n, x_{n+1}), b(x_{n+1}, x_{n+2})\}}.\end{aligned} \tag{59}$$

Assuming that there exists $p_0 \in \mathbb{N}$ such that $\max\{b(x_{p_0}, x_{p_0+1}), b(x_{p_0+1}, x_{p_0+2})\} = b(x_{p_0+1}, x_{p_0+2})$, we have

$$\begin{aligned}0 &< R_3\left(x_{p_0}, x_{p_0+1}\right)\\&\leq (\alpha_1 + \alpha_2 + \alpha_3)b\left(x_{p_0+1}, x_{p_0+2}\right) + \alpha_5 \frac{2s\cdot\left(b\left(x_{p_0+1}, x_{p_0+2}\right)\right)^2}{1 + sb\left(x_{p_0+1}, x_{p_0+2}\right)}\\&\leq (\alpha_1 + \alpha_2 + \alpha_3)b\left(x_{p_0+1}, x_{p_0+2}\right) + 2\alpha_5 b\left(x_{p_0+1}, x_{p_0+2}\right) = \phi.\end{aligned} \tag{60}$$

Therefore, by (58) and (59), together with (p_1), we get

$$\begin{aligned}\psi\left(s^\beta b\left(x_{p_0+1}, x_{p_0+2}\right)\right) &= \psi\left(s^\beta b\left(Tx_{p_0}, Tx_{p_0+1}\right)\right)\\&< \phi\left(R_3\left(x_{p_0}, x_{p_0+1}\right)\right)\\&< \psi\left((\alpha_1 + \alpha_2 + \alpha_3)b\left(x_{p_0+1}, x_{p_0+2}\right)\right.\\&\quad\left. + 2\alpha_5 b\left(x_{p_0+1}, x_{p_0+2}\right)\right),\end{aligned} \tag{61}$$

and taking (p_2) into account, it follows

$$s^\beta b\left(x_{p_0+1}, x_{p_0+2}\right) < (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_5)b\left(x_{p_0+1}, x_{p_0+2}\right), \tag{62}$$

which is a contradiction.

Consequently, $b(x_n, x_{n+1}) > b(x_{n+1}, x_{n+2})$, for any $n \in \mathbb{N}$, and $\{b(x_n, x_{n+1})\}$ is a nonincreasing sequence; so, we can find $\rho \geq 0$ such that $\lim_{n\longrightarrow\infty} b(x_n, x_{n+1}) = \rho$. Moreover,

$$\begin{aligned}0 < R_3(x_n, x_{n+1}) &\leq \alpha_1 b(x_n, x_{n+1}) + \alpha_2 b(x_n, x_{n+1}) + \alpha_3 b(x_{n+1}, x_{n+2})\\&\quad ++\alpha_4 b(x_n, x_{n+1}) + \alpha_5 \frac{b(x_n, x_{n+1})b(x_n, x_{n+2})}{1 + sb(x_n, x_{n+1})},\\&\leq (\alpha_1 + \alpha_2 + \alpha_4)b(x_n, x_{n+1}) + \alpha_3 b(x_{n+1}, x_{n+2})\\&\quad ++\alpha_5 \frac{s\cdot b(x_n, x_{n+1})[b(x_n, x_{n+1}) + b(x_{n+1}, x_{n+2})]}{1 + sb(x_n, x_{n+1})}\\&\leq (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)b(x_n, x_{n+1}) + (\alpha_3 + \alpha_5)b(x_{n+1}, x_{n+2}),\end{aligned} \tag{63}$$

and then, from (58) and (p_2),

$$\begin{aligned}s^\beta b(x_{n+1}, x_{n+2}) &< R_3(x_n, x_{n+1})\\&< (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)b(x_n, x_{n+1})\\&\quad + (\alpha_3 + \alpha_5)b(x_{n+1}, x_{n+2}),\end{aligned} \tag{64}$$

which leads us to

$$b(x_{n+1}, x_{n+2}) < \frac{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5}{s^\beta - \alpha_3 - \alpha_5} b(x_n, x_{n+1}). \tag{65}$$

Letting $\kappa_1 = (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5)/(s^\beta - \alpha_3 - \alpha_5) < 1$, we get $b(x_{n+1}, x_{n+2}) < \kappa_1 b(x_n, x_{n+1})$, for any $n \in \mathbb{N}$. Thus, Lemma 6 ensure that the sequence $\{x_n\}$ is Cauchy, that is,

$$\lim_{n,m\longrightarrow\infty} b(x_n, x_m) = 0. \tag{66}$$

Moreover, the b-metric space $(\mathfrak{X}, b, s)$ is supposed to be complete, so, we can find $\omega \in \mathfrak{X}$ such that

$$\lim_{m\longrightarrow\infty} x_m = \omega. \tag{67}$$

Further, from the proof of Theorem 13, we know that at least one of (39) or (40) holds, and for this reason, there exists a subsequence $\{x_k\}$ of $\{x_n\}$ such that

$$\frac{1}{2s}\min\{x_k, Tx_k), b(\omega, T\omega)\} \leq \frac{1}{2s} b(x_k, x_{k+1}) \leq b(x_k, \omega), \tag{68}$$

which implies

$$\zeta_b\left(\psi\left(s^\beta b(Tx_k, T\omega)\right), \phi(R_3(x_k, \omega))\right) \geq 0. \tag{69}$$

Therefore,

$$\psi\left(s^\beta b(Tx_k, T\omega)\right) < \phi(R_3(x_k, \omega)) < \psi(R_3(x_k, \omega)), \tag{70}$$

and, by (p_2),

$$s^\beta b(Tx_k, T\omega) < R_3(x_k, \omega). \tag{71}$$

Now, since

$$R_3(x_k,\omega) = \alpha_1 b(x_k,\omega) + \alpha_2 b(x_k,x_{k+1}) + \alpha_3 b(\omega,T\omega) + \alpha_4 \frac{b(x_k,x_{k+1})b(x_k,\omega) + b(\omega,T\omega)b(\omega,x_{k+1})}{1+\max\{b(x_k,T\omega), b(\omega,x_{k+1})\}} + +\alpha_4 \frac{b(x_k,x_{k+1})b(x_k,\omega) + b(\omega,T\omega)b(\omega,x_{k+1})}{1+s\max\{b(x_k,x_{k+1}), b(\omega,T\omega)\}}, \tag{72}$$

taking into account (66) and (67),

$$\limsup_{k\longrightarrow\infty} R_3(x_k,\omega) \le \alpha_3 b(\omega,T\omega) < b(\omega,T\omega). \tag{73}$$

But,

$$\begin{aligned} b(\omega,T\omega) &\le s[b(\omega,Tx_k) + b(Tx_k,T\omega)] \\ &\le sb(\omega,Tx_k) + s^\beta b(Tx_k,T\omega) \\ &< sb(\omega,Tx_k) + R_3(x_k,\omega), \end{aligned} \tag{74}$$

which combined with (73) showing that

$$b(\omega,T\omega) \le \limsup_{k\longrightarrow\infty} R_3(x_k,\omega) \le \alpha_3 b(\omega,T\omega). \tag{75}$$

But, this is a contradiction, so, $T\omega=\omega$.

We claim that ω is the only fixed point of T. Suppose that, on the contrary, there exists $\upsilon \in \mathfrak{X}$, such that $T\upsilon=\upsilon$ and $b(\upsilon,\omega)>0$. Thus,

$$\begin{aligned} 0 &= \frac{1}{2s}\min\{b(\upsilon,T\upsilon), b(\omega,T\omega)\} \\ &< b(\upsilon,\omega) \Longrightarrow \zeta_b\left(\psi\left(s^\beta b(T\upsilon,T\omega)\right), \phi(R_3(\upsilon,\omega))\right) \ge 0, \end{aligned} \tag{76}$$

and moreover,

$$\begin{aligned} \psi\left(s^\beta b(\upsilon,\omega)\right) &= \psi\left(s^\beta b(T\upsilon,T\omega)\right) \\ &< \phi(R_3(\upsilon,\omega)) \\ &= \phi(\alpha_1 b(\upsilon,\omega)) \\ &< \psi(\alpha_1 b(\upsilon,\omega)), \end{aligned} \tag{77}$$

which is a contradiction. □

Example 3. Let $\mathfrak{X}=\{q_1,q_2,q_3,q_4\}$ and a function $b:\mathfrak{X}\times\mathfrak{X} \longrightarrow [0,\infty)$, defined as follows:

$$\begin{array}{ccccc} b(x,y) & q_1 & q_2 & q_3 & q_4 \\ q_1 & 0 & \frac{1}{4} & \frac{5}{4} & 3 \\ q_2 & \frac{1}{4} & 0 & 2 & 3 \\ q_3 & \frac{5}{4} & 2 & 0 & 2 \\ q_4 & 3 & 3 & 2 & 0 \end{array}. \tag{78}$$

It is easy to check that b is a b-metric, with $s=2$. Let the mapping $T:\mathfrak{X}\longrightarrow\mathfrak{X}$, where

$$\begin{array}{ccccc} x & q_1 & q_2 & q_3 & q_4 \\ Tx & q_1 & q_1 & q_1 & q_2 \end{array}. \tag{79}$$

Let the pair $(\psi,\phi)\in\mathscr{P}$, where $\psi(u)=e^u$, $\phi(u)=1+\ln(1+u)$, for any $u>0$, and $\zeta_b\in\mathscr{C}_b$, $\zeta_b(r,t)=(11t/12)-r$. Choosing $\beta=1$ and $\alpha_1=\alpha_2=\alpha_4=\alpha_5=1/6$ and $\alpha_3=8/9$, we have

$$\begin{aligned} &\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right), \phi(R_3(x,y))\right) \\ &= \frac{11}{12}\phi(R_3(x,y)) - \psi(2b(Tx,Ty)) \\ &= \frac{11}{12}(1+\ln(1+R_3(x,y))) - e^{2b(Tx,Ty)} \\ &= \frac{11}{12}\left[1+\ln\left(1+\frac{1}{6}(b(x,y)+b(x,Tx) + \frac{b(x,Tx)b(x,Ty)+b(y,Ty)b(y,Tx)}{1+\max\{b(x,Ty),b(y,Tx)\}} + +\frac{b(x,Tx)\mathrm{b}(x,Ty)+b(y,Ty)b(y,Tx)}{1+2\max\{b(x,Tx),b(y,Ty)\}}\right) + \frac{8}{9}b(y,Ty))\right] - e^{2b(Tx,Ty)}. \end{aligned} \tag{80}$$

We consider the following cases (which respect the condition $b(Tx,Ty)>0$):

(i) $x=q_j$, $y=q_4$, $j\in\{1,2\}$,

$$\begin{aligned} &\frac{1}{4}\min\left\{b\left(q_j,Tq_j\right), b(q_4,Tq_4)\right\} < 3 = b\left(q_j,q_4\right) \\ &\Longrightarrow \zeta_b\left(\psi\left(s^\beta b\left(Tq_j,Tq_4\right)\right), \phi\left(R_3\left(q_j,q_4\right)\right)\right) \ge 0, \end{aligned} \tag{81}$$

which means

$$\begin{aligned} e^{2b\left(Tq_j,Tq_4\right)} &= e^{2b(q_1,q_2)} \\ &= \sqrt{e} < \frac{11}{12}\left(1+\ln\frac{11}{3}\right) \\ &= \frac{11}{12}\left(1+\ln\left(1+\alpha_3 b(q_4,Tq_4)\right)\right) \\ &\le \frac{11}{12}\left(1+\ln\left(1+R_3\left(q_j,q_4\right)\right)\right). \end{aligned} \tag{82}$$

(ii) $x = q_3$, $y = q_4$,

$$\begin{aligned} &\frac{1}{4}\min\{b(q_3,Tq_3), b(q_4,Tq_4)\} < 2 = b(q_3,q_4) \\ &\quad\Longrightarrow \zeta_b\left(\psi\left(s^\beta b(Tq_3,Tq_4)\right), \phi(R_3(q_3,q_4))\right) \ge 0, \end{aligned} \tag{83}$$

which means

$$\begin{aligned} e^{2b(Tq_3,Tq_4)} &= e^{2b(q_1,q_2)} \\ &= \sqrt{e} < \frac{11}{12}\left(1+\ln\frac{11}{3}\right) \\ &= \frac{11}{12}\left(1+\ln\left(1+\alpha_3 b(q_4,Tq_4)\right)\right) \\ &\le \frac{11}{12}\left(1+\ln\left(1+R_3(q_3,q_4)\right)\right). \end{aligned} \tag{84}$$

Consequently, the mapping T is a Proinov-$\mathscr{C}_b$-contraction mapping of type R_3 and, by Theorem 15, it follows that T has a unique fixed point.

Corollary 16. *Let $(\mathfrak{X}, b, s)$ be a complete b-metric space and $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ be a c mapping such that there exist $(\psi, \phi) \in \mathscr{P}$, $\zeta \in \mathscr{C}_b$, a number $\beta \ge 1$, and nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that for all $x, y \in \mathfrak{X}$ with $b(Tx, Ty) > 0$, we have*

$$\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right), \phi(R_3(x,y))\right) \ge 0, \tag{85}$$

where

$$\begin{aligned} R_3(x,y) = {} & \alpha_1 b(x,y) + \alpha_2 b(x,Tx) + \alpha_3 b(y,Ty) \\ & + \alpha_4 \frac{b(x,Tx)b(x,Ty) + b(y,Ty)b(y,Tx)}{1+\max\{b(x,Ty), b(y,Tx)\}} \\ & + +\alpha_5 \frac{(b(x,Tx))b(x,Ty) + b(y,Ty)b(y,Tx)}{1+s\max\{b(x,Tx), b(y,Ty)\}}. \end{aligned} \tag{86}$$

Then, T has a unique fixed point provided that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 < s^\beta$ and $\alpha_3 < 1$.

3. Conclusion

In this paper, we extend the renowned Proinov's result [15] in several directions: First of all, we investigate the contractions involving interesting rational forms. Secondly, the abstracted structure is chosen as a b-metric space that is one of the natural and novel generalizations of the concept of metric spaces. Thirdly, we use auxiliary simulation functions to improve Proinov's results [15].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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