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Research article

Extension of aggregation operators to site selection for solid waste management under neutrosophic hypersoft set

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Abstract: With the fast growth of the economy and rapid urbanization, the waste produced by the urban population also rises as the population increases. Due to communal, ecological, and financial constrictions, indicating a landfill site has become perplexing. Also, the choice of the landfill site is oppressed with vagueness and complexity due to the deficiency of information from experts and the existence of indeterminate data in the decision-making (DM) process. The neutrosophic hypersoft set (NHSS) is the most generalized form of the neutrosophic soft set, which deals with the multi-sub-attributes of the alternatives. The NHSS accurately judges the insufficiencies, concerns, and hesitation in the DM process compared to IFHSS and PFHSS, considering the truthiness, falsity, and indeterminacy of each sub-attribute of given parameters. This research extant the operational laws for neutrosophic hypersoft numbers (NHSNs). Furthermore, we introduce the aggregation operators (AOs) for NHSS, such as neutrosophic hypersoft weighted average (NHSWA) and neutrosophic hypersoft weighted geometric (NHSWG) operators, with their necessary properties. Also, a novel multi-criteria decision-making (MCDM) approach has been developed for site selection of solid waste management (SWM). Moreover, a numerical description is presented to confirm the reliability and usability of the proposed technique. The output of the advocated algorithm is compared with the related models already established to regulate the favorable features of the planned study.

Keywords: neutrosophic soft set; neutrosophic hypersoft set; NHSWA operator; NHSWG operator; MCDM; SWM
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1. Introduction

MCDM is the most efficient approach to determine an adequate alternative among all viable options. Maximum assessments are made when objectives and limitations in everyday situations are often imprecise or blurred. Zadeh [1] introduced the fuzzy set (FS) theory to address this ambiguity and concern, an essential tool for solving unnecessary and hesitant problems in DM. Experts consider primarily membership (MD) and non-membership degree (NMD) that FS cannot lead. Atanassov [2] overcame these limitations and introduced the intuitionistic fuzzy set (IFS). Wang and Liu [3] presented some operations such as Einstein summation, Einstein product, etc., and the AOs of the IFS. However, the available IFS cannot understand the infrequent and disturbing details because it is intentionally assumed to be a linear inequality between MD and NMD. If the expert selects MD = 0.6 and NMD=0.7, then IFS cannot handle it as 0.6+0.7≥1. Yager [4] presented a Pythagorean fuzzy set (PFS) to precise these deficiencies by revising the basic state $\kappa + \delta \leq 1$ to $\kappa^2 + \delta^2 \leq 1$. Xiao and Ding [5] considered the Jensen–Shannon divergence and presented an inventive divergence measure for PFS.

Thao and Smarandache [6] proposed entropy measures of PFS and developed the MCDM method. Zhang et al. [7] introduced some new SMs for PFS and proved that their protracted SMs are capable compared to existing SMs. Rahman et al. [8] scheduled the geometric AO of PFS and demonstrated the multi-attribute group decision-making (MAGDM) approach by their proposed AO. Lin et al. [9] extended the partitioned Bonferroni mean operators for linguistic PFS and developed the MAGDM using their established operators. Zhang and Xu [10] prolonged the order of preference by similarity to the ideal solution (TOPSIS) to solve MCDM developments. Wei and Lu [11] delivered the basic properties of Pythagorean fuzzy power AOs and developed the multi-attribute decision-making (MADM) approach to resolve the DM obstacles. Wang and Li [12] revealed the interaction operational laws of Pythagorean fuzzy numbers (PFNs) and developed the power Bonferroni mean operators. Lin et al. [13] introduced a new probability density-based ordered weighted average operator, which was proposed based on the input values' data distribution features. Zhang [14] proposed an effective DM procedure based on SMs to overcome the MCGDM constraints under the PFS environment. Peng and Yuan [15] presented the AOs of PFS and validated the DM method by their scheduled method. Lin et al. [16] proposed the interactional partitioned Heronian mean AOs for linguistic q-rung orthopair fuzzy set and planned a MAGDM model based on their developed AOs. Zulqarnain et al. [17] presented the TOPSIS method for NS to crack MCDM issues. Lin et al. [18] proposed interactional operational laws to compute the picture fuzzy numbers. All these theories cannot handle the indeterminacy of the alternates. Smarandache [19] protracted the neutrosophic set (NS) theory capably deals with imprecise information considering the alternatives' truth, false, and indeterminacy.

All of the above approaches have a wide range of applications, but parametric chemistry has some limitations due to the incompetence of these theories. Molodtsov [20] offered the idea of soft sets (SS) and illuminated some basic operations and their properties to deal with misunderstandings and ambiguities. Maji et al. [21] extended the concept of SS and developed numerous basic operations. Cagman and Enginoglu [22] presented fuzzy parametric SS with fundamental operations. Ali et al. [23] extended the notion of SS and explained numerous basic operations of SS and their basic features. Maji et al. [24] introduced fuzzy soft sets (FSS) by associating them with two prevailing theories of FS and SS. Moreover, Roy and Maji [25] developed a unique DM technique for FSS to categorize inadequate facts. Cagman et al. [26] introduced the AOs of FSS and anticipated the DM technique via their projected AOs. Feng et al. [27] presented amendable agendas in FSS and delivered weighted FSS with their applications. Maji et al. [28] introduced the intuitionistic fuzzy soft sets (IFSS) with some fundamental operations and their basic features. Arora and Garg [29] prolonged the AOs for IFSS and delivered the MCDM technique to the operators they had settled. Çağman and Karataş [30] established a DM methodology for IFSS built on their demonstrated operations. Muthukumar and Krishnan [31] scheduled a DM technique for IFSS employing their developed SMs and weighted SMs.

Peng et al. [32] protracted the Pythagorean fuzzy soft set (PFSS), the most authentic leeway of IFSS, by including two prevalent models, PFS and SS. Zulqarnain et al. [33] presented the algebraic operational laws for PFSS and introduced the AOs for PFSS. Athira et al. [34] defined Hamming and Euclidean distances and established entropy measures for PFSS. Zulqarnain et al. [35] prolonged the Einstein AOs for PFSS with their properties to resolve MAGDM problems. Athira et al. [36] developed the entropy measures of PFSS. Zulqarnain et al. [37-38] extended the Einstein-ordered AOs for PFSS based on their developed Einstein operational laws. They also developed the DM approaches to resolve composite real-life impediments. Naeem et al. [39] proposed the TOPSIS and VIKOR methodologies utilizing the AOs of PFSS to resolve MCGDM complications. Maji et al. [40] projected the Neutrosophic soft set (NSS) by unifying the SS and NS. The notion of the possibility NSS was settled by Karaaslan [41], who presented a DM technique to resolve those complications, which enclose hesitation.

Broumi [42] discussed the fundamental operations with their properties for NSS and scheduled a DM model to resolve real-life complications. Deli and Subas [43] presented the cut sets for single-valued neutrosophic numbers (SVNNs) and settled on an MCDM approach to resolving DM obstacles. Wang et al. [44] developed the CC for SVNNs. Ye [45] offered simplified NS and presented some operational rules and AOs. Smarandache [46] planned the theory of a hypersoft set (HSS), which integrates numerous multi-sub-parameters of the considered set of parameters in the attribution function f. Smarandache HSS is the most influential theory for dealing with multi-sub-parameters of supposed parameters compared to extensions of SS theory. Different researchers in the world promote numerous HSS extensions and their DM methods. Rahman et al. [47] settled a novel extension of IFHSS known as possibility IFHSS. Zulqarnain et al. [48] extended the CC for IFHSS and developed the TOPSIS approach and AOs for IFHSS. Zulqarnain et al. [49] prolonged the impression of IFHSS to PFHSS with essential operations. Siddique et al. [50] presented the AOs for PFHSS and projected an MCDM technique utilizing their anticipated AOs. Sunthrayuth et al. [51] anticipated the Einstein weighted average aggregation operator for PFHSS to resolve MCDM complications. Zulqarnain et al. [52] presented the Einstein weighted geometric aggregation operator for PFHSS and used it for material selection. Zulqarnain et al. [53] developed the Einstein-ordered AOs for PFHSS and developed an MCDM approach to resolve DM complexities. Khan et al. [54] introduced the q-ROFHSS with some basic operations. q-ROFHSS is a combination rational formation of q-ROFSS. Gurmani et al. [55] extended the TOPSIS method using q-ROFHSS information to develop a MAGDM technique. Khan et al. [56] projected the AOs for q-ROFHSS and utilized their presented AOs to analyze the cryptocurrency market.

The population is growing rapidly and significantly impacts the environment, threatening the planet's sustainability. One of the most noticeable concerns of population growth is urbanization. Urbanization disrupts the balance of the usual atmosphere. In every metropolitan zone, a massive quantity of solid waste is produced. Developing states face substantial challenges in choosing and handling solid waste dumping spots for numerous reasons, including urban enlargement, fluctuations in local use land use, environmental influences, and financial and prospect-premeditated scheduling and supervision. The Waste face-to-face is presented by Alkaradaghi et al. [57]. Authors [58–60] discussed that the unexpected resident growth, rapid economic growth, abandoned urbanization, modification of solid waste, and misconduct are critical aspects of the rise in solid waste generation, causing humiliation to the 'environment. In South Asia, Pakistan, as a developing state, does not have a suitable waste management policy, thus producing 20 million tons of solid waste every year with an increased rate of about 2%, estimated to be 2.4%. According to Naqvi et al. [61–62], it is expected to be between 0.283 and 0.612 kg/capita/day annually. Currently, waste landfills in developing republics, comprising Pakistan, are not designated based on appropriate features and robust dealings, which is consequential in the enterprise and careless environmental organizations [63–65]. Ignore rules dispensed by Landfill Management Barzehkar et al. [66] and Kamdar et al. [67] stated that landfill is an essential part of the waste management chain, comprising waste bargain, recycling, reusing, composting, and lastly, landfilling. Basar et al. [68] introduced the REGIME method for PFS and used it for waste disposal site selection. Li et al. [69] introduced the GM (1, 1) model for PFNs and utilized their proposed model for site selection in SWM. Ren et al. [70] developed the novel MADM approach to finding a suitable site for SWM using the power Muirhead mean operator for a q-rung orthopair probabilistic hesitant fuzzy set. But, as mentioned above, these models cannot deal with the parameterized and multi-parameterized values of the considered alternatives. Karasan and Bolturk [71] proposed the combined compromise solution technique for disposal site selection under neutrosophic sets. They deliberated the truthiness, falsity, and indeterminacy of the considered alternatives. But, the combined compromise solution technique can also not accommodate the parameterized and multi-parameterized values of the alternatives. On the other hand, our presented model in this research competently deals with the parameterized and multi-parameterized values of alternatives considering the truthiness, falsity, and indeterminacy. So, defining solid waste disposal and disposal sites is one of the most significant and problematic steps.

1.1. Motivation

The NHSS is a competent amalgam of the NS and HSS, a dominant scientific tool for conciliation with ambiguous, inconsistent, and incomplete facts. It has been found that AOs perform a crucial role in DM, so cooperative evaluation data from disparate origins can be designed into specific verdicts. To our expertise, there is no application of AOs in the literature under the NHSS setting. Still, not all of the above methods are suitable for summarizing neutrosophic hypersoft numbers (NHSNs), and they cannot be intentionally associated with the multi sub-attributes of the parameters MD, NMD, and indeterminacy. We can say that the results of MD or other degrees of NMD and the effect of indeterminacy on the corresponding average and geometric AOs do not interrupt the entire procedure. So, the consequences of the present models are obscene and do not effectively order predilection for substitutes. The techniques defined in [48,50,55,56] are inappropriate for assessing facts in determining the ability to attain improved ideas and precise outcomes. All these existing theories cannot handle the indeterminacy of the multi sub-attributes of the alternatives. Meanwhile, Smarandache presented NHSS capably accommodating mentioned above obstacles, a hybrid tool to address the truthiness, falsity, and indeterminacy of the sub-attribute values. A boosted organizing procedure attracts investigators to flaw impenetrable and insufficient data. NHSS plays a vital part in DM concerning the exploration consequences by assembling various sources into a single value. Thus, to encourage the present investigation on NHSS, we will state AOs with their desirable properties built on irregular information.

1.2. Significant Contributions

We instigated a strategy to use NHSS information to select the right location for SWM to address these shortcomings. A rich organizational process leads investigators to poorly indescribable and scarce data to correct these shortcomings. The NHSS has played an essential role in interpreting the DM of exploration results by combining rich resources into a single value. The prevailing AOs for IFHSS and PFHSS cannot cope with the state when the facts of any multi-sub attribute comprise indeterminacy. It is an ingenious hybrid structure to cope with unpredicted problems during the DM practice. So, to initiate the current exploration of NHSS, we will state AOs based on rough data. The main intentions of the extant study are given as follows:

(1)NHSS can integrate several aspects of the problem concerning several sub-attributes of parameters considered in the DM system. To maintain the benefits of this absorption, we extended the AOs of NHSS.

(2)The AOs for NHSS are well-known gorgeous assessment AOs. In some situations, the prevalent AOs aspect is unresponsive to marking the precise finding over the DM method. To stun these particular obstacles, the existing AOs necessary to be revised. We regulate innovative operational laws for neutrosophic hypersoft numbers (NHSNs).

(3)NHSWA and NHSWG operators have been presented with their crucial properties with settled algebraic operational laws.

(4)Establish a new algorithm based on plan operators to solve the MCDM problem under the NHSS setting.

(5)Site selection for SWM is a subservient feature of urbanization as it realizes the concrete circumstances for all aspects. SWM is an arduous but significant stage in the certified improvement

(6)A comparative study of the advanced MCDM method and prevailing approaches has been offered to contemplate usefulness and dominance.

The rest of the study can be summed up like this. Section 2 discusses fundamental notions such as SS, NSS, and NHSS, which comfort us in constructing the structure of the consequent study. Section 3 defines some operating laws for the NHSS and develops some AOs, such as NHSWA and NHSWG operators, with existing operational laws with the required properties. Section 4 uses a DM technique to resolve the MCDM issue using the planned operators. A mathematical illustration is given to certify the practicability of the established DM technique. Also, we employ some of the present techniques for comparative analysis with our scheduled method. Furthermore, we extant the assistances, easiness, tractability, and efficacy of the proposed strategy in Section 5. An inclusive debate is organized between some existing methods and our established approach in the same section.

2. Preliminaries

In the subsequent section, we recollected necessary notions that assisted us in growing the present article's configuration, such as SS, NS, NSS, HSS, FHSS, and NHSS.

Definition 2.1. [20] Let $\mathcal{U}$ and $\mathcal{E}$ be the universe of discourse and set of attributes, respectively, and $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathsf{A}\subseteq \mathcal{E}$. A pair $(\mathcal{F},\ \mathsf{A})$ is called a soft set over $\mathcal{U}$, where $\mathcal{F}$ is a mapping:

$$\mathcal{F}:\mathsf{A}\rightarrow \mathcal{P}(\mathcal{U})$$

Also, it can be defined as follows:

$$(\mathcal{F},\mathsf{A}) = \{\mathcal{F}(e) \in \mathcal{P}(\mathcal{U}): e \in \mathcal{E}, \mathcal{F}(e) = \emptyset \ if\ e \notin \mathsf{A}\}$$

Definition 2.2. [19] Let $\boldsymbol{\mathcal{U}}$ be a universe of discourse and A be an neutrosophic set on $\boldsymbol{\mathcal{U}}$ is defined as $\mathsf{A}=\{\boldsymbol{v}, (\boldsymbol{\mathcal{T}}_{\mathsf{A}}(\boldsymbol{v}), \boldsymbol{\mathcal{I}}_{\mathsf{A}}(\boldsymbol{v}), \boldsymbol{\mathfrak{C}}_{\mathsf{A}}(\boldsymbol{v})): \boldsymbol{v} \in \boldsymbol{\mathcal{U}}\}$, where $\boldsymbol{\mathcal{T}}$, $\boldsymbol{\mathcal{I}}$, $\boldsymbol{\mathfrak{C}}: \boldsymbol{\mathcal{U}} \rightarrow]\mathbf{0}^-,\ \mathbf{1}^+[$ and $\mathbf{0}^- \leq \boldsymbol{\mathcal{T}}_{\mathsf{A}}(\boldsymbol{v}) + \boldsymbol{\mathcal{I}}_{\mathsf{A}}(\boldsymbol{v}) + \boldsymbol{\mathfrak{C}}_{\mathsf{A}}(\boldsymbol{v}) \leq \mathbf{3}^+$.

Definition 2.3. [40] Let $\mathcal{U}$ be a universe of discourse and $\mathcal{E}$ be a collection of parameters regarding $\mathcal{U}$ and $\mathcal{P}(\mathcal{U})$ be a collection of all neutrosophic subsets of $\mathcal{U}$ and $\mathsf{A}\subseteq \mathcal{E}$. A pair $(\mathcal{F},\mathsf{A})$ is known as a neutrosophic soft set over $\mathcal{U}$, where $\mathcal{F}$ is a mapping:

$$\mathcal{F}:\mathsf{A}\rightarrow \mathcal{P}(\mathcal{U})$$

Definition 2.4. [46] Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k=\{k_1, k_2, k_3,..., k_n\}$,(n≥1) and K_i signified the set of parameters with their conforming sub-parameters, such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \epsilon \{1,2,3\ldots n\}$. Suppose $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ be a collection of sub-parameters, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathbb{N}$. Then the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n = (\mathcal{F}, \ddot{\mathsf{A}})$ is known as a hypersoft set, where $\mathcal{F}$ is a mapping:

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow \mathcal{P}(\mathcal{U}).$$

It is also defined as

$$(\mathcal{F},\ddot{\mathsf{A}}) = \{\check{d}, \mathcal{F}_{\ddot{\mathsf{A}}}(\check{d}): \check{d} \in \ddot{\mathsf{A}},\ \mathcal{F}_{\ddot{\mathsf{A}}}(\check{d}) \in \mathcal{P}(\mathcal{U})\}.$$

Definition 2.5. [46] Let $\mathcal{U}$ be a universe of discourse and $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$ and $k=\{k_1,k_2,k_3,...,k_n\}$,(n≥1) and K_i signified the set of parameters with their conforming sub-parameters, such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and $i, j \epsilon \{1,2,3\ldots n\}$. Suppose $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ be a collection of multi-sub-attributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, and $1 \leq l \leq \gamma$, and $\alpha, \beta, \gamma \in \mathbb{N}$, and $NS^{\mathcal{U}}$ represents neutrosophic subsets over $\mathcal{U}$. Then, $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \ldots \times K_n\ \ddot{\mathsf{A}})$ is called neutrosophic hypersoft set, where $\mathcal{F}$ is a mapping:

$$\mathcal{F}: K_1 \times K_2 \times K_3 \times \ldots \times K_n = \ddot{\mathsf{A}} \rightarrow NS^{\mathcal{U}}.$$

It is also defined as

$$(\mathcal{F},\ddot{\mathsf{A}}) = \{(\check{d}, \mathcal{F}_{\ddot{\mathsf{A}}}(\check{d})): \check{d} \in \ddot{\mathsf{A}},\ \mathcal{F}_{\ddot{\mathsf{A}}}(\check{d}) \in NS^{\mathcal{U}}\},$$

where $\mathcal{F}_{Ä}(\check{d})=\left\{\langle v, \mathcal{T}_{\mathcal{F}(\check{d})}(v), \mathcal{I}_{\mathcal{F}(\check{d})}(v), \mathfrak{C}_{\mathcal{F}(\check{d})}(v)\rangle : v \in \mathcal{U}\right\}$, where $\mathcal{T}_{\mathcal{F}(\check{d})}(v)$, $\mathcal{I}_{\mathcal{F}(\check{d})}(v)$, and $\mathfrak{C}_{\mathcal{F}(\check{d})}(v)$ symbolize the truthness, indeterminacy, and false values for attributes such as $\mathcal{T}_{\mathcal{F}(\check{d})}(v)$, $\mathcal{I}_{\mathcal{F}(\check{d})}(v)$, $\mathfrak{C}_{\mathcal{F}(\check{d})}(v) \in [0,1]$, and $0 \leq \mathcal{T}_{\mathcal{F}(\check{d})}(v)+\mathcal{I}_{\mathcal{F}(\check{d})}(v)+\mathfrak{C}_{\mathcal{F}(\check{d})}(v) \leq 3$.

Simply an NHSN can be stated as $\mathcal{F}=\left\{\left(\mathcal{T}_{\mathcal{F}(\check{d})}(v), \mathcal{I}_{\mathcal{F}(\check{d})}(v), \mathfrak{C}_{\mathcal{F}(\check{d})}(v)\right)\right\}$, where $0 \leq \mathcal{T}_{\mathcal{F}(\check{d})}(v) + \mathcal{I}_{\mathcal{F}(\check{d})}(v) + \mathfrak{C}_{\mathcal{F}(\check{d})}(v) \leq 3$.

Example 2.1. Consider $\mathcal{U} = \{v_1, v_2\}$ be a universe of discourse and $\mathfrak{L}= \{L_1 = Teaching\ methdology, L_2 = Subjects, L_3 = Classes\}$ are the deliberated aspects, and their corresponding n-tuple sub-attributes are given as follows: $L_1 = \{a_{11} = project\ base, a_{12} = class\ discussion\}$, $L_2=\{a_{21} = Mathematics, a_{22} = Computer\ Science, a_{23} = Statistics\}$, and Classes=$L_3$=$\{a_{31} = Masters, a_{32} = Doctorol\}$. Let Ä=$L_1 \times L_2 \times L_3$ be a set of attributes

$$Ä=L_1 \times L_2 \times L_3=\{d_{11}, d_{12}\} \times \{d_{21}, d_{22}, d_{23}\} \times \{d_{31}, d_{32}\}$$

$$=\begin{Bmatrix}(d_{11}, d_{21}, d_{31}), (d_{11}, d_{21}, d_{32}), (d_{11}, d_{22}, d_{31}), (d_{11}, d_{22}, d_{32}), (d_{11}, d_{23}, d_{31}), (d_{11}, d_{23}, d_{32}), \\ (d_{12}, d_{21}, d_{31}), (d_{12}, d_{21}, d_{32}), (d_{12}, d_{22}, d_{31}), (d_{12}, d_{22}, d_{32}), (d_{12}, d_{23}, d_{31}), (d_{12}, d_{23}, d_{32}),\end{Bmatrix}$$

$$Ä=\{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4, \check{d}_5, \check{d}_6, \check{d}_7, \check{d}_8, \check{d}_9, \check{d}_{10}, \check{d}_{11}, \check{d}_{12}\}$$

Then the NHSS over $\mathcal{U}$ is given as follows:

$$(\mathcal{F},Ä)=\left\{\begin{matrix} \left(\check{d}_1, (\delta_1, (.6, .3, .8)), (\delta_2, (.9, .3, .5))\right), \left(\check{d}_2, (\delta_1, (.5, .2, .7)), (\delta_2, (.7, .1, .5))\right), \left(\check{d}_3, (\delta_1, (.5, .2, .8)), (\delta_2, (.4, .3, .4))\right), \\ \left(\check{d}_4, (\delta_1, (.2, .5, .6)), (\delta_2, (.5, .1, .6))\right), \left(\check{d}_5, (\delta_1, (.8, .4, .3)), (\delta_2, (.2, .3, .5))\right), \left(\check{d}_6, (\delta_1, (.9, .6, .4)), (\delta_2, (.7, .6, .8))\right), \\ \left(\check{d}_7, (\delta_1, (.6. .5, .3)), (\delta_2, (.4, .2, .8))\right), \left(\check{d}_8, (\delta_1, (.8, .2, .5)), (\delta_2, (.6, .8, .4))\right), \left(\check{d}_9, (\delta_1, (.7, .4, .9)), (\delta_2, (.7. .3, .5))\right), \\ \left(\check{d}_{10}, (\delta_1, (.8, .4, .6)), (\delta_2, (.7, .2, .9))\right), \left(\check{d}_{11}, (\delta_1, (.8, .4, .5)), (\delta_2, (.4, .2, .5))\right), \left(\check{d}_{12}, (\delta_1, (.7, .5, .8)), (\delta_2, (.7, .5, .9))\right). \end{matrix}\right\}$$

For simplicity, we will express $\mathcal{F}_{v_i}(\check{d}_j) = \left\{\left\{\left(\mathcal{T}_{\mathcal{F}(\check{d})}(v_i), \mathcal{I}_{\mathcal{F}(\check{d})}(v_i), \mathfrak{C}_{\mathcal{F}(\check{d})}(v_i)\right)\right\} \mid v_i \in \mathcal{U}\right\}$ as $\mathfrak{J}_{\check{d}_{ij}} = \left\langle \mathcal{T}_{\mathcal{F}(\check{d}_{ij})}, \mathcal{J}_{\mathcal{F}(\check{d}_{ij})}, \mathfrak{C}_{\mathcal{F}(\check{d}_{ij})} \right\rangle$ is called NHSN, where $0 \leq \mathcal{T}_{\mathcal{F}(\check{d}_{ij})} + \mathcal{J}_{\mathcal{F}(\check{d}_{ij})} + \mathfrak{C}_{\mathcal{F}(\check{d}_{ij})} \leq 3$, and $\mathcal{T}_{\mathcal{F}(\check{d}_{ij})}, \mathcal{J}_{\mathcal{F}(\check{d}_{ij})}, \mathfrak{C}_{\mathcal{F}(\check{d}_{ij})} \in [0,1]$. The score function for NHSNs $\mathfrak{J}_{\check{d}_{ij}}$ is defined as follows:

$$\mathcal{S}(\mathfrak{J}_{\check{d}_{ij}})= \quad \mathcal{T}_{\mathcal{F}(\check{d}_{ij})} - \mathfrak{C}_{\mathcal{F}(\check{d}_{ij})}. \tag{2.1}$$

Where $\mathcal{S}(\mathfrak{J}_{\check{d}_{ij}}) \in [-1,1]$, sometimes the score function is unable to compare any two NHSNs. like $\mathfrak{J}_{\check{d}_{11}} = \langle 0.6, 0.2, 0.2\rangle$ and $\mathfrak{J}_{\check{d}_{12}} = \langle 0.5, 0.1, 0.1\rangle$. Where $\mathcal{S}(\mathfrak{J}_{\check{d}_{11}})$ =0.4= $\mathcal{S}(\mathfrak{J}_{\check{d}_{12}})$ in such cases, choosing which alternate is more applicable is challenging. To grasp such facts, we must present the accuracy function for NHSNs.

$$H(\mathfrak{J}_{\check{d}_{ij}})=\mathcal{T}_{\mathcal{F}(\check{d}_{ij})} + \mathcal{J}_{\mathcal{F}(\check{d}_{ij})} + \mathfrak{C}_{\mathcal{F}(\check{d}_{ij})}, H(\mathfrak{J}_{\check{d}_{ij}}) \in [0,1]. \tag{2.2}$$

In the succeeding, we extant the comparison laws to relate NHSNs $\mathfrak{J}_{\check{d}_{ij}}$ and $\mathfrak{T}_{\check{d}_{ij}}$such as

(1) If $\mathcal{S}(\mathfrak{J}_{\check{d}_{ij}}) > \mathcal{S}(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$.

(2) If $\mathcal{S}(\mathfrak{J}_{\check{d}_{ij}}) = \mathcal{S}(\mathfrak{T}_{\check{d}_{ij}})$, then

If $H\left(\mathfrak{J}_{\check{d}_{ij}}\right) > H\left(\mathfrak{T}_{\check{d}_{ij}}\right)$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$;

If $H\left(\mathfrak{J}_{\check{d}_{ij}}\right) = H(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{T}_{\check{d}_{ij}}$.

3. Aggregation operators for neutrosophic hypersoft numbers

In the following section, we will introduce the AOs, $i.e.$, NHSWA, and NHSWG operators for NHSNs. We also discuss the fundamental properties of our developed AOs.

Definition 3.1. Let $\mathfrak{I}_{\check{d}_k}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}} \right\rangle$, $\mathfrak{I}_{\check{d}_{11}}=\langle \mathcal{T}_{\check{d}_{11}}, \mathcal{J}_{\check{d}_{11}}, \mathfrak{C}_{\check{d}_{11}} \rangle$, and $\mathfrak{I}_{\check{d}_{12}}=\langle \mathcal{T}_{\check{d}_{12}}, \mathcal{J}_{\check{d}_{12}}, \mathfrak{C}_{\check{d}_{12}} \rangle$ be three NHSNs and $\alpha > 0$, and then the algebraic operational laws are defined as:

(1) $\mathfrak{I}_{\check{d}_{11}} \oplus \mathfrak{I}_{\check{d}_{12}}=\langle \mathcal{T}_{\check{d}_{11}} + \mathcal{T}_{\check{d}_{12}} - \mathcal{T}_{\check{d}_{11}}\mathcal{T}_{\check{d}_{12}}, \mathcal{J}_{\check{d}_{11}}\mathcal{J}_{\check{d}_{12}},\ \mathfrak{C}_{\check{d}_{11}}\mathfrak{C}_{\check{d}_{12}} \rangle$,

(2) $\mathfrak{I}_{\check{d}_{11}} \otimes \mathfrak{I}_{\check{d}_{12}}=\langle \mathcal{T}_{\check{d}_{11}}\mathcal{T}_{\check{d}_{12}}, \mathcal{J}_{\check{d}_{11}} + \mathcal{J}_{\check{d}_{12}} - \mathcal{J}_{\check{d}_{11}}\mathcal{J}_{\check{d}_{12}},\ \mathfrak{C}_{\check{d}_{11}} + \mathfrak{C}_{\check{d}_{12}} - \mathfrak{C}_{\check{d}_{11}}\mathfrak{C}_{\check{d}_{12}} \rangle$,

(3) $\alpha\mathfrak{I}_{\check{d}_k}=\left\langle 1-\left(1-\mathcal{T}_{\check{d}_k}\right)^{\alpha}, {\mathcal{J}_{\check{d}_k}}^{\alpha}, {\mathfrak{C}_{\check{d}_k}}^{\alpha} \right\rangle$,

(4) $\mathfrak{I}_{\check{d}_k}^{\alpha}=\left\langle {\mathcal{T}_{\check{d}_k}}^{\alpha}, 1-\left(1-\mathcal{J}_{\check{d}_k}\right)^{\alpha}, 1-\left(1-\mathfrak{C}_{\check{d}_k}\right)^{\alpha} \right\rangle$.

Using the above presented algebraic operational laws, the AOs for NHSNs can be defined as follows.

Definition 3.2. $\mathfrak{I}_{\check{d}_k}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}} \right\rangle$ be an NHSN, Ω_i and γ_j be the weights for experts and multi sub-attributes of the deliberated attributes consistently along with indicated circumstances $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i=1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j=1$. Then NHSWA operator can be defined as NHSWA: $\Delta^n \rightarrow \Delta$ and expressed as follows:

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right)=\oplus_{j=1}^{m}\gamma_j\left(\oplus_{i=1}^{n}\Omega_i\mathfrak{I}_{\check{d}_{ij}}\right). \tag{3.1}$$

Theorem 3.1. $\mathfrak{I}_{\check{d}_k}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}} \right\rangle$ be an NHSN, where $(i = 1, 2, \ldots, n, and\ j = 1, 2, \ldots, m)$. Then, obtained aggregated values using Eq 3.1 is also an NHSN and

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right)=$$

$$\left\langle 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \right\rangle. \tag{3.2}$$

Ω_i and γ_j be the weights for specialists and multi sub-attributes of the intended parameters, respectively, along with indicated conditions $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i=1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j=1$.

Proof. Using the principle of mathematical induction NHSWA operator can be proved by utilizing the following steps:

For $n = 1$, we get $\Omega_1=1$. Then, we have

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{1m}}\right)=\oplus_{j=1}^{m}\gamma_j\mathfrak{I}_{\check{d}_{1j}}$$

$$=\left\langle 1-\prod_{j=1}^{m}\left(1-\mathcal{T}_{\check{d}_{1j}}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\mathcal{J}_{\check{d}_{1j}}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\mathfrak{C}_{\check{d}_{1j}}\right)^{\gamma_j} \right\rangle$$

$$=\left\langle 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \right\rangle.$$

For $m = 1$, we get $\gamma_1=1$. Then, we have

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{21}}, \ldots, \mathfrak{I}_{\check{d}_{n1}}\right)=\oplus_{i=1}^{n}\Omega_i\mathfrak{I}_{\check{d}_{i1}}$$

$$= \left\langle 1-\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{i1}}\right)^{\Omega_i}, \prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{i1}}\right)^{\Omega_i}, \prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{i1}}\right)^{\Omega_i} \right\rangle$$

$$=\left\langle 1-\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle.$$

So, for $n=1$ and $m=1$, Eq 3.2 satisfied. Consider Eq 3.2 holds for $m=\beta_1+1$, $n=\beta_2$ and $m=\beta_1$, $n=\beta_2+1$, such as:

$$\oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2}\Omega_i\mathfrak{I}_{\check{d}_{ij}}\right)=$$

$$\left\langle 1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle,$$

$$\oplus_{j=1}^{\beta_1}\gamma_j\left(\oplus_{i=1}^{\beta_2+1}\Omega_i\mathfrak{I}_{\check{d}_{ij}}\right)=$$

$$\left\langle 1-\prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle.$$

For $m=\beta_1+1$ and $n=\beta_2+1$, we have

$$\oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2+1}\Omega_i\mathfrak{I}_{\check{d}_{ij}}\right)=\oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2}\Omega_i\mathfrak{I}_{\check{d}_{ij}}\oplus\Omega_{\beta_2+1}\mathfrak{I}_{\check{d}_{(\beta_2+1)j}}\right)$$

$$\oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2+1}\Omega_i\mathfrak{I}_{\check{d}_{ij}}\right)=\oplus_{j=1}^{\beta_1+1}\oplus_{i=1}^{\beta_2}\gamma_j\Omega_i\mathfrak{I}_{\check{d}_{ij}}\ \oplus_{j=1}^{\beta_1+1}\gamma_j\Omega_{\beta_2+1}\mathfrak{I}_{\check{d}_{(\beta_2+1)j}}$$

$$=\left\langle \begin{array}{c} 1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\oplus 1-\prod_{j=1}^{\beta_1+1}\left(\left(1-\mathcal{T}_{(\beta_2+1)j}{}^{2}\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j}, \\ \prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\oplus\prod_{j=1}^{\beta_1+1}\left(\left(\mathcal{J}_{(\beta_2+1)j}\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j},\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\oplus\prod_{j=1}^{\beta_1+1}\left(\left(\mathfrak{C}_{(\beta_2+1)j}\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j}\end{array}\right\rangle$$

$$=\left\langle 1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle.$$

Therefore, it is correct for $m=\beta_1+1$ and $n=\beta_2+1$.

Example 3.1. Let $\mathcal{U}=\{\kappa_1, \kappa_2, \kappa_3\}$ be a collection of experts with their weights $\Omega_i=(0.143, 0.514, 0.343)^T$. The group of specialists is operational to precise the attractiveness of a firm using a defined set of attributes $\mathfrak{L}'=\{d_1=lawn, d_2=security\ system\}$ with their conforming sub-attributes; Lawn $=d_1=\{d_{11}=with\ grass, d_{12}=without\ grass\}$, security system $=d_2=\{d_{21}=guards, d_{22}=cameras\}$. Let $\mathfrak{L}'=d_1\times d_2$ be a set of multi sub-attributes

$$\mathfrak{L}'=d_1\times d_2=\{d_{11}, d_{12}\}\times\{d_{21}, d_{22}\}=\{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}.$$

$\mathfrak{L}'=\{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ with weights $\gamma_j=(.35, .15, .2, .3)^T$. Specialists' estimation for each multi-sub-attribute in the form of NHSNs $(\mathfrak{I}, \mathfrak{L}')=\left\langle\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle_{3\times 4}$ given as follows:

$$(\mathfrak{I}, \mathfrak{L}')=\begin{bmatrix}(.3,.8,.6) & (.4,.6,.3) & (.8,.3,.6) & (.2,.5,.6)\\(.8,.3,.7) & (.5,.7,.4) & (.1,.7,.3) & (.9,.4,.8)\\(.3,.6,.5) & (.8,.5,.7) & (.2,.6,.5) & (.8,.5,.4)\end{bmatrix}.$$

Using Eq 3.2.

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{34}}\right)$$

$$=\left\langle 1-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle$$

$$=\left\langle\begin{array}{l}1-(\{(.7)^{.143}(.2)^{.514}(.7)^{.343}\}^{.35}\{(.6)^{.143}(.5)^{.514}(.2)^{.343}\}^{.15}\{(.2)^{.143}(.9)^{.514}(.8)^{.343}\}^{.2}\{(.8)^{.143}(.1)^{.514}(.2)^{.343}\}^{.3}),\\ (\{(.8)^{.143}(.3)^{.514}(.6)^{.343}\}^{.35}\{(.6)^{.143}(.7)^{.514}(.5)^{.343}\}^{.15}\{(.3)^{.143}(.7)^{.514}(.6)^{.343}\}^{.2},\{(.5)^{.143}(.4)^{.514}(.5)^{.343}\}^{.3}),\\ (\{(.6)^{.143}(.7)^{.514}(.5)^{.343}\}^{.35}\{(.3)^{.143}(.4)^{.514}(.7)^{.343}\}^{.15}\{(.6)^{.143}(.3)^{.514}(.5)^{.343}\}^{.2},\{(.6)^{.143}(.8)^{.514}(.4)^{.343}\}^{.3})\end{array}\right\rangle$$

$$=\left\langle\begin{array}{l}1-(\{.35478\}^{.35}\{.35613\}^{.15}\{.59346\}^{.2}\{.16699\}^{.3}),\\ (\{.42815\}^{.35}\{.57972\}^{.15}\{.52147\}^{.2},\{.41596\}^{.3}),\\ (\{.57972\}^{.35}\{.41235\}^{.15}\{.42092\}^{.2},\{.57516\}^{.3}),\end{array}\right\rangle$$

$=\langle .68615, .46206, .51546\rangle$.

3.1. Properties of NHSWA operator

3.1.1. Idempotency

If $\mathfrak{I}_{\check{d}_{ij}}=\mathfrak{I}_{\check{d}}=\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle\ \forall i,j$. Then,

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\dots,\mathfrak{I}_{\check{d}_{nm}}\right)=\mathfrak{I}_{\check{d}}.$$

Proof. As we know that all $\mathfrak{I}_{\check{d}_{ij}}=\mathfrak{I}_{\check{d}}=\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle\ \forall i,j$. Then, using Eq 3.2

$NHSWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\dots,\mathfrak{I}_{\check{d}_{nm}}\right)$

$$=\left\langle 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle$$

$$=\left\langle 1-\left(\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j},\left(\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j},\left(\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}\right\rangle$$

$$=\left\langle 1-\left(1-\mathcal{T}_{\check{d}_{ij}}\right),\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle=\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle=\mathfrak{I}_{\check{d}}.$$

3.1.2. Boundedness

Let $\mathfrak{I}_{\check{d}_{ij}}=\backslash\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle$ be a collection of NHSNs.

$\mathfrak{I}_{\check{d}_{ij}}{}^{-}=\left\langle\min_j\min_i\left\{\mathcal{T}_{\check{d}_{ij}}\right\},\max_j\max_i\left\{\mathcal{J}_{\check{d}_{ij}}\right\},\max_j\max_i\left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right\rangle$ and

$\mathfrak{I}_{\check{d}_{ij}}{}^{+}=\left\langle\max_j\max_i\left\{\mathcal{T}_{\check{d}_{ij}}\right\},\min_j\min_i\left\{\mathcal{J}_{\check{d}_{ij}}\right\},\min_j\min_i\left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right\rangle$, then

$$\mathfrak{I}_{\check{d}_{ij}}{}^{-}\leq NHSWA\left(\mathfrak{I}_{\check{d}_{11}},\mathfrak{I}_{\check{d}_{12}},\dots,\mathfrak{I}_{\check{d}_{nm}}\right)\leq\mathfrak{I}_{\check{d}_{ij}}{}^{+}.$$

Proof. Since $\mathfrak{I}_{\check{d}_{ij}}=\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle$ be a collection of NHSNs$\forall i,j$. Then,

$$\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \le \mathcal{T}_{\check{d}_{ij}} \le \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$$

$$\Rightarrow 1-\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \le 1-\mathcal{T}_{\check{d}_{ij}} \le 1-\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$$

$$\Leftrightarrow \left(1-\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\Omega_i} \le \left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \le \left(1-\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\Omega_i}$$

$$\Leftrightarrow \left(1-\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \le \prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \le \left(1-\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$$

$$\Leftrightarrow \left(1-\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \left(1-\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Leftrightarrow 1-\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le 1-\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$$

$$\Leftrightarrow \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \le 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}. \tag{3.3}$$

Again,

$$\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \le \mathcal{J}_{\check{d}_{ij}} \le \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}$$

$$\Rightarrow \left(\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \le \prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \le \left(\max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$$

$$\Leftrightarrow \min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \le \prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \le \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\gamma_j} \le \left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \left(\max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\gamma_j}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \left(\max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j}.$$

So, we get

$$\Leftrightarrow \min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}, \tag{3.4}$$

and

$$\min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\} \le \mathfrak{C}_{\check{d}_{ij}} \le \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}$$

$$\Rightarrow \left(\min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \leq \prod_{i=1}^{n}\left(\mathfrak{C}_{\breve{d}_{ij}}\right)^{\Omega_i} \leq \left(\max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$$

$$\Leftrightarrow \min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\} \leq \prod_{i=1}^{n}\left(\mathfrak{C}_{\breve{d}_{ij}}\right)^{\Omega_i} \leq \max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}\right)^{\gamma_j} \leq \left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\breve{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \left(\max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}\right)^{\gamma_j}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\breve{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \left(\max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j}.$$

So, we get

$$\Leftrightarrow \min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\breve{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}. \tag{3.5}$$

Let $NHSWA\left(\mathfrak{J}_{\breve{d}_{11}}, \mathfrak{J}_{\breve{d}_{12}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}}\right) = \langle \mathcal{T}_{\breve{\delta}}, \mathcal{J}_{\breve{\delta}}, \mathfrak{C}_{\breve{\delta}} \rangle = \mathfrak{J}_{\breve{\delta}}$, then inequalities (3.3)–(3.5) can be transformed into the following form: $\min_j \min_i \left\{\mathcal{T}_{\breve{d}_{ij}}\right\} \leq \mathcal{T}_{\breve{\delta}} \leq \max_j \max_i \left\{\mathcal{T}_{\breve{d}_{ij}}\right\}$, $\min_j \min_i \left\{\mathcal{J}_{\breve{d}_{ij}}\right\} \leq \mathcal{J}_{\breve{\delta}} \leq \max_j \max_i \left\{\mathcal{J}_{\breve{d}_{ij}}\right\}$, and $\min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\} \leq \mathfrak{C}_{\breve{\delta}} \leq \max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}$ respectively.

Using Eq 2.1, we get the following:

$$\mathcal{S}(\mathfrak{J}_{\breve{\delta}})=\mathcal{T}_{\breve{\delta}} - \mathfrak{C}_{\breve{\delta}} \leq \max_j \max_i \left\{\mathcal{T}_{\breve{d}_{ij}}\right\} - \min_j \min_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}=\mathcal{S}\left(\mathfrak{J}_{\breve{d}_{ij}}{}^{+}\right),$$

$$\mathcal{S}(\mathfrak{J}_{\breve{\delta}})=\mathcal{T}_{\breve{\delta}} - \mathfrak{C}_{\breve{\delta}} \geq \min_j \min_i \left\{\mathcal{T}_{\breve{d}_{ij}}\right\} - \max_j \max_i \left\{\mathfrak{C}_{\breve{d}_{ij}}\right\}=\mathcal{S}\left(\mathfrak{J}_{\breve{d}_{ij}}{}^{-}\right).$$

Then, order relation to two NHSNs, we have

$$\mathfrak{J}_{\breve{d}_{ij}}{}^{-} \leq NHSWA\left(\mathfrak{J}_{\breve{d}_{11}}, \mathfrak{J}_{\breve{d}_{12}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}}\right) \leq \mathfrak{J}_{\breve{d}_{ij}}{}^{+}.$$

3.1.3. Shift Invariance

If $\mathfrak{J}_{\breve{d}}=\langle \mathcal{T}_{\breve{d}}, \mathcal{J}_{\breve{d}}, \mathfrak{C}_{\breve{d}} \rangle$ be an NHSN. Then,

$$NHSWA\left(\mathfrak{J}_{\breve{d}_{11}} \oplus \mathfrak{J}_{\breve{d}}, \mathfrak{J}_{\breve{d}_{12}} \oplus \mathfrak{J}_{\breve{d}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}} \oplus \mathfrak{J}_{\breve{d}}\right)=NHSWA\left(\mathfrak{J}_{\breve{d}_{11}}, \mathfrak{J}_{\breve{d}_{12}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}}\right) \oplus \mathfrak{J}_{\breve{d}}.$$

Proof. Assume $\mathfrak{J}_{\breve{d}}$ be an NHSN and $\mathfrak{J}_{\breve{d}_{ij}}$ be a collection of NHSNs. Utilizing operational laws, we have:

$$\mathfrak{J}_{\breve{d}} \oplus \mathfrak{J}_{\breve{d}_{nm}}=\left\langle \mathcal{T}_{\breve{d}} + \mathcal{T}_{\breve{d}_{ij}} - \mathcal{T}_{\breve{d}}\mathcal{T}_{\breve{d}_{ij}}, \mathcal{J}_{\breve{d}}\mathcal{J}_{\breve{d}_{ij}}, \mathfrak{C}_{\breve{d}}\mathfrak{C}_{\breve{d}_{ij}} \right\rangle.$$

So

$$NHSWA\left(\mathfrak{J}_{\breve{d}_{11}} \oplus \mathfrak{J}_{\breve{d}}, \mathfrak{J}_{\breve{d}_{12}} \oplus \mathfrak{J}_{\breve{d}}, \ldots, \mathfrak{J}_{\breve{d}_{nm}} \oplus \mathfrak{J}_{\breve{d}}\right)=\oplus_{j=1}^{m} \gamma_j \left(\oplus_{i=1}^{n} \Omega_i \left(\mathfrak{J}_{\breve{d}_{ij}} \oplus \mathfrak{J}_{\breve{d}}\right)\right)$$

$$=\left\langle \begin{array}{c} 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}(1-\mathcal{T}_{\check{d}})^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}(\mathcal{J}_{\check{d}})^{\Omega_i}\right)^{\gamma_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}(\mathfrak{C}_{\check{d}})^{\Omega_i}\right)^{\gamma_j} \end{array}\right\rangle$$

$$=\left\langle \begin{array}{c} 1-(1-\mathcal{T}_{\check{d}})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \mathcal{J}_{\check{d}}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \\ \mathfrak{C}_{\check{d}}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \end{array}\right\rangle$$

$$=\left\langle \begin{array}{c} 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \end{array}\right\rangle \oplus \langle \mathcal{T}_{\check{d}}, \mathcal{J}_{\check{d}}, \mathfrak{C}_{\check{d}}\rangle$$

$$=NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right) \oplus \mathfrak{I}_{\check{d}}.$$

3.1.4. Homogeneity

Let $\alpha > 0$; then, we have to Prove that $NHSWA\left(\alpha\mathfrak{I}_{\check{d}_{11}}, \alpha\mathfrak{I}_{\check{d}_{12}} \dots, \alpha\mathfrak{I}_{\check{d}_{nm}}\right)=\alpha NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right)$.

Proof. Let $\mathfrak{I}_{\check{d}_{ij}}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle$ be a collection of NHSNs$\forall i, j$, and α is any positive real number. Then,

$$\alpha\mathfrak{I}_{\check{d}_{ij}}=\left\langle 1-\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\alpha}, \mathcal{J}_{\check{d}_{ij}}{}^{\alpha}, \mathfrak{C}_{\check{d}_{ij}}{}^{\alpha}\right\rangle.$$

So,

$NHSWA\left(\alpha\mathfrak{I}_{\check{d}_{11}}, \alpha\mathfrak{I}_{\check{d}_{12}} \dots, \alpha\mathfrak{I}_{\check{d}_{nm}}\right)$

$$=\left\langle 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}\right\rangle$$

$$=\left\langle 1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}, \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}, \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}\right\rangle$$

$$=\alpha NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right).$$

3.1.5. Monotonicity

Let $\mathfrak{I}_{\check{d}_{ij}}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle$ and $\mathfrak{I}^*_{\check{d}_{ij}}=\left\langle \mathcal{T}^*_{\check{d}_{ij}}, \mathcal{J}^*_{\check{d}_{ij}}, \mathfrak{C}^*_{\check{d}_{ij}}\right\rangle$ be the collection of NHSNs. Then

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right) \le NHSWA\left(\mathfrak{I}^*_{\check{d}_{11}}, \mathfrak{I}^*_{\check{d}_{12}}, \dots, \mathfrak{I}^*_{\check{d}_{nm}}\right), \text{ if } \mathfrak{I}_{\check{d}_{ij}} \le \mathfrak{I}^*_{\check{d}_{ij}} \forall i, j.$$

Proof. Let $f(x) = \frac{1-x}{1+x}$, $x \in [0, 1]$, then $\frac{d}{dx}\left(f(x)\right) = \frac{-2}{(1+x)^2} < 0$. So, $f(x)$ is decreasing function on $]0, 1]$. If $\mathcal{T}_{\check{d}_{ij}} \le \mathcal{T}^*_{\check{d}_{ij}}$, then $f\left(\mathcal{T}^*_{\check{d}_{ij}}\right) \le f\left(\mathcal{T}_{\check{d}_{ij}}\right) \forall i, j$.i.e.,

$$1-\mathcal{T}^{*}_{\check{d}_{ij}} \leq 1-\mathcal{T}_{\check{d}_{ij}}.$$

Where, Ω_i >0, $\sum_{i=1}^{n}\Omega_i$=1, γ_j >0, $\sum_{j=1}^{m}\gamma_j$=1. So,

$$\left(\left(1-\mathcal{T}^{*}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \left(\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Rightarrow \left(\prod_{i=1}^{n}\left(1-\mathcal{T}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}$$

$$\Rightarrow 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \geq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}.$$

Again let $g(y)=\frac{2-y}{y}$, $y\in[0,1]$, then $\frac{d}{dy}\left(g(y)\right)=\frac{-2}{y^2}<0$. So, $g(y)$ is decreasing function on $]0,1]$. If $\mathcal{J}^{*}_{\check{d}_{ij}} \leq \mathcal{J}_{\check{d}_{ij}}$, then $g\left(\mathcal{J}^{*}_{\check{d}_{ij}}\right) \geq g\left(\mathcal{J}_{\check{d}_{ij}}\right) \forall i,j$. i.e.,

$$\mathcal{J}^{*}_{\check{d}_{ij}} \leq \mathcal{J}_{\check{d}_{ij}}.$$

As we know that $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i$= , γ_j >0, $\sum_{j=1}^{m}\gamma_j$=1. So,

$$\left(\left(\mathcal{J}^{*}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \left(\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Rightarrow \left(\prod_{i=1}^{n}\left(\mathcal{J}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}.$$

Again let $h(t)=\frac{1}{t}$, $t\in[0,1]$, then $\frac{d}{dt}\left(h(t)\right)=\frac{-1}{t^2}<0$. So, $h(t)$ is decreasing function on $]0,1]$. If $\mathfrak{C}^{*}_{\check{d}_{ij}} \leq \mathfrak{C}_{\check{d}_{ij}}$, then $h\left(\mathfrak{C}^{*}_{\check{d}_{ij}}\right) \geq h\left(\mathfrak{C}_{\check{d}_{ij}}\right) \forall i,j$. i.e.,

$$\mathfrak{C}^{*}_{\check{d}_{ij}} \leq \mathfrak{C}_{\check{d}_{ij}}.$$

As we know that $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j = 1$. So,

$$\left(\left(\mathfrak{C}^{*}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \left(\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Rightarrow \left(\prod_{i=1}^{n}\left(\mathfrak{C}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}^{*}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}.$$

So, it is proven that

$$NHSWA\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \leq NHSWA\left(\mathfrak{I}^{*}_{\check{d}_{11}}, \mathfrak{I}^{*}_{\check{d}_{12}}, \ldots, \mathfrak{I}^{*}_{\check{d}_{nm}}\right).$$

Definition 3.3. $\mathfrak{I}_{\check{d}_k} = \left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}} \right\rangle$ be an NHSN, Ω_i and γ_j be weights for experts and multi sub-attributes of the deliberated attributes, respectively, along with indicated surroundings $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j = 1$. Then NHSWG operator can be demarcated as follows: NHSWG: $\Delta^n \rightarrow \Delta$ defined as follows

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\mathfrak{I}^{\Omega_i}_{\check{d}_{nm}}\right)^{\gamma_j}. \quad (3.6)$$

Theorem 3.2. $\mathfrak{I}_{\check{d}_k} = \left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}} \right\rangle$ be an NHSN, where $(i = 1, 2, \ldots, n, and\ j = 1, 2, \ldots, m)$. Then, obtained aggregated values using Eq 3.6 is also an NHSN and

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) =$$

$$\left\langle \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \right\rangle. \quad (3.7)$$

Ω_i and γ_j be the weights for specialists and multi sub-attributes of the intended parameters, respectively, along with indicated conditions $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, $\sum_{j=1}^{m}\gamma_j = 1$.

Proof. Using the principle of mathematical induction NHSWG operator can be proved by utilizing the following steps

For $n = 1$, we get $\Omega_1 = 1$. Then, we have

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{1m}}\right) = \otimes_{j=1}^{m}\mathfrak{I}^{\gamma_j}_{\check{d}_{1j}}$$

$$= \left\langle \prod_{j=1}^{m}\left(\mathcal{T}_{\check{d}_{1j}}\right)^{\gamma_j}, 1 - \prod_{j=1}^{m}\left(1 - \mathcal{J}_{\check{d}_{1j}}\right)^{\gamma_j}, 1 - \prod_{j=1}^{m}\left(1 - \mathfrak{C}_{\check{d}_{1j}}\right)^{\gamma_j} \right\rangle$$

$$= \left\langle \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \right\rangle.$$

For $m = 1$, we get $\gamma_1 = 1$. Then, we have

$NHSWG(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{21}}, \ldots, \mathfrak{I}_{\check{d}_{n1}}) = \otimes_{i=1}^{n} (\mathfrak{I}_{\check{d}_{i1}})^{\Omega_i}$

$= \left\langle \prod_{i=1}^{n} (\mathcal{T}_{\check{d}_{i1}})^{\Omega_i}, 1 - \prod_{i=1}^{n} (1 - \mathcal{J}_{\check{d}_{i1}})^{\Omega_i}, 1 - \prod_{i=1}^{n} (1 - \mathfrak{C}_{\check{d}_{i1}})^{\Omega_i} \right\rangle$

$= \left\langle \prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \right\rangle.$

For $n = 1$ and $m = 1$, Eq 3.7 satisfied the NHSWG operator. Let Eq 3.7 holds for $m = \beta_1 + 1$, $n = \beta_2$ and $m = \beta_1$, $n = \beta_2 + 1$, such as:

$\otimes_{j=1}^{\beta_1+1} \left(\left(\otimes_{i=1}^{\beta_2} \left(\mathfrak{I}_{\check{d}_{ij}}\right)^{\Omega_i} \right) \right)^{\gamma_j} =$

$\left\langle \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \right\rangle,$

$\otimes_{j=1}^{\beta_1} \left(\left(\otimes_{i=1}^{\beta_2+1} \left(\mathfrak{I}_{\check{d}_{ij}}\right)^{\Omega_i} \right) \right)^{\gamma_j} =$

$\left\langle \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} \left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{\beta_1} \left(\prod_{i=1}^{\beta_2+1} \left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \right\rangle.$

For $m = \beta_1 + 1$ and $n = \beta_2 + 1$, we have

$\otimes_{j=1}^{\beta_1+1} \left(\left(\otimes_{i=1}^{\beta_2+1} \left(\mathfrak{I}_{\check{d}_{ij}}\right)^{\Omega_i} \right) \right)^{\gamma_j} = \otimes_{j=1}^{\beta_1+1} \left(\left(\otimes_{i=1}^{\beta_2+1} \left(\mathfrak{I}_{\check{d}_{ij}}\right)^{\Omega_i} \otimes \left(\mathfrak{I}_{\check{d}_{(\beta_2+1)j}}\right)^{\Omega_{\beta_2+1}} \right) \right)^{\gamma_j}$

$\otimes_{j=1}^{\beta_1+1} \left(\left(\otimes_{i=1}^{\beta_2+1} \left(\mathfrak{I}_{\check{d}_{ij}}\right)^{\Omega_i} \right) \right)^{\gamma_j} = \otimes_{j=1}^{\beta_1+1} \left(\otimes_{i=1}^{\beta_2} \left(\mathfrak{I}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \otimes_{j=1}^{\beta_1+1} \left(\left(\mathfrak{I}_{\check{d}_{(\beta_2+1)j}}\right)^{\Omega_{\beta_2+1}} \right)^{\gamma_j}$

$$= \left\langle \begin{array}{c} \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \otimes \prod_{j=1}^{\beta_1+1} \left(\left({\mathcal{T}_{(\beta_2+1)j}}^{2}\right)^{\Omega_{\beta_2+1}} \right)^{\gamma_j}, \\ 1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \otimes 1 - \prod_{j=1}^{\beta_1+1} \left(\left(1 - \mathcal{J}_{(\beta_2+1)j}\right)^{\Omega_{\beta_2+1}} \right)^{\gamma_j}, \\ 1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2} \left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \otimes 1 - \prod_{j=1}^{\beta_1+1} \left(\left(1 - \mathfrak{C}_{(\beta_2+1)j}\right)^{\Omega_{\beta_2+1}} \right)^{\gamma_j} \end{array} \right\rangle$$

$= \left\langle \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} \left(1 - \mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j}, 1 - \prod_{j=1}^{\beta_1+1} \left(\prod_{i=1}^{\beta_2+1} \left(1 - \mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i} \right)^{\gamma_j} \right\rangle.$

Hence, it is true for $m = \beta_1 + 1$ and $n = \beta_2 + 1$.

Example 3.2. Let $\mathcal{U} = \{\kappa_1, \kappa_2, \kappa_3\}$ be a collection of specialists with their weights $\Omega_i = (0.143, 0.514, 0.343)^T$. The team of experts is operational to precise the attractiveness of a firm using a defined set of attributes $\mathfrak{L}' = \{d_1 = lawn, d_2 = security\ system\}$ with their conforming sub-attributes, Lawn $= d_1 = \{d_{11} = with\ grass, d_{12} = without\ grass\}$, security system $= d_2 = \{d_{21} = guards, d_{22} = cameras\}$. Let $\mathfrak{L}' = d_1 \times d_2$ be a set of multi sub-attributes

$$\mathfrak{L}'=d_1\times d_2=\{d_{11},d_{12}\}\times\{d_{21},d_{22}\}=\{(d_{11},d_{21}),(d_{11},d_{22}),(d_{12},d_{21}),(d_{12},d_{22})\}.$$

$\mathfrak{L}'=\{\check{d}_1,\check{d}_2,\check{d}_3,\check{d}_4\}$ with weights $\gamma_j=(.35,.15,.2,.3)^T$. Specialists' estimation for each multi-sub-attribute in the term of NHSNs $(\mathfrak{J},\mathfrak{L}')=\left\langle \mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle_{3\times 4}$ given as follows:

$$(\mathfrak{J},\mathfrak{L}')=\begin{bmatrix}(.3,.8,.6) & (.4,.6,.3) & (.8,.3,.6) & (.2,.5,.6)\\ (.8,.3,.7) & (.5,.7,.4) & (.1,.7,.3) & (.9,.4,.8)\\ (.3,.6,.5) & (.8,.5,.7) & (.2,.6,.5) & (.8,.5,.4)\end{bmatrix}.$$

Using Eq 3.7.,

$NHSWG\left(\mathfrak{J}_{\check{d}_{11}},\mathfrak{J}_{\check{d}_{12}},\dots,\mathfrak{J}_{\check{d}_{34}}\right)=$

$$\left\langle \prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},1-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},1-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle$$

$$=\left\langle \begin{matrix}\left(\begin{matrix}\{(.3)^{.143}(.8)^{.514}(.3)^{.343}\}^{.35}\{(.4)^{.143}(.5)^{.514}(.8)^{.343}\}^{.15}\\ \{(.8)^{.143}(.1)^{.514}(.2)^{.343}\}^{.2}\{(.2)^{.143}(.9)^{.514}(.8)^{.343}\}^{.3}\end{matrix}\right),\\ 1-\left(\begin{matrix}\{(.2)^{.143}(.7)^{.514}(.4)^{.343}\}^{.35}\{(.4)^{.143}(.3)^{.514}(.5)^{.343}\}^{.15}\\ \{(.7)^{.143}(.3)^{.514}(.4)^{.343}\}^{.2},\{(.5)^{.143}(.6)^{.514}(.5)^{.343}\}^{.3}\end{matrix}\right),\\ 1-\left(\begin{matrix}\{(.4)^{.143}(.3)^{.514}(.5)^{.343}\}^{.35}\{(.7)^{.143}(.6)^{.514}(.3)^{.343}\}^{.15}\\ \{(.4)^{.143}(.7)^{.514}(.5)^{.343}\}^{.2},\{(.4)^{.143}(.2)^{.514}(.6)^{.343}\}^{.3}\end{matrix}\right)\end{matrix}\right\rangle$$

$$=\langle 0.40661,0.60939,0.84213\rangle.$$

3.2. Properties of NHSWG operator

3.2.1. Idempotency

If $\mathfrak{J}_{\check{d}_{ij}}=\mathfrak{J}_{\check{d}}=\langle\mathcal{T}_{\check{d}},\mathcal{J}_{\check{d}},\mathfrak{C}_{\check{d}}\rangle\forall i,j$. Then,

$$NHSWG\left(\mathfrak{J}_{\check{d}_{11}},\mathfrak{J}_{\check{d}_{12}},\dots,\mathfrak{J}_{\check{d}_{nm}}\right)=\mathfrak{J}_{\check{d}}.$$

Proof. As we know that all $\mathfrak{J}_{\check{d}_{ij}}=\mathfrak{J}_{\check{d}}=\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{J}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle\forall i,j$. Then, using Eq 3.7.

$NHSWG\left(\mathfrak{J}_{\check{d}_{11}},\mathfrak{J}_{\check{d}_{12}},\dots,\mathfrak{J}_{\check{d}_{nm}}\right)=$

$$\left\langle \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j},1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right\rangle$$

$NHSWG\left(\mathfrak{J}_{\check{d}_{11}},\mathfrak{J}_{\check{d}_{12}},\dots,\mathfrak{J}_{\check{d}_{nm}}\right)=$

$$\left\langle \left(\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j},1-\left(\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j},1-\left(\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}\right\rangle$$

$$=\left\langle \mathcal{T}_{\check{d}_{ij}}, 1-\left(1-\mathcal{J}_{\check{d}_{ij}}\right), 1-\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)\right\rangle$$

$$=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle=\mathfrak{J}_{\check{d}}.$$

3.2.2. Boundedness

Let $\mathfrak{J}_{\check{d}_{ij}}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle$ be a collection of NHSNs

$$\mathfrak{J}_{\check{d}_{ij}}{}^{-}=\left\langle \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}, \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}, \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right\rangle$$

and

$$\mathfrak{J}_{\check{d}_{ij}}{}^{+}=\left\langle \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}, \min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}, \min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right\rangle,$$

then

$$\mathfrak{J}_{\check{d}_{ij}}{}^{-} \leq NHSWG\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \leq \mathfrak{J}_{\check{d}_{ij}}{}^{+}.$$

Proof. Since $\mathfrak{J}_{\check{d}_{ij}}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle$ be a collection of NHSNs$\forall i, j$. Then,

$$\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \leq \mathcal{T}_{\check{d}_{ij}} \leq \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$$

$$\Rightarrow \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \leq \mathcal{T}_{\check{d}_{ij}} \leq \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\Omega_i} \leq \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \leq \left(\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\Omega_i}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \leq \prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i} \leq \left(\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$$

$$\Leftrightarrow \left(\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \left(\max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Leftrightarrow \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}. \tag{3.8}$$

Again,

$$\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \leq \mathcal{J}_{\check{d}_{ij}} \leq \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}$$

$$1-\max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \leq 1-\mathcal{J}_{\check{d}_{ij}} \leq 1-\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}$$

$$\Rightarrow 1-\left(\max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \leq \prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i} \leq 1-\left(\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$$

$$\Leftrightarrow 1-\left(\max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le 1-\left(\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Leftrightarrow \min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \le 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}.$$

So, we get

$$\Leftrightarrow \min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \le 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}, \tag{3.9}$$

and,

$$\min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\} \le \mathfrak{C}_{\check{d}_{ij}} \le \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}$$

$$1-\max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\} \le 1-\mathfrak{C}_{\check{d}_{ij}} \le 1-\min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}$$

$$\Rightarrow 1-\left(\max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i} \le \prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i} \le 1-\left(\min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^{n}\Omega_i}$$

$$\Leftrightarrow 1-\left(\max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le 1-\left(\min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^{m}\gamma_j}$$

$$\Leftrightarrow \min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\} \le 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}$$

So, we get

$$\Leftrightarrow \min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\} \le 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \le \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}. \tag{3.10}$$

Let $NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right)=\langle \mathcal{T}_{\check{\delta}}, \mathcal{J}_{\check{\delta}}, \mathfrak{C}_{\check{\delta}}\rangle=\mathfrak{I}_{\check{\delta}}$, then inequalities (3.8)–(3.10) can be transformed into the following form: $\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \le \mathcal{T}_{\check{\delta}} \le \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$, $\min_j \min_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\} \le \mathcal{J}_{\check{\delta}} \le \max_j \max_i \left\{\mathcal{J}_{\check{d}_{ij}}\right\}$, and $\min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\} \le \mathfrak{C}_{\check{\delta}} \le \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}$ respectively.

Using Eq 2.1, we get the following:

$$\mathcal{S}(\mathfrak{I}_{\check{\delta}})=\mathcal{T}_{\check{\delta}}-\mathfrak{C}_{\check{\delta}} \le \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} - \min_j \min_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}=\mathcal{S}\left(\mathfrak{I}_{\check{d}_{ij}}{}^{+}\right),$$

$$\mathcal{S}(\mathfrak{I}_{\check{\delta}})=\mathcal{T}_{\check{\delta}}-\mathfrak{C}_{\check{\delta}} \ge \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} - \max_j \max_i \left\{\mathfrak{C}_{\check{d}_{ij}}\right\}=\mathcal{S}\left(\mathfrak{I}_{\check{d}_{ij}}{}^{-}\right).$$

Then, order relation to two NHSNs, we have

$$\mathfrak{I}_{\check{d}_{ij}}{}^{-} \le NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \le \mathfrak{I}_{\check{d}_{ij}}{}^{+}.$$

3.2.3. Shift Invariance

If $\mathfrak{I}_{\check{d}}=\langle \mathcal{T}_{\check{d}}, \mathcal{J}_{\check{d}}, \mathfrak{C}_{\check{d}}\rangle$ be an NHSN. Then,

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}} \otimes \mathfrak{I}_{\check{d}}, \mathfrak{I}_{\check{d}_{12}} \otimes \mathfrak{I}_{\check{d}}, \ldots, \mathfrak{I}_{\check{d}_{nm}} \otimes \mathfrak{I}_{\check{d}}\right)=NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \otimes \mathfrak{I}_{\check{d}}.$$

Proof. Assume $\mathfrak{I}_{\check{d}}$ be an NHSN and $\mathfrak{I}_{\check{d}_{ij}}$ be a collection of NHSNs. Utilizing operational laws, we have:

$$\mathfrak{I}_{\check{d}_{11}} \otimes \mathfrak{I}_{\check{d}_{12}} = \left\langle \mathcal{T}_{\check{d}_{11}}\mathcal{T}_{\check{d}_{12}}, \mathcal{J}_{\check{d}_{11}} + \mathcal{J}_{\check{d}_{12}} - \mathcal{J}_{\check{d}_{11}}\mathcal{J}_{\check{d}_{12}},\ \mathfrak{C}_{\check{d}_{11}} + \mathfrak{C}_{\check{d}_{12}} - \mathfrak{C}_{\check{d}_{11}}\mathfrak{C}_{\check{d}_{12}} \right\rangle.$$

So,

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}} \otimes \mathfrak{I}_{\check{d}}, \mathfrak{I}_{\check{d}_{12}} \otimes \mathfrak{I}_{\check{d}}, \dots, \mathfrak{I}_{\check{d}_{nm}} \otimes \mathfrak{I}_{\check{d}}\right) = \otimes_{j=1}^{m} \gamma_j \left(\otimes_{i=1}^{n} \Omega_i \left(\mathfrak{I}_{\check{d}_{ij}} \otimes \mathfrak{I}_{\check{d}}\right)\right)$$

$$= \left\langle \begin{array}{c} \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}(\mathcal{T}_{\check{d}})^{\Omega_i}\right)^{\gamma_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}(1-\mathcal{J}_{\check{d}})^{\Omega_i}\right)^{\gamma_j}, \\ 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}(1-\mathfrak{C}_{\check{d}})^{\Omega_i}\right)^{\gamma_j} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \mathcal{T}_{\check{d}}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1-(1-\mathcal{J}_{\check{d}})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \\ 1-(1-\mathfrak{C}_{\check{d}})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \mathcal{T}_{\check{d}}\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1-(1-\mathcal{J}_{\check{d}})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \\ 1-(1-\mathfrak{C}_{\check{d}})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \\ 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \end{array} \right\rangle \otimes \langle \mathcal{T}_{\check{d}}, \mathcal{J}_{\check{d}}, \mathfrak{C}_{\check{d}} \rangle$$

$$= NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right) \otimes \mathfrak{I}_{\check{d}}.$$

3.2.4. Homogeneity

Let $\alpha > 0$, where α is any positive real number. Then, we will prove that

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}}{}^{\alpha}, \mathfrak{I}_{\check{d}_{12}}{}^{\alpha}, \dots, \mathfrak{I}_{\check{d}_{nm}}{}^{\alpha}\right) = \left(NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right)\right)^{\alpha}.$$

Proof. Let $\mathfrak{I}_{\check{d}_{ij}} = \left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}} \right\rangle$ be a collection of NHSNs $\forall i, j$, and α is any positive real number. Then,

$$\mathfrak{I}_{\check{d}_k}^{\alpha} = \left\langle \mathcal{T}_{\check{d}_k}{}^{\alpha}, 1-\left(1-\mathcal{J}_{\check{d}_k}\right)^{\alpha}, 1-\left(1-\mathfrak{C}_{\check{d}_k}\right)^{\alpha} \right\rangle,$$

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \dots, \mathfrak{I}_{\check{d}_{nm}}\right)$$

$$=\left\langle \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}, 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}\right\rangle$$

$$=\left\langle \begin{array}{c}\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}, 1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}, \\ 1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathfrak{C}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}\end{array}\right\rangle$$

$$=\left(NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right)\right)^{\alpha}.$$

3.2.5. Monotonicity

Let $\mathfrak{I}_{\check{d}_{ij}}=\left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}}, \mathfrak{C}_{\check{d}_{ij}}\right\rangle$ and $\mathfrak{I}^{*}_{\check{d}_{ij}}=\left\langle \mathcal{T}^{*}_{\check{d}_{ij}}, \mathcal{J}^{*}_{\check{d}_{ij}}, \mathfrak{C}^{*}_{\check{d}_{ij}}\right\rangle$ be the collection of NHSNs. Then

$$NHSWG\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) \leq NHSWG\left(\mathfrak{I}^{*}_{\check{d}_{11}}, \mathfrak{I}^{*}_{\check{d}_{12}}, \ldots, \mathfrak{I}^{*}_{\check{d}_{nm}}\right), \text{ if } \mathfrak{I}_{\check{d}_{ij}} \leq \mathfrak{I}^{*}_{\check{d}_{ij}}\ \forall\ i, j.$$

Proof. Similar to 3.1.5.

4. MCDM approach using developed NHSWA and NHSWG operators

In the subsequent section, we will introduce the MCDM methodology by the anticipated NHSWA and NHSWG operators in the NHSS setting.

4.1. Proposed decision-making approach

Let $\mathrm{C}=\{\mathrm{C}^1, \mathrm{C}^2, \mathrm{C}^3, \ldots, \mathrm{C}^s\}$ be a set of s alternatives and $\mathcal{U}=\{\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_n\}$ be a set of n experts. The weights of experts are given as $\Omega=(\Omega_1, \Omega_1, \ldots, \Omega_n)^T$ and $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$. Let $\mathfrak{L}=\{d_1, d_2, \ldots, d_m\}$ be a set of attributes with their corresponding multi sub-attributes such as $\mathfrak{L}' = \{(d_{1\rho} \times d_{2\rho} \times \ldots \times d_{m\rho}) \text{ for all } \rho \in \{1, 2, \ldots, t\}\}$ with weights $\gamma = (\gamma_{1\rho}, \gamma_{2\rho}, \gamma_{3\rho}, \ldots, \gamma_{m\rho})^T$ such as $\gamma_\rho > 0$, $\sum_{\rho=1}^{t}\gamma_\rho = 1$ and can be specified as $\mathfrak{L}' = \left\{\check{d}_\partial : \partial \in \{1, 2, \ldots, k\}\right\}$. The team of experts $\{\kappa^i : i = 1, 2, \ldots, n\}$ appraise the substitutes $\{\mathrm{C}^{(z)} : z = 1, 2, \ldots, s\}$ under the anticipated sub-attributes of the considered parameters $\{\check{d}_\partial : \partial = 1, 2, \ldots, k\}$ given in the form of NHSNs, such as $\left(\mathrm{C}^{(z)}_{\check{d}_{ik}}\right)_{n\times\partial} = \left(\mathcal{T}^{(z)}_{\check{d}_{ik}}, \mathcal{J}^{(z)}_{\check{d}_{ik}}, \mathfrak{C}^{(z)}_{\check{d}_{ik}}\right)_{n\times\partial}$, where $0 \leq \mathcal{T}^{(z)}_{\check{d}_{ik}}, \mathcal{J}^{(z)}_{\check{d}_{ik}}, \mathfrak{C}^{(z)}_{\check{d}_{ik}} \leq 1$ and $0 \leq \mathcal{T}^{(z)}_{\check{d}_{ik}} + \mathcal{J}^{(z)}_{\check{d}_{ik}} + \mathfrak{C}^{(z)}_{\check{d}_{ik}} \leq 3$ for all i, k. Applying the anticipated NHSWA and NHSWG operators grow accumulated NHSNs $\mathcal{L}_{\Phi}$ for each substitute conferring to the specialist's predilections. Use Eq 2.1 to calculate the score value for each alternative. The above methods can be summarized as follows:

Step 1. Develop decision matrices $\left(\complement_{\check{d}_{ik}}^{(z)}\right)_{n\times\partial}=\left(\mathcal{T}_{\check{d}_{ik}}^{(z)},\mathcal{I}_{\check{d}_{ik}}^{(z)},\mathfrak{C}_{\check{d}_{ik}}^{(z)}\right)_{n\times\partial}$ in the form of NHSNs for each alternative.

$$\left(\complement^{(z)},\mathfrak{L}'\right)_{n\times\partial}=\begin{matrix} \\ \kappa^1\\ \kappa^2\\ \vdots\\ \kappa^{n}\end{matrix}\begin{matrix} \check{d}_1 & \check{d}_2 & \dots & \check{d}_\partial \\ \end{matrix}\begin{pmatrix}\left(\mathcal{T}_{\check{d}_{11}}^{(z)},\mathcal{I}_{\check{d}_{11}}^{(z)},\mathfrak{C}_{\check{d}_{11}}^{(z)}\right) & \left(\mathcal{T}_{\check{d}_{12}}^{(z)},\mathcal{I}_{\check{d}_{12}}^{(z)},\mathfrak{C}_{\check{d}_{12}}^{(z)}\right) & \cdots & \left(\mathcal{T}_{\check{d}_{1\partial}}^{(z)},\mathcal{I}_{\check{d}_{1\partial}}^{(z)},\mathfrak{C}_{\check{d}_{1\partial}}^{(z)}\right)\\ \left(\mathcal{T}_{\check{d}_{21}}^{(z)},\mathcal{I}_{\check{d}_{21}}^{(z)},\mathfrak{C}_{\check{d}_{21}}^{(z)}\right) & \left(\mathcal{T}_{\check{d}_{22}}^{(z)},\mathcal{I}_{\check{d}_{22}}^{(z)},\mathfrak{C}_{\check{d}_{22}}^{(z)}\right) & \cdots & \left(\mathcal{T}_{\check{d}_{2\partial}}^{(z)},\mathcal{I}_{\check{d}_{2\partial}}^{(z)},\mathfrak{C}_{\check{d}_{2\partial}}^{(z)}\right)\\ \vdots & \vdots & \vdots & \vdots\\ \left(\mathcal{T}_{\check{d}_{n1}}^{(z)},\mathcal{I}_{\check{d}_{n1}}^{(z)},\mathfrak{C}_{\check{d}_{n1}}^{(z)}\right) & \left(\mathcal{T}_{\check{d}_{n2}}^{(z)},\mathcal{I}_{\check{d}_{n2}}^{(z)},\mathfrak{C}_{\check{d}_{n2}}^{(z)}\right) & \cdots & \left(\mathcal{T}_{\check{d}_{n\partial}}^{(z)},\mathcal{I}_{\check{d}_{n\partial}}^{(z)},\mathfrak{C}_{\check{d}_{n\partial}}^{(z)}\right)\end{pmatrix}.$$

Step 2. Using the normalization rule converts the cost type sub-attributes to benefit type sub-attribute.

$$\chi_{ij}=\begin{cases}\mathfrak{I}_{\check{d}_{ij}}^{c}; & \text{cost type parameter}\\ \mathfrak{I}_{\check{d}_{ij}}; & \text{benefit type parameter}\end{cases}.$$

Where $\mathfrak{I}_{\check{d}_{ij}}^{c}=\left\langle 1-\mathfrak{C}_{\check{d}_{ij}},\mathcal{I}_{\check{d}_{ij}},\mathcal{T}_{\check{d}_{ij}}\right\rangle$, which represents the complement of $\left\langle\mathcal{T}_{\check{d}_{ij}},\mathcal{I}_{\check{d}_{ij}},\mathfrak{C}_{\check{d}_{ij}}\right\rangle$.

Step 3. Aggregate the NHSNs $\mathfrak{I}_{\check{d}_{ij}}$ for each alternative $\complement=\{\complement^1,\complement^2,\complement^3,\dots,\complement^s\}$ into a collective decision matrix $\mathcal{L}_\Phi$ using settled NHSWA or NHSWG operators (or Eqs 3.2 and 3.7, respectively).

Step 4. If $\complement=\{\complement^1,\complement^2,\complement^3,\dots,\complement^s\}$ be an assortment of deliberated alternatives. Then, compute the score values $\mathcal{L}_\Phi$ for each alternative using Eq 2.1.

Step 5. Choose the most appropriate alternate with a maximum score value $\mathcal{L}_\Phi$.

Step 6. Rank the substitutes.

4.2. Application of the proposed MCDM technique for site selection of solid waste management

Finding new landfills needs numerous features, data, investigation, and queries. These aspects mostly contain community health alarms, ecological aspects, the landscape of the zone, geology, hydrology, drainage and climate in the region, the accessibility of landfills in the area to refuge waste, contiguity to housing and industrialized regions, traveling from city, drainage system, cost, and present/coming land usage in the region [72]. It also involves a widespread valuation of spot surroundings and probable influence on the atmosphere. This comprises societies' geography, drainage, geology, hydrogeology, air quality, surface water, and the coldest.

The slope is one of the influential topographic aspects of landfill spot choice. The bottom and moderate slopes are mostly appropriate for detecting landfill spots, whereas the average slope is more proper than other slope curriculums for landfill spot choice. In this investigation, a slope with 0-2% is deliberated as low, 2-8% is very high, 8-15% is high, and >30% is very low aptness. The central fragment of the metropolis is not appropriate for landfill spot selection since sloping to intensely sloping topography is more soaked in this area than in other regions. While: the southern and northern landfills of the metropolis have a good part. The landfill zone must not be situated on a mountain with a rocky slope and landslide zones. Areas located on mildly sloping are favored over smooth ground and extremely sloping regions, as indicated in [73], because extremely sloping zones can source an incurable slide, particularly when there is rain or high-water escape. Also, abrupt inclines can cause carriage difficulty in production or flattening exertion, which can be a surplus charge; it can also decrease the steadiness of the adjacent slopes and raise the threat of landslides. Landfill spot is not endorsed in immediacy to defrayal regions. The developed regions that contain saleable areas, administrative and private associations, colleges, health centers, spiritual associations, informative associations, homes, and other societal service areas are deliberated as reimbursement extents in this study. The landfill site is generally endorsed to be one kilometer far from residential regions [72,74–78], with a set distance from residential to be 3000 m, 500-2000 m, and 3000 m multi-ring barriers.

The present and upcoming land usage are imperative standards when choosing landfill spots. Generally, land with fewer socio-economic, ecological, and dogmatic worth or cost is suggested as a removal spot. As it is a substance of community health anxiety, seeing the extraordinary proportion of growth, one must proceed with the long-term land usage organization of land into deliberation to detect the removal region. Additionally, the current and upcoming waste wagons transportation must be reflected [72]. The MCDM approach is used for allocating measures weights for each aspect plan. The spatial multi-criteria analysis is the best frequently employed technique when it originates from spotting selection difficulties or appropriateness replicas. Different weights were specified in partiality of individual aspects compared to other aspects. The literature recommends numerous criterion-weighting processes built on experts' decisions in the MCDM [79,80]. To address this problem, we supposed four distinct geographic regions associated with the key factors we discussed.

Let $\mathsf{C} = \{\mathsf{C}^1, \mathsf{C}^2, \mathsf{C}^3, \mathsf{C}^4\}$ be a set of alternatives (geographic regions) and $\mathcal{U} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ be a team of four experts with weights $(.1, .25, .35, .3)^T$ for choosing a suitable alternate, the most appropriate spot for solid waste. The group of specialists selects the parameters for the assortment of areas for solid waste, such as $\mathfrak{L} = \{\ell_1 = \text{ slope}, \ell_2 = \text{ distance from settlement}, \ell_3 = \text{Land use}\}$ with their corresponding sub-attribute: Slope $= \ell_1 = \{d_{11} = 0 - 2\% \text{ and more than } 30\% \text{ is low and very low}, \ d_{12} = 2\text{–}8\% \text{ is very high, and } 8 - 15\% \text{ is high}\}$ Distance and settlement $= \ell_2 = \{d_{21} = \mathit{distance\ from\ resedential\ area\ is}\ 500 - 2000\text{ m}, d_{22} = \text{more than 3000 m}\}$, and Land use $= \ell_3 = \{d_{31} = \text{Land use}\}$. Let $\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3$ shows the multi sub-attributes

$$\mathfrak{L}' = \ell_1 \times \ell_2 \times \ell_3 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} \times \{d_{31}\}$$

$$= \{(d_{11}, d_{21}, d_{31}),\ (d_{11}, d_{22}, d_{31}), (d_{12}, d_{21}, d_{31}), (d_{12}, d_{22}, d_{31})\}.$$

$\mathfrak{L}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ with weights $(0.2, 0.1, 0.4, 0.3)^T$. Experts deliver their opinion for each geographical region in the form of NHSNs following multi sub-attributes of deliberated attributes.

A flow chart of the above-presented approach to finding the most suitable site for SWM under the NHSS environment is given in the following Figure 1.

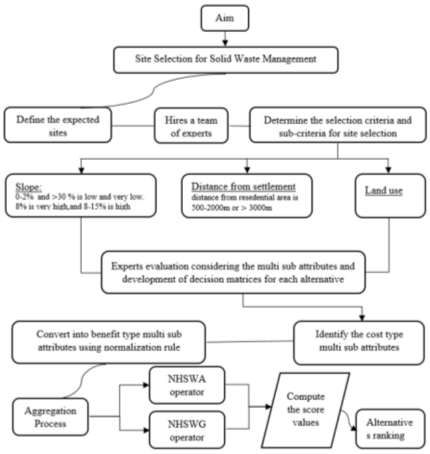

Figure 1. Flow chart of site selection for SWM using AOs under NHSS.

4.3.NHSWA operator

Step 1. The specialists investigated all considered sites for solid waste and gave their opinion in the form of NHSNs. The summary of their score values for each alternative is shown in Tables 1–4.

Table 1. Decision matrix for alternative $\complement^1$.

$\complement^1$	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
κ_1	(0.9,0.2,0.1)	(0.3, 0.3, 0.7)	(0.6, 0.4, 0.2)	(0.7,0.1,0.3)
κ_2	(0.8, 0.3,0.2)	(0.6,0.2, 0.6)	(0.8, 0.3, 0.1)	(0.2, 0.6, 0.8)
κ_3	(0.6,0.1,0.3)	(0.6,0.1,0.3)	(0.8, 0.2, 0.1)	(0.6, 0.3, 0.4)
κ_4	(0.9, 0.1, 0.1)	(0.9,0.1,0.1)	(0.8, 0.1, 0.1)	(0.9,0.1,0.2)

Table 2. Decision matrix for alternative $\complement^2$.

$\complement^2$	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
κ_1	(0.3,0.3,0.7)	(0.9, 0.2, 0.1)	(0.6, 0.1, 0.3)	(0.3,0.6,0.2)
κ_2	(0.8, 0.2,0.1)	(0.8,0.3, 0.2)	(0.9, 0.1, 0.1)	(0.8, 0.3, 0.1)
κ_3	(0.6,0.3,0.4)	(0.8,0.1,0.2)	(0.9, 0.1, 0.1)	(0.2, 0.3, 0.8)
κ_4	(0.9, 0.1, 0.2)	(0.8,0.1,0.1)	(0.7, 0.1, 0.3)	(0.6,0.3,0.4)

Table 3. Decision matrix for alternative $\complement^3$.

$\complement^3$	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
κ_1	(0.6,0.3,0.4)	(0.2, 0.3, 0.8)	(0.3, 0.6, 0.2)	(0.3,0.6,0.2)
κ_2	(0.9, 0.1,0.1)	(0.9,0.1, 0.1)	(0.9, 0.1, 0.1)	(0.8, 0.3, 0.2)
κ_3	(0.8,0.3,0.2)	(0.9,0.2,0.1)	(0.9, 0.1, 0.1)	(0.2, 0.3, 0.8)
κ_4	(0.3, 0.3, 0.7)	(0.9,0.1,0.2)	(0.7, 0.1, 0.3)	(0.6,0.3,0.4)

Table 4. Decision matrix for alternative C^4.

C^4	$\breve{d}_1$	$\breve{d}_2$	$\breve{d}_3$	$\breve{d}_4$
κ_1	(0.9,0.1,0.1)	(0.9, 0.1, 0.2)	(0.8, 0.2, 0.1)	(0.3,0.6,0.2)
κ_2	(0.8, 0.2,0.1)	(0.8,0.2, 0.1)	(0.6, 0.3, 0.4)	(0.8, 0.3, 0.2)
κ_3	(0.8,0.1,0.1)	(0.8,0.1,0.2)	(0.9, 0.1, 0.1)	(0.3, 0.6, 0.2)
κ_4	(0.9, 0.1, 0.2)	(0.3,0.3,0.7)	(0.8, 0.3, 0.2)	(0.9,0.1,0.1)

Step 2. The considered parameters are identical, so there is no need to normalize.
Step 3. Expert's opinions can be précised as follows using Eq 3.2:
$\mathcal{L}_1 = \langle .76040, .18389, .26121 \rangle$, $\mathcal{L}_2 = \langle .76041, .16792, .21357 \rangle$, $\mathcal{L}_3 = \langle .75965, .18628, .22480 \rangle$, and $\mathcal{L}_4 = \langle .78886, .19121, .16463 \rangle$.
Step 4. Utilizing Eq 2.1, calculate the score values as follows:
$\mathcal{S}(\mathcal{L}_1) = 0.49919$, $\mathcal{S}(\mathcal{L}_2) = 0.54684$, $\mathcal{S}(\mathcal{L}_3) = 0.53485$, and $\mathcal{S}(\mathcal{L}_4) = 0.62423$.
Step 5. The alternative C^4 has a maximum score value, so C^4 is the most acceptable alternative.
Step 6. The obtained ranking of alternatives using the NHSWA operator is given as follows:
$\mathcal{S}(\mathcal{L}_4) > \mathcal{S}(\mathcal{L}_2) > \mathcal{S}(\mathcal{L}_3) > \mathcal{S}(\mathcal{L}_1)$. So, $C^{(4)} > C^{(2)} > C^{(3)} > C^{(1)}$.

4.4. NHSWG operator

Step 1 and **step 2** are the same as 4.3.
Step 3. Expert's opinions can be summarized as follows using Eq 3.7:
$\mathcal{L}_1 = \langle .67268, .33949, .29462 \rangle$, $\mathcal{L}_2 = \langle .63457, .17526, .33252 \rangle$, $\mathcal{L}_3 = \langle .59871, .23992, .36100 \rangle$, and $\mathcal{L}_4 = \langle .68656, .25923, .20554 \rangle$.
Step 4. Calculate the score values using Eq 2.1.
$\mathcal{S}(\mathcal{L}_1) = 0.37806$, $\mathcal{S}(\mathcal{L}_2) = 0.30205$, $\mathcal{S}(\mathcal{L}_3) = 0.23771$, and $\mathcal{S}(\mathcal{L}_4) = 0.48102$.
Step 5. The alternative C^4 has a maximum score value, so C^4 is the most acceptable alternative.
Step 6. The obtained ranking of alternatives using the NHSWG operator is given as follows:
$\mathcal{S}(\mathcal{L}_4) > \mathcal{S}(\mathcal{L}_1) > \mathcal{S}(\mathcal{L}_2) > \mathcal{S}(\mathcal{L}_3)$. So, $C^{(4)} > C^{(1)} > C^{(2)} > C^{(3)}$.

A graphical representation of the obtained results is given in the following Figure 2.

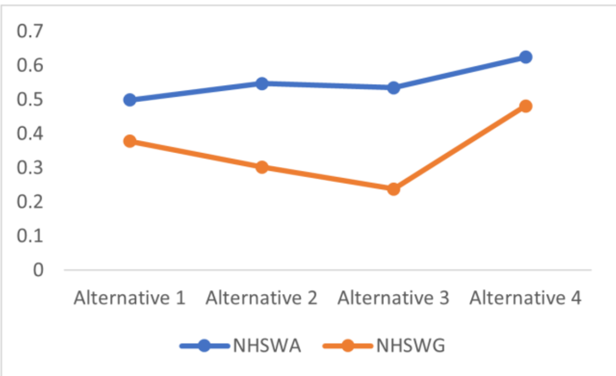

Figure 2. Alternatives ranking.

5. Comparative analysis and discussion

In the subsequent section, we present the practicality, easiness, and manageability, and a short assessment of the undermentioned: the intentional method and some prevalent approaches.

5.1. Superiority of the proposed approach

From this research and evaluation, it can be concluded that the importance gained through the planning technique is more corporate than any other technique. Overall, DM and popular DM techniques provide additional data to signal uncertainties. In addition, many FS hybrid structures have become a particular feature of NHSS, and some appropriate parameters have been added. General item information can be presented in a precise and analytical manner, see Table 5. Therefore, combining incorrect and inexplicit data in the DM procedure is an appropriate technique. Thus, the proposed procedure goes beyond the unique hybrid structure of fuzzy sets.

Table 5. Structural analysis between NHSS and some prevailing models

	Set	Truthiness	Indeterminacy	Falsity	Parameterization	Attributes	Sub-attributes
Zadeh [1]	FS	✓	×	×	×	✓	×
Atanassov [2]	IFS	✓	×	✓	×	✓	×
Yager [4]	PFS	✓	×	✓	×	✓	×
Smarandache [19]	NS	✓	✓	✓	×	✓	×
Maji et al. [24]	FSS	✓	×	×	✓	✓	×
Maji et al. [28]	IFSS	✓	×	✓	✓	✓	×
Peng et al. [32]	PFSS	✓	×	✓	✓	✓	×
Maji [40]	NSS	✓	✓	✓	✓	✓	×
Smarandache [46]	IFHSS	✓	×	✓	✓	✓	✓
Zulqarnain et al. [49]	PFHSS	✓	×	✓	✓	✓	✓
Khan et al. [54]	q-ROFHSS	✓	×	✓	✓	✓	✓
Proposed approach	NHSS	✓	✓	✓	✓	✓	✓

5.2. Discussion

Zadeh's FS [1] addressed inaccurate and misleading information using attributes considered by MD for each alternative. But FS does not indicate the NMD of the parameters under study. Atanasov's IFS [2] adjusts dark and dim substances using MD and NMD. But, IFS cannot handle the environments when $MD + NMD \geq 1$. On the other hand, our concept deals with these problems competently. At the same time, these theories contain no information about attribute uncertainty. To be astounded by such challenges, Smarandache [19] proposed the idea of NS. Maji et al. [24] suggested the concept of FSS considering the parameterized values of the objects, which creates hesitation because of the MD of parameters. But, the proposed FSS does not deliver any facts about the NMD of the object. To remedy the proposed error, Maji et al. [28] introduced the theory of IFSS. The scheduled idea grips hesitation more precisely by using MD and NMD of features with their parameterization, and the combination of MD and NMD does not exceed 1. To grip this consequence, Peng et al. [32] anticipated the idea of PFSS by amending the state $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$ with their parametrization. The PFSS is unable to deal with the indeterminacy of the attributes. Maji [40] introduced the concept of NSS, in which decision-makers competently solve the DM problems compared to the above-studied theories using truthiness, falsity, and indeterminacy of the object. But all the studies stated above have no material about the sub-attributes of the deliberated parameters. So the theories discussed above cannot handle the consequence when parameters have their conforming sub-attributes. Utilizing the MD and NMD, Smarandache [46] protracted the IFSS to IFHSS in which $MD + NMD \leq 1$ for each sub-attribute. But IFHSS cannot carry any data on the NMD of the sub-attribute of the deliberated attribute. Zulqarnain et al. [49] anticipated another comprehensive idea of PFHSS relative to IFHSS. The PFHSS accommodates more uncertainty compared to IFHSS by updating the condition $MD + NMD \leq 1$ to $\left(\mathcal{T}_{\mathcal{F}(\breve{d})}(\delta)\right)^2 + \left(\mathcal{J}_{\mathcal{F}(\breve{d})}(\delta)\right)^2 \leq 1$. The q-ROFHSS [54] handles more hesitation compared to IFHSS and PFHSS by improving the conditions $MD + NMD \leq 1$ and $\left(\mathcal{T}_{\mathcal{F}(\breve{d})}(\delta)\right)^2 + \left(\mathcal{J}_{\mathcal{F}(\breve{d})}(\delta)\right)^2 \leq 1$ to $\left(\mathcal{T}_{\mathcal{F}(\breve{d})}(\delta)\right)^q + \left(\mathcal{J}_{\mathcal{F}(\breve{d})}(\delta)\right)^q \leq 1$. Not all prevailing hybrid configurations of FS can deal with the indeterminacy of sub-attributes of deliberated n-tuple attributes. On the other hand, developed aggregation operators can accommodate the sub-attributes of considered attributes using truthiness, indeterminacy, and falsity objects of sub-attributes with the following condition $0 \leq \mathcal{T}_{\mathcal{F}(\breve{a})}(v) + \mathcal{I}_{\mathcal{F}(\breve{a})}(v) + \mathfrak{C}_{\mathcal{F}(\breve{a})}(v) \leq 3$. It can be seen that the best choice of the proposed approach is an oral resemblance to one's method, which ensures accountability and effectiveness of the proposed approach.

5.3. Comparative analysis

In the following section, we endorse another algorithmic rule under NHSS operating the progressed NHSWA and NHSWG operators. Subsequently, we use the suggested algorithm for a realistic problem: the appropriate site selection for solid waste. The overall consequences demonstrate that the developed algorithm is appreciated and concrete. It can be observed that C^4 is the most acceptable alternative for the disposal of solid waste management. The projected method can be equated to other existing approaches. The investigation determines that the outcomes achieved in an intended process are more capable than those of prevailing theories. Therefore, established AOs deal with indeterminate and confusing information more efficiently than prevalent techniques. However, under the modern DM approach, the core benefit of the intentional method is that it can incorporate extra information into the data compared to the current process. It is also helpful for determining deception and knowledge in the DM process. The benefit of the scheduled technique for prevailing approaches and attendant measures is to avoid consequences based on undesirable causes. For further assistance, please see the following Table 6.

Table 6. Comparative studies between proposed AOs and existing AOs under considered data in subsection 4.2

AOs	Parameterization domain	Approximation range	Approximation function type	Ranking order under the HSS setting	Remarks
IFSWA [29]	IFS	IFSS	Simple Parameters	N/A	Ignored the sub-attributes
IFSWG [29]	IFS	IFSS	Simple Parameters	N/A	Ignored the sub-attributes
PFSWA[33]	PFS	PFSS	Simple Parameters	N/A	Ignored the sub-attributes
PFSWG [33]	PFS	PFSS	Simple Parameters	N/A	Ignored the sub-attributes
PFSEWA [35]	PFS	PFSS	Simple Parameters	N/A	Ignored the sub-attributes
PFSEWG [35]	PFS	PFSS	Simple Parameters	N/A	Ignored the sub-attributes
IFHSWA [48]	IFS	IFHSS	multiple sub-parameter	Applicable	Ignored the indeterminacy of the sub-attributes
IFHSWG [48]	IFS	IFHSS	multiple sub-parameter	Applicable	Ignored the indeterminacy of the sub-attributes
PFHSWA [50]	PFS	PFHSS	multiple sub-parameter	Applicable	Ignored the indeterminacy of the sub-attributes
PFHSWG [50]	PFS	PFHSS	multiple sub-parameter	Applicable	Ignored the indeterminacy of the sub-attributes
q-ROFHSWA [56]	q-ROFS	q-ROFHSS	multiple sub-parameter	Applicable	Ignored the indeterminacy of the sub-attributes
q-ROFHSWG [56]	q-ROFS	q-ROFHSS	multiple sub-parameter	Applicable	Ignored the indeterminacy of the sub-attributes
Proposed NHSWA	NS	NHSS	multiple sub-parameter	Applicable	Delivered the most precise information comparative to IFHSWA and PFHSWA operators considering the indeterminacy
Proposed NHSWG	NS	NHSS	multiple sub-parameter	Applicable	Delivered the most precise information comparative to IFHSWG and PFHSWG operators considering the indeterminacy

5.4. Advantages of Proposed Research

In the following subsection, we will describe the planned approach's advantages.

- The planned method practices the idea of parameterization in conjunction with NHSS to address the importance of DM obstacles. Consistency-parameterized neutrosophic degree imitates the prospect that there is a level of salutation and justification. With these aspects, this correspondence holds astonishing capacity in practical computing demonstrations in the interpolation universe.
- Because the model emphasizes in-depth surveillance of parameters and the set of values of their respective sub-parameters, it supports decision-makers make balanced and consistent judgments over DM.
- It authenticates all the forms and features of predominant theories, so it's not unreasonable to contemplate it as a general system of existing ideas.

6. Conclusions

To deal with partial, uncertain, indeterminate, or inaccurate content, NHSS is a more efficient mathematical model. It is an ordinary model to deal with inexact and puzzling information compared to FS, IFS, PFS, etc. The NHSS to eliminate less conversant, unclear, and inequitable information around the degree of truth, falsity, and indeterminacy of n-tuples subattribute of the examined attributes. This study intends novel AOs for NHSS, such as NHSWA and NHSWG operators, with their fundamental properties. Waste dumping is a significant part of town governance, and constructing new landfills is an imperative tactic to report environmental pollution that is affected by the absence of waste dumping capability. Landfill structure must support its comforts with the atmosphere, citizens, and administration. The systematic and realistic technique of site selection can exploit securities. The choice of committal spots is subjective in several aspects, and people's intellectual capability is inadequate, so it is incredible to evaluate a detailed analysis. The analytical approach of the landfill site selection structure based on FS and IFS has effectually resolved this [79,80]. Still, it does not proceed into the interpretation of the decision-makers reluctance. So, to competently address the decision-makers hesitancy in the DM process, an MCDM algorithm has been proposed based on NHSWA and NHSWG operators. Furthermore, a comparative analysis is conceded to certify the anticipated method's usefulness and proficiency. Lastly, based on the consequences achieved, it can be resolute that the scheduled method exhibits extraordinary steadiness and achievability in the DM process. Future studies will focus on existing DM tools used by several other operators as part of the NHSS. Also, numerous other structures can be established and projected, such as interaction AOs, Einstein AOs, Bonferroni mean AOs and dombi AOs, etc., with their DM techniques for the NHSS environment.

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Conflict of interest

The authors declare no conflicts of interest.

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