Global stability, periodicity, and bifurcation analysis of a difference equation

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ABSTRACT

This research aims to discuss the existence, global stability, periodicity, and bifurcation analysis of a modified version of the ecological model proposed by Tilman and Wedlin [Nature **353**, 653–655 (1991)].

I. INTRODUCTION

Difference equations containing exponential terms have many applications in biology (see Ref. 2).[3–8] In Ref. 10, the authors modeled an ecological condition as a difference equation

$$B_{t+1} = cN\frac{e^{a-bL_t}}{1+e^{a-bL_t}},\ L_{t+1} = \frac{L_t^2}{L_t+d} + kB_{t+1}. \tag{1}$$

Here, B_t is the living biomass, L_t the litter mass, N the total soil nitrogen, t the time, and constants $a, b, c, d > 0$ and $0 < k < 1$. Chaotic nature of (1) was observed by a bifurcation diagram with the varying parameter, such as soil nitrogen.

In Ref. 9, the authors modified (1) as

$$x_{n+1} = \frac{ax_n^2}{x_n+b} + c\frac{e^{k-dx_n}}{1+e^{k-dx_n}},\ x_0 \in \mathbb{R}, \tag{2}$$

where $0 < a < 1, b, c, d, k \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$. Here, in (2), the constant c, which was ck times the soil nitrogen N in (1), and the inclusion of parameter a are the main modification they made and its dynamical analysis was discussed with proof.

In Ref. 1, the authors considered the base of the exponent of a difference equation as λ and discussed various stability properties.

In this paper, we generalized (2) by changing the exponent base e to $\lambda \in (1, \infty)$ and investigated the global attractivity and periodicity of the solutions of the difference equations

$$y_{n+1} = \frac{\alpha y_n^2}{\beta + y_n} + \gamma\frac{\lambda^{\eta-\delta y_n}}{1+\lambda^{\eta-\delta y_n}}, \quad n \in \mathbb{N}. \tag{3}$$

Here, $\lambda > 1, 0 < \alpha < 1$, and $\beta, \gamma, \delta, \eta \in \mathbb{R}^+$ and y_0 is the initial arbitrary non-negative numbers.

Moreover, to know the importance of λ, we did bifurcation analysis by varying the base parameter l and observed the nature of solutions, such as periodic or chaotic, by keeping α, β, γ, δ, and η fixed.

II. UNIQUENESS, GLOBAL STABILITY, AND PERIODICITY

Here, we study the existence of invariant intervals, uniqueness, global stability, and periodic solutions of Eq. (3).

Theorem II.1. (i) *Let* $I = \left[\frac{\gamma\lambda^{\eta-(\gamma\delta/(1-\alpha)))}}{1+\lambda^{\eta-(\gamma\delta/(1-\alpha)))}}, \frac{\gamma}{1-\alpha}\right]$. *If* $y_n \in I$ *for all* $n = 0, 1, 2, \ldots$, *I is an invariant interval for (3).*

(ii) *For a positive constant a, let* $I = \left[\frac{\gamma\lambda^{\eta-(\gamma\delta/(1-\alpha)))-\delta a}}{1+\lambda^{\eta-(\gamma\delta/(1-\alpha)))-\delta a}}, \frac{\gamma}{1-\alpha} + a\right]$. *Then, there exists an* $n_0 \in \{1, 2, \ldots\}$ *for all* $n \geq n_0, y_n \in I$.

Proof. (i) Let $y_n > 0$ and $y_0 \in I$. Since $0 < \alpha < 1$, we have

$$y_1 = \frac{\alpha y_0^2}{\beta + y_0} + \gamma \frac{\lambda^{\eta-\delta y_0}}{1 + \lambda^{\eta-\delta y_0}} < \alpha y_0 + \gamma \leq \frac{\alpha\gamma}{1-\alpha} + \gamma = \frac{\gamma}{1-\alpha}.$$

Then,

$$y_n \leq \frac{\gamma}{1-\alpha}, \quad n = 0, 1, 2, \ldots. \tag{4}$$

Let

$$h(y) = \frac{\gamma\lambda^{\eta-\delta y}}{1 + \lambda^{\eta-\delta y}}. \tag{5}$$

We see that h decreases since $h'(y) = -\frac{\gamma\delta(\ln \lambda)\lambda^{\eta-\delta y}}{(1+\lambda^{\eta-\delta y})^2} < 0$.

Therefore, from (4), we have

$$y_n \geq \frac{\gamma\lambda^{\eta-(\gamma\delta/(1-\alpha))}}{1 + \lambda^{\eta-(\gamma\delta/(1-\alpha))}}. \tag{6}$$

Hence the proof.

(ii) Let $x_n > 0$ be a solution of (3). From (2.4) and (2.6) of Ref. 9 for an $n_0 \in \{1, 2, \ldots\}$ such that

$$y_n \leq a + \frac{\gamma}{1-\alpha}, \ n \geq n_0. \tag{7}$$

Since h decreases, we get

$$y_n \geq \frac{\gamma\lambda^{\eta-(\gamma\delta/(1-\alpha))-\delta a}}{1 + \lambda^{\eta-(\gamma\delta/(1-\alpha))-\delta a}}, \ n \geq n_0. \tag{8}$$

Equations (7) and (8) complete the proof.

Example II.2. Let $\alpha = 0.5, \beta = 3, \gamma = 8, \delta = 4, \eta = 7$, *and* $\lambda = 3$. *We see by computation that* $I = [5.0954 \times 10^{-27}, 16]$ *and by taking initial value* $y_0 \in I$, *we have* $y_n \in I$.

Theorem II.3. *(3) has a unique positive equilibrium solution.*

Proof. Let $F(y) = \frac{\alpha y^2}{\beta+y} + \gamma\frac{\lambda^{\eta-\delta y}}{1+\lambda^{\eta-\delta y}} - y$,

$$\Rightarrow F(y) = -\frac{(1-\alpha)y^2 + \beta y}{\beta + y} + \gamma\frac{\lambda^{\eta-\delta y}}{1 + \lambda^{\eta-\delta y}} \tag{9}$$

and $F(0) = \frac{\gamma\lambda^\eta}{1+\lambda^\eta} > 0$.

Taking $\lim_{y\to\infty} F(y) = -\infty$, we get from (3)

$$\begin{aligned} F'(y) &= \frac{\alpha y^2 + 2\alpha\beta y}{(\beta + y)^2} - \frac{\gamma\delta(\ln \lambda)\lambda^{\eta-\delta y}}{(1 + \lambda^{\eta-\delta y})^2} - 1 \\ &= -\frac{(1-\alpha)y^2 + 2(1-\alpha)\beta y + \beta^2}{(\beta + y)^2} - \frac{\gamma\delta(\ln \lambda)\lambda^{\eta-\delta y}}{(1 + \lambda^{\eta-\delta y})^2} < 0, \end{aligned}$$

which results in the decrease in F. Therefore, the solution is unique.

Theorem II.4. *Suppose that*

$$\gamma\delta \ln \lambda < 2(1-\alpha). \tag{10}$$

Then, the following statements are true:

(i) *Every positive solution of (3) tends to the unique positive equilibrium* $\bar{y}$.
(ii) *The unique* $\bar{y} > 0$ *of (3) is globally asymptotically stable.*

Proof. (i) Let

$$f(x, y) = \frac{\alpha x^2}{\beta + x} + \frac{\gamma\lambda^{\eta-\delta y}}{1 + \lambda^{\eta-\delta y}}. \tag{11}$$

Let I be the same as defined in Theorem II.1.

Let $x, y \in I$.

We have

$$\begin{aligned} f(x,y) &\leq \alpha x + \gamma \leq \alpha\left(\frac{\gamma}{1-\alpha} + a\right) + \gamma \\ &= \frac{\alpha\gamma + \gamma - \alpha\gamma}{1-\alpha} + a\alpha \leq \frac{\gamma}{1-\alpha} + a. \end{aligned} \tag{12}$$

From (5), h decreases, and hence we obtain

$$f(x, y) \geq \frac{\gamma\lambda^{\eta-(\gamma\delta/(1-\alpha))-\delta a}}{1 + \lambda^{\eta-(\gamma\delta/(1-\alpha))-\delta a}}. \tag{13}$$

Equations (12) and (13) imply that $f : I \times I \to I$,

$$\frac{\partial f}{\partial x} = \frac{\alpha x^2 + 2\alpha\beta x}{(\beta + x)^2} > 0, \ \frac{\partial f}{\partial y} = -\frac{\gamma\delta\lambda^{\eta-\delta y}}{(1 + \lambda^{\eta-\delta y})} < 0.$$

Here, for every x, the function f decreases with respect to y and increases for every y with respect to x.

If (3) has a solution y_n, Theorem II.1 (ii) gives the following:

Let $\ell, L > 0$ such that

$$L = f(L, \ell) = \frac{\alpha L^2}{\beta + L} + \frac{\gamma\lambda^{\eta-\delta\ell}}{1 + \lambda^{\eta-\delta\ell}}, \tag{14}$$

$$\ell = f(\ell, L) = \frac{\alpha\ell^2}{\beta + \ell} + \frac{\gamma\lambda^{\eta-\delta L}}{1 + \lambda^{\eta-\delta L}}, \ n \geq n_0, \ y_n \in I. \tag{15}$$

Consider the system of equations

$$\begin{aligned} z = f(z, w) &= \frac{\alpha z^2}{\beta + z} + \frac{\gamma\lambda^{\eta-\delta w}}{1 + \lambda^{\eta-\delta w}}, \\ w = f(w, z) &= \frac{\alpha w^2}{\beta + w} + \frac{\gamma\lambda^{\eta-\delta z}}{1 + \lambda^{\eta-\delta z}}. \end{aligned} \tag{16}$$

Considering $z = z(w)$, we obtain

$$\frac{(1-\alpha)z^2 + \beta z}{\beta + z} = \frac{\gamma\lambda^{\eta-\delta w}}{1 + \lambda^{\eta-\delta w}}$$

and

$$z'(w) = -\frac{\gamma\delta(z+\beta)^2(\ln \lambda)\lambda^{\eta-\delta w}}{(1 + \lambda^{\eta-\delta w})^2((1-\alpha)z^2 + 2(1-\alpha)\beta z + \beta^2)}. \tag{17}$$

Let $G(w) = \frac{(1-\alpha)w^2+\beta w}{\beta+w} - \frac{\gamma\lambda^{\eta-\delta z(w)}}{1+\lambda^{\eta-\delta z(w)}}$.

From (16), we obtain

$$G'(w) = \frac{(1-\alpha)w^2 + 2(1-\alpha)\beta w + \beta^2}{(\beta+w)^2} + \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta z(w)}}{(1+\lambda^{\eta-\delta z(w)})^2} z'(w). \tag{18}$$

From (17), we get

$$G'(w) = \frac{(1-\alpha)w^2 + 2(1-\alpha)\beta w + \beta^2}{(\beta+w)^2} - \frac{\gamma^2\delta^2(z+\beta)^2(\ln\lambda)^2\lambda^{\eta-\delta w}\lambda^{\eta-\delta z(w)}}{(1+\lambda^{\eta-\delta w})^2(1+\lambda^{\eta-\delta z(w)})^2((1-\alpha)z^2+2(1-\alpha)\beta z+\beta^2)}. \tag{19}$$

Now, the following relations hold:

$$\frac{\lambda^{\eta-\delta z(w)}}{(1+\lambda^{\eta-\delta z(w)})^2} < \frac{1}{2}, \quad \frac{\lambda^{\eta-\delta w}}{(1+\lambda^{\eta-\delta w})^2} < \frac{1}{2},$$

$$\frac{(z+\beta)^2}{(1-\alpha)z^2 + 2(1-\alpha)\beta z + \beta^2} < \frac{1}{1-\alpha},$$

$$\frac{(1-\alpha)w^2 + 2(1-\alpha)\beta w + \beta^2}{(w+\beta)^2} > 1-\alpha.$$

Then, by the above-mentioned relations and Theorem (II.4), we have

$$G'(w) = 1 - \alpha - \gamma^2\delta^2\frac{1}{4(1-\alpha)} > 0.$$

Therefore, function G increases. We have from (15), $G(w) = 0$ has solutions ℓ, L. Therefore, $L = \ell$. From Lemma 3.1 of Ref. 9, every y_n of (3) contains the unique equilibrium $\bar{y}$ when $n \to \infty$.

(ii) The linearized equation about the equilibrium $\bar{y}$ is

$$y_{n+1} = \left(\frac{\alpha\bar{y}^2 + 2\alpha\beta\bar{y}}{(\beta+\bar{y})^2} - \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2}\right) y_n.$$

We prove that

$$\left|\frac{\alpha\bar{y}^2 + 2\alpha\beta\bar{y}}{(\beta+\bar{y})^2} - \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2}\right| < 1 \tag{20}$$

or

$$-1 < \frac{\alpha\bar{y}^2 + 2\alpha\beta\bar{y}}{(\beta+\bar{y})^2} - \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} < 1. \tag{21}$$

Since $0 < \alpha < 1$, it is obvious that

$$-\frac{(1-\alpha)\bar{y}^2 + 2(1-\alpha)\beta\bar{y} + \beta^2}{(\beta+\bar{y})^2} < \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2}.$$

From Theorem (II.4), we get $\gamma\delta\ln\lambda < 2(1-\alpha) < 2$,

$$\frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} < \frac{2\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} < 1 < 1 + \frac{\alpha\bar{y}^2 + 2\alpha\beta\bar{y}}{(\beta+\bar{y})^2},$$

and so (21) is true. Hence, (20) holds, and we see that from Theorem 1.3.7 of Ref. 11, $\bar{y}$ is locally asymptotically *stable*. From (i), every y_n of (3) tends to $\bar{y}$ as $n \to \infty$. Therefore, $\bar{y}$ is globally asymptotically stable.

Example II.5. For $\alpha = 0.5, \beta = 3, \gamma = 0.8, \delta = 0.4, \eta = 7, \lambda = 3$, and Theorem (II.4) holds, then every positive solution of (3) tends to the unique positive equilibrium $\bar{y}$ and $\bar{y} > 0$ is globally asymptotically stable.

Theorem II.6. *Let $\mu > 0$ and $\delta > \frac{(1+\alpha)(1+\lambda^\eta)}{\mu(1-\alpha)\ln\lambda}$, $\gamma > \frac{((1-\alpha)\mu^2+\beta\mu)(1+\lambda^{\eta-\delta\mu})}{(\mu+\beta)\lambda^{\eta-\delta\mu}}$. Then, (3) has period-2 solutions.*

Proof. Solution of (3) is of period two if $y_2 = y_0$. Then,

$$\frac{\alpha y_1^2}{\beta+y_1} + \gamma\frac{\lambda^{\eta-\delta y_1}}{1+\lambda^{\eta-\delta y_1}} = y_0, \quad \frac{\alpha y_0^2}{\beta+y_0} + \gamma\frac{\lambda^{\eta-\delta y_0}}{1+\lambda^{\eta-\delta y_0}} = y_1. \tag{22}$$

Let

$$H(y) = \frac{\alpha h(y)^2}{\beta+h(y)} + \gamma\frac{\lambda^{\eta-\delta h(y)}}{1+\lambda^{\eta-\delta h(y)}} - y, \quad h(y) = \frac{\alpha y^2}{\beta+y} + \gamma\frac{\lambda^{\eta-\delta y}}{1+\lambda^{\eta-\delta y}}. \tag{23}$$

We prove that $H'(\bar{y}) > 0$.

From (23), we get

$$H'(y) = h'(y)\left(\frac{\alpha h(y)^2 + 2\alpha\beta h(y)}{(\beta+h(y))^2} - \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta h(y)}}{(1+\lambda^{\eta-\delta h(y)})^2}\right) - 1,$$
$$h'(y) = \frac{\alpha y^2 + 2\alpha\beta y}{(\beta+y)^2} - \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta y}}{(1+\lambda^{\eta-\delta y})^2}. \tag{24}$$

From Theorem II.4, we have

$$h(\bar{y}) = \bar{y}. \tag{25}$$

From (24) and (25), we get

$$H'(\bar{y}) = \left(-\frac{\alpha\bar{y}^2 + 2\alpha\beta\bar{y}}{(\beta+\bar{y})^2} + \frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2}\right)^2 - 1.$$

We prove that

$$\frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} > 1 + \frac{\alpha\bar{y}^2 + 2\alpha\beta\bar{y}}{(\beta+\bar{y})^2}. \tag{26}$$

We have from (3)

$$\frac{\gamma\lambda^{\eta-\delta\bar{y}}}{1+\lambda^{\eta-\delta\bar{y}}} = \frac{(1-\alpha)\bar{y}^2 + \beta\bar{y}}{\beta+\bar{y}}. \tag{27}$$

We get

$$\frac{((1-\alpha)\bar{y}^2 + \beta\bar{y})(1+\lambda^{\eta-\delta\bar{y}})}{(\beta+\bar{y})\lambda^{\eta-\delta\bar{y}}} = \gamma. \tag{28}$$

Let $G(y) = \frac{((1-\alpha)y^2+\beta y)(1+\lambda^{\eta-\delta y})}{(\beta+y)\lambda^{\eta-\delta y}}$.

From the hypothesis and (28), we get $G(\bar{y}) = c > G(\mu)$. G is an increasing function since $G'(y) > 0$. Therefore, we have $\bar{y} > \mu$.

From (27), we have

$$\frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} = \frac{\delta(\ln\lambda)((1-\alpha)\bar{y}^2+\beta\bar{y})}{(\beta+\bar{y})(1+\lambda^{\eta-\delta\bar{y}})} > \frac{\delta(1-\alpha)(\ln\lambda)\bar{y}}{1+\lambda^{\eta}},$$

which implies

$$\frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} > \frac{\delta(1-\alpha)\mu\ln\lambda}{1+\lambda^{\eta}}.$$

From the hypothesis, we get

$$\frac{\gamma\delta(\ln\lambda)\lambda^{\eta-\delta\bar{y}}}{(1+\lambda^{\eta-\delta\bar{y}})^2} > 1+\alpha.$$

Therefore, (26) is true. Hence, $H'(\bar{y}) > 0$. There exists a positive constant ε such that H is an increasing function for $y \in (\bar{y}-\varepsilon, \bar{y}+\varepsilon)$.

From (23), we can prove that $H(\bar{y}) = 0, H(0) > 0$. So, there exists a $\Psi < \bar{y}$ such that $H(\Psi) = 0$.

Hence, we have y_n with $y_0 = \Psi$ is a solution of (3) with period-2.

Example II.7. For $\alpha = 0.05, \beta = 3, \gamma = 222, \delta = 50, \eta = 7, \lambda = 2, \mu = 0.25$, since the conditions of Theorem II.6 *holds, then (3) has a periodic solution of prime period-2.*

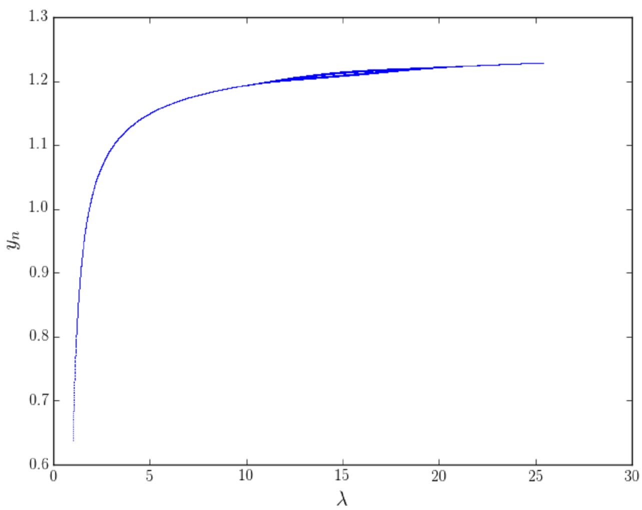

FIG. 1. $\gamma = 1.2277$.

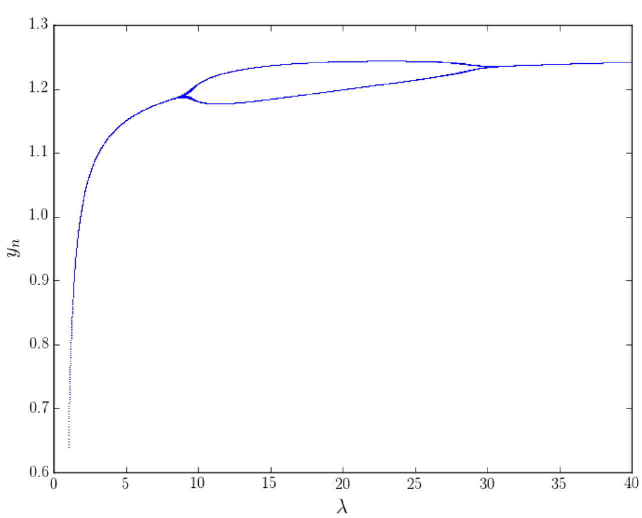

FIG. 2. $\gamma = 1.23$.

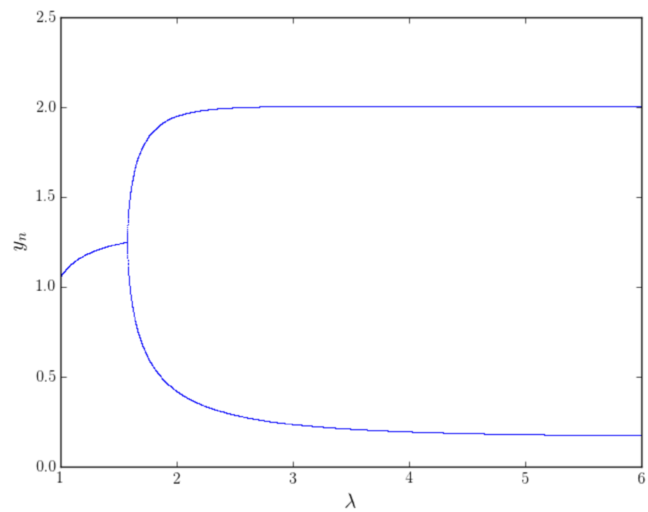

FIG. 3. $\gamma = 2$.

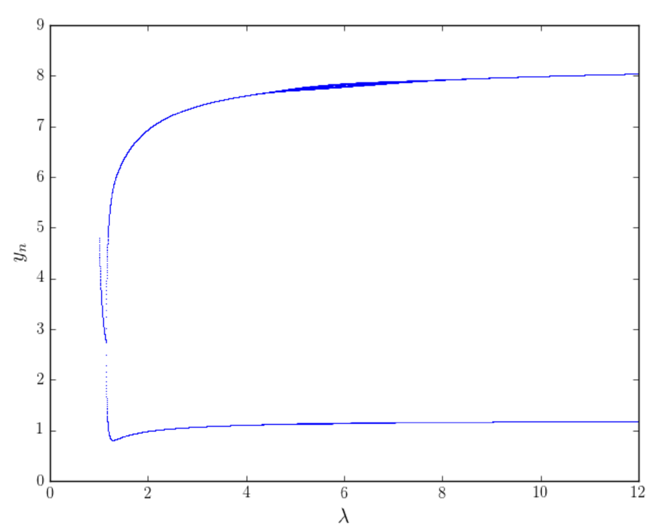

FIG. 4. $\gamma = 8.4$.

III. BIFURCATION ANALYSIS

In this section, we present the bifurcation analysis of the qualitative changes that happen for the solutions of the difference equation (3). Here, we consider λ as the variation parameter, which is the base of the exponent $\eta - \delta y_n$ in (3).

We fix $\alpha = 0.2, \beta = 3, \delta = 5, \eta = 7$, and $y_0 = 3$.

For $\gamma < 1.2$ and for any λ, the solution is asymptotically stable. Therefore, no bifurcation occurs. When $\gamma = 1.2277$, we observe from Fig. 1 a small periodic bubble that originates at $\lambda \approx 12$, and so, period-2 solution occurs approximately in the interval $[12, 17]$.

In Fig. 2, we observe that the bubble expands to a certain stage when $\gamma = 1.23$. Hence, we confirm the existence of period-2 solution for $9 < \lambda < 30$. On increasing γ, the bubble vanishes and bifurcation

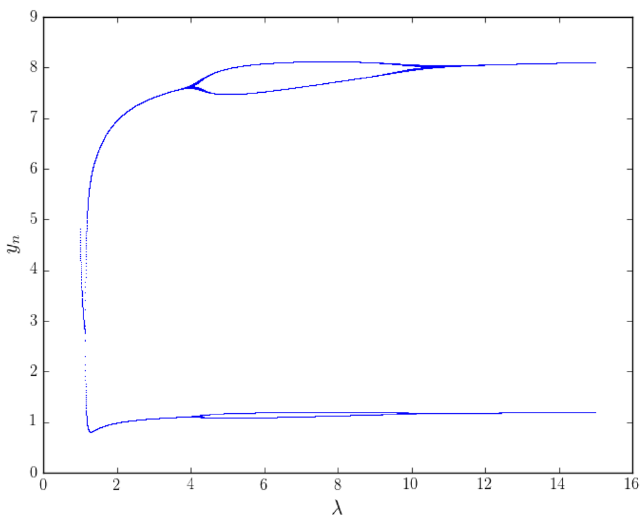

FIG. 5. $\gamma = 8.43$.

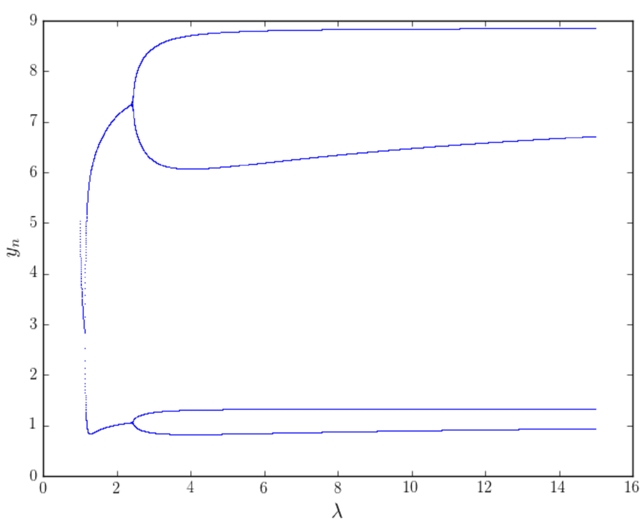

FIG. 6. $\gamma = 8.8$.

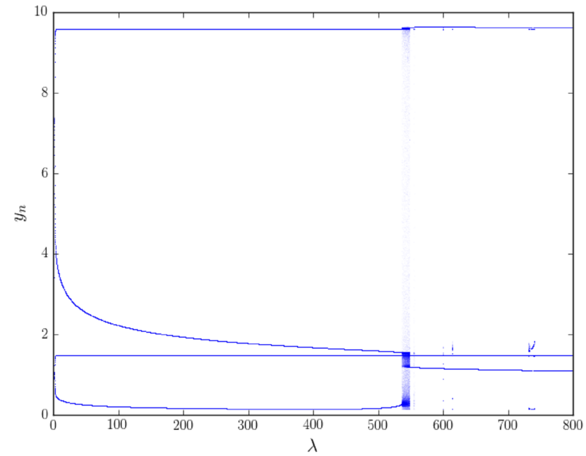

FIG. 7. $\gamma = 9.555$.

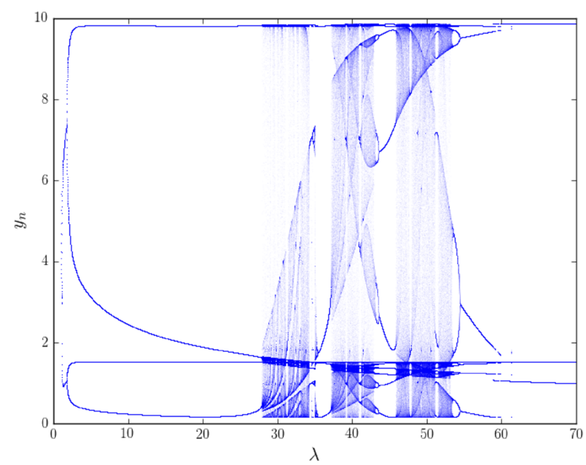

FIG. 8. $\gamma = 9.8$.

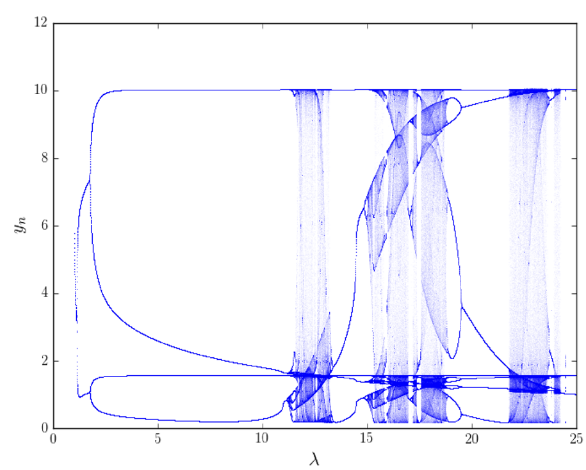

FIG. 9. $\gamma = 10$.

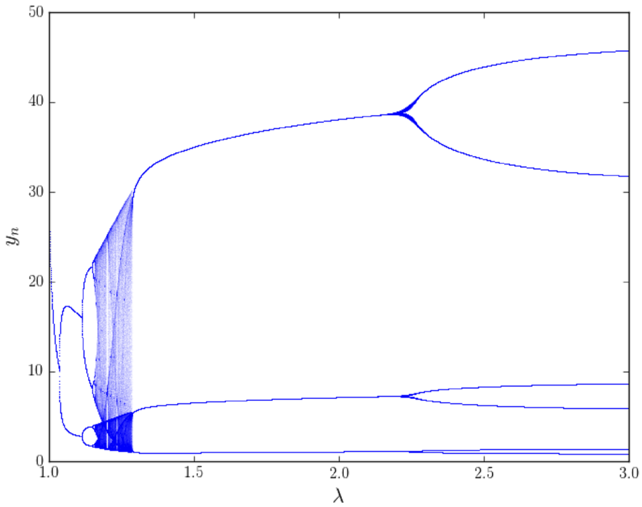

FIG. 10. $\gamma = 47$.

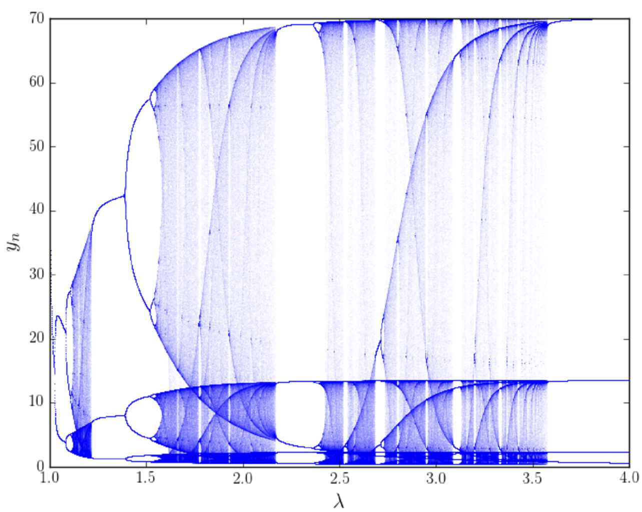

FIG. 11. $\gamma = 70$.

appears like a fork that guarantees a period-2 solutions after bifurcation (see Fig. 3). In Fig. 4, we observe the blooming of two new bubbles when $\gamma = 8.4$. As shown in Fig. 2, the bubbles expand and period doubling also occurs (see Figs. 5 and 6).

When $\gamma = 9.555$ and around $\lambda = 540$, in Fig. 7, we observe the possibility of chaotic solutions. After the chaos period-3, solutions occur. Figures 8 and 9 show the increase in chaotic solutions. When we increase γ, chaotic solutions occur for small λ, and simultaneously, three bubbles are formed, which guarantees the period-3 solution and bifurcates to period-6 followed by chaos and period-4 solutions and so on (see Figs. 10–12).

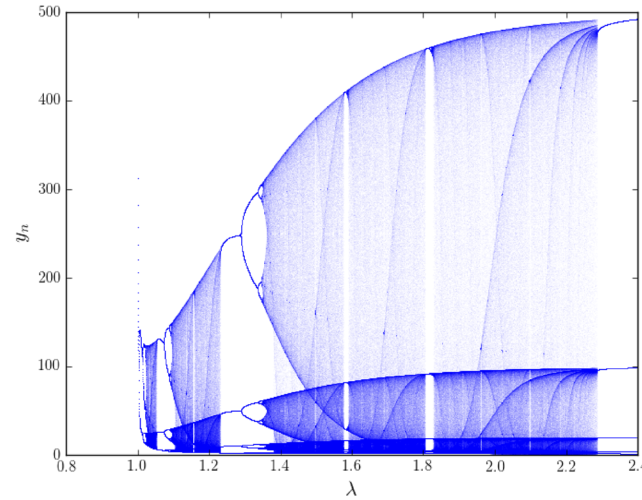

FIG. 12. $\gamma = 500$.

IV. CONCLUSION

In this research, we argued the existence of unique positive equilibrium solution, its convergence, global asymptotic stability, and conditions for the existence of periodic solutions. The importance of the generalized base λ of the exponent was discussed using the bifurcation diagram by varying it. For various λ, we observed the existence of asymptotic, periodic, and chaotic solutions.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

J. Leo Amalraj: Investigation (equal); Writing – original draft (equal). **M. Maria Susai Manuel**: Methodology (equal); Supervision (equal). **Dumitru Baleanu**: Funding acquisition (equal); Validation (equal). **D. S. Dilip**: Investigation (equal); Visualization (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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