

ARTICLE

Minimal Doubly Resolving Sets of Certain Families of Toeplitz Graph

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ABSTRACT

The doubly resolving sets are a natural tool to identify where diffusion occurs in a complicated network. Many real-world phenomena, such as rumour spreading on social networks, the spread of infectious diseases, and the spread of the virus on the internet, may be modelled using information diffusion in networks. It is obviously impractical to monitor every node due to cost and overhead limits because there are too many nodes in the network, some of which may be unable or unwilling to send information about their state. As a result, the source localization problem is to find the number of nodes in the network that best explains the observed diffusion. This problem can be successfully solved by using its relationship with the well-studied related minimal doubly resolving set problem, which minimizes the number of observers required for accurate detection. This paper aims to investigate the minimal doubly resolving set for certain families of Toeplitz graph $T_n(1, t)$, for $t \geq 2$ and $n \geq t + 2$. We come to the conclusion that for $T_n(1, 2)$, the metric and double metric dimensions are equal and for $T_n(1, 4)$, the double metric dimension is exactly one more than the metric dimension. Also, the double metric dimension for $T_n(1, 3)$ is equal to the metric dimension for $n = 5, 6, 7$ and one greater than the metric dimension for $n \geq 8$.

KEYWORDS

Family of Toeplitz graph; resolving sets; metric dimension; doubly resolving sets; double metric dimension

1 Introduction and Preliminaries

To understand the abstract structure of graphs, graph invariants can be a powerful tool. Metric generators in graphs have evolved into a variety of distinct types based on their popularity or usefulness. As an example, virus propagation in complicated network difficulties can be detected using the double metric dimension (DMD).

We consider a finite, undirected, and connected graph Γ with the edge set E_Γ and the vertex set V_Γ . In a connected graph Γ , the distance between a pair of distinct vertices, say k and l in Γ , is the smallest path length among the lengths of all paths between them and is represented by $d(k, l)$. For any two distinct vertices j and k in V_Γ , a vertex $l \in V_\Gamma$ resolves the vertices j and k if the condition $d(j, l) \neq d(k, l)$ holds true. Let k be a vertex of Γ and $H_\Gamma = \{h_\mu | 1 \leq \mu \leq \rho\}$ be an ordered subset of the vertex set V_Γ , then $r(k, H_\Gamma)$ is the representation of k with respect to H_Γ which is the ρ -vector



$\left(d(k, h_\mu)\right)_{\mu=1}^p$, also known as vector of metric coordinates. If each vertex of graph Γ possesses a unique vector of metric coordinates with respect to H_Γ then the subset H_Γ is called a resolving set for graph Γ . If the subset H_Γ is the resolving set with the minimum number of elements in Γ , then its count becomes the metric dimension (MD) of Γ and can be denoted as $\dim(\Gamma)$.

It is common in computer science, chemistry, biology, and operations research to structure graph theory models. In graph theory, the computation of the resolving sets and MD for general graphs is complex. In various disciplines, such as optimization, pattern recognition, and loran detecting devices, the MD has gained all the interest due to its applications.

Slater [1], a mathematician, was the first to discover the term MD. This notion was used to describe the study of the Loran or sonar model station. Another mathematician, Harary, independently described the concept of MD with the help of Harary et al. [2]. Additionally, this concept cleared the path for finding the unique recipient of a message on a network. Since then, several studies have been done on resolving sets, including [3,4]. Many different tasks can be accomplished with resolving sets, like network discovery and verification [5], strategies for the mastermind games [6], as well as digital geometry, pattern recognition, and processing of images [7]. The minimum order of a resolving set of Hamming graphs closely relates to the problem of weighing the coins discussed in [8,9]. Also, there are numerous applications in different fields, such as chemical structures [10], robotics [11], combinatorial optimization [12], geographic routing protocols [13] for the theoretical study of the MD.

The MD of any graph is a computationally difficult problem to solve. So, valuable bounds for different types of graphs have been found. The MD of all connected graphs, as well as the MD of certain well-known graph families, such as trees, pathways, and complete graphs, were established by Chartrand et al. [10,14]. For some families of generalized Petersen graphs, MD bounds were studied in [15]. The MD for chorded cycles and kayak paddles graphs was found to be constant by Ahmad et al. [16]. Many authors have extensively studied the MD of path-related graphs. The MD of the Kenser graph was computed by Hui et al. [17]. On the other hand, Alholi et al. [18] also contributed to this area by finding that the MD is constant. Buczkowski and others [19] presented a generic concept of k -dimensional graphs. The NP-completeness of the MD for general graphs was demonstrated by Khuller et al. [11]. The minimum order of a resolving set of Jahangir graphs has also been discussed in detail by Tomescu et al. [20].

Caceres et al. [21] developed the concept of doubly resolving sets (DRSs) of a graph Γ , demonstrating that the MD of the Cartesian product $\Gamma \square \Gamma$ is closely linked to the minimal doubly resolving sets (MDRSs) of Γ . Consider the vertices $k_1, k_2, l_1, l_2 \in V_\Gamma$, then any pair of vertices k_1, k_2 is said to be doubly resolved by the vertices l_1, l_2 if the condition $d(k_1, l_1) - d(k_1, l_2) \neq d(k_2, l_1) - d(k_2, l_2)$, holds true. A subset $N_\Gamma \in V_\Gamma$ is said to be a DRS, if any two distinct vertices of graph Γ are doubly resolved by some two vertices of the subset N_Γ . A MDRS is a DRS having minimum order. The order of MDRS is denoted by $\psi(\Gamma)$ and is termed the DMD of Γ . Note that if the vertices l_1, l_2 doubly resolve the vertices k_1, k_2 then $d(k_1, l_1) - d(k_2, l_1) \neq 0$ or $d(k_1, l_2) - d(k_2, l_2) \neq 0$. For all graphs Γ , any DRS is obviously a resolving set, with $\dim(\Gamma) \leq \psi(\Gamma)$. The MDRS problem has been proven to be NP-hard in the context of general graphs [22]. Furthermore, it was shown in [21] that the upper bound on the MD of the Cartesian product of Γ and Ω can be stated as; $\dim(\Gamma \square \Omega) \leq \dim(\Gamma) + \psi(\Omega) - 1$.

Therefore, DRSs are essential for studying the MD of Cartesian product graphs. These results have inspired us to work on finding the MDRSs. Also, MDRSs themselves have a unique combinatorial nature that can be seen in their integer programming model, which was shown in [22]. There are several families of graphs for which the problem of finding the MDRS has been examined. For example,

MDRSs of prisms [23], convex polytopes [24], and Hamming graphs [25] have all been studied in the literature. The family of circulant graphs in [26] was found to have the same MD and MDRS. The MDRSs for the line graphs of prisms and sunlet graphs were found in [27]. For the first time, Chen et al. explicitly approximated the upper and lower bounds for the MDRS problem [28]. In [29], the minimal resolving sets and DMD of the line graph of chorded cycles were examined. Authors demonstrated that the DMD of $L(C'_n)$ is exactly one greater than its MD. Ahmad et al. discussed the problem of finding the minimal resolving set and MDRS for line graphs of kayak paddles graphs [30]. The authors in [31,32] presented some families of convex polytopes with constant DMD. While solving the MDRS problem, certain families of Harary graphs, layer-sun graphs have also been investigated in [33,34], respectively. Cocktail, jellyfish and necklace graphs and their line graphs have also been studied for the MD and MDRS problems, see [35–37].

Our research is the continuation of the literature work mentioned earlier. We have investigated the DMD for some particular classes of Toeplitz graphs to contribute to our knowledge of this distance-based parameter in graphs. However, this variant is helpful in many fields, for instance, diffusion over the network, epidemics in human beings, the origin of a disease outbreak, etc. Using MDRSs, it is possible to identify the source of diffusion in complicated networks. Attempting to find the source of an infection in an extensive network might be difficult, but it is still an exciting challenge. For example, to detect the spread of the virus throughout a network, we need to know when specific nodes in the network are infected. However, maintaining observer nodes that can report their infection time may be costly. In addition, the position of a node in the network affects how much information it contains. Which nodes should we use as observers in order to increase our chances of correctly identifying the source? Because of its close relationship to another well-known DRS problem, we can reduce the number of observers required to achieve perfect detection even if the initial time at which an epidemic spreads is unknown [38,39].

Liu et al. [40] determined the MD of some families of Toeplitz graph $T_n(1, t)$ for $n \geq t + 2$ given in the following theorems:

Theorem 1.1. For $n \geq 4$, the metric dimension of Toeplitz graph $T_n(1, 2) = 2$.

Theorem 1.2. For $n \geq 6$, the metric dimension of Toeplitz graph $T_n(1, 4) = 2$.

Theorem 1.3. For $n \geq 5$, the metric dimension of Toeplitz graph $T_n(1, 3) = 3$.

Detailed discussion of the rest of the article is provided in the following sections:

- Our discussion of Toeplitz graphs in Section 2 included the computation of MDRSs for the family of Toeplitz graphs $T_n(1, 2)$ for $n \geq 4$.
- The MDRSs for the family of Toeplitz graphs $T_n(1, 4)$, for $n \geq 6$ were conjectured in Section 3.
- The MDRSs for the family of Toeplitz graph $T_n(1, 3)$ for $n \geq 5$, were discussed in Section 4.
- In Section 5, we conclude this article by expressing an opinion.

2 Minimal Doubly Resolving Sets for the Family of Toeplitz Graph $T_n(1, 2)$

For a graph Γ with n vertices labelled as $\{1, 2, \dots, n\}$ its adjacency matrix A is $n \times n$ matrix whose jk th entry is 1 if the vertex j and vertex k joined by an edge and 0 otherwise. An $n \times n$ matrix $B = b_{jk}$ is known as Toeplitz matrix if $b_{jk} = b_{j+1, k+1}$ for each $\{j, k = 1, \dots, n - 1\}$. A simple undirected graph T_n is Toeplitz graph if matrix $n \times n$ which is $B = b_{jk}$ is the symmetric Toeplitz matrix and for all $j, k = \{1, \dots, n\}$ satisfied the following: edge j, k is in E_Γ iff $b_{jk} = b_{kj} = 1$. An $n \times n$ matrix B will be labelled 0, 1, $\dots, n - 1$ which has n distinct diagonals. The main diagonal has $b_{jj} = 0$ for all $j = 1, \dots, n$, so Toeplitz graph has no loop. The diagonals $\{x_1, x_2, \dots, x_q\}$ containing ones $0 < x_1 < x_2 < \dots < x_q < n$.

The Toeplitz graph $T_n < x_1, x_2, \dots, x_q >$ with vertex set $\{1, 2, \dots, n\}$ has edge j, k for $1 \leq j \leq k \leq n$, occurs iff $k - j = x_p$ for some $p, 1 \leq p \leq q$. Toeplitz graphs are defined as graphs formed from Toeplitz matrices. Specifically, in this part, we computed the MDRSs for the family of Toeplitz graph $T_n(1, 2)$. Fig. 1 illustrates the graph $T_{11}(1, 2)$.

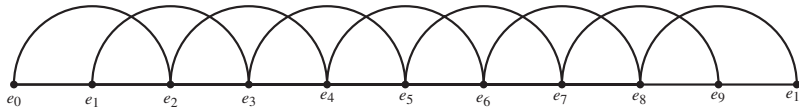


Figure 1: The Toeplitz graph $T_{11}(1, 2)$

It follows from Theorem 1.1 that $\psi(T_n(1, 2)) \geq 2$, for $n \geq 4$. We will also calculate that $\psi(T_n(1, 2)) = 2$, for $n \geq 4$. To compute the distances between the vertices of the Toeplitz graph $T_n(1, 2)$: Define $S_\omega(e_0) = \{e \in T_n(1, 2) : d(e_0, e) = \omega\}$, which is the vertex subset in $V_{T_n(1,2)}$ at distance ω from e_0 . The Table 1 shows the sets $S_\omega(e_0)$ for Toeplitz graph $T_n(1, 2)$, where $n \geq 4$.

Table 1: $S_\omega(e_0)$ for $T_n(1, 2)$

| n | ω | $S_\omega(e_0)$ |
|-------------|----------------------------|----------------------------------|
| | $1 \leq \omega \leq l - 1$ | $\{e_{2\omega-1}, e_{2\omega}\}$ |
| <i>even</i> | l | $\{e_{2l-1}\}$ |
| <i>odd</i> | l | $\{e_{2l-1}, e_{2l}\}$ |

Due to the symmetry of $T_n(1, 2)$, where $n \geq 4$: $d(e_\omega, e_\nu) = d(e_0, e_{|\nu-\omega|})$ if $0 \leq |\nu - \omega| \leq n - 1$.

Due to the fact that, in order to calculate the distance between any pair of vertices in $V_{T_n(1,2)}$, we must know the distance $d(e_0, e)$ for each $e \in T_n(1, 2)$.

Lemma 2.1. Let $T_n(1, 2)$ be the family of Toeplitz graph, then $\psi(T_n(1, 2)) = 2$ for any even positive integer $n \geq 4$.

Proof. Consider the case $n = 2l$, where $l \geq 2$. We must prove that $\psi(T_n(1, 2)) \leq 2$ for any even positive integer $n \geq 4$. So, finding a DRS of order 2 is sufficient. The Table 2 displays the metric coordinate vectors for all vertices of $T_n(1, 2)$ with respect to the set $N_{T_n(1,2)} = \{e_1, e_{n-1}\}$.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 2 is 1. Using Table 2, we may check that there exist a pair of vertices $f_1, f_2 \in S_\omega(e_0)$, for some $\omega \in \{1, 2, \dots, l\}$, such as the condition $r(f_1, N_{T_n(1,2)}) - r(f_2, N_{T_n(1,2)}) \neq 0$, holds true. Also, there exist the vertices $f_1 \in S_\omega(e_0)$ and $f_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l\}$, such as $\omega \neq \nu$, then the condition $r(f_1, N_{T_n(1,2)}) - r(f_2, N_{T_n(1,2)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,2)} = \{e_1, e_{n-1}\}$ is the MDRS. As a result, the Lemma 2.1 holds.

Table 2: The metric coordinate vectors for $T_n(1, 2)$, for any even positive integer $n \geq 4$

| ω | $S_\omega(e_0)$ | $N_{T_n(1,2)} = \{e_1, e_{n-1}\}$ |
|----------------------------|-----------------|-----------------------------------|
| 0 | e_0 | $(1, l)$ |
| $1 \leq \omega \leq l - 1$ | $e_{2\omega-1}$ | $(\omega - 1, l - \omega)$ |
| | $e_{2\omega}$ | $(\omega, l - \omega)$ |
| l | e_{2l-1} | $(l - 1, 0)$ |

Lemma 2.2. Let $T_n(1, 2)$ be the family of Toeplitz graph, then $\psi(T_n(1, 2)) = 2$ for any odd positive integer $n \geq 5$.

Proof. Consider the case $n = 2l + 1$, where $l \geq 2$. We must prove that $\psi(T_n(1, 2)) \leq 2$ for any odd positive integer $n \geq 5$. So, finding a DRS of order 2 is sufficient. The Table 3 displays the metric coordinate vectors for all vertices of $T_n(1, 2)$ with respect to the set $N_{T_n(1,2)} = \{e_0, e_{n-1}\}$.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 3 is 0. Using Table 3, we may check that there exist a pair of vertices $f_1, f_2 \in S_\omega(e_0)$, for some $\omega \in \{1, 2, \dots, l\}$, such as the condition $r(f_1, N_{T_n(1,2)}) - r(f_2, N_{T_n(1,2)}) \neq 0$, holds true. Also, there exist the vertices $f_1 \in S_\omega(e_0)$ and $f_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l\}$, such as $\omega \neq \nu$, then the condition $r(f_1, N_{T_n(1,2)}) - r(f_2, N_{T_n(1,2)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,2)} = \{e_0, e_{n-1}\}$ is the MDRS. As a result, the Lemma 2.2 holds.

The main theorem is stated below by using Lemmas 2.1 and 2.2.

Table 3: The metric coordinate vectors for $T_n(1, 2)$, for any odd positive integer $n \geq 5$

| ω | $S_\omega(e_0)$ | $N_{T_n(1,2)} = \{e_0, e_{n-1}\}$ |
|----------------------------|----------------------------------|--|
| 0 | e_0 | (0, l) |
| $1 \leq \omega \leq l - 1$ | $e_{2\omega-1}$ $e_{2\omega}$ | ($\omega, l - \omega + 1$) ($\omega, l - \omega$) |
| l | e_{2l-1} e_{2l} | ($l, 1$) ($l, 0$) |

Theorem 2.1. Let $T_n(1, 2)$ be the family of Toeplitz graph. Then $\psi(T_n(1, 2)) = 2$, for $n \geq 4$.

Example 2.1. For the Toeplitz graph $T_n(1, 2)$, where $n \geq 4$, let us consider the set $N_{T_8(1,2)} = \{e_1, e_7\}$. Now, the vectors of metric coordinates for $T_8(1, 2)$ with respect to the set $N_{T_8(1,2)}$ are: $r(e_0|N_{T_8(1,2)}) = (1, 4)$, $r(e_1|N_{T_8(1,2)}) = (0, 3)$, $r(e_2|N_{T_8(1,2)}) = (1, 3)$, $r(e_3|N_{T_8(1,2)}) = (1, 2)$, $r(e_4|N_{T_8(1,2)}) = (2, 2)$, $r(e_5|N_{T_8(1,2)}) = (2, 1)$, $r(e_6|N_{T_8(1,2)}) = (3, 1)$, $r(e_7|N_{T_8(1,2)}) = (3, 0)$. Thus, the set $N_{T_8(1,2)}$ is clearly a DRS, as can be observed.

3 Minimal Doubly Resolving Sets for the Family of Toeplitz Graph $T_n(1, 4)$

Specifically, in this part, we computed the MDRSs for the family of Toeplitz graph $T_n(1, 4)$. Fig. 2 illustrates the graph $T_{11}(1, 4)$.

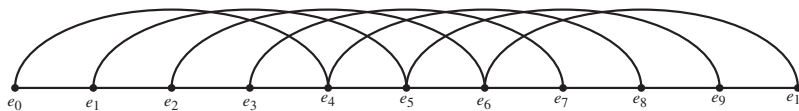


Figure 2: The Toeplitz graph $T_{11}(1, 4)$

It follows from Theorem 1.2 that $\psi(T_n(1, 4)) \geq 2$, for $n \geq 6$. We will also calculate that $\psi(T_n(1, 4)) = 3$, for $n \geq 6$. To compute the distances between the vertices of the Toeplitz graph $T_n(1, 4)$: Define $S_\omega(e_0) = \{e \in T_n(1, 4) : d(e_0, e) = \omega\}$, which is the vertex subset in $V_{T_n(1,4)}$ at distance ω from e_0 . The Table 4 shows the sets $S_\omega(e_0)$ for Toeplitz graph $T_n(1, 4)$, where $n \geq 6$.

Table 4: $S_\omega(e_0)$ for $T_n(1, 4)$

| n | ω | $S_\omega(e_0)$ |
|----------|----------------------------|--|
| ≥ 6 | 1 | $\{e_1, e_4\}$ |
| | $2 \leq \omega \leq l - 1$ | $\{e_{4\omega-6}, e_{4\omega-5}, e_{4\omega-3}, e_{4\omega}\}$ |
| | l | $= \begin{cases} \{e_{4l-6}, e_{4l-5}, e_{4l-3}\}, & \text{if } n \equiv 0 \pmod{4}; \\ \{e_{4l-6}, e_{4l-5}, e_{4l-3}, e_{4l}\}, & \text{if } n \equiv 1, 2, 3 \pmod{4}. \end{cases}$ |
| | $l + 1$ | $= \begin{cases} \{e_{4l-2}, e_{4l-1}\}, & \text{if } n \equiv 0, 1 \pmod{4}; \\ \{e_{4l-2}, e_{4l-1}, e_{4l+1}\}, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$ |
| | $l + 2$ | $\{e_{4l+2}\}, \text{ if } n \equiv 3 \pmod{4}$ |

Due to the symmetry of $T_n(1, 4)$, where $n \geq 6$: $d(e_\omega, e_\nu) = d(e_0, e_{|v-\omega|})$ if $0 \leq |v - \omega| \leq n - 1$. Due to the fact that, in order to calculate the distance between any pair of vertices in $V_{T_n(1,4)}$, we must know the distance $d(e_0, e)$ for each $e \in T_n(1, 4)$.

Lemma 3.1. $\psi(T_n(1, 4)) > 2$, for all $n \geq 6$.

Proof. We already know that $\psi(T_n(1, 4)) \geq 2$, for $n \geq 6$. As a result, it suffices to demonstrate that no subset $N_{T_n(1,4)} \subseteq V_{T_n(1,4)}$ of order 2 is a DRS for $T_n(1, 4)$. We can assume that $e_0 \in N_{T_n(1,4)}$ because of the symmetry of $T_n(1, 4)$. Table 5 lists the non-doubly resolved pair of vertices from $V_{T_n(1,4)}$ that correspond to each of the five distinct forms of set $N_{T_n(1,4)}$. Let us explain why the vertices e_{n-2}, e_{n-3} cannot be doubly resolved by any two vertices of the set $N_{T_n(1,4)} = \{e_0, e_\omega; \omega = n - 1\}$. It is clear that for $\omega = n - 1$, we have $d(e_0, e_{n-2}) = d(e_0, e_{|n-2|}) = l$, $d(e_0, e_{n-3}) = d(e_0, e_{|n-3|}) = l + 1$, $d(e_\omega, e_{n-2}) = d(e_0, e_{|n-2-\omega|}) = 1$ and $d(e_\omega, e_{n-3}) = d(e_0, e_{|n-3-\omega|}) = 2$. So, $d(e_0, e_{n-2}) - d(e_0, e_{n-3}) = d(e_\omega, e_{n-2}) - d(e_\omega, e_{n-3}) = -1$, that is, the set $N_{T_n(1,4)} = \{e_0, e_\omega; \omega = n - 1\}$ is not a DRS of $T_n(1, 4)$. The non-doubly resolved pairs of vertices for all the other types of set $N_{T_n(1,4)}$ listed in Table 5 can be verified in the same way.

Table 5: Non-doubly resolved pairs for $T_n(1, 4)$, where $n \geq 6$

| $N_{T_n(1,4)}$ | Non-doubly resolved pairs |
|---|---|
| $\{e_0, e_\omega\}, 1 \leq \omega \leq n - 4$ | $\{e_{n-5}, e_{n-1}\}, \text{ for all } n$ |
| $\{e_0, e_\omega\}, \omega = n - 3$ | $\{e_{n-3}, e_{n-2}\}, \text{ if } n \equiv 0 \pmod{4}$ |
| $\{e_0, e_\omega\}, n - 2 \leq \omega \leq n - 1$ | $\{e_{n-6}, e_{n-3}\}, \text{ if } n \equiv 0 \pmod{4}$ |
| $\{e_0, e_\omega\}, n - 3 \leq \omega \leq n - 2$ | $= \begin{cases} \{e_{n-6}, e_{n-1}\}, & \text{if } n \equiv 1 \pmod{4}; \\ \{e_{n-2}, e_{n-1}\}, & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$ |
| $\{e_0, e_\omega\}, \omega = n - 1$ | $= \begin{cases} \{e_{n-2}, e_{n-1}\}, & \text{if } n \equiv 1 \pmod{4}; \\ \{e_{n-3}, e_{n-2}\}, & \text{if } n \equiv 2 \pmod{4}; \\ \{e_{n-5}, e_{n-2}\}, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$ |

Lemma 3.2. Let $n \equiv 0(mod 4)$ and $n \geq 8$, we have $\psi(T_n(1, 4)) = 3$.

Proof. Suppose that $n \equiv 0(mod 4)$ and $n \geq 8$. We need to prove that $\psi(T_n(1, 4)) \leq 3$, for $n \geq 8$. So, finding a DRS having order 3 is sufficient. Now, from Table 4 using the sets $S_\omega(e_0)$, the following Table 6 illustrates the metric coordinate vectors for all vertices of $T_n(1, 4)$ with respect to the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$, where $n = 4l$ and $l \geq 2$ be an integer.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 6 is 0. Using Table 6, we may check that there exist a pair of vertices $g_1, g_2 \in S_\omega(e_0)$, for some $\omega \in \{1, 2, \dots, l + 1\}$, such as the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq 0$, holds true. Also, there exist the vertices $g_1 \in S_\omega(e_0)$ and $g_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 1\}$ such as $\omega \neq \nu$, then the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ is the MDRS. As a result, the Lemma 3.2 holds.

Table 6: The metric coordinate vectors for $T_n(1, 4)$, where $n = 4l, l \geq 2$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ |
|---------------------|----------------------------|-----------------|--|
| $n \equiv 0(mod 4)$ | 0 | e_0 | $(0, 2, l)$ |
| | 1 | e_1 | $(1, 1, l - 1)$ |
| | | e_4 | $(1, 2, l - 1)$ |
| | $2 \leq \omega \leq l - 1$ | $e_{4\omega-6}$ | $(\omega, \omega - 2, l - \omega + 2)$ |
| | | $e_{4\omega-5}$ | $(\omega, \omega - 1, l - \omega + 1)$ |
| | | $e_{4\omega-3}$ | $(\omega, \omega, l - \omega + 2)$ |
| | | $e_{4\omega}$ | $(\omega, \omega + 1, l - \omega + 1)$ |
| | | l | e_{4l-3} |
| | e_{4l-5} | | $(l, l - 1, 2)$ |
| | e_{4l-6} | | $(l, l - 2, 2)$ |
| | $l + 1$ | e_{4l-2} | $(l + 1, l - 1, 1)$ |
| | | e_{4l-1} | $(l + 1, l, 2)$ |

Lemma 3.3. Let $n \equiv 1(mod 4)$ and $n \geq 9$, we have $\psi(T_n(1, 4)) = 3$.

Proof. Suppose that $n \equiv 1(mod 4)$ and $n \geq 9$. We need to prove that $\psi(T_n(1, 4)) \leq 3$, for $n \geq 9$. So, finding a DRS having order 3 is sufficient. Now, from Table 4 using the sets $S_\omega(e_0)$, the following Table 7 illustrates the metric coordinate vectors for all vertices of $T_n(1, 4)$ in relation to the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$, where $n = 4l + 1$ and $l \geq 2$ be an integer.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 7 is 0. Using Table 7, we may check that there exist a pair of vertices $g_1, g_2 \in S_\omega(e_0)$, for some $\omega \in \{1, 2, \dots, l + 1\}$, such as the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq 0$, holds true. Also, there exist the vertices $g_1 \in S_\omega(e_0)$ and $g_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 1\}$ such as $\omega \neq \nu$, then the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq \omega - \nu$, holds true. Therefore, the set $N_\Gamma = \{e_0, e_2, e_{n-3}\}$ is the MDRS. As a result, the Lemma 3.3 holds.

Table 7: The metric coordinate vectors for $T_n(1, 4)$, where $n = 4l + 1, l \geq 2$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ |
|---------------------|------------------------|-----------------|--|
| $n \equiv 1(mod 4)$ | 0 | e_0 | $(0, 2, l + 1)$ |
| | 1 | e_1 | $(1, 1, l)$ |
| | | e_4 | $(1, 2, l)$ |
| | $2 \leq \omega \leq l$ | $e_{4\omega-6}$ | $(\omega, \omega - 2, l - \omega + 1)$ |
| | | $e_{4\omega-5}$ | $(\omega, \omega - 1, l - \omega + 2)$ |
| | | $e_{4\omega-3}$ | $(\omega, \omega, l - \omega + 1)$ |
| | | $e_{4\omega}$ | $= \begin{cases} (\omega, \omega + 1, l - \omega + 1), & \text{if } \omega \leq l - 1; \\ (l, l + 1, 2), & \text{if } \omega = l. \end{cases}$ |
| | $l + 1$ | e_{4l-2} | $(l + 1, l - 1, 0)$ |
| | | e_{4l-1} | $(l + 1, l, 1)$ |

Lemma 3.4. Let $n \equiv 2(mod 4)$ and $n \geq 6$, we have $\psi(T_n(1, 4)) = 3$.

Proof. Suppose that $n \equiv 2(mod 4)$ and $n \geq 6$. We need to prove that $\psi(T_n(1, 4)) \leq 3$, for $n \geq 6$. So, finding a DRS having order 3 is sufficient. Now, from Table 4 using the sets $S_\omega(e_0)$, the following Table 8 illustrates the metric coordinate vectors for all vertices of $T_n(1, 4)$ with respect to the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$, where $n = 4l + 2$ and $l \geq 1$ be an integer.

Table 8: The metric coordinate vectors for $T_n(1, 4)$, where $n = 4l + 2, l \geq 1$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ | |
|---------------------|----------------------------|-----------------|--|---------------------|
| $n \equiv 2(mod 4)$ | 0 | e_0 | $(0, 2, l + 1)$ | |
| | 1 | e_1 | $(1, 1, l - 1)$ | |
| | | e_4 | $(1, 2, l)$ | |
| | $2 \leq \omega \leq l - 1$ | $e_{4\omega-6}$ | $(\omega, \omega - 2, l - \omega + 2)$ | |
| | | $e_{4\omega-5}$ | $(\omega, \omega - 1, l - \omega + 2)$ | |
| | | $e_{4\omega-3}$ | $(\omega, \omega, l - \omega)$ | |
| | | $e_{4\omega}$ | $(\omega, \omega + 1, l - \omega)$ | |
| | l | e_{4l} | $(l, l + 1, 1)$ | |
| | | e_{4l-3} | $(l, l, 2)$ | |
| | | e_{4l-5} | $(l, l - 1, 1)$ | |
| | | e_{4l-6} | $(l, l - 2, 2)$ | |
| | | $l + 1$ | e_{4l+1} | $(l + 1, l + 1, 2)$ |
| | | | e_{4l-2} | $(l + 1, l - 1, 1)$ |
| | e_{4l-1} | | $(l + 1, l, 0)$ | |

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 8 is 0. Using Table 8, we may check that there exist a pair of vertices $g_1, g_2 \in S_\omega(e_0)$, for some $\omega \in \{1, 2, \dots, l + 1\}$, such as the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq 0$, holds true. Also, there exist the vertices $g_1 \in S_\omega(e_0)$ and $g_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 1\}$ such as $\omega \neq \nu$, then the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ is the MDRS. As a result, the Lemma 3.4 holds.

Lemma 3.5. Let $n \equiv 3(mod 4)$ and $n \geq 7$, we have $\psi(T_n(1, 4)) = 3$.

Proof. Suppose that $n \equiv 3(mod 4)$ and $n \geq 7$. We need to prove that $\psi(T_n(1, 4)) \leq 3$, for $n \geq 7$. So, finding a DRS having order 3 is sufficient. Now, from Table 4 using the sets $S_\omega(e_0)$, the following Table 9 illustrates the metric coordinate vectors for all vertices of $T_n(1, 4)$ with respect to the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$, where $n = 4l + 3$ and $l \geq 1$ be an integer.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 9 is 0. From Table 9, we may check that there exist a pair of vertices $g_1, g_2 \in S_\omega(e_0)$, for some $\omega \in \{1, 2, \dots, l + 2\}$, such as the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq 0$, holds true. Also, there exist the vertices $g_1 \in S_\omega(e_0)$ and $g_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 2\}$ such as $\omega \neq \nu$, then the condition $r(g_1, N_{T_n(1,4)}) - r(g_2, N_{T_n(1,4)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ is the MDRS. As a result, the Lemma 3.5 holds.

The main theorem is stated below by using Lemmas 3.2–3.5.

Table 9: The metric coordinate vectors for $T_n(1, 4)$, where $n = 4l + 3, l \geq 1$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,4)} = \{e_0, e_2, e_{n-3}\}$ | |
|---------------------|------------|------------------------|--|--|
| $n \equiv 3(mod 4)$ | 0 | e_0 | $(0, 2, l)$ | |
| | 1 | e_1 | $(1, 1, l + 1)$ | |
| | | e_4 | $(1, 2, l - 1)$ | |
| | | $2 \leq \omega \leq l$ | $e_{4\omega-6}$ | $(\omega, \omega - 2, l - \omega + 3)$ |
| | | | $e_{4\omega-5}$ | $(\omega, \omega - 1, l - \omega + 2)$ |
| | | | $e_{4\omega-3}$ | $(\omega, \omega, l - \omega + 2)$ |
| | | | $e_{4\omega}$ | $(\omega, \omega + 1, l - \omega)$ |
| | $l + 1$ | e_{4l+1} | $(l + 1, l + 1, 1)$ | |
| | | e_{4l-1} | $(l + 1, l, 1)$ | |
| | | e_{4l-2} | $(l + 1, l - 1, 2)$ | |
| $l + 2$ | e_{4l+2} | $(l + 2, l, 2)$ | | |

Theorem 3.1. Let $T_n(1, 4)$ be the family of Toeplitz graph. Then $\psi(T_n(1, 4)) = 3$, for $n \geq 6$.

4 Minimal Doubly Resolving Sets for the Family of Toeplitz Graph $T_n(1, 3)$

Specifically, in this part, we computed the MDRSs for the family of Toeplitz graph $T_n(1, 3)$. Fig. 3 illustrates the graph $T_{11}(1, 3)$.

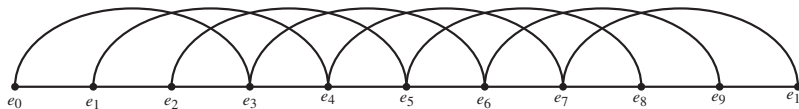


Figure 3: The Toeplitz graph $T_{11}(1, 3)$

It follows from Theorem 1.3 that $\psi(T_n(1, 3)) \geq 3$, for $n \geq 5$. We will also calculate that

$$\psi(T_n(1, 3)) = \begin{cases} 3, & \text{for } 5 \leq n \leq 7; \\ 4, & \text{for } n \geq 8. \end{cases}$$

To compute the distances between the vertices of the Toeplitz graph $T_n(1, 3)$: Define $S_\omega(e_0) = \{e \in T_n(1, 3) : d(e_0, e) = \omega\}$, which is the vertex subset in $V_{T_n(1,3)}$ at distance ω from e_0 . The Table 10 shows the sets $S_\omega(e_0)$ for Toeplitz graph $T_n(1, 3)$, where $n \geq 5$.

Table 10: $S_\omega(e_0)$ for $T_n(1, 3)$

| n | ω | $S_\omega(e_0)$ |
|----------|----------------------------|---|
| ≥ 5 | 1 | $\{e_1, e_3\}$ |
| | $2 \leq \omega \leq l - 1$ | $\{e_{3\omega-4}, e_{3\omega-2}, e_{3\omega}\}$ |
| | l | $= \begin{cases} \{e_{3l-4}, e_{3l-2}\}, & \text{if } n \equiv 0(\text{mod } 3); \\ \{e_{3l-4}, e_{3l-2}, e_{3l}\}, & \text{if } n \equiv 1(\text{mod } 3); \\ \{e_{3l-4}, e_{3l-2}, e_{3l}\}, & \text{if } n \equiv 2(\text{mod } 3). \end{cases}$ |
| | $l + 1$ | $= \begin{cases} \{e_{3l-1}\}, & \text{if } n \equiv 0(\text{mod } 3); \\ \{e_{3l-1}\}, & \text{if } n \equiv 1(\text{mod } 3); \\ \{e_{3l-1}, e_{3l+1}\}, & \text{if } n \equiv 2(\text{mod } 3). \end{cases}$ |

Due to the symmetry of $T_n(1, 3)$, where $n \geq 5$: $d(e_\omega, e_\nu) = d(e_0, e_{|\nu-\omega|})$ if $0 \leq |\nu - \omega| \leq n - 1$. Due to the fact that, in order to calculate the distance between any pair of vertices in $V_{T_n(1,3)}$, we must know the distance $d(e_0, e)$ for each $e \in T_n(1, 3)$.

Lemma 4.1. $\psi(T_n(1, 3)) > 3$, for all $n \geq 5$.

Proof. We already know that $\psi(T_n(1, 3)) \geq 3$, for $n \geq 5$. As a result, it suffices to demonstrate that no subset $N_{T_n(1,3)} \subseteq V_{T_n(1,3)}$ of order 3 is a DRS for $T_n(1, 3)$. We can assume that $e_0 \in N_{T_n(1,3)}$ because of the symmetry of $T_n(1, 3)$. Table 11 lists the non-doubly resolved pair of vertices from $V_{T_n(1,3)}$ that correspond to each of the seven distinct forms of set $N_{T_n(1,3)}$. Let us explain why the vertices e_{n-2}, e_{n-1} cannot be doubly resolved by any two vertices of the set $N_{T_n(1,3)} = \{e_0, e_\omega, e_\nu; \omega = 1, \nu = n - 2\}$. It is clear that for $\omega = 1, \nu = n - 2$, we have $d(e_0, e_{n-2}) = d(e_0, e_{|n-2|}) = l, d(e_0, e_{n-1}) = d(e_0, e_{|n-1|}) = l + 1, d(e_\omega, e_{n-2}) = d(e_0, e_{|n-2-\omega|}) = l - 1, d(e_\omega, e_{n-1}) = d(e_0, e_{|n-1-\omega|}) = l, d(e_\nu, e_{n-2}) = d(e_0, e_{|n-2-\nu|}) = 0$ and $d(e_\nu, e_{n-1}) = d(e_0, e_{|n-1-\nu|}) = 1$. So, $d(e_0, e_{n-2}) - d(e_0, e_{n-1}) = d(e_\omega, e_{n-2}) - d(e_\omega, e_{n-1}) = d(e_\nu, e_{n-2}) - d(e_\nu, e_{n-1}) = -1$, that is, the set $N_{T_n(1,3)} = \{e_0, e_\omega, e_\nu; \omega = 1, \nu = n - 2\}$ is not a DRS of $T_n(1, 3)$. The non-doubly resolved pairs of vertices for all the other types of set $N_{T_n(1,3)}$ listed in Table 11 can be verified in the same way.

Table 11: Non-doubly resolved pairs for $T_n(1, 3)$, where $n \geq 5$

| $N_{T_n(1,3)}$ | Non-doubly resolved pairs |
|---|---------------------------|
| $\{e_0, e_\omega, e_\nu\}, 1 \leq \omega < \nu \leq n - 3$ | $\{e_{n-4}, e_{n-1}\}$ |
| $\{e_0, e_\omega, e_\nu\}, 2 \leq \omega \leq 3, n - 2 \leq \nu \leq n - 1$ | $\{e_{n-5}, e_{n-3}\}$ |
| $\{e_0, e_\omega, e_\nu\}, \omega = 1, \nu = n - 2$ | $\{e_{n-2}, e_{n-1}\}$ |
| $\{e_0, e_\omega, e_\nu\}, \omega = 1, \nu = n - 1$ | $\{e_{n-4}, e_{n-2}\}$ |
| $\{e_0, e_\omega, e_\nu\}, \omega = 2m - 2$ (where $3 \leq m \leq 4$), $\nu = n - 2$ | $\{e_{n-2}, e_{n-1}\}$ |

(Continued)

Table 11 (continued)

| $N_{T_n(1,3)}$ | Non-doubly resolved pairs |
|--|---------------------------|
| $\{e_0, e_\omega, e_\nu\},$ $\omega = 2m - 1$ (where $3 \leq m \leq 4$), $n - 2 \leq \nu \leq n - 1$ | $\{e_{n-5}, e_{n-3}\}$ |
| $\{e_0, e_\omega, e_\nu\}, \omega = 2m - 2$ (where $3 \leq m \leq 4$), $\nu = n - 1$ | $\{e_{n-4}, e_{n-2}\}$ |

Lemma 4.2. Let $n \equiv 0(mod 3)$ for $n \geq 6$, we have $\psi(T_n(1, 3)) = \begin{cases} 3, & \text{for } n = 6; \\ 4, & \text{for } n > 6. \end{cases}$

Proof. Suppose that $n \equiv 0(mod 3)$ and $n \geq 6$. Then, in the case $n \equiv 0(mod 3)$ and $n = 6$, the MDRS for $T_6(1, 3)$ is $N_{T_n(1,3)} = \{e_0, e_1, e_2\}$. Now, for the case $n \equiv 0(mod 3)$ and $n > 6$, we need to prove that $\psi(T_n(1, 3)) \leq 4$. So, finding a DRS having order 4 is sufficient. From Table 10 using the sets $S_\omega(e_0)$, the following Table 12 illustrates the metric coordinate vectors for all vertices of $T_n(1, 3)$ with respect to the set $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$, where $n = 3l$, and $l \geq 3$ be an integer.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 12 is 0. Using Table 12, we may check that there exist a pair of vertices $h_1, h_2 \in S_\omega(e_0)$ for some $\omega \in \{1, 2, \dots, l + 1\}$ such as the condition $r(h_1, N_{T_n(1,3)}) - r(h_2, N_{T_n(1,3)}) \neq 0$, holds true. Also, there exist the vertices $h_1 \in S_\omega(e_0)$ and $h_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 1\}$ such as $\omega \neq \nu$, then the condition $r(h_1, N_{T_n(1,3)}) - r(h_2, N_{T_n(1,3)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$ is the MDRS. AS a result, the Lemma 4.2 holds.

Table 12: The metric coordinate vectors for $T_n(1, 3)$, where $n = 3l, l \geq 3$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$ |
|---------------------|----------------------------|-----------------|--|
| $n \equiv 0(mod 3)$ | 0 | e_0 | $(0, 1, 2, l)$ |
| | 1 | e_1 | $(1, 0, 1, l - 1)$ |
| | | e_3 | $(1, 2, 1, l - 1)$ |
| | $2 \leq \omega \leq l - 1$ | $e_{3\omega-4}$ | $(\omega, \omega - 1, \omega - 2, l - \omega + 2)$ |
| | | $e_{3\omega-2}$ | $(\omega, \omega - 1, \omega, l - \omega)$ |
| | | $e_{3\omega}$ | $(\omega, \omega + 1, \omega, l - \omega)$ |
| | l | e_{3l-2} | $(l, l - 1, l, 0)$ |
| | | e_{3l-4} | $(l, l - 1, l - 2, 2)$ |
| | $l + 1$ | e_{3l-1} | $(l + 1, l, l - 1, 1)$ |

Lemma 4.3. Let $n \equiv 1(mod 3)$ for $n \geq 7$, we have

$$\psi(T_n(1, 3)) = \begin{cases} 3, & \text{for } n = 7; \\ 4, & \text{for } n > 7. \end{cases}$$

Proof. Suppose that $n \equiv 1(mod 3)$ and $n \geq 7$. Then, in the case $n \equiv 1(mod 3)$ and $n = 7$, the MDRS for $T_7(1, 3)$ is $N_{T_n(1,3)} = \{e_1, e_2, e_5\}$. Now, for the case $n \equiv 1(mod 3)$ and $n > 7$, we need to prove that $\psi(T_n(1, 3)) \leq 4$. So, finding a DRS having order 4 is sufficient. From Table 10 using the sets $S_\omega(e_0)$, the following Table 13 illustrates the metric coordinate vectors for all vertices of $T_n(1, 3)$ with respect to the set $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$, where $n = 3l + 1$, and $l \geq 3$ be an integer.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 13 is 0. Using Table 13, we may check that there exist a pair of vertices $h_1, h_2 \in S_\omega(e_0)$ for some $\omega \in \{1, 2, \dots, l + 1\}$ such as the condition $r(h_1, N_{T_n(1,3)}) - r(h_2, N_{T_n(1,3)}) \neq 0$, holds true. Also, there exist the vertices $h_1 \in S_\omega(e_0)$ and $h_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 1\}$ such as $\omega \neq \nu$, then the condition $r(h_1, N_{T_n(1,3)}) - r(h_2, N_{T_n(1,3)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$ is the MDRS. AS a result, the Lemma 4.3 holds.

Table 13: The metric coordinate vectors for $T_n(1, 3)$, where $n = 3l + 1, l \geq 3$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$ | |
|---------------------|----------------------------|-----------------|---|--|
| $n \equiv 1(mod 3)$ | 0 | e_0 | $(0, 1, 2, l + 1)$ | |
| | 1 | e_1 | $(1, 0, 1, l)$ | |
| | | e_3 | $(1, 2, 1, l)$ | |
| | $2 \leq \omega \leq l - 1$ | | $e_{3\omega-4}$ | $(\omega, \omega - 1, \omega - 2, l - \omega + 1)$ |
| | | | $e_{3\omega-2}$ | $(\omega, \omega - 1, \omega, l - \omega + 1)$ |
| | | | $e_{3\omega}$ | $(\omega, \omega + 1, \omega, l - \omega + 1)$ |
| | l | | e_{3l} | $(l, l + 1, l, 1)$ |
| | | | e_{3l-2} | $(l, l - 1, l, 1)$ |
| | | | e_{3l-4} | $(l, l - 1, l - 2, 1)$ |
| | | $l + 1$ | e_{3l-1} | $(l + 1, l, l - 1, 0)$ |

Lemma 4.4. Let $n \equiv 2(mod 3)$ for $n \geq 5$, we have

$$\psi(T_n(1, 3)) = \begin{cases} 3, & \text{for } n = 5; \\ 4, & \text{for } n > 5. \end{cases}$$

Proof. Suppose that $n \equiv 2(mod 3)$ and $n \geq 5$. Then, in the case $n \equiv 1(mod 3)$ and $n = 5$, the MDRS for $T_5(1, 3)$ is $N_{T_n(1,3)} = \{e_1, e_2, e_4\}$. Now, for the case $n \equiv 2(mod 3)$ and $n > 5$, we need to prove that $\psi(T_n(1, 3)) \leq 4$. So, finding a DRS having order 4 is sufficient. From Table 10 using the sets $S_\omega(e_0)$, the following Table 14 illustrates the metric coordinate vectors for all vertices of $T_n(1, 3)$ with respect to the set $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$, where $n = 3l + 2$, and $l \geq 2$ be an integer.

It turns out that the value of the first metric coordinate of the vertex e_0 from $S_\omega(e_0)$ in Table 14 is 0. Using Table 14, we may check that there exist a pair of vertices $h_1, h_2 \in S_\omega(e_0)$ for some $\omega \in \{1, 2, \dots, l + 1\}$ such as the condition $r(h_1, N_{T_n(1,3)}) - r(h_2, N_{T_n(1,3)}) \neq 0$, holds true. Also, there exist the vertices $h_1 \in S_\omega(e_0)$ and $h_2 \in S_\nu(e_0)$ for any $\omega, \nu \in \{1, 2, \dots, l + 1\}$ such as $\omega \neq \nu$, then the condition $r(h_1, N_{T_n(1,3)}) - r(h_2, N_{T_n(1,3)}) \neq \omega - \nu$, holds true. Therefore, the set $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$ is the MDRS. AS a result, the Lemma 4.4 holds.

The main theorem is stated below by using Lemmas 4.2–4.4.

Theorem 4.1. Let $T_n(1, 3)$ be the family of Toeplitz graph, then

$$\psi(T_n(1, 3)) = \begin{cases} 3, & \text{for } 5 \leq n \leq 7; \\ 4, & \text{for } n \geq 8. \end{cases}$$

Table 14: The metric coordinate vectors for $T_n(1, 3)$, where $n = 3l + 2, l \geq 2$

| n | ω | $S_\omega(e_0)$ | $N_{T_n(1,3)} = \{e_0, e_1, e_2, e_{n-2}\}$ |
|---------------------|----------|----------------------------|--|
| $n \equiv 2(mod 3)$ | 0 | e_0 | $(0, 1, 2, l)$ |
| | 1 | e_1 | $(1, 0, 1, l + 1)$ |
| | | e_3 | $(1, 2, 1, l - 1)$ |
| | | $2 \leq \omega \leq l - 1$ | $e_{3\omega-4}$ |
| | | $e_{3\omega-2}$ | $(\omega, \omega - 1, \omega, l - \omega + 2)$ |
| | | $e_{3\omega}$ | $(\omega, \omega + 1, \omega, l - \omega)$ |
| | l | e_{3l} | $(l, l + 1, l, 0)$ |
| | | e_{3l-2} | $(l, l - 1, l, 2)$ |
| | | e_{3l-4} | $(l, l - 1, l - 2, 2)$ |
| | $l + 1$ | e_{3l+1} | $(l + 1, l, l + 1, 1)$ |
| | | e_{3l-1} | $(l + 1, l, l - 1, 1)$ |

5 Application

Recently, some applications of DRSs can be seen to localizing the epidemic source in different complex networks such as social networks, epidemics in human beings and the origin of a disease outbreak, etc. (see [38,39]). In particular, we consider a network to reduce the number of observers using the DRSs in order to achieve the perfect detection of an epidemic source.

Let us assume, a social network arranged in the form of Toeplitz network $T_{11}(1, 3)$, where the node set $V_{T_{11}(1,3)} = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ represents the people and the edge set $E_{T_{11}(1,3)} = \{e_0e_1, e_1e_2, e_2e_3, e_3e_4, e_4e_5, e_5e_6, e_6e_7, e_7e_8, e_8e_9, e_9e_{10}\} \cup \{e_0e_3, e_1e_4, e_2e_5, e_3e_6, e_4e_7, e_5e_8, e_6e_9, e_7e_{10}\}$ between the nodes represents the links between the people. If the observers are placed throughout the network, and the inter-node distances are reliable and known, a direct solution can be found. But, the entire process would be very costly and time taking.

So, what are the fewest number of observers required to account epidemic source with an unknown starting time and irregular transmission delays among the nodes? Such that each node has a unique representation depending upon minimum distances from the observer nodes. In order to reduce the number of observers, we employed the MDRS $N_{T_{11}(1,3)} = \{e_0, e_1, e_2, e_9\}$ of the Toeplitz network $T_{11}(1, 3)$.

By using the Lemma 4.4 and Theorem 4.1, we placed the observers only on the nodes e_0, e_1, e_2, e_9 . It is clear from Table 14 and Fig. 4 that each node has a unique representation and epidemic propagation will be optimal by placing observers at the nodes e_0, e_1, e_2, e_9 . This scenario explains the applicability of DRSs in source localization problems.

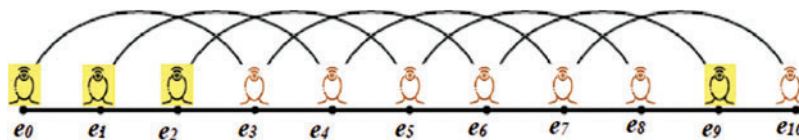


Figure 4: DRS in $T_{11}(1, 3)$

6 Conclusion

This study is concerned with the concept of calculating MDRSs of graphs using an integer linear programming formulation that has been proposed earlier in the literature. In this article, we theoretically establish the MDRSs for the certain families of Toeplitz graphs $T_n(1, t)$ for $t = 2, 3, 4$ and $n \geq t + 2$. We observed that the MD and DMD are equal for the Toeplitz graphs $T_n(1, 2)$. Also, the DMD for the Toeplitz graphs $T_n(1, 4)$ is exactly one greater than its MD. In the case of $T_n(1, 3)$, the DMD is equal to the MD for $n = 5, 6, 7$ and is exactly one greater than its MD for $n \geq 8$. In future work, certain classes of subdivision graphs of circulant graphs $C_n(1, k)$ will be investigated for the MDRS problem.

This research work may lead to the following open problem.

Open Problem 6.1. Investigate the minimal doubly resolving sets for the family of the generalized Toeplitz graphs $T_n(1, t)$ for all values of t and $n \geq t + 2$.

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