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ORIGINAL ARTICLE

Modified Atangana-Baleanu fractional operators involving generalized Mittag-Leffler function

Wen-Hua Huang [a], **Muhammad Samraiz** [b], **Ahsan Mehmood** [b], **Dumitru Baleanu** [c,d,e,*], **Gauhar Rahman** [f], **Saima Naheed** [b]

[a] *School of Science, Huzhou University, Huzhou 313000, PR China*
[b] *Department of Mathematics, University of Sargodha, P.O. Box 40100, Sargodha, Pakistan*
[c] *Department of Mathematics, Cankaya University, 06530 Ankara, D.B., Turkey*
[d] *Department of Medical Research, China Medical University, Taichung 40402, Taiwan*
[e] *Instiute of Space Sciences, Magurele-Bucharest, 077125 Magurele, Romania*
[f] *Department of Mathematics and Statistics, Hazara University, Mansehra, Pakistan*

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Abstract In this paper, we are going to deal with fractional operators (FOs) with non-singular kernels which is not an easy task because of its restriction at the origin. In this work, we first show the boundedness of the extended form of the modified Atangana-Baleanu (A-B) Caputo fractional derivative operator. The generalized Laplace transform is evaluated for the introduced operator. By using the generalized Laplace transform, we solve some fractional differential equations. The corresponding form of the Atangana-Baleanu Caputo fractional integral operator is also established. This integral operator is proved bounded and obtained its Laplace transform. The existence and Hyers-Ulam stability is explored. In the last results, we studied the relation between our defined operators. The operators in the literature are obtained as special cases for these newly explored FOs.

1. Introduction and preliminaries

Despite the commendable 325-year history of fractional calculus, there are still a lot of unresolved issues from both a theoretical and an applied perspective. Abel use the fractional calculus in resolving the tautochrone problem [1]. This work creates a big attention of researchers towards the applications of fractional calculus in differential and integral equations [2]. Many other books are written by different researchers which contain different theories, applications of FOs in [3–6]. A

Abbreviations: FOs, Fractional Operators; A-B, Atangana-Baleanu; M-L, Mittag-Leffler.

* Corresponding author at: Department of Mathematics, Cankaya University, 06530 Ankara, Turkey.
E-mail addresses: whhuanghz@zjhu.edu.cn (W.-H. Huang), muhammad.samraiz@uos.edu.pk (M. Samraiz), mehmoodahsan154@gmail.com (A. Mehmood), dumitru@cankaya.edu.tr (D. Baleanu), drgauhar.rahman@hu.edu.pk (G. Rahman), saima.naheed@uos.edu.pk (S. Naheed).

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variety of FO classifications that have been put out in recent years. Some researchers came to the conclusion that a single FO, such as the Caputo ones, cannot be used to represent all types of complex processes in science and engineering. Since more experimental data are needed to confirm the validity of the fractional models [7,8].

Numerous mathematicians and experts have paid close attention to the field of fractional differential and integral equations in recent years [9,10]. The fractional order derivatives reflect physical models of various phenomena in various disciplines, including biology, physics, mechanics, and dynamical systems. Analysts are paying close attention to the existence theory of solutions, which is one of the top research areas in fractional order differential equations. Finding an exact solution to a fractional order differential equation is extremely challenging.

In fractional calculus, dealing with the non singular kernel is not an easy task. We can define the generalization of FOs by generalizing their kernels [11–13]. In [12], Samraiz et al. introduced the (k,s) form of FOs with a non-singular kernel and their applications in physics. Using such a form of FOs, they established the Cauchy problems and obtained their solutions. The Hilfer-Prabhakar fractional derivative was inroduced and used to solve physical problems in [14]. In [15,16] the weighted generalized form of FOs were introduced, containing the Mittag–Leffler (M-L) function as a non-singular kernel. These operators were used to detect Cauchy problems, which are used in continuous time random walk theory. The authors of [17] discuss the generalisation of FOs with the generalized (k,s) form of the multivariate M-L function as a non-singular kernel. For more studies of FOs containing non-singular kernels and their applications, we refer the reader to [18,19,21,20,22–24].

Let's recall the following basic definitions. The following definitions of Gamma and Beta functions presented in [25].

Definition 1.1. The definition of Gamma function is characterized by

$$\Gamma(\eta_1) = \int_0^\infty u^{\eta_1-1}e^{-u}d\eta_1,\ Re(\eta_1) > 0.$$

Definition 1.2. The Beta function $B(\mu_1,\nu_1)$ is defined by the formula

$$B(\mu_1,\nu_1) = \int_0^1 \tau^{\mu_1-1}(1-\tau)^{\nu_1-1}d\tau,\ \Re(\mu_1) > 0,\ \Re(\nu_1) > 0$$

and the identity which relates it with Γ function is

$$B(\mu_1,\nu_1) = \frac{\Gamma(\mu_1)\Gamma(\nu_1)}{\Gamma(\mu_1+\nu_1)}.$$

Prabhakar [26] introduced the following definition of three parameter M-L function that generalized the previous forms exists in literature.

Definition 1.3. Let $n \in N$ and $\eta_1,\varrho_1,\gamma_1 \in \mathbb{C}, Re(\varrho_1) > 0$, $Re(\eta_1) > 0$, then the M-L function is given by the following expression

$$\mathscr{E}^{\gamma_1}_{\varrho_1,\eta_1}(\vartheta) = \sum_{n=0}^{\infty}\frac{(\gamma_1)_n\vartheta^n}{\Gamma(\varrho_1 n+\eta_1)n!}.$$

In 1888, Leonard James Rogers derived Hölder's inequality and later in 1889, it was given differently by Otto Ludwig Hölder. The definition of Hölder inequality is given by the following.

Definition 1.4. [27] Let p and q be real numbers such that $p,q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, then the Hölder inequality for integrals states that

$$\int_a^b |f(\ell)g(\ell)|d\ell \leqslant \left(\int_a^b |f(\ell)|^p d\ell\right)^{\frac{1}{p}}\left(\int_a^b |g(\ell)|^q d\ell\right)^{\frac{1}{q}},$$

where $f,g \in C^1[a,b]$.

The Lebesgue measurable functions with norm is defined by Kilbas et al. in [2] by the following way.

Definition 1.5. Let f be a function defined on $[c,d]$. The space $\chi^q(c,d), 1 \leqslant q \leqslant \infty$ of Lebesgue measurable functions for which $\|\varphi\|_{\chi^q} < \infty$, i.e.,

$$\|\varphi\|_{\chi^q} = \left[\int_a^b |\varphi(t)|^q dt\right]^{\frac{1}{q}},\ 1 \leqslant q < \infty,$$

$$\|\varphi\|_{\chi^\infty} = ess\,sup_{c\leqslant t\leqslant d}|\varphi(t)| < \infty.$$

In the modern era, fractional calculus theory faces many unresolved problems, as the Riemann and Caputo fractional operators are insufficient to solve both theoretical and physical problems. To address these gaps, researchers have developed their own operators, but the theory still has many shortcomings. To overcome these gaps, Atangana and Baleanu introduced the well-known Atangana-Baleanu fractional operators in both the Riemann and Caputo senses, which are useful in solving many theoretical and physical problems. For example, Panda et al. [29] utilized the A-B fractional operator in their study of the Willis aneurysm system, employing it to solve a nonlinear singularity perturbed boundary value problem. The A-B fractional operators have also found practical applications, as Panda et al. in [30] also used these operators to discuss the prevalence of COVID-19 in the United States, Italy, and France. He presented new insights into the existence and uniqueness of solutions for the 2019-nCOV models using fractional and fractal-fractional operators. Additionally, the solution of the complex valued A-B integral operator is discussed in [31] and in [32] discuss the solution of the A-B fractional and L^p-Fredholm integral equations. A new form of fractional operators in the Caputo sense are presented by the following definition.

Definition 1.6. [28] Let $f\prime \in H^1(0,T)$, then ABC-fractional derivative operator of order $0 < \sigma < 1$ is defined by

$${}_{ABC}\mathfrak{D}^\sigma_0 f(\ell) = \frac{Q(\sigma)}{1-\sigma}\int_0^\ell \mathscr{E}_\sigma(-\mu_\sigma(\ell-t)^\sigma)f\prime(t)dt,\quad t \geqslant 0$$

and the corresponding integral operator is given by

$${}_{ABC}\mathfrak{I}^\sigma_0 f(\ell) = \frac{1-\sigma}{Q(\sigma)}f(\ell) + \frac{Q(\sigma)}{1-\sigma}\int_0^\ell (\ell-t)^\sigma f(t)dt,\quad t \geqslant 0. \tag{1.1}$$

The above mentioned operators are used to solve many theoretical results. The spaces play an important role in the applications of operators as explained by Al-Refai et al. in [22]. For example, we choose $f(x) \in C^1(0,T)$ then the differential equation $_{ABC}\mathfrak{D}_0^\sigma f(x) = \lambda f(x)$, gives the trivial solution i.e $_{ABC}\mathfrak{D}_0^\sigma f(0) = 0$ and fractional equation $_{ABC}\mathfrak{D}_0^\sigma f(x) = -\lambda f(x) + h(x)$ gives the solution $\lambda f(0) + h(0) = 0$. But here the space is restrictive for the Caputo derivative. If we choose the space $\chi(f) = \{f : f\prime \in L^1[0,1]\}$, then for the fractional initial value problem

$$_{ABC}\mathfrak{D}_0^\sigma f(x) = \begin{cases} -\lambda f(x) + h(x), & x \in (0,T); \\ f_0, & x = 0, \end{cases}$$

with $0 < \sigma < 1$, we get the unique solution

$$f(x) = f_0 \mathscr{E}_{\sigma,1}(-\lambda x^\sigma) + \int_0^x (x-s)^{\sigma-1} \times \mathscr{E}_{\sigma,\sigma}(-\lambda(x-s)^\sigma) h(s) ds.$$

and the corresponding homogenous equation also obtain non trivial solution $f(x) = f_0 \mathscr{E}_{\sigma,1}(-\lambda x^\sigma)$. This proves the role of space is considerable. To overcome this difficulty, Al-Refai et al. presented an article [22] in which more wider space is chosen to avoid from extra condition. In [33] weighted form of A-B FOs are defined and discussed its applications in differential equations.

Definition 1.7. Let $f' \in L^1(0,T)$, then the weighted ABC-fractional derivative operator of order $0 < \sigma < 1$ is defined as follows:

$$_{ABC}\mathfrak{D}_0^\sigma f(\ell) = \frac{Q(\sigma)}{1-\sigma} \int_0^\ell \mathscr{E}_\sigma(-\mu_\sigma(\ell - t)^\sigma)(w(t)f(t))' dt, \quad t \geqslant 0.$$

and the corresponding integral operator is defined by

$$_{ABC}\mathfrak{I}_0^\sigma f(\ell) = \frac{1-\sigma}{Q(\sigma)} f(\ell) + \frac{Q(\sigma)}{1-\sigma} w^{-1}(t) \int_0^\ell (\ell - t)^\sigma w(t) f(t) dt, \quad t \geqslant 0,$$

where $Q(\sigma)$, is a normalized function having property $Q(0) = Q(1) = 1$.

Definition 1.8. Let f be a continuous function and $f' \in L^1(0,T)$, then the generalized form of MABC-fractional derivative operator of order $0 < \sigma < 1$ is defined as follows:

$$^{MABC}\mathfrak{D}_0^\sigma f(\ell) = \frac{Q(\sigma)}{1-\sigma} \int_0^\ell \mathscr{E}_\sigma(-\mu_\sigma(\ell - t)^\sigma) f'(t) dt, \quad t \geqslant 0.$$

By using integration by parts on (2.1) leads to

$$^{MABC}\mathfrak{D}_0^\sigma f(\ell) = \tfrac{Q(\sigma)}{1-\sigma}\Big[f(\ell) - \mathscr{E}_\sigma(-\mu_\sigma(\ell)^\sigma) f(0) \\ -\mu_\sigma \int_0^\ell (\ell - t)^{\sigma-1} \mathscr{E}_{\sigma,\sigma}(-\mu_\sigma(\ell - t)^\sigma)\Omega'(t) f(t) dt\Big], \quad t \geqslant 0,$$

where $\mu_\sigma = \frac{\sigma}{1-\sigma}$ and $Q(\sigma)$, is a normalized function having property $Q(0) = Q(1) = 1$, and integral operator is given by

$$_{ABC}\mathfrak{I}_0^\sigma f(\ell) = \frac{1-\sigma}{Q(\sigma)} f(\ell) + \frac{Q(\sigma)}{1-\sigma} \int_0^\ell (\ell - t)^\sigma f(t) dt, \quad t \geqslant 0.$$

Inspired by recent research in formulating fractional differential equations and determining their exact solutions through diverse methods, we will present the fractional operator as a tool to model several differential equations. To obtain precise solutions for the investigated problems, we will apply the generalized Laplace transform. The findings of our study are broader in scope than those reported in existing literature.

2. The Modified Form of A-B Fractional Derivative in Caputo Sense Involving Generalized M-L Function in its Kernel

Atangana-Baleanu fractional integral operators still have lots of problems in initialization. To avoid such problems, researchers defined modified A-B fractional operators in which the M-L function plays a key role as a non-singular kernel. Many problems are resolved through these operators. The M-L function is generalized in many ways by extending the number of parameters. In our present work, we use the generalized M-L function as a kernel. In the modified Atangana-Baleanu fractional operator, the author used the difference of two linear functions as a non-singular kernel, but in the present work, we use the difference of generalized functions that can be both linear and non-linear as a kernel. The generalized version of the modified A-B Caputo (MABC) fractional derivative operator, which incorporates a generalized M-L function in its kernel is defined in the following definition. (see Fig. 1)

Definition 2.1. Let f be a continuous function and $f' \in L^1(0,T)$, then the generalized form of MABC-fractional derivative operator of order $0 < \sigma < 1$, with respect to $\aleph$, is defined as follows:

$$\mathfrak{D}_{a^+}^\sigma f(u) = \frac{Q(\sigma)}{1-\sigma} \int_{a^+}^u \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u) - \aleph(t))^\sigma) f'(t) dt, \quad t, a \geqslant 0. \tag{2.1}$$

Integrating by parts leads to

$$^{MABC}_{\aleph}\mathfrak{D}_{a^+}^\sigma f(u) = \tfrac{Q(\sigma)}{1-\sigma}\Big[f(u) - \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u) - \aleph(a^+))^\sigma) f(a^+) \\ -\mu_\sigma \int_{a^+}^u (\aleph(u) - \aleph(t))^{\sigma-1} \mathscr{E}_{\sigma,\sigma}(-\mu_\sigma(\aleph(u) - \aleph(t))^\sigma)\aleph'(t) f(t) dt\Big],$$

where $t \geqslant 0, \aleph$, a strictly increasing function, $\mu_\sigma = \frac{\sigma}{1-\sigma}$ and $Q(\sigma)$, a normalized function having property $Q(0) = Q(1) = 1$.

Remark 2.1. If we substitute $\aleph(x) = x$ in (2.1) then we get Definition 1.8.

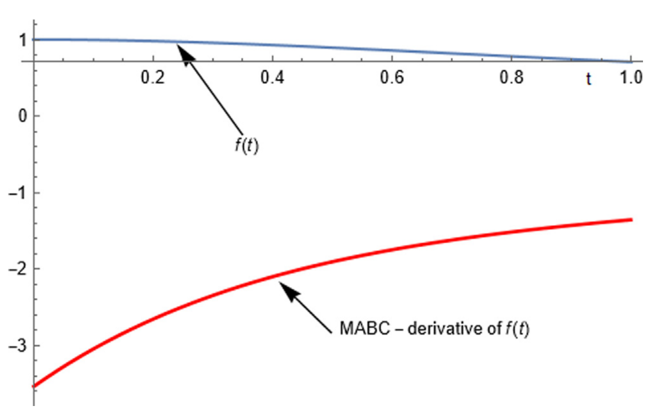

Fig. 1 For $N(t) = t^2 + 1$, we get Fig. 1, where $0^< t < 1$.

In first result of this section, we prove the operator given by the Definition 2.1 is bounded.

Theorem 2.1. Let $f \in X^p(0,T)$, and $\aleph, \mu_\sigma = \frac{\sigma}{1-\sigma}$ and $Q(\sigma)$, admit the same properties as mentioned in Definition 2.1, then the inequality

$$\begin{aligned} &\left|\left|\mathfrak{D}^{\sigma}_{a^+}f(u)\right|\right|_{X^p} \leqslant \tfrac{Q(\sigma)}{1-\sigma}||f(u)||_{X^p} \\ &+||\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a^+))^\sigma)f(a^+)||_{X^p} \\ &\quad+\tfrac{\mathscr{E}_{\sigma,\sigma}\left(-\mu_\sigma(\aleph(r)-\aleph(a^+))^\sigma\right)}{(\aleph(r)-\aleph(a^+))(\sigma n-1)}||f||_{X^p}, \end{aligned} \tag{2.2}$$

holds for $1 \leqslant p < \infty$.

Proof. By using the Definition 2.1, we have

$$\begin{aligned} \left\|{}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{a^+}f(u)\right\|_{X^p} &\leqslant \tfrac{Q(\sigma)}{1-\sigma}\|f(u)-\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a^+))^\sigma)f(a^+)\|_{X^p} \\ &+\left\|\mu_\sigma\textstyle\int_{a^+}^u(\aleph(u)-\aleph(t))^{\sigma-1}\mathscr{E}_{\sigma,\sigma}(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)\aleph'(t)f(t)dt\right\|_{X^p} \\ &\leqslant \tfrac{Q(\sigma)}{1-\sigma}\|f(u)\|_{X^p}+\|\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a^+))^\sigma)f(a^+)\|_{X^p} \\ &+\left\|\mu_\sigma\textstyle\int_{a^+}^u(\aleph(u)-\aleph(t))^{\sigma-1}\mathscr{E}_{\sigma,\sigma}(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)\aleph'(t)f(t)dt\right\|_{X^p}. \end{aligned} \tag{2.3}$$

Consider

$$\begin{aligned} &\left\|\textstyle\int_0^u(\aleph(u)-\aleph(t))^{\sigma-1}\mathscr{E}_{\sigma,\sigma}(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)\aleph'(t)f(t)dt\right\|_{X^p} \\ &=\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left\|\textstyle\int_0^u(\aleph(u)-\aleph(t))^{\sigma n-1}\aleph'(t)f(t)dt\right\|_{X^p} \\ &=\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left(\textstyle\int_{a^+}^r\left|\int_{a^+}^u(\aleph(u)-\aleph(t))^{\sigma n-1}\aleph'(t)f(t)dt\right|^p\aleph'(u)du\right)^{\frac{1}{p}}. \end{aligned}$$

By substituting $\theta = \aleph(u)$ and $\beta = \aleph(t)$, we have

$$=\sum_{n=0}^{\infty}\frac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left(\int_{\aleph(a^+)}^{\aleph(r)}\left|\int_{\aleph(a^+)}^{\aleph(u)}(\theta-\beta)^{\sigma n-1}f(t)d\beta\right|^p d\theta\right)^{\frac{1}{p}}.$$

Using the generalized Minkowski's inequality, we have

$$\begin{aligned} &\leqslant\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left(\textstyle\int_{\aleph(a^+)}^{\aleph(r)}|f(\aleph^{-1}(\beta))|^p\int_\beta^{\aleph(r)}(\theta-\beta)^{(\sigma n-1)p}d\theta\right)^{\frac{1}{p}}d\beta \\ &=\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left(\textstyle\int_{\aleph(a^+)}^{\aleph(r)}|f(\aleph^{-1}(\beta))|^p\tfrac{(\aleph(r)-\beta)^{(\sigma n-1)p+1}}{(\sigma n-1)p+1}\right)^{\frac{1}{p}}d\beta. \end{aligned}$$

Using Hölder inequality, we have

$$\begin{aligned} &\leqslant\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left(\textstyle\int_{\aleph(a^+)}^{\aleph(r)}|f(\aleph^{-1}(\beta))|^p d\beta\right)^{\frac{1}{p}}\left(\int_{\aleph(a^+)}^{\aleph(r)}\left(\tfrac{(\aleph(r)-\beta)^{(\sigma n-1)p+1}}{(\sigma n-1)p+1}\right)^{\frac{q}{p}}d\beta\right)^{\frac{1}{q}} \\ &\leqslant\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\left(\textstyle\int_{\aleph(0)}^{\aleph(r)}|f(\aleph^{-1}(\beta))|^p d\beta\right)^{\frac{1}{p}}\tfrac{(\aleph(r)-\aleph(a^+))^{(\sigma n-1)}}{(\sigma n-1)}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. By substituting $\aleph^{-1}(\beta)=t$, we obtain

$$\begin{aligned} &=\sum_{n=0}^{\infty}\tfrac{|\mu_\sigma|^n}{|\Gamma(\sigma n+\sigma)|}\tfrac{(\aleph(r)-\aleph(a^+))^{(\sigma n-1)}}{(\sigma n-1)}\textstyle\int_{\aleph(a^+)}^{\aleph(r)}|f(t)|^p\aleph'(t)dt \\ &=\tfrac{\mathscr{E}_{\sigma,\sigma}\left(\mu_\sigma(\aleph(r)-\aleph(a^+))^\sigma\right)}{(\aleph(r)-\aleph(a^+))(\sigma n-1)}||f||_{X^p}. \end{aligned}$$

By using this values in (2.3), we have the result (2.2).

Now, we present some illustrative examples of new fractional derivative operator.

Example 2.1. Let we have a constant function $f(t) = C$ and $0 < \sigma < 1$. By using the Definition 2.1, we can write

$$\begin{aligned} \left({}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{a^+}C\right)(t) &= \tfrac{Q(\sigma)}{1-\sigma}\Big[C-\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a^+))^\sigma)C \\ &-\mu_\sigma\textstyle\int_{a^+}^u(\aleph(u)-\aleph(t))^{\sigma-1}\mathscr{E}_{\sigma,\sigma}(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)\aleph'(t)Cdt\Big] \\ &= \tfrac{Q(\sigma)}{1-\sigma}[C-\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a^+))^\sigma)C \\ &-\mu_\sigma\left(\tfrac{-1}{\mu_\sigma}(\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a^+))^\sigma)-1)\right)C \\ \left({}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{0}C\right)(t) &= 0 \end{aligned}$$

i.e., the derivative of constant is zero.

Example 2.2. Consider the piecewise continuous function f

$$f(t)=\begin{cases}\aleph^{\frac{-1}{2}}(t), & t\neq 0;\\ A, & t=0.\end{cases} \tag{2.4}$$

Now by choosing $A \in \mathbb{R}\setminus\{0\}$, $\sigma = \frac{1}{2}, u_\sigma = 1$ and $Q(\sigma) = 1$ in Definition 2.1, we obtain

$$\begin{aligned} {}^{MABC}_{\aleph}\mathfrak{D}^{\frac{1}{2}}_{0^+}f(u) &= 2\Big[f(u)-A\mathscr{E}_{\frac{1}{2}}\left(-(\aleph(u)-\aleph(0^+))^{\frac{1}{2}}\right) \\ &-\mu_{\frac{1}{2}}\textstyle\int_{0^+}^u(\aleph(u)-\aleph(t))^{-\frac{1}{2}}\mathscr{E}_{\frac{1}{2},\frac{1}{2}}\left(-(\aleph(u)-\aleph(t))^{\frac{1}{2}}\right)\aleph'(t)\aleph^{\frac{-1}{2}}(t)dt\Big] \\ &=2\Big[f(u)-A\mathscr{E}_{\frac{1}{2}}\left(-(\aleph(u)-\aleph(0^+))^{\frac{1}{2}}\right)-\sqrt{\pi}\mathscr{E}_{\frac{1}{2}}\left(-(\aleph(u)-\aleph(0^+))^{\frac{1}{2}}\right) \\ &=2\Big[f(u)-(A+\sqrt{\pi})\mathscr{E}_{\frac{1}{2}}\left(-(\aleph(u)-\aleph(0^+))^{\frac{1}{2}}\right)\Big]. \end{aligned} \tag{2.5}$$

If we choose $\mathscr{E}_{\frac{1}{2}}\left(-(\aleph(u)-\aleph(0^+))^{\frac{1}{2}}\right)=0$, then we get

${}^{MABC}_{\aleph}\mathfrak{D}^{\frac{1}{2}}_{0}f(t) = -2\sqrt{\pi} \neq 0,$

which is solution of problem presented by Al-Refai et al. in [22, Example 2].

For the graphical representation, we have the following special case.

Example 2.3. If we choose $N(t) = t^2 + 1$ in Eq. (2.5), the derivative of given function is

$${}^{MABC}_{\aleph}\mathfrak{D}^{\frac{1}{2}}_{0^+}f(u)=2\left[(u^2+1)^{\frac{-1}{2}}-(1+\sqrt{\pi})\sum_{n=0}^{\infty}(-u)^n\right]. \tag{2.6}$$

The comparison of graph of function $N(t) = t^2 + 1$ and of derivative presented in (2.6) is given as

The generalized Laplace transform and convolution defined in [20], can also be written by the following definitions with a choice of weight $w(t) = 1$.

Definition 2.2. Let ψ and $\aleph$ be defined on $[a,\infty)$, where $\aleph$ a monotonically increasing, then the Laplace transform of ψ is given by

$$L_{\aleph}(\psi)(s)=\int_a^{\infty}e^{-s(\aleph(u)-\aleph(a))}\aleph'(u)\psi(u)du$$

such that for all values of s, the Eq. (2.2) is true.

Definition 2.3. The convolution of ϕ and ϖ with respect to ψ is defined as

$$(\phi*_{\psi}\varpi)(x)=\int_a^x\phi\big(\psi^{-1}(\psi(x)+\psi(a)-\psi(t))\big)\varpi(t)\psi'(t)dt.$$

The convolution form of FO in Definition 2.1 is given by

$$\begin{aligned}{}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{a^+}f(u)&=\tfrac{Q(\sigma)}{1-\sigma}\Big[f(u)-\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a^+))^{\sigma})f(a^+)\\&-\mu_{\sigma}\int_{a^+}^{u}(\aleph(u)-\aleph(t))^{\sigma-1}\mathscr{E}_{\sigma,\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(t))^{\sigma})\aleph'(t)f(t)dt\Big]\\&=\tfrac{Q(\sigma)}{1-\sigma}\Big[f(u)-\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a^+))^{\sigma})f(a^+)\\&-\mu_{\sigma}(\aleph(u)-\aleph(a^+))^{\sigma-1}\mathscr{E}_{\sigma,\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a^+))^{\sigma})\ *\ f(t)\Big].\end{aligned}\tag{2.7}$$

Next, we evaluate the Laplace transform of M-L function involved in new defined operator given by Definition 2.1 in the following lemma.

Lemma 2.1. If $\aleph$ is the increasing function and $0<\sigma<1$, then we have

$$L_{\aleph}(\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a^+))^{\sigma}))(s)=\frac{s^{\sigma-1}}{s^{\sigma}+\mu_{\sigma}},\left|\frac{\mu_{\sigma}}{s^{\sigma}}\right|<1.$$

Proof. By using Definition 2.2, we have

$$\begin{aligned}L_{\aleph}(\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a))^{\sigma}))(s)&=\int_a^{\infty}e^{-s(\aleph(u)-\aleph(a))}\aleph'(u)\\&\quad\times\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a))^{\sigma})du\\&=\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^n}{\Gamma(\sigma n+1)}\int_a^{\infty}e^{-s(\aleph(u)-\aleph(a))}\aleph'(u)(\aleph(u)-\aleph(a))^{\sigma n}du.\end{aligned}$$

Substituting $(\aleph(u)-\aleph(a))=t$, we obtain

$$\begin{aligned}&=\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^n}{\Gamma(\sigma n+1)}\int_a^{\infty}e^{-st}t^{\sigma n}dt=\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^n}{\Gamma(\sigma n+1)}L\{t^{\sigma n}\}\\&=\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^{\sigma n}}{s^{\sigma n+1}}=\tfrac{s^{\sigma-1}}{s^{\sigma}+\mu_{\sigma}}.\end{aligned}$$

Hence the result is proved.

Theorem 2.2. Let f be a continuous function and $f'\in L^1(0,T)$, then the generalized Laplace transform of the MABC-fractional derivative operator in Definition 2.1 with $0<\sigma<1$, be given as

$$L_{\aleph}\big({}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{a^+}f(u)\big)(s)=\frac{Q(\sigma)}{1-\sigma}\frac{s^{\sigma}L_{\aleph}\{f(u)\}-s^{\sigma-1}f(a^+)}{s^{\sigma}+\mu_{\sigma}},$$

where $\left|\frac{\mu_{\sigma}}{s^{\sigma}}\right|<1$.

Proof. By using the Eq. (2.7), we have

$$\begin{aligned}L_{\aleph}\{\mathfrak{D}^{\sigma}_{a^+}f(u)\}(s)&=\tfrac{Q(\sigma)}{1-\sigma}L_{\aleph}\{f(u)\}-f(a^+)\times L_{\aleph}(\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a))^{\sigma}))\\&-\mu_{\sigma}L_{\aleph}\Big((\aleph(u)-\aleph(a))^{\sigma-1}\mathscr{E}_{\sigma}(-\mu_{\sigma}(\aleph(u)-\aleph(a))^{\sigma})\ *\ f(u)\Big)\\&=\tfrac{Q(\sigma)}{1-\sigma}L_{\aleph}\{f(u)\}-f(a^+)\tfrac{s^{\sigma-1}}{s^{\sigma}+\mu_{\sigma}}-\mu_{\sigma}\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^n}{\Gamma(\sigma n+\sigma)}\\&\times L_{\aleph}\Big((\aleph(u)-\aleph(a))^{\sigma}(\aleph(u)-\aleph(a))^{\sigma n-1}\Big)L_{\aleph}\{f(u)\}\\&=\tfrac{Q(\sigma)}{1-\sigma}L_{\aleph}\{f(u)\}-f(a^+)\tfrac{s^{\sigma-1}}{s^{\sigma}+\mu_{\sigma}}--\mu_{\sigma}\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^n}{\Gamma(\sigma n+\sigma)}\\&\times L_{\aleph}\Big((\aleph(u)-\aleph(a))^{\sigma(n+1)-1}\Big)L_{\aleph}\{f(u)\}\\&=\tfrac{Q(\sigma)}{1-\sigma}\Big(L_{\aleph}\{f(u)\}-f(a^+)\tfrac{s^{\sigma-1}}{s^{\sigma}+\mu_{\sigma}}+\sum_{n=0}^{\infty}\tfrac{(-\mu_{\sigma})^{n+1}}{s^{\sigma(n+1)}}L_{\aleph}\{f(u)\}\Big)\\&=\tfrac{Q(\sigma)}{1-\sigma}\Big(L_{\aleph}\{f(u)\}-f(a^+)\tfrac{s^{\sigma-1}}{s^{\sigma}+\mu_{\sigma}}-\tfrac{\mu_{\sigma}}{s^{\sigma}+\mu_{\sigma}}L_{\aleph}\{f(u)\}\Big)\\&=\tfrac{Q(\sigma)}{1-\sigma}\tfrac{s^{\sigma}L_{\aleph}\{f(u)\}-s^{\sigma-1}f(a^+)}{s^{\sigma}+\mu_{\sigma}}.\end{aligned}$$

Hence the result is proved.

Example 2.4. For the choice of parameters $0<\sigma<1$, with the condition $\left|\frac{\mu_{\sigma}}{s^{\sigma}}\right|<1$, the solution to the equation

$${}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{0}f(u)=C$$

is given by

$$f(t)=\begin{cases}\frac{C(1-\sigma)}{Q(\sigma)}\Big(1+\mu_{\sigma}\frac{t^{\sigma}}{\Gamma(\sigma+1)}\Big), & t\neq 0;\\ 0, & t=0.\end{cases}$$

Proof. Since $f(0)=0$, therefore for $t>0$

$$\begin{aligned}L_{\aleph}\{{}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{0}f(s)\}&==\tfrac{Q(\sigma)}{1-\sigma}\tfrac{s^{\sigma}}{s^{\sigma}+\mu_{\sigma}}(L_{\aleph}\{f(t)\}(s))\\&=C\tfrac{s^{\sigma}}{s^{\sigma}+\mu_{\sigma}}L_{\aleph}\Big(1+\mu_{\sigma}\tfrac{t^{\sigma}}{\Gamma(\sigma+1)}\Big)(s)\\&=C\tfrac{s^{\sigma}}{s^{\sigma}+\mu_{\sigma}}\big(\tfrac{1}{s}+\tfrac{\mu_{\sigma}}{s^{\sigma+1}}\big)\\&=\tfrac{C}{s}.\end{aligned}$$

This can be written as

$$L_{\aleph}\{{}^{MABC}_{\aleph}\mathfrak{D}^{\sigma}_{0}f(s)\}=L_{\aleph}(C).$$

Hence the result is proved.

3. The Modified Form of A-B Fractional Integral in Caputo Sense Involving Generalzed M-L Function in its Kernel

In this section, we introduce a revised version of the A-B fractional integral in the Caputo sense that includes the generalized M-L function.

Definition 3.1. For $f\in L^1(0,T]$ the modified form of MABC-fractional integral operator of order $0<\sigma<1$ with respect to $\aleph$ is defined as

$${}^{MABC}_{\aleph}\mathfrak{I}^{\sigma}_{a^+}f(u)=\frac{1-\sigma}{Q(\sigma)}f(u)+\frac{\sigma}{Q(\sigma)\Gamma(\sigma)}\times\int_{a^+}^{u}(\aleph(u)-\aleph(t))^{\sigma-1}\aleph'(t)f(t)dt.\tag{3.1}$$

Firstly, we prove this operator is bounded.

Theorem 3.1. Let $f \in X^p(0,T), \aleph$, be a strictly increasing function, $\mu_\sigma = \frac{\sigma}{1-\sigma}$ and $Q(\sigma)$, is a normalized function having property $Q(0) = Q(1) = 1$, then the following inequality holds for $1 \leqslant p < \infty$.

$$\left\|{}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right\|_{X^p} \leqslant \frac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \frac{\sigma}{Q(\sigma)\Gamma(\sigma)}\|f\|_{X^p} \times \frac{(\aleph(r)-\aleph(a^+))^{\sigma-1}}{\sigma-1}. \tag{3.2}$$

Proof. By using the Definition 3.1, we have

$$\left\|{}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right\|_{X^p} \leqslant \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}$$
$$\left\|\int_{a^+}^{u}(\aleph(u)-\aleph(t))^{\sigma-1}\aleph'(t)f(t)dt\right\|_{X^p}$$
$$= \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}\left(\int_0^r\left|\int_{a^+}^{u}(\aleph(u)-\aleph(t))^{\sigma-1}\aleph'(t)f(t)dt\right|^p\aleph'(u)du\right)^{\frac{1}{p}}.$$

Substituting $\vartheta = \aleph(u)$ and $\beta = \aleph(t)$, we have

$$\left\|{}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right\|_{X^p} \leqslant \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}$$
$$\times\left(\int_{\aleph(a^+)}^{\aleph(r)}\left|\int_{\aleph(0)}^{\vartheta}(\vartheta-\beta)^{\sigma-1}f(\aleph^{-1}(\beta))d\beta\right|^p d\vartheta\right)^{\frac{1}{p}}$$
$$= \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}$$
$$\times\int_{\aleph(a^+)}^{\aleph(r)}\left(|f(\aleph^{-1}(\beta))|^p\left|\int_{\beta}^{\aleph(r)}(\theta-\beta)^{\sigma-1}\right|^p d\vartheta\right)^{\frac{1}{p}}d\beta$$
$$= \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}$$
$$\times\int_{\aleph(a^+)}^{\aleph(r)}|f(\aleph^{-1}(\beta))|\left(\int_{\beta}^{\aleph(r)}(\theta-\beta)^{p(\sigma-1)}d\vartheta\right)^{\frac{1}{p}}d\beta$$
$$= \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}$$
$$\times\int_{\aleph(a^+)}^{\aleph(r)}|f(\aleph^{-1}(\beta))|\left(\tfrac{(\aleph(r)-\beta)^{p(\sigma-1)+1}}{p(\sigma-1)+1}\right)^{\frac{1}{p}}d\beta.$$

By using Hölder, inequality we have

$$\left\|{}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right\|_{X^p} \leqslant \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}\int_{\aleph(a^+)}^{\aleph(r)}\left(|f(\aleph^{-1}(\beta))|^p d\beta\right)^{\frac{1}{p}}$$
$$\times\left(\left(\int_{\aleph(a^+)}^{\aleph(r)}\tfrac{(\aleph(r)-\beta)^{p(\sigma-1)+1}}{p(\sigma-1)+1}\right)^{\frac{q}{p}}d\beta\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$, Now substituting $\aleph^{-1}(\beta)=t$, we have

$$\left\|{}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right\|_{X^p} \leqslant \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}\int_{a^+}^{r}\left(|f(t)|^p\aleph'(t)dt\right)^{\frac{1}{p}}$$
$$\times\left(\left(\int_{\aleph(a^+)}^{\aleph(r)}\tfrac{(\aleph(r)-\aleph(t))^{p(\sigma-1)+1}}{p(\sigma-1)+1}\right)^{\frac{q}{p}}\aleph'(t)dt\right)^{\frac{1}{q}}$$
$$= \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}\int_{\aleph(a^+)}^{\aleph(r)}\left(|f(t)|^p\aleph'(t)dt\right)^{\frac{1}{p}}$$
$$\times\left(\left(\int_{\aleph(a^+)}^{\aleph(r)}\tfrac{(\aleph(r)-\aleph(t))^{p(\sigma-1)+1}}{p(\sigma-1)+1}\right)^{\frac{q}{p}}\aleph'(t)dt\right)^{\frac{1}{q}}$$
$$\leqslant \tfrac{1-\sigma}{Q(\sigma)}\|f(u)\|_{X^p} + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)} + \|f\|_{X^p}\tfrac{(\aleph(r)-\aleph(a^+))^{\sigma-1}}{\sigma-1}.$$

Next, we find the Laplace transform of our generalized fractional integral operator.

Theorem 3.2. For $f \in L^1(0,T]$, the modified form of MABC-fractional integral involving generalized M-L function as kernel of order $0 < \sigma < 1$, with respect to $\aleph$, is defined as

$$L_{\aleph}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right) = \frac{1-\sigma}{Q(\sigma)}\frac{s^\sigma+\mu_\sigma}{s^\sigma}L_{\aleph}(f(u)).$$

Proof.

By using the Definition 3.1, we have

$$L_{\aleph}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}f(u)\right) = \tfrac{1-\sigma}{Q(\sigma)}L_{\aleph}(f(u)) + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}L_{\aleph}\left((\aleph(u)-\aleph(a))^{\sigma-1} * f(u)\right)$$
$$= \tfrac{1-\sigma}{Q(\sigma)}L_{\aleph}(f(u)) + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}L_{\aleph}\left((\aleph(u)-\aleph(a))^{\sigma-1}\right)L_{\aleph}(f(u))$$
$$= \tfrac{1-\sigma}{Q(\sigma)}L_{\aleph}(f(u)) + \tfrac{\sigma}{Q(\sigma)s^\sigma}L_{\aleph}(f(u))$$
$$= \tfrac{1-\sigma}{Q(\sigma)}\tfrac{s^\sigma+\mu_\sigma}{s^\sigma}L_{\aleph}(f(u)).$$

This completes the proof.

Theorem 3.3. Let $f' \in L^1(0,T)$, and $0 < \sigma < 1$, then we have the following result.

$${}_{\aleph}^{MABC}\mathfrak{D}_{a^+}^{\sigma}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a^+}^{\sigma}\right)f(u) = f(u) - f(0)\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a))^\sigma). \tag{3.3}$$

Proof. By using the Definition of Laplace transform, we have

$$L_{\aleph}\left({}_{\aleph}^{MABC}\mathfrak{D}_{a^+}^{\sigma}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a}^{\sigma}\right)f(u)\right)$$
$$= \tfrac{Q(\sigma)}{1-\sigma}\tfrac{s^\sigma}{s^\sigma+\mu_\sigma}L_{\aleph}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a}^{\sigma}f(u)\right) - \tfrac{Q(\sigma)}{1-\sigma}\tfrac{s^{\sigma-1}}{s^\sigma+\mu_\sigma}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a}^{\sigma}f(0)\right)$$
$$= \tfrac{s^\sigma}{s^\sigma+\mu_\sigma}\left(\tfrac{s^\sigma+\mu_\sigma}{s^\sigma}\right)L_{\aleph}(f(u)) - \tfrac{s^{\sigma-1}}{s^\sigma+\mu_\sigma}f(0).$$

By using inverse Laplace transform, we obtain

$$\left({}_{\aleph}^{MABC}\mathfrak{D}_{a}^{\sigma}\left({}_{\aleph}^{MABC}\mathfrak{I}_{a}^{\sigma}\right)f(u)\right) = f(u) - f(0)\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(a))^\sigma).$$

Hence the result is proved.

Theorem 3.4. Let $f' \in L^1(0,T), a = 0$ and $0 < \sigma < 1$, then we have the following result.

$${}_{\aleph}^{MABC}\mathfrak{I}_{0}^{\sigma}\left({}_{\aleph}^{MABC}\mathfrak{D}_{0}^{\sigma}\right)f(u) = f(u) - f(0).$$

Proof. By using the Definition 2.1 and 3.1, we have

$${}_{\aleph}^{MABC}\mathfrak{I}_{0}^{\sigma}\left({}_{\aleph}^{MABC}\mathfrak{D}_{0}^{\sigma}\right)f(u) = \tfrac{1-\sigma}{Q(\sigma)}\left({}_{\aleph}^{MABC}\mathfrak{D}_{0}^{\sigma}\right)f(u) + \tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}\int_0^t(\aleph(u)-\aleph(t))^{\sigma-1}$$
$$\times\aleph'(t)\left({}_{\aleph}^{MABC}\mathfrak{D}_{0}^{\sigma}\right)f(t)dt$$
$$= \int_0^u\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt + \tfrac{Q(\sigma)}{1-\sigma}\tfrac{\sigma}{Q(\sigma)\Gamma(\sigma)}\int_0^u(\aleph(u)-\aleph(t))^{\sigma-1}$$
$$\times\aleph'(t)\int_0^t\mathscr{E}_\sigma(-\mu_\sigma(\aleph(t)-\aleph(z))^\sigma)f'(z)dz$$
$$= \int_0^u\mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt + \tfrac{\sigma}{(1-\sigma)\Gamma(\sigma)}$$

$$\times \int_0^u f'(z) \int_z^u (\aleph(u)-\aleph(t))^{\sigma-1}\aleph'(t)\mathscr{E}_\sigma(-\mu_\sigma(\aleph(t)-\aleph(z))^\sigma)dtdz$$

$$= \int_0^u \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt + \tfrac{\sigma}{(1-\sigma)\Gamma(\sigma)}\sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma n+1)}$$

$$\times \int_0^u f'(z)\int_z^u (\aleph(u)-\aleph(t))^{\sigma-1}\aleph'(t)(\aleph(t)-\aleph(z))^{\sigma n}dtdz$$

$$= \int_0^u \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt + \tfrac{\sigma}{(1-\sigma)\Gamma(\sigma)}\sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma n+1)}$$

$$\times \int_0^u f'(z)(\aleph(u)-\aleph(z))^{\sigma(n+1)-1}\int_z^u \left(\tfrac{\aleph(u)-\aleph(t)}{(\aleph(u)-\aleph(z))}\right)^{\sigma-1}\left(\tfrac{\aleph(t)-\aleph(z)}{(\aleph(u)-\aleph(z))}\right)^{\sigma n}\aleph'(t)dtdz$$

$$= \int_0^u \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt + \tfrac{\sigma}{(1-\sigma)\Gamma(\sigma)}\sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma n+1)}$$

$$\times B(\sigma n+1,\sigma)\int_0^u f'(z)(\aleph(u)-\aleph(z))^{\sigma(n+1)}dz$$

$$= \int_0^u \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt + \tfrac{\mu_\sigma}{\Gamma(\sigma)}\sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma(n+1))}$$

$$\times \tfrac{\Gamma(\sigma n+1)\Gamma(\sigma)}{\Gamma(\sigma(n+1)+1)}\int_0^u f'(z)(\aleph(u)-\aleph(z))^{\sigma(n+1)}dz$$

$$= \int_0^u \mathscr{E}_\sigma(-\mu_\sigma(\aleph(u)-\aleph(t))^\sigma)f'(t)dt - \sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^{n+1}}{\Gamma(\sigma(n+1)+1)}$$

$$\times \int_0^u f'(z)(\aleph(u)-\aleph(z))^{\sigma(n+1)}dz$$

$$= \sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma n+1)}\int_0^u f'(t)(\aleph(u)-\aleph(t))^{\sigma n}dt$$

$$-\sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^{n+1}}{\Gamma(\sigma(n+1)+1)}\int_0^u f'(t)(\aleph(u)-\aleph(t))^{\sigma(n+1)}dt$$

$$= \sum_{n=0}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma n+1)}\int_0^u f'(t)(\aleph(u)-\aleph(t))^{\sigma n}dt$$

$$-\sum_{n=1}^{\infty}\tfrac{(-\mu_\sigma)^n}{\Gamma(\sigma(n)+1)}\int_0^u f'(t)(\aleph(u)-\aleph(t))^{\sigma n}dt$$

$$= \int_0^u f'(t)dt = f(u)-f(0).$$

Hence the result is proved.

4. Hyers-Ulam Stability of MABC Fractional Operator Involving Generalized M-L Function

In this section, we will explore the necessary and sufficient conditions for solving fractional differential equations (FDEs) in hybrid systems that incorporate MABC fractional operators with a generalized M-L function as the kernel. Our approach differs from the previous research in [34], which mainly focused on solving FDEs in hybrid systems that include MABC fractional operators and established Hyers-Ulam stability criteria. Instead of using MABC fractional operators, we employ fractional operators that involve a generalized M-L function as the kernel. We will not only find solutions to FDEs but also establish existence, uniqueness and Hyers-Ulam stability criteria for the introduced operators.

$$ {}^{MABC}_{\aleph}\mathfrak{D}_0^{\sigma_i}\left[\psi_i(\upsilon)-\sum_{i=1}^{n}\mathscr{G}_i(\upsilon,\psi_i(\upsilon))\right] = -\chi_i^*(\upsilon,\psi_i(\upsilon)), \upsilon\in I=[0,1] \tag{4.1}$$

$$\psi_i(0)=\zeta_i, \mathscr{G}_i(\upsilon,\psi_i(\upsilon))|_{\upsilon=0}=0, \tag{4.2}$$

where $0<\sigma_i<1, \zeta_i\in\mathbb{R}$, the functions $\psi_i\in C(I)$, with $i=1,2,3,\ldots,n, \chi_i^*, G_i: I\times\mathbb{R}\to\mathbb{R}, (i=1,2,3,\ldots,m)$ are continuous and satisfy the Caratheodory assumptions. ${}^{MABC}_{\aleph}\mathfrak{D}_0^{\sigma_i}$, the MABC fractional operators involving generalized M-L function for $i=1,2,\ldots,n$.

Lemma 4.1. The solution to the n-coupled system of hybrid fractional differential equations containing MABC fractional operators with generalized M-L function given in (4.1) can be expressed as follows:

$$\psi_i(\upsilon)=\zeta_i+\sum_{i=1}^{n}\mathscr{G}_i(\upsilon,\psi_i(\upsilon))+\frac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon,\psi_i(\upsilon))+\frac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}$$
$$\times\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s)ds.$$

Proof. By applying ${}^{MABC}_{\aleph}\mathfrak{I}_0^{\sigma_i}$ to the system of differential equations presented in (4.1) for $i=1,2,\ldots,n$, and utilizing Theorem 3.4, we obtain the following

$$\psi_i(\upsilon)-\sum_{i=1}^{n}\mathscr{G}_i(\upsilon,\psi_i(\upsilon))-\psi_i(0)=({}^{MABC}_{\aleph}\mathfrak{I}_0^{\sigma_i})\chi_i^*(\upsilon,\psi_i(\upsilon)), i$$
$$=1,2,\ldots,n.$$

By the using the conditions $\psi_i(0)=\zeta_i$, we get the following

$$\psi_i(\upsilon)=\zeta_i+\sum_{i=1}^{n}\mathscr{G}_i(\upsilon,\psi_i(\upsilon))+({}^{MABC}_{\aleph}\mathfrak{I}_0^{\sigma_i})\chi_i^*(\upsilon,\psi_i(\upsilon))$$
$$=\zeta_i+\sum_{i=1}^{n}\mathscr{G}_i(\upsilon,\psi_i(\upsilon))+\tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon,\psi_i(\upsilon))$$
$$+\tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s)ds.$$

Hence the result is proved.

To proceed with the primary outcomes of this paper, we assume Banach space.

$\mathbb{B}=\{\psi_i(\upsilon):\psi_i(\upsilon)\in\mathbb{C}([0,1],\mathbb{R}),\upsilon\in[0,1]\}$,

with the norm

$||\psi_i||=\max|\psi_i(\upsilon)|, i=1,2,\ldots,n.$

Assume that $T_i:\mathbb{C}([0,1],\mathbb{R})\to\mathbb{C}([0,1],\mathbb{R})$, with operators for $i=1,2,\ldots,n$, where

$$T_i(\psi_i(\upsilon))=\zeta_i+\sum_{i=1}^{n}\mathscr{G}_i(\upsilon,\psi_i(\upsilon))+\tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon,\psi_i(\upsilon))$$
$$+\tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s))ds. \tag{4.3}$$

Lemma 4.2. Assume that for some $\zeta_i^1,\zeta_i^2\in\mathbb{R}$, and $\psi_i,\bar{\psi}_i\in\mathbb{C}, t\in[0,k]$, we have

$$|\chi_i^*(\upsilon,\psi_i)-\chi_i^*(\upsilon,\bar{\psi}_i)|\leqslant\zeta_i^1|\psi_i-\bar{\psi}_i|,$$
$$|\mathscr{G}_i(\upsilon,\psi_i)-\mathscr{G}_i(\upsilon,\bar{\psi}_i)|\leqslant\zeta_i^2|\psi_i-\bar{\psi}_i|$$

and

$$\eta_i=\sum_{i=1}^{n}\zeta_i^2+\frac{\zeta_i^1}{B(\sigma_i)\Gamma(\sigma_i)}, \tag{4.4}$$

where $\eta_i<1$, for all $i=1,2,\ldots,n$. Now we can say that the solution to the n-coupled hybrid-system of MABC-FDEs given by (4.1) and represented by (4.3) is unique.

Proof. Assume that $\sup_{\upsilon\in[0,k]}|\chi_i^*(\upsilon)| = \rho_1 < \infty$, and $\sup_{\upsilon\in[0,k]}|\mathscr{G}_i(\upsilon,0)| = \rho_2 < \infty$, $S_{\eta_i} = \{\psi_i \in \mathbb{C}([0,k],\mathbb{R}) : ||\psi_i|| < \eta_i\}$, for $k \geqslant 1$ and $i = 1,2,\ldots,n$. For $\psi_i \in S_{\eta_i}$ and $t \in [0,k]$, we have from [34, Lemma 2.2]

$$|\chi_i^*(\upsilon, w(\upsilon))| \leqslant \zeta_i^1\eta_i + \rho_1, \tag{4.5}$$

and for $\psi_i \in S_{\eta_i}, \upsilon \in [0,k]$, we have

$$|\mathscr{G}_i^*(\upsilon, w(\upsilon))| \leqslant \zeta_i^2\eta_i + \rho_2, \tag{4.6}$$

and from 4.3, for $\upsilon \geqslant s$, we have

$$\begin{aligned}|T_i(\psi_i(\upsilon))| = \Bigg|&\zeta_i + \sum_{i=1}^n \mathscr{G}_i(\upsilon,\psi_i(v)) + \tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon,\psi_i(\upsilon))\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s))ds\Bigg|\\ &\leqslant \zeta_i + n(\zeta_i^2\eta_i+\rho_2) + \tfrac{1-\sigma_i}{Q(\sigma_i)}(\zeta_i^1\eta_i+\rho_1)\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s) + (\zeta_i^1\eta_i+\rho_1)ds\\ &\leqslant \zeta_i + n(\zeta_i^2\eta_i+\rho_2) + \left(\tfrac{1-\sigma_i}{Q(\sigma_i)} + \tfrac{1}{Q(\sigma_i)\Gamma(\sigma_i)}\right)(\zeta_i^1\eta_i+\rho_1).\end{aligned}$$

From the given statement, we can conclude that $T_i(S_{\eta_i})$ is a subset of S_{η_i}, where T_i is defined as the mapping from S_{η_i} to itself. Additionally, we assume that ψ_i and φ_i are complex-valued functions belonging to $\mathbb{C}([0,1],\mathbb{R})$, and k is a positive integer. Moreover, it is true that for $\upsilon \geqslant s \in [0,1]$, we have the following inequality:

$$\begin{aligned}|T_i\psi_i(\upsilon) - T_i\varphi_i(\upsilon)| = \Bigg|&\zeta_i + \sum_{i=1}^n \mathscr{G}_i(\upsilon,\psi_i(\upsilon)) + \tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon,\psi_i(\upsilon))\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s))ds\\ &-\zeta_i + \sum_{i=1}^n \mathscr{G}_i(\upsilon,\varphi_i(\upsilon)) + \tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon,\varphi_i(\upsilon))\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon}(\aleph(\upsilon)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i^*(s))ds\Bigg|\\ &\leqslant \sum_{i=1}^n \zeta_i^2|\psi_i-\varphi_i| + \tfrac{1-\sigma_i}{Q(\sigma_i)}\zeta_i^1|\psi_i-\varphi_i| + \tfrac{1}{B(\sigma_i)\Gamma(\sigma_i)}\zeta_i^1|\psi_i-\varphi_i|.\end{aligned}$$

Given that η_i's defined in (4.4) are less than 1, the operators T_i are contractions. Using the Banach fixed point theorem, we can conclude that the n-coupled hybrid system of MABC-FDEs given by (4.1) has a unique solution, which can be obtained as fixed points of the operators T_i, where $i = 1,2,\ldots,n$.

Theorem 4.1. Suppose that the condition of the Lemma 4.2 holds then, the hybrid m-coupled-system MABC-FDEs (4.1) has a solution (4.3).

Proof. Based on the assumptions made in Lemma 4.2, we can conclude that the operators T_i are bounded for $i = 1,2,\ldots,n$, and for $\upsilon_1,\upsilon_2 \in [0,k]$ with $\upsilon_2 > \upsilon_1$, where $k \leq 1$. Now consider the following:

$$\begin{aligned}|T_i\psi_i(\upsilon_2) - T_i\psi_i(\upsilon_1)| = \Bigg|&\zeta_i + \sum_{i=1}^n \mathscr{G}_i(\upsilon_2,\psi_i(\upsilon_2)) + \tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon_2,\psi_i(\upsilon_2))\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon_2}(\aleph(\upsilon_2)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s))ds\\ &-\zeta_i + \sum_{i=1}^n \mathscr{G}_i(\upsilon_1,\psi_i(\upsilon_1)) + \tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon_1,\psi_i(\upsilon_1))\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon_1}(\aleph(\upsilon_1)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i^*(s))ds\Bigg|\\ \leqslant \sum_{i=1}^n&|\mathscr{G}_i(\upsilon_2,\psi_i(\upsilon_2)) - \mathscr{G}_i(\upsilon_1,\psi_i(\upsilon_1))| + \tfrac{1-\sigma_i}{Q(\sigma_i)}|\chi_i^*(\upsilon_2,\psi_i(\upsilon_2)) - \chi_i^*(\upsilon_1,\psi_i(\upsilon_1))|\\ &+ \tfrac{1}{Q(\sigma_i)\Gamma(\sigma_i)}|(\aleph(\upsilon_2)-\aleph(a^+))^{\sigma} - (\aleph(\upsilon_1)-\aleph(a^+))^{\sigma_i}|(\zeta_i^1|\psi_i-\varphi_i).\end{aligned}$$

As $\upsilon_2 \to \upsilon_1$, we have $T_i\psi_i(\upsilon_2) - T_i\psi_i(\upsilon_1) = 0$. Therefore, $|T_i\psi_i(\upsilon_2) - T_i\psi_i(\upsilon_1)| \to 0$, as $\upsilon_2 \to \upsilon_1$. Hence, we can say that the operators T_i are equicontinuous for $i = 1,2,\ldots,n$ and for $s \leqslant t$. Moreover, for $u \in u \in \mathbb{C}([0,k],\mathbb{R}) : u = \hbar T_i(u)$, *for*, $\hbar \in [0,1]$, we have the following:

$$\begin{aligned}||\psi_i|| = \max_{t\in I}|T_i\psi| = \Bigg|&\zeta_i + \sum_{i=1}^n \mathscr{G}_i(\upsilon_2,\psi_i(\upsilon_2)) + \tfrac{1-\sigma_i}{Q(\sigma_i)}\chi_i^*(\upsilon_2,\psi_i(\upsilon_2))\\ &+ \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)}\int_{a^+}^{\upsilon_2}(\aleph(\upsilon_2)-\aleph(s))^{\sigma_i-1}\aleph'(s)\chi_i^*(s,\psi_i(s))ds\Bigg|\\ &\leqslant \zeta_i + \sum_{i=1}^n(\zeta_i^2||\psi_i|| + \rho_2) + \tfrac{1-\sigma_i}{Q(\sigma_i)}(\zeta_i^1||\psi_i|| + \rho_1)\\ &+ \tfrac{(\zeta_i^1||\psi_i||+\rho_1)}{Q(\sigma_i)\Gamma(\sigma_i)}\\ &= \chi_i^1 + \chi_i^2||\psi_i||,\end{aligned} \tag{4.7}$$

where

$$\chi_i^1 = \zeta_i + \rho_2 + \frac{1-\sigma_i}{Q(\sigma_i)}\rho_1 + \frac{\rho_1}{Q(\sigma_i)\Gamma(\sigma_i)}$$

and

$$\chi_i^2 = \sum_{i=1}^n \zeta_i^2 + \frac{1-\sigma_i}{Q(\sigma_i)}\zeta_i^1 + \frac{\zeta_i^1}{Q(\sigma_i)\Gamma(\sigma_i)}.$$

For $i = 1,2,\ldots,n$. From (4.7), we have

$$||\psi_i|| \leqslant \frac{\chi_i^1}{1-\chi_i^2}, \quad i = 1,2,\ldots,n. \tag{4.8}$$

Therefore, we can apply Leray–Schauder's alternative theorem and conclude that (4.1) has a solution.

Every fixed point of T_i corresponds to a solution of the system of differential equations given in (4.1).

Next, we will establish the Hyers-Ulam stability (HU stability) criteria for our operator. To do this, we will use the following definition from [34].

Definition 4.1. The coupled integral system (4.3) is considered to be HU stable if, for some $\zeta_i > 0$, we have $\Delta_i > 0$ and ψ_i satisfies

$$||\psi_i - T\psi_i|| < \Delta_i, \tag{4.9}$$

where

$$\bar{\psi}_i(\upsilon) = T\bar{\psi}_i(\upsilon) \tag{4.10}$$

and

$$||\psi_i(\upsilon) - \bar{\psi}_i(\upsilon)|| < \Delta_i\zeta_i,\ i = 12,\ldots,n. \tag{4.11}$$

Theorem 4.2. Assuming that the conditions of Lemma 4.2 hold, we can conclude that the hybrid system of MABC-FDEs given by (4.1) is stable. Alternatively, we can say that the HU is stable.

Proof. Assuming that $\psi_i \in \mathbb{C}$ for $i = 1, 2, \ldots$with the property (4.9), and let $\psi_i^* \in \mathbb{C}$ be the solution for the coupled-system (4.1) satisfying (4.3). Then, we can conclude that

$$\begin{aligned} |T_i\psi_i(\upsilon) - T_i\psi_i^*(\upsilon)| = \Bigg| & \zeta_i + \sum_{i=1}^{n} \mathscr{G}_i(\upsilon, \psi_i(\upsilon)) + \tfrac{1-\sigma_i}{Q(\sigma_i)} \chi_i^*(\upsilon, \psi_i(\upsilon)) \\ & + \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)} \int_{a^+}^{\upsilon} (\aleph(\upsilon) - \aleph(s))^{\sigma_i - 1} \aleph'(s) \chi_i^*(s, \psi_i(s)) ds \\ & - \zeta_i + \sum_{i=1}^{n} \mathscr{G}_i(\upsilon, \psi_i^*(\upsilon)) + \tfrac{1-\sigma_i}{Q(\sigma_i)} \chi_i^*(\upsilon, \psi_i^*(\upsilon)) \\ & + \tfrac{\sigma_i}{Q(\sigma_i)\Gamma(\sigma_i)} \int_{a^+}^{\upsilon} (\aleph(\upsilon) - \aleph(s))^{\sigma_i - 1} \aleph'(s) \chi_i^*(s, \psi_i^*(s)) ds \Bigg| \\ \leqslant & \sum_{i=1}^{n} \zeta_i^2 |\psi_i - \psi_i^*| + \tfrac{1-\sigma_i}{Q(\sigma_i)} \zeta_i^1 |\psi_i - \psi_i^*| + \tfrac{1}{B(\sigma_i)\Gamma(\sigma_i)} \zeta_i^1 |\psi_i - \psi_i^*|. \end{aligned}$$

For $\eta_i < 1$, where $\eta_i' s$ are given by (4.4), for $i = 1, 2, \ldots, n$. By the (4.9), (4.10) and (4.11), consider the following norm

$$\begin{aligned} ||\psi_i - \bar{\psi}_i^*|| &= ||\psi_i - T_i\psi_i + T_i\psi_i - \bar{\psi}_i^*|| \\ &\leqslant ||\psi_i - T_i\psi_i|| + ||T_i\psi_i - \bar{\psi}_i^*|| \\ &\leqslant \Delta_i + \eta_i ||\psi_i - \bar{\psi}_i^*||, \end{aligned}$$

where $i = 1, 2, \ldots, m$. Furthermore

$$||\psi_i - \bar{\psi}_i^*|| \leqslant \frac{\Delta_i}{1 - \eta_i}$$

with $\zeta_i = \frac{1}{1-\eta_i}$. Therefore, we can conclude that the coupled system (4.3) is stable, which further implies the stability of the coupled hybrid MABC-FDEs system (4.1).

5. Conclusion

The FOs introduced in this are the extended forms of MAB fractional integral and derivative in Caputo sense. The space of these operators is more wider than Hölder space. By using these new operators, we established some differential equations and explored their solutions by using generalized Laplace transform. Such differential equations are not solvable with ABC fractional derivative. The defined operators are proved bounded in X^p with norm. The Laplace transform of both the FOs is evaluated. The inverse property of the operators exists with a condition $f(0) = 0$. The existence, uniqueness and stability in Hyers-Ulam sense for the Cauchy model involving generalized MABC fractional derivative operator is proved. The presented work proved the importance of the role of space in fractional calculus. This work motivate the researchers to explore such modified forms that are helpful to model new differential equations and find out the ways to explore their solutions.

6. Availability of data and material

Not data were used to this study.

7. Funding

Not applicable.

8. Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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