AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 5303–5317.
DOI: 10.3934/math.2023266
Received: 27 October 2022
Revised: 08 December 2022
Accepted: 08 December 2022
Published: 14 December 2022

Research article

Monotonicity and extremality analysis of difference operators in Riemann-Liouville family

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Abstract: In this paper, we will discuss the monotone decreasing and increasing of a discrete nonpositive and nonnegative function defined on $\mathbb{N}_{r_0+1}$, respectively, which come from analysing the discrete Riemann-Liouville differences together with two necessary conditions (see Lemmas 2.1 and 2.3). Then, the relative minimum and relative maximum will be obtained in view of these results combined with another condition (see Theorems 2.1 and 2.2). We will modify and reform the main two lemmas by replacing the main condition with a new simpler and stronger condition. For these new lemmas, we will establish similar results related to the relative minimum and relative maximum again. Finally, some examples, figures and tables are reported to demonstrate the applicability of the main lemmas. Furthermore, we will clarify that the first condition in the main first two lemmas is solely not sufficient for the function to be monotone decreasing or increasing.

Keywords: Riemann-Liouville discrete operators; monotonicity analysis; extremality analysis
Mathematics Subject Classification: 39A12, 39B62, 33B10, 26A48, 26A51

1. Introduction

For $r_0 \in \mathbb{R}$ consider the function f from $\{r_0, r_0+1, r_0+2, \ldots\}$ to $\mathbb{R}$. Recall that the nabla first order difference operator is defined by

$$(\nabla f)(t) := f(t) - f(t-1), \, t \in \{r_0+1, a+2, \ldots\}.$$

It is then almost a triviality that the following implication holds:

$$f \text{ is increasing on } \{r_0, r_0+1, r_0+2, \ldots\} \Longleftrightarrow (\nabla f)(t) \geqq 0 \text{ for } t \in \{r_0+1, r_0+2, \ldots\}. \tag{1.1}$$

Consequently, in this instance there exists a clear connection between the sign of the difference and the monotone behavior (decreasing or increasing) of the function on which the difference acts. Note that in view of (1.1), there is a relationship between the positivity and monotonicity of the function f as $(\nabla f)(t) \geqq 0$, we have that f is increasing. Moreover, there is a relationship between the positivity and convexity of the function f as $\left(\nabla^2 f\right)(t) \geqq 0$, we have that f is convex. However, there is no such a relationship between the monotonicity and convexity of a function because monotonicity is defined based on ∇ whereas convexity is defined based on ∇^2. For more details please see the article [1].

In 2007, Atici and Eloe [2–4] as well as the subsequent work of Abdeljawad et al. [5], Abdeljawad and Atici [6], Abdeljawad and Baleanu [7], Mohammed et al. [8], Chen et al. [9], Ferreira [10], Holm [11], and Wu and Baleanu [12] employed difference operators to develop the concept of discrete fractional calculus. In particular, there has been increasing interest in a nonlocal version of the difference calculus, that is, “discrete fractional calculus”. For this reason and a wealth of additional information on a variety of nonlocal discrete operators and their properties, we refer to the great monograph in [13] by Goodrich and Peterson.

A particularly curious and mathematically nontrivial aspect of this theory is that there is not a clean correlation between the sign of a discrete fractional operator and the monotone (or positive or convex) behavior of the function on which the operator acts. In fact, as has been shown time and time again, there is a highly complex and subtle relationship. This mathematically rich behavior was first documented in the monotonicity case by Dahal and Goodrich [14] in 2014. Since their initial work, numerous other studies have been published, including those by Atici and Uyanik [15], Jia et al. [16], Bravo et al. [17, 18], Dahal and Goodrich [19], Ahrendt et al. [20], Erbe et al. [21], Goodrich [22, 23], Goodrich et al. [24], Goodrich and Lizama [25, 26], Goodrich et al. [27], Goodrich and Muellner [28], Mohammed et al. [29, 30], Liu et al. [31] and Guirao et al. [32]. These papers investigate a variety of questions surrounding the qualitative properties inferred from the sign of a fractional difference acting on a function.

Above and beyond the pure mathematical interest in this type of problem, there exists a compelling practical reason to care. In the application of both the continuous and the discrete calculus, the ability of the difference (or derivative) to detect when a function is increasing or decreasing is of paramount importance. Specifically, discrete operators have favorable shape-preserving properties and studying them is crucial for applications in monotonicity analysis. Thus, clarifying this aspect of the theory of fractional difference operators is important. This is particularly the case since there have been some initial attempts to apply discrete fractional calculus to biological modeling; see, for example, Atici and Şengül in [33], Atici et al. in [34], and Atici et al. in [35].

The aim of the present work is to analyse the discrete nabla fractional difference operators of Riemann-Liouville type and to obtain the monotone increasing and decreasing as outcome of the analyses. These allow us to establish the relative minimum and maximum of the functions at certain points. Furthermore, we will modify the main lemmas by finding a new stronger and simpler condition, and then we will rearrange the relative minimum and maximum results by considering the new lemmas. Later, we will discuss our main results with two examples presented via tables and figures.

The layout of this paper is as follows. We recall in Section 2 the main principles to make the paper self-contained and we also divide the main results of this article into two subsections. Subsection 2.1 explores the monotone decreasing and relative minimum of the discrete operators and studies main lemmas. In Subsection 2.2 we examine the monotone increasing and relative maximum of the discrete operators. We discuss the application of the main lemmas in Section 3 including two examples. Our final section is dedicated to the concluding remarks on the main results, and we state an open problem for the interested readers.

2. Preliminaries and main results

Let us start with recalling some definitions and facts whose applications and further detailed accounts can be found in [13, 31, 36, 37].

Let f be a discrete function on $\mathbb{N}_{r_0} := \{r_0, r_0 + 1, r_0 + 2, \ldots\}$, $0 < \alpha$ be the order of discrete nabla operators, and a be a real number. Then, the nabla fractional sum operator is defined by the following identity:

$$\left({}_{r_0}\nabla^{-\alpha} f\right)(t) := \sum_{r=r_0+1}^{t} \frac{(t-r+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(r), \quad \text{for } t \in \mathbb{N}_{r_0+1}, \tag{2.1}$$

respectively, where $t^{\overline{\alpha}}$ is defined by

$$t^{\overline{\alpha}} := \Gamma(t+\alpha)/\Gamma(t),$$

for those values of t and α such that $\Gamma(t+\alpha)/\Gamma(t)$ is well defined.

Furthermore, for $\ell - 1 < \alpha < \ell$, the nabla fractional differences of Riemann-Liouville type of order α can be expressed as follows (see [31, Lemma 2.1]):

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{\mathrm{r}=r_0+1}^{t} (t-\mathrm{r}+1)^{\overline{-\alpha-1}} f(\mathrm{r}), \tag{2.2}$$

for $t \in \mathbb{N}_{r_0+\ell}$.

Next, we have our main results for the monotone decreasing and monotone increasing functions.

2.1. Decreasing part

Lemma 2.1. *For $0 < \alpha < 1$ and a nonnegative function f on $\mathbb{N}_{r_0}$, let the following conditions hold:*

(i) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0 + 2) \leq 0,$

$$\text{(ii)} \qquad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \le \left(\sum_{k=1}^{t-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0+1), \quad t \in \mathbb{N}_3.$$

Then, $(\nabla f)(t) \le 0$ *for each* $t \in \mathbb{N}_{r_0+2}$*, i.e.,* f *is decreasing on* $\mathbb{N}_{r_0+2}$*.*

Proof. Firstly, we see that $\sum_{k=1}^{t-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}$ is negative for each $t \in \mathbb{N}_3$, as follows:

$$\frac{\Gamma(k+1-\alpha)}{(k+1)!\Gamma(-\alpha)} = \frac{\overbrace{(k-\alpha)}^{>0}\overbrace{(k-1-\alpha)}^{>0}\cdots\overbrace{(1-\alpha)}^{>0}\overbrace{(-\alpha)}^{<0}}{(k+1)!} < 0. \tag{2.3}$$

Compute the first condition to have

$$\begin{aligned}\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) &= \frac{1}{\Gamma(-\alpha)} \sum_{\mathrm{r}=r_0+1}^{r_0+2} (r_0+3-\mathrm{r})^{\overline{-\alpha-1}} f(\mathrm{r}) \\ &= f(r_0+2) - \alpha f(r_0+1).\end{aligned}$$

From this we conclude that

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) - (\nabla f)(r_0+2) = (1-\alpha) f(r_0+1) \ge 0,$$

and this leads to

$$(\nabla f)(r_0+2) \le \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) \underset{\text{condition (i)}}{\overset{by}{\le}} 0. \tag{2.4}$$

Assume that $(\nabla f)(r_0+t) \le 0$ for all $t \in \mathbb{N}_2^{K_0+1}$, when $K_0 \in \mathbb{N}_1$. Then we are planning to show that $(\nabla f)(r_0+K_0+2) \le 0$. To do this, we consider the definition (2.2) to have

$$\begin{aligned}\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) =& \frac{1}{\Gamma(-\alpha)} \sum_{\mathrm{r}=r_0+1}^{r_0+K_0+2} (r_0+K_0+3-\mathrm{r})^{\overline{-\alpha-1}} f(\mathrm{r}) \\ =& \frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!} f(r_0+1) + \frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!} f(r_0+2) \\ &+ \cdots + \frac{(-\alpha)(1-\alpha)}{2} f(r_0+K_0) + (-\alpha) f(r_0+K_0+1) + f(r_0+K_0+2).\end{aligned}$$

It follows that

$$\begin{aligned}&\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) - (\nabla f)(r_0+K_0+2) \\ =& \frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!} f(r_0+1) + \frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!} f(r_0+2) + \cdots \\ &+ \frac{(-\alpha)(1-\alpha)}{2} f(r_0+K_0) + (1-\alpha) f(r_0+K_0+1),\end{aligned} \tag{2.5}$$

and consequently,

$$
\begin{aligned}
(\nabla f)(r_0+K_0+2) \leq & \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) - \frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!} f(r_0+1) - \frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!} f(r_0+1) + \cdots \\
& - \frac{(-\alpha)(1-\alpha)}{2} f(r_0+1) \underbrace{-(1-\alpha)f(r_0+1)}_{\leq 0} \\
\leq & \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) - \left(\sum_{k=1}^{(K_0+2)-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0+1) \\
\leq & 0, \qquad (2.6)
\end{aligned}
$$

where we have used condition (ii), (2.3), $t = K_0 + 2 \in N_3$, and the hypothesis nonnegativity of f:

$$f(r_0+K_0+1) \leq f(r_0+K_0) \leq \cdots \leq f(r_0+1).$$

Hence, the inequality (2.4) combined with (2.6) gives us the required result. □

As a consequence of Lemma 2.1, we obtain the following relativity (min) result.

Theorem 2.1. *For* $0 < \alpha < 1$ *and a nonnegative function* f *on* $\mathbb{N}_{r_0}$*, let the following conditions hold:*

(i) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) \leq 0,$

(ii) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \leq \left(\sum_{k=1}^{t-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0+1), \quad t \in \mathbb{N}_3^{K_0},$

(iii) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) \geq (1-\alpha) f(r_0+K_0+1),$

for a fixed $K_0 \in \mathbb{N}_4$*. Then* f *is a relative minimum at* $r_0 + K_0 + 1$*.*

Proof. The conditions (i) and (ii) tell us that $(\nabla f)(t) \leq 0$ for each $t \in \mathbb{N}_{r_0+2}^{r_0+K_0+1}$ when $K_0 \geq 1$, as according to Lemma 2.1. Particularly, we have $(\nabla f)(r_0+K_0+1) \leq 0$. To achieve our required result, we claim that $(\nabla f)(r_0+K_0+2) \geq 0$, i.e., f is a relative minimum at r_0+K_0+1. Rearrange (2.5) again to have

$$
\begin{aligned}
& (\nabla f)(r_0+K_0+2) - \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) \\
= & - \frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!} f(r_0+1) - \frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!} f(r_0+2) - \cdots \\
& - \frac{(-\alpha)(1-\alpha)}{2} f(r_0+K_0) - (1-\alpha) f(r_0+K_0+1) \\
\geq & -(1-\alpha) f(r_0+K_0+1), \qquad (2.7)
\end{aligned}
$$

where we have used

$$- \frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!} f(r_0+1) - \frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!} f(r_0+2) - \cdots - \frac{(-\alpha)(1-\alpha)}{2} f(r_0+K_0) \geq 0,$$

by using (2.3) and the nonnegativity of f. In other words, we can write (2.7) as follows:

$$(\nabla f)(r_0+K_0+2) \geq \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) - (1-\alpha) f(r_0+K_0+1) \underset{\text{(iii)}}{\overset{by}{\geq}} 0,$$

which is the result as claimed. Hence, f is a relative minimum at $r_0 + K_0 + 1$. □

We always look for a stronger condition than the existing one. For this reason, in the lemma below, we use another condition instead of condition (ii) of Lemma 2.1, which is simpler than (ii) as well.

Lemma 2.2. *Assume that* $0 < \alpha < 1$ *and* f *is a nonnegative function on* $\mathbb{N}_{r_0}$. *Then, condition (ii) of Lemma 2.1 leads to the following new condition:*

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \leq \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1),$$

for $t \in \mathbb{N}_3$.

Proof. First, for $k = 2$, we see that $\frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}$ leads to

$$\frac{\Gamma(3-\alpha)}{\Gamma(-\alpha)(3!)} = \frac{(-\alpha)(1-\alpha)(2-\alpha)}{6} > \frac{(-\alpha)(1-\alpha)}{2},$$

for $\alpha > -1$. We proceed with it to show that

$$0 > \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!} > \frac{(-\alpha)(1-\alpha)}{2}, \tag{2.8}$$

for all $k \geq 2$. The first inequality is clear given (2.3). For the second one, we assume that

$$\frac{\Gamma(k_0+1-\alpha)}{\Gamma(-\alpha)(k_0+1)!} > \frac{(-\alpha)(1-\alpha)}{2},$$

for all $k_0 \geq 2$. Then, we see that

$$\begin{aligned} \frac{\Gamma(k_0+2-\alpha)}{\Gamma(-\alpha)(k_0+2)!} &> \frac{k_0+1-\alpha}{k_0+2} \cdot \frac{\Gamma(k_0+1-\alpha)}{\Gamma(-\alpha)(k_0+1)!} \\ &\geq \underbrace{\frac{k_0+1-\alpha}{k_0+2}}_{0<\uparrow<1} \cdot \underbrace{\frac{(-\alpha)(1-\alpha)}{2}}_{<0} \\ &> \frac{(-\alpha)(1-\alpha)}{2}. \end{aligned}$$

Therefore, the inequalities expressed in (2.8) are valid for all $k \geq 2$ as we claimed. We know that $f(r_0+1) > 0$, and hence,

$$\begin{aligned} 0 &> \left(\sum_{k=1}^{t-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0+1) \\ &> \left(\sum_{k=1}^{t-2} \frac{(-\alpha)(1-\alpha)}{2}\right) f(r_0+1) \\ &= \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1). \end{aligned}$$

Thus, we can deduce that if

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \leq \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1), \quad t \in \mathbb{N}_3,$$

then, condition (ii) of Lemma 2.1 will be satisfied. Hence, the proof is complete. □

Corollary 2.1. *For* $0 < \alpha < 1$ *and a nonnegative function* f *on* $\mathbb{N}_{r_0}$*, let the following conditions hold:*

$$\begin{aligned}
&\text{(i)} \qquad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) \le 0,\\
&\text{(ii)} \qquad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \le \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1), \quad t \in \mathbb{N}_3^{K_0},\\
&\text{(iii)} \qquad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) \ge (1-\alpha)f(r_0+K_0+1),
\end{aligned}$$

for a fixed $K_0 \in \mathbb{N}_4$*. Then,* f *is a relative minimum at* $r_0 + K_0 + 1$.

Proof. The proof obviously follows from Theorem 2.1 and Lemma 2.2 since the condition (ii) of Lemma 2.1 is implied by the condition (ii) of this corollary. □

2.2. Increasing part

Lemma 2.3. *For* $0 < \alpha < 1$ *and a nonpositive function* f *on* $\mathbb{N}_{r_0}$*, let the following conditions hold:*

$$\begin{aligned}
&\text{(i)} \qquad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) \ge 0,\\
&\text{(ii)} \qquad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \ge \left(\sum_{k=1}^{t-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0+1), \quad t \in \mathbb{N}_3.
\end{aligned}$$

Then, $(\nabla f)(t) \ge 0$ *for each* $t \in \mathbb{N}_{r_0+2}$*, i.e.,* f *is increasing on* $\mathbb{N}_{r_0+2}$.

Proof. First, we compute the first condition to get

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) = f(r_0+2) - \alpha f(r_0+1).$$

This gives that

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) - (\nabla f)(r_0+2) = (1-\alpha) f(r_0+1) \le 0,$$

and it follows that

$$(\nabla f)(r_0+2) \ge \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) \underset{\text{condition (i)}}{\overset{by}{\ge}} 0. \tag{2.9}$$

Now we assume that $(\nabla f)(r_0+t) \ge 0$ for all $t \in \mathbb{N}_2^{K_0+1}$ when $K_0 \in \mathbb{N}_1$. Then, we will try to show that $(\nabla f)(r_0+K_0+2) \le 0$. To do this, we recall the identity (2.5), which leads to

$$\begin{aligned}
&\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) - (\nabla f)(r_0+K_0+2)\\
=&\frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!} f(r_0+1) + \frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!} f(r_0+2) + \cdots\\
&+ \frac{(-\alpha)(1-\alpha)}{2} f(r_0+K_0) + (1-\alpha) f(r_0+K_0+1),
\end{aligned} \tag{2.10}$$

and the nonpositivity of f, which leads to

$$f(r_0+K_0+1) \le f(r_0+K_0) \le \cdots \le f(r_0+1).$$

Combining these we can deduce that

$$\begin{aligned}(\nabla f)(r_0+K_0+2) \geq & \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) \\ & -\frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!}f(r_0+1)-\frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!}f(r_0+1)+\cdots \\ & -\frac{(-\alpha)(1-\alpha)}{2}f(r_0+1)\underbrace{-(1-\alpha)f(r_0+1)}_{\geq 0} \\ \geq & \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2)-\left(\sum_{k=1}^{(K_0+2)-2}\frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right)f(r_0+1) \\ \geq & 0, \end{aligned} \tag{2.11}$$

where we have used condition (ii). Therefore, the inequality (2.9) combined with (2.11) when $t = K_0+2 \in N_3$, proves the desired result. □

As a consequence of Lemma 2.3, we obtain the following relativity (max) result.

Theorem 2.2. *For* $0<\alpha<1$ *and a nonpositive function* f *on* $\mathbb{N}_{r_0}$*, let the following conditions hold:*

(i) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+2) \geq 0,$

(ii) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+t) \geq \left(\sum_{k=1}^{t-2}\frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right)f(r_0+1), \quad t\in\mathbb{N}_3^{K_0},$

(iii) $\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) \leq (1-\alpha)f(r_0+K_0+1),$

for a fixed $K_0\in\mathbb{N}_4$*. Then,* f *is a relative max at* r_0+K_0+1*.*

Proof. The conditions (i) and (ii) enable us to have $(\nabla f)(t) \geq 0$ for each $t \in \mathbb{N}_{r_0+2}^{r_0+K_0+1}$, when $K_0 \geq 1$, according to Lemma 2.3. Therefore, $(\nabla f)(r_0+K_0+1) \geq 0$. Let us claim that $(\nabla f)(r_0+K_0+2) \geq 0$, i.e., f is a relative max at r_0+K_0+1. We rearrange (2.10); we have

$$\begin{aligned} & (\nabla f)(r_0+K_0+2)-\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2) \\ = & -\frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!}f(r_0+1)-\frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!}f(r_0+2)-\cdots \\ & -\frac{(-\alpha)(1-\alpha)}{2}f(r_0+K_0)-(1-\alpha)f(r_0+K_0+1) \\ \leq & -(1-\alpha)f(r_0+K_0+1), \end{aligned} \tag{2.12}$$

where by using (2.3) and the nonpositivity of f the following is used:

$$-\frac{\Gamma(K_0+1-\alpha)}{\Gamma(-\alpha)(K_0+1)!}f(r_0+1)-\frac{\Gamma(K_0-\alpha)}{\Gamma(-\alpha)K_0!}f(r_0+2)-\cdots-\frac{(-\alpha)(1-\alpha)}{2}f(r_0+K_0) \leq 0.$$

Then, we can express (2.12) as follows:

$$(\nabla f)(r_0+K_0+2) \leq \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0+K_0+2)-(1-\alpha)f(r_0+K_0+1) \underset{\text{(iii)}}{\overset{by}{\leq}} 0,$$

which is our result as claimed. Therefore, f is a relative max at r_0+K_0+1. □

As in Lemma 2.2, we obtain a simpler and stronger condition in the following lemma.

Lemma 2.4. *Assume that* $0 < \alpha < 1$ *and* f *is a nonpositive function on* $\mathbb{N}_{r_0}$. *Then, condition (ii) of Lemma 2.2 enable the following condition:*

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0 + t) \geq \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1),$$

for $t \in \mathbb{N}_3$.

Proof. From (2.8), the following inequalities are obvious:

$$0 > \frac{\Gamma(k+1-\alpha)}{(k+1)!\Gamma(-\alpha)} > \frac{(1-\alpha)(-\alpha)}{2}, \tag{2.13}$$

for all $k \geq 2$. The first inequality is given by (2.3). For the second one, we assume that

$$\frac{\Gamma(k_0+1-\alpha)}{\Gamma(-\alpha)(k_0+1)!} > \frac{(-\alpha)(1-\alpha)}{2},$$

for all $k_0 \geq 2$. We know that $f(r_0+1) < 0$ from the hypothesis; thus,

$$\begin{aligned} 0 &< \left(\sum_{k=1}^{t-2} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0+1) \\ &< \left(\sum_{k=1}^{t-2} \frac{(-\alpha)(1-\alpha)}{2}\right) f(r_0+1) \\ &= \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1). \end{aligned}$$

Therefore, we can deduce that if

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0 + t) \geq \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1), \quad t \in \mathbb{N}_3,$$

then condition (ii) of Lemma 2.2 holds true. This ends the proof. □

Corollary 2.2. *For* $0 < \alpha < 1$ *and a nonpositive function* f *on* $\mathbb{N}_{r_0}$, *let the following conditions hold:*

$$\begin{aligned} &\text{(i)} \quad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0 + 2) \geq 0, \\ &\text{(ii)} \quad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0 + t) \geq \frac{(-\alpha)(1-\alpha)}{2}(t-2)f(r_0+1), \quad t \in \mathbb{N}_3^{K_0}, \\ &\text{(iii)} \quad \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(r_0 + K_0 + 2) \leq (1-\alpha)f(r_0+K_0+1), \end{aligned}$$

for a fixed $K_0 \in \mathbb{N}_4$. *Then,* f *is a relative max at* $r_0 + K_0 + 1$.

Proof. The proof follows from Theorem 2.2 and Lemma 2.4 immediately because the condition (ii) of Lemma 2.3 is implied by the condition (ii) of this corollary. □

3. Example explanations

In this section, we report the application results by using the main findings of Subsections 2.1 and 2.2. Throughout this section, we use the following notations:

$$A(t) := \left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(t + r_0),$$
$$B(t) := \left(\sum_{k=1}^{t-1} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(r_0 + 1).$$

Example 3.1. Let $\alpha = 0.9$, $r_0 = 0$, and f be defined by

$$f(t) = \left(\frac{1}{2}\right)^{t-r_0}, \quad \text{for } t \in \mathbb{N}_{r_0+2}.$$

It is obvious that $f(t)$ is nonnegative and

$$\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(2) = \frac{1}{\Gamma(-0.9)} \sum_{\mathrm{r}=1}^{2} (3-\mathrm{r})^{\overline{-1.9}} f(\mathrm{r}) = -0.9 f(1) + f(2) = \frac{-1}{5} \leq 0.$$

Furthermore, according to the numerical results reported in Table 1 and Figure 1, we see that

$$\left({}^{\mathrm{RL}}_{0}\nabla^{\alpha} f\right)(t) \leq \left(\sum_{k=1}^{t-1} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!}\right) f(1),$$

for $t = 3, 4, 5$. Thus, all conditions of the statement of Theorem 2.1 are verified; hence, the function will be decreasing on $\{2, 3, 4, 5\}$.

Table 1. Comparison of $A(t)$ and $B(t)$ values.

	$t = r_0 + 2$	$t = r_0 + 3$	$t = r_0 + 4$	$t = r_0 + 5 \cdots$
$A(t)$	0	$-\frac{49}{400}$	$-\frac{139}{2000}$	$-\frac{114}{2917} \cdots$
$B(t)$	0	$-\frac{3}{400}$	$-\frac{123}{4000}$	$-\frac{95}{2708} \cdots$

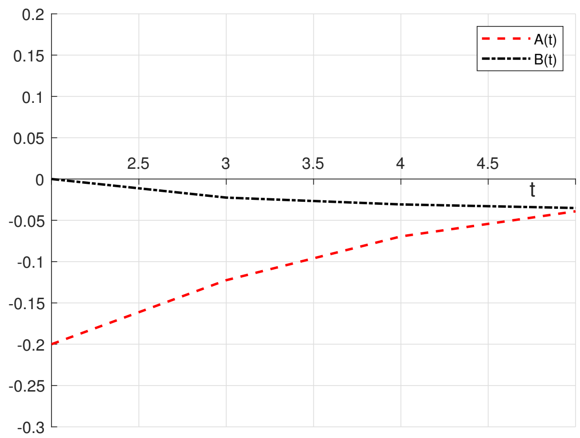

Figure 1. Graph of $A(t)$ and $B(t)$ for different values of t.

Example 3.2. Let $\alpha = 0.85$, $r_0 = 0$, and f be defined by

$$f(t) = \left(\frac{-3}{4}\right)^{t-r_0}, \quad \text{for } t \in \mathbb{N}_{r_0+2}.$$

It is obvious that $f(t)$ is nonnegative and

$$\begin{aligned}\left({}^{\mathrm{RL}}_{r_0}\nabla^{\alpha} f\right)(2) &= \frac{1}{\Gamma(-0.85)} \sum_{\mathrm{r}=1}^{2} (3-\mathrm{r})^{\overline{-1.85}} f(\mathrm{r}) \\ &= -0.85 f(1) + f(2) = \frac{3}{40} \geq 0.\end{aligned}$$

Moreover, the numerical results reported in Table 2 and Figure 2 tell us that

$$\left({}^{\mathrm{RL}}_{0}\nabla^{\alpha} f\right)(t) \leq \left(\sum_{k=1}^{t-1} \frac{\Gamma(k+1-\alpha)}{\Gamma(-\alpha)(k+1)!} \right) f(1),$$

for $t = 3, 4, 5$. Therefore, all conditions of the statement of Theorem 2.2 are verified. Hence, the function will be decreasing on the set $\{2, 3, 4, 5\}$.

Table 2. Comparison of $A(t)$ and $B(t)$ values.

	$t = r_0 + 2$	$t = r_0 + 3$	$t = r_0 + 4$	$t = r_0 + 5 \cdots$
$A(t)$	$\frac{3}{40}$	$\frac{333}{3200}$	$\frac{771}{8000}$	$\frac{57}{694} \cdots$
$B(t)$	0	$\frac{153}{3200}$	$\frac{285}{4309}$	$\frac{113}{1487} \cdots$

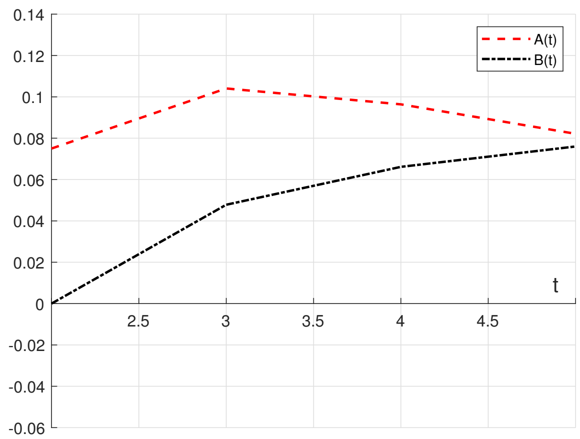

Figure 2. Graph of $A(t)$ and $B(t)$ for different values of t.

Remark 3.1. *It is important to point out that the condition (i) in both Lemmas 2.1 and 2.3 is not a sufficient condition to guarantee that f is monotone decreasing and increasing, respectively. We clarify this point in the following examples.*

- *Let* $f(1) = -1/8$, $f(2) = -1/15$, $a = 0$ *and* $\alpha = 1/2$. *Then we see that*

$$\left({}^{\mathrm{RL}}_{0}\nabla^{\frac{1}{2}} f\right)(2) = -0.0042 < 0.$$

However, f *is not decreasing on* $\mathbb{N}_2$ *because* $(\nabla f)(2) = \frac{7}{120} > 0$.

- *If* $f(1) = 1/8$, $f(2) = 1/15$, $a = 0$ *and* $\alpha = 1/2$, *then we have*

$$\left({}^{\mathrm{RL}}_{0}\nabla^{\frac{1}{2}} f\right)(2) = 0.0042 > 0.$$

But, f *is not increasing on* $\mathbb{N}_2$ *since* $(\nabla f)(2) = -\frac{7}{120} < 0$.

4. Conclusions

Based on the nabla fractional differences for Riemann-Liouville operators including two necessary conditions, we have proved the monotone decreasing behavior of a discrete nonpositive function f defined on $\mathbb{N}_{r_0+1}$. At the same time, we have established the monotone increasing of a discrete nonpositive function f defined on $\mathbb{N}_{r_0+1}$ under two different conditions. The increasing and decreasing results together with extra conditions allow us to obtain the relative minimum and relative maximum of the function f. Furthermore, we have found alternative conditions corresponding to the main conditions (condition (ii)) of Lemmas 2.1 and 2.3 which are simpler and stronger than the existing ones in both of the lemmas. For more clarification, we have expressed the relative minimum and relative maximum results by using the new simple conditions. Also, we have explained that the condition (i) in Lemmas 2.1 and 2.3 is solely not sufficient for the function to be monotone decreasing or increasing.

There is interesting future work for the researchers after reading our article that we have left as an open problem. This will be obtaining similar results for the nabla Caputo operators.

Acknowledgements

This research was supported by the Taif University Researchers Supporting Project (No. TURSP-2020/155), Taif University, Taif, Saudi Arabia. Also, the third author would like to thank Prince Sultan University for the support through the TAS research lab.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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