Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 22097–22111.
DOI: 10.3934/math.20231126
Received: 24 April 2023
Revised: 05 June 2023
Accepted: 12 June 2023
Published: 12 July 2023

Research article

On Hardy-Hilbert-type inequalities with α-fractional derivatives

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Abstract: In the current manuscript, new alpha delta dynamic Hardy-Hilbert inequalities on time scales are discussed. These inequalities combine and expand a number of continuous inequalities and their corresponding discrete analogues in the literature. We shall illustrate our results using Hölder's inequality on time scales and a few algebraic inequalities.

Keywords: Steffensen's inequality; dynamic inequality; dynamic integral; time scale
Mathematics Subject Classification: 26D10, 26D15, 26D20, 34A12, 34A40

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1. Introduction

Hardy-Hilbert's double-series theorem [1] states:

Theorem 1.1. If $\nu, \varpi > 1$ are such that $\frac{1}{\nu} + \frac{1}{\varpi} \leqslant 1$ and $0 < \lambda = 2 - \frac{1}{\nu} - \frac{1}{\varpi} = \frac{1}{\nu'} + \frac{1}{\varpi'} \leqslant 1$, such that ν' and ϖ' present the exponents conjugate, then

$$\sum_{\jmath=1}^{\infty}\sum_{\imath=1}^{\infty}\frac{\vartheta_{\jmath}\pi_{\imath}}{(\jmath+\imath)^{\lambda}} \leqslant K\Big(\sum_{\jmath=1}^{\infty}\vartheta_{\jmath}^{\nu}\Big)^{\frac{1}{\nu}}\sum_{\imath=1}^{\infty}\pi_{\imath}^{\varpi}\Big)^{\frac{1}{\varpi}}, \tag{1.1}$$

where $K = K(\nu, \varpi)$ depends on ν and ϖ only.
You may find the integral analogue of Theorem 1.1 in [1].
Theorem 1.2. Let ν, ϖ, ν', ϖ' and λ be as in Theorem 1.1. If $\vartheta \in L^{\nu}(0,\infty)$ and $\theta \in L^{\varpi}(0,\infty)$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{\vartheta(\iota)\theta(\varsigma)}{(\iota+\varsigma)^{\lambda}}d\iota d\varsigma \leqslant K\Big(\int_0^{\infty}\vartheta^{\nu}(\iota)d\iota\Big)^{\frac{1}{\nu}}\Big(\int_0^{\infty}\theta^{\varpi}(\varsigma)d\varsigma\Big)^{\frac{1}{\varpi}}, \tag{1.2}$$

where $K = K(\nu, \varpi)$ depends on ν and ϖ only.

In 2000, Pachpatte [2] established different inequalities from inequality (1.1) but, to a certain extent, having a glimpse of inequality (1.1) as follows:
Theorem 1.3. Let ν, ϖ, $a(\mathfrak{I})$, $b(\delta)$, $a(0)$, $b(0)$, $\Delta^{\alpha}a(\mathfrak{I})$ and $\Delta^{\alpha}b(\delta)$ be as in [2] then

$$\sum_{\mathfrak{I}=1}^{m}\sum_{\delta=1}^{n}\frac{|a(\mathfrak{I})||b(\delta)|}{qs^{\nu-1}+pt^{\varpi-1}} \leqslant \frac{1}{pq}m^{\frac{\nu-1}{\nu}}n^{\frac{\varpi-1}{\varpi}}\Big(\sum_{\mathfrak{I}=1}^{m}(m-\mathfrak{I}+1)|\Delta^{\alpha}a(\mathfrak{I})|^{\nu}\Big)^{\frac{1}{\nu}}\Big(\sum_{\delta=1}^{n}(n-\delta+1)|\Delta^{\alpha}b(\delta)|^{\varpi}\Big)^{\frac{1}{\varpi}}. \tag{1.3}$$

In the same paper [2], Pachpatte also established more advanced versions of inequality (1.2) as follows:
Theorem 1.4. Let ν, ϖ, $\vartheta(\mathfrak{I})$, $\theta(\delta)$, $\vartheta(0)$, $\theta(0)$, $\vartheta'(\mathfrak{I})$ and $\theta'(\delta)$ be as in [2], then

$$\int_0^{\iota}\int_0^{\varsigma}\frac{|\vartheta(\mathfrak{I})||\theta(\delta)|}{qs^{\nu-1}+pt^{\varpi-1}}d\mathfrak{I}d\delta \leqslant \frac{1}{pq}\iota^{\frac{\nu-1}{\nu}}\varsigma^{\frac{\varpi-1}{\varpi}}\Big(\int_0^{\iota}(\iota-\mathfrak{I})|\vartheta'(\mathfrak{I})|^{\nu}d\mathfrak{I}\Big)^{\frac{1}{\nu}}\Big(\int_0^{\varsigma}(\varsigma-\delta)|\theta'(\delta)|^{\varpi}d\delta\Big)^{\frac{1}{\varpi}}. \tag{1.4}$$

In 2011, Zhao et al. [3] proposed a new inequality similar to Theorem 1.2.
Theorem 1.5. Let $h_i \geqslant 1$, $\nu_i > 1$ be constants and $\frac{1}{\nu_i}+\frac{1}{\varpi_i} = 1$. Let the differentiable fun. $\vartheta_i(\mathfrak{I}_i)$ on $[0,\iota_i)$, where $\iota_i \in (0,\infty)$ and we use ϑ_i' as differentiation of ϑ_i. Suppose $\vartheta_i(0) = 0$ for $(i = 1,\dots,n)$. Then

$$\int_0^{\iota_1}\int_0^{\iota_2}\cdots\int_0^{\iota_n}\frac{\prod_{i=1}^{n}|\vartheta_i^{h_i}(\mathfrak{I}_i)|}{\Big(\sum_{i=1}^{n}\frac{\mathfrak{I}_i}{\varpi_i}\Big)^{\sum_{i=1}^{n}\frac{1}{\varpi_i}}}d\mathfrak{I}_n d\mathfrak{I}_{n-1}\dots d\mathfrak{I}_1$$
$$\leqslant K\prod_{i=1}^{n}\Big(\int_0^{\iota_i}(\iota_i-\mathfrak{I}_i)\Big|\vartheta_i^{h_i-1}(\mathfrak{I}_i)\vartheta_i'(\mathfrak{I}_i)\Big|^{\nu_i}d\mathfrak{I}_i\Big)^{\frac{1}{\nu_i}},$$

where

$$K = \Big(n-\sum_{i=1}^{n}\frac{1}{\nu_i}\Big)^{\sum_{i=1}^{n}\frac{1}{\nu_i}-n}\prod_{i=1}^{n}h_i\iota_i^{\frac{1}{\varpi_i}}.$$

Also in 2012, Zhao and Chung [4] proved the following theorem:
Theorem 1.6. Let $\nu_i > 1$, be constants and $\frac{1}{\nu_i}+\frac{1}{\varpi_i} = 1$. Let $\vartheta_i(\tau_{1i},\dots,\tau_{ni})$ be real valued nth differentiable functions defined on $[0,\iota_{1i})\times\cdots\times[0,\iota_{ni})$, where $0 \leqslant \iota_{ji} \leqslant \delta_{ji}$, $\delta_{ji} \in (0,\infty)$ and $i, j = 1,\dots,n$. Suppose

$$\vartheta_i(\iota_{1i},\dots,\iota_{ni}) = \int_0^{\iota_{1i}}\cdots\int_0^{\iota_{ni}}\frac{\partial^n}{\partial\tau_{1i}\dots\partial\tau_{ni}}\vartheta_i(\tau_{1i},\dots,\tau_{ni})d\tau_{ni}\dots d\tau_{1i},$$

then

$$\int_0^{\delta_{11}} \cdots \int_0^{\delta_{n1}} \int_0^{\delta_{12}} \cdots \int_0^{\delta_{n2}} \cdots \int_0^{\delta_{1n}} \cdots \int_0^{\delta_{nn}} \tag{1.5}$$

$$\frac{\prod_{i=1}^{n} \left(\int_0^{\iota_{1i}} \cdots \int_0^{\iota_{ni}} \left| \frac{\partial^n}{\partial \tau_{1i} \ldots \partial \tau_{ni}} \vartheta_i(\tau_{1i}, \ldots, \tau_{ni}) \right|^{\nu_i} d\tau_{ni} \ldots d\tau_{1i} \right)^{\frac{1}{\nu_i}}}{\left(\sum_{i=1}^{n} \frac{\left[\iota_{1i} \ldots \iota_{ni}\right]}{\varpi_i} \right)^{\sum_{i=1}^{n} \frac{1}{\varpi_i}}}$$

$$d\iota_{11} \ldots d\iota_{n1} d\iota_{12} \ldots d\iota_{n2} \ldots d\iota_{1n} \ldots d\iota_{nn}$$

$$\leqslant N \prod_{i=1}^{n} \left(\int_0^{\delta_{1i}} \cdots \int_0^{\delta_{ni}} \prod_{j=1}^{n} (\delta_{ji} - \iota_{ji}) \left| \frac{\partial^n}{\partial \iota_{1i} \ldots \partial \iota_{ni}} \vartheta_i(\iota_{1i}, \ldots, \iota_{ni}) \right|^{\nu_i} d\iota_{ni} \ldots d\iota_{1i} \right)^{\frac{1}{\nu_i}},$$

where

$$N = \left(n - \sum_{i=1}^{n} \frac{1}{\nu_i} \right)^{\sum_{i=1}^{n} \frac{1}{\nu_i} - n} \prod_{i=1}^{n} \left[\delta_{1i} \ldots \delta_{ni} \right]^{\frac{1}{\varpi_i}}.$$

Fractional calculus on fractal sets has been widely applied in science and engineering because it is better able to describe natural events. Many academics have recently used fractal sets and fractional calculus to study various classical inequalities.

All of the aforementioned findings hold true for both continuous and discrete domains. The purpose of the current research is to provide new, more general conclusions to the time-scale-based disparities previously established. Supreme outcomes, from which many other previous and current results may be taken, would be produced in this way. See the publications for various dynamic inequalities, integrals of Hilbert's kind, and other categories of inequalities on time scales [5–8, 11–20].

A very important question here is: Is it possible to prove the time scales version via α-conformable derivative of the inequalities of Hardy-Hilbert-type presented in [3] due to Zhao et al. and also the inequalities presented in [4] due to Zhao and Chung?

In this manuscript, we intend to address the question above, and establish a few novel delta fractional dynamic inequalities of the Hardy-Hilbert type on time scales, which are studied in [3, 4]. We also extract the discrete counterparts of the continuous Hilbert inequalities that are present in some special situations of our results. The present article is arranged as follows: In Section 2, some basic concepts of the α-fractional calculus on time scales and useful lemmas are introduced. In Section 3, we state and prove our main results. We end with Section 4 of conclusion.

2. Preliminaries on time scales

Now, we present the fundamental results about the fractional time scales calculus. The results are adapted from [9, 10, 21–24]. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(\zeta) := \inf\{s \in \mathbb{T} : s > \zeta\}, \qquad \zeta \in \mathbb{T}, \tag{2.1}$$

and the backward jump operator $\rho : \mathbb{T} :\to \mathbb{T}$ is defined by

$$\rho(\zeta) := \sup\{s \in \mathbb{T} : s < \zeta\}, \qquad \zeta \in \mathbb{T}. \tag{2.2}$$

In the previous two definitions, we set $\inf \emptyset = \sup \mathbb{T}$ (i.e., if ζ is the minimum of $\mathbb{T}$, then $\sigma(\zeta) = \zeta$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if ζ is the maximum of ζ, then $\rho(\zeta) = \zeta$), where $\emptyset$ is the empty set.

A point $\zeta \in \mathbb{T}$ with $\inf \mathbb{T} < \zeta < \sup \mathbb{T}$, is said to be right-dense if $\sigma(\zeta) = \zeta$, left-dense if $\rho(\zeta) = \zeta$, right-scattered if $\sigma(\zeta) > \zeta$, and left-scattered if $\rho(\zeta) < \zeta$. Points that are simultaneously right-dense and left-dense are called dense points, and points that are simultaneously right-scattered and left-scattered are called isolated points. The forward and backward graininess functions μ and ν for a time scale $\mathbb{T}$ are defined by $\mu(\zeta) := \sigma(\zeta) - \zeta$ and $\nu(\zeta) := \zeta - \rho(\zeta)$, respectively.

Definition 2.1. The number $T_\alpha^\Delta(f)(\zeta)$ (provided it exists) of the function $f : \mathbb{T} \to \mathbb{R}$, for $\zeta > 0$ and $\alpha \in (0, 1]$ is the number which has the property that for any $\epsilon > 0$, there exists a neighbrhood U of ζ such that

$$|[f(\sigma(\zeta)) - f(s)]\zeta^{1-\alpha} - T_\alpha^\Delta(f(\zeta))[\sigma(\zeta) - s]| \leq \varepsilon|\sigma(\zeta) - s|,$$

for all $s \in U$. $T_\alpha^\Delta(f(\zeta))$ is called conformable $\alpha-$ fractional derivative of function f of order α at ζ, for conformable fractional derivative on $\mathbb{T}$ at 0, we define it with $T_\alpha^\Delta(f(0)) = \lim_{\zeta\longrightarrow 0} T_\alpha^\Delta(f(\zeta))$.

Remark 2.1. If $\alpha = 1$ then we obtain from Definition 2.1 the delta derivative of time scales. The conformable fractional derivative of order zero is defined by the identity operator: $T_0^\Delta(\eta) = \eta$.

Remark 2.2. Along the work, we also use the notation $(\eta(\zeta))^{\Delta_\alpha} = T_\alpha^\Delta(\eta)$.

Theorem 2.1. Let $\alpha \in (0, 1]$ and $\mathbb{T}$ be a time scale. Assume $\eta : \mathbb{T} \to \mathbb{R}$ and $\zeta \in \mathbb{T}^\kappa$. The following properties hold.

(i) If η is conformal fractional differentiable of order α a at $\zeta > 0$, then η is continuous at ζ.

(ii) If η is continuous at ζ and ζ is right-scattered, then η is conformable fractional differentiable of order α at ζ with

$$T_\alpha^\Delta(\eta)(\zeta) = \frac{\eta(\sigma(\zeta)) - \eta(\zeta)}{\mu(\zeta)}\zeta^{1-\alpha}.$$

(iii) If ζ is right-dense, then η is conformable fractional differentiable of order α at ζ if, and only if, the limit $\lim_{s\longrightarrow\zeta} \frac{\eta(\zeta)-\eta(s)}{\zeta-s}\zeta^{1-\alpha}$ exists as a finite number. In this case,

$$T_\alpha^\Delta(\eta)(\zeta) = \lim_{s\longrightarrow\zeta} \frac{\eta(\zeta) - \eta(s)}{\zeta - s}\zeta^{1-\alpha}.$$

(iv) If η is fractional differentiable of order α at ζ, then

$$\eta(\sigma(\zeta)) = \eta(\zeta) + (\mu(\zeta)\zeta^{\alpha-1}T_\alpha^\Delta(\eta)(\zeta)).$$

Remark 2.3. In a time scale $\mathbb{T}$, due to the inherited topology of the real numbers, a function η is always continuous at any isolated point $\zeta \in \mathbb{T}$.

Example 2.1. Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$. Then $\sigma(\zeta) = \zeta + h$ and $\mu(\zeta) = h$ for all $\zeta \in \mathbb{T}$. For function $\eta : \mathbb{T} \longrightarrow \mathbb{R}$, $\eta(\zeta) = \zeta^2$ we have $T_\alpha^\Delta(\zeta^2) = (2\zeta + h)\zeta^{1-\alpha}$.

Example 2.2. Let $q > 1$ and $\mathbb{T} = q^{\overline{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$ with $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$. In this time scale

$$\sigma(\zeta) = \begin{cases} qt, & \text{if } \zeta \neq 0; \\ 0, & \text{if } \zeta = 0; \end{cases} \quad \text{and} \quad \mu(\zeta) = \begin{cases} (q-1)\zeta, & \text{if } \zeta \neq 0; \\ 0, & \text{if } \zeta = 0. \end{cases}$$

Here 0 is a right-dense minimum and every other point in $\mathbb{T}$ is isolated. Now consider the square function of Example 2.1. It follows that

$$T_\alpha^\Delta(\eta)(\zeta) = T_\alpha^\Delta(\zeta^2) = \begin{cases} (q+1)\zeta^{2-\alpha}, & \text{if } \zeta \neq 0; \\ 0, & \text{if } \zeta = 0. \end{cases}$$

The conformable fractional derivative has the following properties

Theorem 2.2. Let $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ be conformable fractional derivative of order $\alpha \in (0, 1]$, the following properties are hold:

(i) The $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ are α-conformable fractional derivative and

$$T_\alpha^\Delta(f + g) = T_\alpha^\Delta(f) + T_\alpha^\Delta(g).$$

(ii) For all $k \in \mathbb{R}$, then $kf : \mathbb{T} \longrightarrow \mathbb{R}$ α-conformable fractional derivative and

$$T_\alpha^\Delta(kf) = kT_\alpha^\Delta(f).$$

(iii) If f and g are α-conformable fractional differentiable, we have $fg : \mathbb{T} \longrightarrow \mathbb{R}$ is α-conformable fractional differentiable and

$$T_\alpha^\Delta(fg) = T_\alpha^\Delta(f)g + f^\sigma T_\alpha^\Delta(g).$$

(iv) If f and g are α-conformable fractional differentiable, then f/g is α-conformable fractional differentiable with

$$T_\alpha^\Delta\left(\frac{f}{g}\right) = \frac{T_\alpha^\Delta(f)g - fT_\alpha^\Delta(g)}{gg^\sigma}$$

valid $\forall \zeta \in \mathbb{T}^\kappa$, where $gg^\sigma \neq 0$.

Theorem 2.3. Let $\varsigma : \mathbb{T} \to \mathbb{R}$ be continuous and α-fractional differentiable at $\zeta \in \mathbb{T}$ for $\alpha \in (0, 1]$ and $\delta : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function. Then there is c in the interval $[\zeta, \sigma(\zeta)]$ such that

$$T_\alpha^\Delta(\delta \circ \varsigma) = \delta'(\varsigma(c))T_\alpha^\Delta(\varsigma(\zeta)). \tag{2.3}$$

Theorem 2.4. Let $\delta : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, $\alpha \in (0, 1]$ and $\varsigma : \mathbb{T} \to \mathbb{R}$ be α-fractional differentiable function. Then $(\delta \circ \varsigma) : \mathbb{T} \to \mathbb{R}$ is α-fractional differentiable and we have

$$T_\alpha^\Delta(\delta \circ \varsigma)(s) = \left\{ \int_0^1 \delta'(\varsigma(s) + h\mu(s)s^{\alpha-1}T_\alpha^\Delta(\varsigma(s)))dh \right\} T_\alpha^\Delta(\varsigma(s)). \tag{2.4}$$

Definition 2.2. Let $0 < \alpha \leqslant 1$, the α-fractional of f, is defined as

$$\int f(s)\Delta_\alpha s = \int f(s)s^{\alpha-1}\Delta s.$$

The conformable fractional integral satisfying the next properties

Theorem 2.5. Assume a, b, $c \in \mathbb{T}$, $\lambda \in \mathbb{R}$. Let $\delta, \varsigma : \mathbb{T} \longrightarrow \mathbb{R}$. Then

(i) $\int_a^b [\varsigma(s) + \delta(s)]\Delta_\alpha s = \int_a^b \varsigma(s)\Delta_\alpha s + \int_a^b \delta(s)\Delta_\alpha s$.
(ii) $\int_a^b \lambda\varsigma(s)\Delta_\alpha s = \lambda \int_a^b \varsigma(s)\Delta_\alpha s$.
(iii) $\int_a^b \varsigma(s)\Delta_\alpha s = -\int_b^a \varsigma(s)\Delta_\alpha s$.
(iv) $\int_a^b \varsigma(s)\Delta_\alpha s = \int_a^c \varsigma(s)\Delta_\alpha s + \int_c^b \varsigma(s)\Delta_\alpha s$.
(v) $\int_a^a \varsigma(s)\Delta_\alpha s = 0$.

Theorem 2.6. (Dynamic Hölder's Inequality [11]) Let $u, v \in \mathbb{T}$ with $u < v$. If $\vartheta, \theta \in CC^1_{rd}([u, v]_{\mathbb{T}} \times [u, v]_{\mathbb{T}}, \mathbb{R})$ be integrable functions and $\frac{1}{\nu} + \frac{1}{\varpi} = 1$ with $\nu > 1$. Then

$$\int_u^v \int_u^v |\vartheta(r,\delta)\theta(r,\delta)|\Delta^\alpha r \Delta^\alpha \delta \leq \left[\int_u^v \int_u^v |\vartheta(r,\delta)|^\nu \Delta^\alpha r \Delta^\alpha \delta\right]^{\frac{1}{\nu}} \times \left[\int_u^v \int_u^v |\theta(r,\delta)|^\varpi \Delta^\alpha r \Delta^\alpha \delta\right]^{\frac{1}{\varpi}}. \quad (2.5)$$

This inequality is reversed if $0 < \nu < 1$ and if $\nu < 0$ or $\varpi < 0$.

3. Main results

Theorem 3.1. Let $\mathbb{T}$ be a time scale with $\delta_0, \iota_i, \Im_i, \delta_i \in \mathbb{T}$, $(i = 1, \ldots, n)$. Let $h_i \geqslant 1$, $\nu_i, \varpi_i > 1$ be constants and $\frac{1}{\nu_i} + \frac{1}{\varpi_i} = 1$. Let Δ^α-differentiable functions $\vartheta_i(\Im_i)$ be decreasing on $[\delta_0, \iota_i)_{\mathbb{T}}$, where $\iota_i \in (0, \infty)$. Suppose $\vartheta_i(\delta_0) = 0$. Then

$$\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\iota_2}\cdots\int_{\delta_0}^{\iota_n} \frac{\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i)|}{\left(\sum_{i=1}^n \frac{(\Im_i - \delta_0)}{\varpi_i}\right)^{\sum_{i=1}^n \frac{1}{\varpi_i}}} \Delta^\alpha \Im_n \Delta^\alpha \Im_{n-1} \ldots \Delta^\alpha \Im_1$$

$$\leqslant K \prod_{i=1}^n \left(\int_{\delta_0}^{\iota_i} (\sigma(\iota_i) - \sigma(\Im_i)) \left|\vartheta_i^{h_i - 1}(\Im_i)\vartheta_i^{\Delta^\alpha}(\Im_i)\right|^{\nu_i} \Delta^\alpha \Im_i\right)^{\frac{1}{\nu_i}}, \quad (3.1)$$

where

$$K = K(\iota_1, \ldots, \iota_n) = \left(n - \sum_{i=1}^n \frac{1}{\nu_i}\right)^{\sum_{i=1}^n \frac{1}{\nu_i} - n} \prod_{i=1}^n h_i (\iota_i - \delta_0)^{\frac{1}{\varpi_i}}.$$

Proof. From Hölder inequality (2.5), one can see that

$$\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i)| \leqslant \prod_{i=1}^n h_i \int_{\delta_0}^{\Im_i} \left|\vartheta_i^{h_i-1}(\tau_i)\vartheta_i^{\Delta^\alpha}(\tau_i)\right| \Delta^\alpha \tau_i \quad (3.2)$$

$$\leqslant \prod_{i=1}^n h_i (\Im_i - \delta_0)^{\frac{1}{\varpi_i}} \left(\int_{\delta_0}^{\Im_i} \left|\vartheta_i^{h_i-1}(\tau_i)\vartheta_i^{\Delta^\alpha}(\tau_i)\right|^{\nu_i} \Delta^\alpha \tau_i\right)^{\frac{1}{\nu_i}}.$$

Using the inequality for the means [25],

$$\left(\prod_{i=1}^n \lambda_i^{\frac{1}{\varpi_i}}\right)^{\frac{1}{\sum_{i=1}^n \frac{1}{\varpi_i}}} \leqslant \frac{1}{\sum_{i=1}^n \frac{1}{\varpi_i}} \sum_{i=1}^n \frac{\lambda_i}{\varpi_i}, \quad \lambda_i > 0 \quad (i = 1, \ldots, n), \quad (3.3)$$

we have

$$\frac{\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i)|}{\left(\sum_{i=1}^n \frac{(\Im_i - \delta_0)}{\varpi_i}\right)^{\sum_{i=1}^n \frac{1}{\varpi_i}}}$$

$$\leqslant \left(n - \sum_{i=1}^n \frac{1}{\nu_i}\right)^{\sum_{i=1}^n \frac{1}{\nu_i} - n} \prod_{i=1}^n h_i \left(\int_{\delta_0}^{\Im_i} \left|\vartheta_i^{h_i-1}(\tau_i)\vartheta_i^{\Delta^\alpha}(\tau_i)\right|^{\nu_i} \Delta^\alpha \tau_i\right)^{\frac{1}{\nu_i}}. \quad (3.4)$$

Using the integration of (3.4) on $\mathfrak{I}_i$ from δ_0 to ι_i $(i = 1, \ldots, n)$ employing the inequality of Hölder's, gets

$$\begin{aligned}
&\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\iota_2}\cdots\int_{\delta_0}^{\iota_n}\frac{\prod_{i=1}^n|\vartheta_i^{h_i}(\mathfrak{I}_i)|}{\left(\sum_{i=1}^n\frac{(\mathfrak{I}_i-\delta_0)}{\varpi_i}\right)^{\sum_{i=1}^n\frac{1}{\varpi_i}}}\Delta^\alpha\mathfrak{I}_n\Delta^\alpha\mathfrak{I}_{n-1}\ldots\Delta^\alpha\mathfrak{I}_1\\
&\leqslant\left(n-\sum_{i=1}^n\frac{1}{\nu_i}\right)^{\sum_{i=1}^n\frac{1}{\nu_i}-n}\prod_{i=1}^n h_i\int_{\delta_0}^{\iota_i}\left(\int_{\delta_0}^{\mathfrak{I}_i}\left|\vartheta_i^{h_i-1}(\tau_i)\vartheta_i^{\Delta^\alpha}(\tau_i)\right|^{\nu_i}\Delta^\alpha\tau_i\right)^{\frac{1}{\nu_i}}\\
&\leqslant K\prod_{i=1}^n\left(\int_{\delta_0}^{\iota_i}\int_{\delta_0}^{\mathfrak{I}_i}\left|\vartheta_i^{h_i-1}(\tau_i)\vartheta_i^{\Delta^\alpha}(\tau_i)\right|^{\nu_i}\Delta^\alpha\tau_i\Delta^\alpha\mathfrak{I}_i\right)^{\frac{1}{\nu_i}}\\
&=K\prod_{i=1}^n\left(\int_{\delta_0}^{\iota_i}(\iota_i-\mathfrak{I}_i)\left|\vartheta_i^{h_i-1}(\mathfrak{I}_i)\vartheta_i^{\Delta^\alpha}(\mathfrak{I}_i)\right|^{\nu_i}\Delta^\alpha\mathfrak{I}_i\right)^{\frac{1}{\nu_i}}. \qquad (3.5)
\end{aligned}$$

By exploiting the fact $\iota_i \leqslant \sigma(\iota_i)$, we get that

$$\begin{aligned}
&\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\iota_2}\cdots\int_{\delta_0}^{\iota_n}\frac{\prod_{i=1}^n|\vartheta_i^{h_i}(\mathfrak{I}_i)|}{\left(\sum_{i=1}^n\frac{(\mathfrak{I}_i-\delta_0)}{\varpi_i}\right)^{\sum_{i=1}^n\frac{1}{\varpi_i}}}\Delta^\alpha\mathfrak{I}_n\Delta^\alpha\mathfrak{I}_{n-1}\ldots\Delta^\alpha\mathfrak{I}_1\\
&\leqslant K\prod_{i=1}^n\left(\int_{\delta_0}^{\iota_i}(\sigma(\iota_i)-\sigma(\mathfrak{I}_i))\left|\vartheta_i^{h_i-1}(\mathfrak{I}_i)\vartheta_i^{\Delta^\alpha}(\mathfrak{I}_i)\right|^{\nu_i}\Delta^\alpha\mathfrak{I}_i\right)^{\frac{1}{\nu_i}}.
\end{aligned}$$

This concludes the evidence. □

Remark 3.1. In Theorem 3.1, taking $\mathbb{T} = \mathbb{Z}, \alpha = 1, h_i = 1$, got results thanks to the authors of [3, Theorem 1.1].

Remark 3.2. In Theorem 3.1, taking $\mathbb{T} = \mathbb{R}, \alpha = 1$, got results thanks to the authors of [3, Theorem 1.3].

Corollary 3.1. In Theorem 3.1, taking $n = 2$, and $h_1 = h_2 = 1$, if $\nu_1, \nu_2 > 1$ are such that $\frac{1}{\nu_1} + \frac{1}{\nu_2} \geqslant 1$ and $0 < \lambda = 2 - \frac{1}{\nu_1} - \frac{1}{\nu_2} = \frac{1}{\varpi_1} + \frac{1}{\varpi_2} \leqslant 1$, inequality (3.1) reduces to

$$\begin{aligned}
&\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\iota_2}\frac{|\vartheta_1(\mathfrak{I}_1)||\vartheta_2(\mathfrak{I}_2)|}{\left(\varpi_2(\mathfrak{I}_1-\delta_0)+\varpi_1(\mathfrak{I}_2-\delta_0)\right)^\lambda}\Delta^\alpha\mathfrak{I}_2\Delta^\alpha\mathfrak{I}_1\leqslant\frac{1}{(\lambda\varpi_1\varpi_2)^\lambda}(\iota_1-\delta_0)^{\frac{1}{\varpi_1}}(\iota_2-\delta_0)^{\frac{1}{\varpi_2}} \qquad (3.6)\\
&\times\left(\int_{\delta_0}^{\iota_1}(\sigma(\iota_1)-\sigma(\mathfrak{I}_1))|\vartheta_1^{\Delta^\alpha}(\mathfrak{I}_1)|^{\nu_1}\Delta^\alpha\mathfrak{I}_1\right)^{\frac{1}{\nu_1}}\left(\int_{\delta_0}^{\iota_2}(\sigma(\iota_2)-\sigma(\mathfrak{I}_2))|\vartheta_2^{\Delta^\alpha}(\mathfrak{I}_2)|^{\nu_2}\Delta^\alpha\mathfrak{I}_2\right)^{\frac{1}{\nu_2}}.
\end{aligned}$$

Remark 3.3. In special case, taking $\mathbb{T} = \mathbb{R}, \alpha = 1$, in (3.6), we have that

$$\begin{aligned}
&\int_0^{\iota_1}\int_0^{\iota_2}\frac{|\vartheta_1(\mathfrak{I}_1)||\vartheta_2(\mathfrak{I}_2)|}{(\varpi_2\mathfrak{I}_1+\varpi_1\mathfrak{I}_2)^\lambda}d\mathfrak{I}_2d\mathfrak{I}_1\leqslant\frac{1}{(\lambda\varpi_1\varpi_2)^\lambda}(\iota_1)^{\frac{1}{\varpi_1}}(\iota_2)^{\frac{1}{\varpi_2}}\\
&\times\left(\int_0^{\iota_1}(\iota_1-\mathfrak{I}_1)|\vartheta_1'(\mathfrak{I}_1)|^{\nu_1}d\mathfrak{I}_1\right)^{\frac{1}{\nu_1}}\left(\int_0^{\iota_2}(\iota_2-\mathfrak{I}_2)|\vartheta_2'(\mathfrak{I}_2)|^{\nu_2}d\mathfrak{I}_2\right)^{\frac{1}{\nu_2}}, \qquad (3.7)
\end{aligned}$$

which is an interesting variation of inequality (1.2).

Remark 3.4. In special case, taking $\mathbb{T}=\mathbb{Z}, \alpha=1$, in (3.6), we have that

$$\sum_{\Im_1=1}^{m_1}\sum_{\Im_2=1}^{m_2}\frac{|a_1(\Im_1)||a_2(\Im_2)|}{(\varpi_2\Im_1+\varpi_1\Im_2)^{\lambda}} \leqslant \frac{1}{(\lambda\varpi_1\varpi_2)^{\lambda}}(m_1)^{\frac{1}{\varpi_1}}(m_2)^{\frac{1}{\varpi_2}} \times\Big(\sum_{\Im_1=1}^{m_1}(m_1-\Im_1+1)|\Delta^{\alpha}a_1(\Im_1)|^{\nu_1}\Big)^{\frac{1}{\nu_1}}\Big(\sum_{\Im_2=1}^{m_2}(m_2-\Im_2+1)|\Delta^{\alpha}a_2(\Im_2)|^{\nu_2}\Big)^{\frac{1}{\nu_2}}, \tag{3.8}$$

which is an interesting variation of inequality (1.1).

Corollary 3.2. In Corollary 3.1, if $\lambda=1$, then $\frac{1}{\nu_1}+\frac{1}{\nu_2}=\frac{1}{\varpi_1}+\frac{1}{\varpi_2}=1$ and take $\nu_1=\varpi_2$, $\nu_2=\varpi_1$. In this case inequality (3.6) reduces to

$$\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\iota_2}\frac{|\vartheta_1(\Im_1)||\vartheta_2(\Im_2)|}{\varpi_2(\Im_1-\delta_0)+\varpi_1(\Im_2-\delta_0)}\Delta^{\alpha}\Im_2\Delta^{\alpha}\Im_1 \leqslant \frac{1}{\nu_1\varpi_1}(\iota_1-\delta_0)^{\frac{\nu_1-1}{\nu_1}}(\iota_2-\delta_0)^{\frac{\varpi_1-1}{\varpi_1}} \times\Big(\int_{\delta_0}^{\iota_1}(\sigma(\iota_1)-\sigma(\Im_1))|\vartheta_1^{\Delta^{\alpha}}(\Im_1)|^{\nu_1}\Delta^{\alpha}\Im_1\Big)^{\frac{1}{\nu_1}}\Big(\int_{\delta_0}^{\iota_2}(\sigma(\iota_2)-\sigma(\Im_2))|\vartheta_2^{\Delta^{\alpha}}(\Im_2)|^{\varpi_1}\Delta^{\alpha}\Im_2\Big)^{\frac{1}{\varpi_1}}. \tag{3.9}$$

Remark 3.5. In Corollary 3.2, if $\mathbb{T}=\mathbb{R}, \alpha=1$ we obtain an equivalent formulation of the inequality that Pachpatte presented in [2, Theorem 2].

Remark 3.6. In Corollary 3.2, if $\mathbb{T}=\mathbb{Z}, \alpha=1$ we obtain an equivalent formulation of the inequality that Pachpatte presented in [2, Theorem 1].

Theorem 3.2. Let $\mathbb{T}$ be a time scale with δ_0, ι_i, ς_i, $\Im_i$, $\delta_i\in\mathbb{T}$, $(i=1,\dots,n)$. Let $h_i\geqslant 1$, $\nu_i,\varpi_i>1$ be constants and $\frac{1}{\nu_i}+\frac{1}{\varpi_i}=1$. Let the Δ^{α}-differentiable fun. $\vartheta_i(\Im_i,\delta_i)$ be decreasing funs. on $[\delta_0,\iota_i)_{\mathbb{T}}\times[\delta_0,\varsigma_i)_{\mathbb{T}}$ and $\vartheta_i(\delta_0,\delta_i)=\vartheta_i(\Im_i,\delta_0)=0$, for $(i=1,\dots,n)$. Partial derivatives of ϑ_i are indicated by $\vartheta_i^{\Delta_1^{\alpha}}$, $\vartheta_i^{\Delta_2^{\alpha}}$, $\vartheta_i^{\Delta_{12}^{\alpha}}=\vartheta_i^{\Delta_{21}^{\alpha}}$. Let

$$(\vartheta_i^{h_i}(\Im_i,\delta_i))^{\Delta_1^{\alpha}\Delta_2^{\alpha}}\leqslant(h_i\vartheta_i^{h_i-1}(\Im_i,\delta_i).\vartheta_i^{\Delta_1^{\alpha}}(\Im_i,\delta_i))^{\Delta_2^{\alpha}}=\vartheta_i^{\Delta_{12}^{\alpha}}(\Im_i,\delta_i).$$

Then

$$\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}\cdots\int_{\delta_0}^{\iota_n}\int_{\delta_0}^{\varsigma_n}\frac{\prod_{i=1}^{n}|\vartheta_i^{h_i}(\Im_i,\delta_i)|}{\Big(\sum_{i=1}^{n}\frac{(\Im_i-\delta_0)(\delta_i-\delta_0)}{\varpi_i}\Big)^{\sum_{i=1}^{n}\frac{1}{\varpi_i}}}\Delta^{\alpha}\delta_n\Delta^{\alpha}\Im_n\dots\Delta^{\alpha}\delta_1\Delta^{\alpha}\Im_1 \leqslant C\prod_{i=1}^{n}\Big(\int_{\delta_0}^{\iota_i}\int_{\delta_0}^{\varsigma_i}(\sigma(\iota_i)-\sigma(\Im_i))(\sigma(\varsigma_i)-\sigma(\delta_i))|\vartheta_i^{\Delta_{12}^{\alpha}}(\Im_i,\delta_i)|^{\nu_i}\Delta^{\alpha}\delta_i\Delta^{\alpha}\Im_i\Big)^{\frac{1}{\nu_i}}, \tag{3.10}$$

where

$$C=C(\iota_1\varsigma_1,\dots,\iota_n\varsigma_n)=\Big(n-\sum_{i=1}^{n}\frac{1}{\nu_i}\Big)^{\sum_{i=1}^{n}\frac{1}{\nu_i}-n}\prod_{i=1}^{n}[(\iota_i-\delta_0)(\varsigma_i-\delta_0)]^{\frac{1}{\varpi_i}}.$$

Proof. We can type

$$\begin{aligned}\vartheta_i^{h_i}(\Im_i,\delta_i) &= \vartheta_i^{h_i}(\Im_i,\delta_i)-\vartheta_i^{h_i}(\delta_0,\delta_i)-\vartheta_i^{h_i}(\Im_i,\delta_0)+\vartheta_i^{h_i}(\delta_0,\delta_0)\\ &= \int_{\delta_0}^{\Im_i}(\vartheta_i^{h_i}(\xi_i,\delta_i))^{\Delta_1^{\alpha}}\Delta_1^{\alpha}\xi_i-\int_{\delta_0}^{\Im_i}(\vartheta_i^{h_i}(\xi_i,\delta_0))^{\Delta_1^{\alpha}}\Delta^{\alpha}\xi_i\end{aligned}$$

$$
\begin{aligned}
&= \int_{\delta_0}^{\Im_i} [(\vartheta_i^{h_i}(\xi_i,\delta_i))^{\Delta_1^\alpha} - (\vartheta_i^{h_i}(\xi_i,\delta_0))^{\Delta_1^\alpha}]\Delta^\alpha \xi_i \\
&\leqslant \int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i} (h_i\vartheta_i^{h_i-1}(\xi_i,\eta_i).\vartheta_i^{\Delta_1^\alpha}(\xi_i,\eta_i))^{\Delta_2^\alpha}\Delta^\alpha\eta_i\Delta^\alpha\xi_i \\
&= \int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i} \vartheta_i^{\Delta_{12}^\alpha}(\xi_i,\eta_i)\Delta^\alpha\eta_i\Delta^\alpha\xi_i. \qquad (3.11)
\end{aligned}
$$

From (3.11) applying (2.5) and (3.12), gets

$$
\begin{aligned}
\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i,\delta_i)| &\leqslant \prod_{i=1}^n \int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i} |\vartheta_i^{\Delta_{12}^\alpha}(\xi_i,\eta_i)|\Delta_1^\alpha\eta_i\Delta_2^\alpha\xi_i \\
&\leqslant \prod_{i=1}^n [(\Im_i-\delta_0)(\delta_i-\delta_0)]^{\frac{1}{\varpi_i}}\Big(\int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i} |\vartheta_i^{\Delta_{12}^\alpha}(\xi_i,\eta_i)|^{\nu_i}\Delta^\alpha\eta_i\Delta^\alpha\xi_i\Big)^{\frac{1}{\nu_i}}.
\end{aligned} \qquad (3.12)
$$

Using inequality (3.3), we get that

$$
\frac{\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i,\delta_i)|}{\Big(\sum_{i=1}^n \frac{(\Im_i-\delta_0)(\delta_i-\delta_0)}{\varpi_i}\Big)^{\sum_{i=1}^n\frac{1}{\varpi_i}}} \leqslant \Big(n-\sum_{i=1}^n\frac{1}{\nu_i}\Big)^{\sum_{i=1}^n\frac{1}{\nu_i}-n}\prod_{i=1}^n\Big(\int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i}|\vartheta_i^{\Delta_{12}^\alpha}(\xi_i,\eta_i)|^{\nu_i}\Delta^\alpha\eta_i\Delta^\alpha\xi_i\Big)^{\frac{1}{\nu_i}}. \qquad (3.13)
$$

Integrating (3.13) on $\Im_i$ and δ_i, applying (2.5) and Fubini's theorem, yields

$$
\begin{aligned}
&\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}\cdots\int_{\delta_0}^{\iota_n}\int_{\delta_0}^{\varsigma_n} \frac{\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i,\delta_i)|}{\Big(\sum_{i=1}^n \frac{(\Im_i-\delta_0)(\delta_i-\delta_0)}{\varpi_i}\Big)^{\sum_{i=1}^n\frac{1}{\varpi_i}}}\Delta^\alpha\delta_n\Delta^\alpha\Im_n\ldots\Delta^\alpha\delta_1\Delta^\alpha\Im_1 \\
&\leqslant \Big(n-\sum_{i=1}^n\frac{1}{\nu_i}\Big)^{\sum_{i=1}^n\frac{1}{\nu_i}-n} \\
&\quad\times\prod_{i=1}^n\Big(\int_{\delta_0}^{\iota_i}\int_{\delta_0}^{\varsigma_i}\Big(\int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i}|\vartheta_i^{\Delta_{12}^\alpha}(\xi_i,\eta_i)|^{\nu_i}\Delta^\alpha\eta_i\Delta^\alpha\xi_i\Big)^{\frac{1}{\nu_i}}\Delta^\alpha\delta_i\Delta^\alpha\Im_i\Big) \\
&\leqslant \Big(n-\sum_{i=1}^n\frac{1}{\nu_i}\Big)^{\sum_{i=1}^n\frac{1}{\nu_i}-n} \\
&\quad\times\prod_{i=1}^n[(\iota_i-\delta_0)(\varsigma_i-\delta_0)]^{\frac{1}{\varpi_i}}\Big(\int_{\delta_0}^{\iota_i}\int_{\delta_0}^{\varsigma_i}\Big(\int_{\delta_0}^{\Im_i}\int_{\delta_0}^{\delta_i}|\vartheta_i^{\Delta_{12}^\alpha}(\xi_i,\eta_i)|^{\nu_i}\Delta^\alpha\eta_i\Delta^\alpha\xi_i\Big)\Delta^\alpha\delta_i\Delta^\alpha\Im_i\Big)^{\frac{1}{\nu_i}} \\
&= C\prod_{i=1}^n\Big(\int_{\delta_0}^{\iota_i}\int_{\delta_0}^{\varsigma_i}(\iota_i-\Im_i)(\varsigma_i-\delta_i)|\vartheta_i^{\Delta_{12}^\alpha}(\Im_i,\delta_i)|^{\nu_i}\Delta^\alpha\delta_i\Delta^\alpha\Im_i\Big)^{\frac{1}{\nu_i}}. \qquad (3.14)
\end{aligned}
$$

By exploiting the fact $\iota_i \leqslant \sigma(\iota_i)$, gives

$$
\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}\cdots\int_{\delta_0}^{\iota_n}\int_{\delta_0}^{\varsigma_n} \frac{\prod_{i=1}^n |\vartheta_i^{h_i}(\Im_i,\delta_i)|}{\Big(\sum_{i=1}^n \frac{(\Im_i-\delta_0)(\delta_i-\delta_0)}{\varpi_i}\Big)^{\sum_{i=1}^n\frac{1}{\varpi_i}}}\Delta^\alpha\delta_n\Delta^\alpha\Im_n\ldots\Delta^\alpha\delta_1\Delta^\alpha\Im_1
$$

$$\leqslant C\prod_{i=1}^{n}\Big(\int_{\delta_0}^{\iota_i}\int_{\delta_0}^{\varsigma_i}(\sigma(\iota_i)-\sigma(\Im_i))(\sigma(\varsigma_i)-\sigma(\delta_i))|\vartheta_i^{\Delta_{12}^{\alpha}}(\Im_i,\delta_i)|^{\nu_i}\Delta^{\alpha}\delta_i\Delta^{\alpha}\Im_i\Big)^{\frac{1}{\nu_i}}.$$

This concludes the evidence. □

Remark 3.7. In Theorem 3.2, if we take $\mathbb{T}=\mathbb{Z},\alpha=1,h_i=1$, got results thanks to the authors [3, Theorem 1.2].

Remark 3.8. In Theorem 3.2, supposing $\mathbb{T}=\mathbb{R},\alpha=1$, got results thanks the authors [3, Theorem 1.4].

Corollary 3.3. Taking $n=2$ and $h_1=h_2=1$ in Theorem 3.2, we have

$$\vartheta_1^{\Delta_{12}^{\alpha}}(\Im_1,\delta_1)=\vartheta^{\Delta_2^{\alpha}\Delta_1^{\alpha}}(\Im_1,\delta_1),\qquad \vartheta_2^{\Delta_{12}^{\alpha}}(\Im_1,\delta_1)=\vartheta^{\Delta_2^{\alpha}\Delta_1^{\alpha}}(\Im_2,\delta_2).$$

Moreover, if $\nu_1,\nu_2>1$ satisfy $\frac{1}{\nu_1}+\frac{1}{\nu_2}\geqslant 1$ and $0<\lambda=2-\frac{1}{\nu_1}-\frac{1}{\nu_2}=\frac{1}{\varpi_1}+\frac{1}{\varpi_2}\leqslant 1$, inequality (3.10) reduces to

$$\begin{aligned}
&\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}\Big(\int_{\delta_0}^{\iota_2}\int_{\delta_0}^{\varsigma_2}\frac{|\vartheta_1(\Im_1,\delta_1)||\vartheta_2(\Im_2,\delta_2)|}{(\nu_1(\Im_1-\delta_0)(\delta_1-\delta_0)+\varpi_1(\Im_2-\delta_0)(\delta_2-\delta_0))^{\lambda}}\Delta^{\alpha}\Im_2\Delta^{\alpha}\delta_2\Big)\Delta^{\alpha}\Im_1\Delta^{\alpha}\delta_1\\
&\leqslant\frac{1}{(\lambda\varpi_1\varpi_2)^{\lambda}}\Big[(\iota_1-\delta_0)(\varsigma_1-\delta_0)\Big]^{\frac{1}{\varpi_1}}\Big[(\iota_2-\delta_0)(\varsigma_2-\delta_0)\Big]^{\frac{\varpi_1-1}{\varpi_1}}\\
&\times\Big(\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}(\sigma(\iota_1)-\sigma(\Im_1))(\sigma(\varsigma_1)-\sigma(\delta_1))|\vartheta^{\Delta_2^{\alpha}\Delta_1^{\alpha}}(\Im_1,\delta_1)|^{\nu_1}\Delta^{\alpha}\Im_1\Delta^{\alpha}\delta_1\Big)^{\frac{1}{\nu_1}}\\
&\Big(\int_{\delta_0}^{\iota_2}\int_{\delta_0}^{\varsigma_2}(\sigma(\iota_2)-\delta_0)(\sigma(\varsigma_2)-\delta_0)|\vartheta^{\Delta_2^{\alpha}\Delta_1^{\alpha}}(\Im_2,\delta_2)|^{\nu_2}\Delta^{\alpha}\Im_2\Delta^{\alpha}\delta_2\Big)^{\frac{1}{\nu_2}}.
\end{aligned}\tag{3.15}$$

Remark 3.9. In a unique scenario, if we take $\mathbb{T}=\mathbb{R}$ in Corollary 3.3, the inequality (3.15) reduces to

$$\begin{aligned}
&\int_{0}^{\iota_1}\int_{0}^{\varsigma_1}\Big(\int_{0}^{\iota_2}\int_{0}^{\varsigma_2}\frac{|\vartheta_1(\Im_1,\delta_1)||\vartheta_2(\Im_2,\delta_2)|}{\Big(\nu_1\Im_1\delta_1+\varpi_1\Im_2\delta_2\Big)^{\lambda}}d\Im_2 d\delta_2\Big)d\Im_1 d\delta_1\\
&\leqslant\frac{1}{(\lambda\varpi_1\varpi_2)^{\lambda}}[\iota_1\varsigma_1]^{\frac{1}{\varpi_1}}[\iota_2\varsigma_2]^{\frac{\varpi_1-1}{\varpi_1}}\\
&\times\Big(\int_{0}^{\iota_1}\int_{0}^{\varsigma_1}(\iota_1-\Im_1)(\varsigma_1-\delta_1)|D_1D_2\vartheta_1(\Im_1,\delta_1)|^{\nu_1}d\Im_1 d\delta_1\Big)^{\frac{1}{\nu_1}}\\
&\times\Big(\int_{0}^{\iota_2}\int_{0}^{\varsigma_2}(\iota_2-\Im_2)(\varsigma_2-\delta_2)|D_1D_2\vartheta_2(\Im_2,\delta_2)|^{\nu_2}d\Im_2 d\delta_2\Big)^{\frac{1}{\nu_2}}.
\end{aligned}\tag{3.16}$$

Remark 3.10. In a unique scenario, if we take $\mathbb{T}=\mathbb{Z}$ in Corollary 3.3, the inequality (3.15) reduces to

$$\begin{aligned}
&\sum_{\Im_1=1}^{m_1}\sum_{\delta_1=1}^{n_1}\Big(\sum_{\Im_2=1}^{m_2}\sum_{\delta_2=1}^{n_2}\frac{|a_1(\Im_1,\delta_1)||a_2(\Im_2,\delta_2)|}{\Big(\nu_1\Im_1\delta_1+\varpi_1\Im_2\delta_2\Big)^{\lambda}}\Big)\\
&\leqslant\frac{1}{(\lambda\varpi_1\varpi_2)^{\lambda}}[m_1n_1]^{\frac{1}{\varpi_1}}[m_2n_2]^{\frac{\varpi_1-1}{\varpi_1}}
\end{aligned}$$

$$\times\Big(\sum_{\Im_1=1}^{m_1}\sum_{\delta_1=1}^{n_1}(n_1-\delta_1)(m_1-\Im_1)|\Delta_1^\alpha\Delta_2^\alpha a_1(\Im_1,\delta_1)|^{\nu_1}\Big)^{\frac{1}{\nu_1}} \tag{3.17}$$
$$\times\quad\Big(\sum_{\Im_2=1}^{m_2}\sum_{\delta_2=1}^{n_2}(n_2-\delta_2)(m_2-\Im_2))|\Delta_1^\alpha\Delta_2^\alpha a_2(\Im_2,\delta_2)|^{\nu_2}\Big)^{\frac{1}{\nu_2}}.$$

Corollary 3.4. In Corollary 3.3, if $\lambda=1$, then $\frac{1}{\nu_1}+\frac{1}{\nu_2}=\frac{1}{\varpi_1}+\frac{1}{\varpi_2}=1$ and take $\nu_1=\varpi_2$, $\nu_2=\varpi_1$. In this case the inequality (3.15) reduces to

$$\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}\Big(\int_{\delta_0}^{\iota_2}\int_{\delta_0}^{\varsigma_2}\frac{|\vartheta_1(\Im_1,\delta_1)||\vartheta_2(\Im_2,\delta_2)|}{(\nu_1(\Im_1-\delta_0)(\delta_1-\delta_0)+\varpi_1(\Im_2-\delta_0)(\delta_2-\delta_0))}\Delta^\alpha\Im_2\Delta^\alpha\delta_2\Big)\Delta^\alpha\Im_1\Delta^\alpha\delta_1$$
$$\leqslant\frac{1}{\nu_1\varpi_1}\Big[(\iota_1-\delta_0)(\varsigma_1-\delta_0)\Big]^{\frac{\nu_1-1}{\nu_1}}\Big[(\iota_2-\delta_0)(\varsigma_2-\delta_0)\Big]^{\frac{\varpi_1-1}{\varpi_1}}$$
$$\times\Big(\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\varsigma_1}(\sigma(\iota_1)-\sigma(\Im_1))(\sigma(\varsigma_1)-\sigma(\delta_1))|\vartheta^{\Delta_2^\alpha\Delta_1^\alpha}(\Im_1,\delta_1)|^{\nu_1}\Delta^\alpha\Im_1\Delta^\alpha\delta_1\Big)^{\frac{1}{\nu_1}} \tag{3.18}$$
$$\Big(\int_{\delta_0}^{\iota_2}\int_{\delta_0}^{\varsigma_2}(\sigma(\iota_2)-\delta_0)(\sigma(\varsigma_2)-\delta_0)|\vartheta^{\Delta_2^\alpha\Delta_1^\alpha}(\Im_2,\delta_2)|^{\nu_2}\Delta^\alpha\Im_2\Delta^\alpha\delta_2\Big)^{\frac{1}{\nu_2}}.$$

Remark 3.11. In Corollary 3.4, if $\mathbb{T}=\mathbb{R},\alpha=1$ we obtain an equivalent formulation of the inequality that Pachpatte presented in [2, Theorem 4].

Remark 3.12. In Corollary 3.4, if $\mathbb{T}=\mathbb{Z},\alpha=1$ we obtain an equivalent formulation of the inequality that Pachpatte presented in [2, Theorem 3].

Theorem 3.3. Let $\mathbb{T}$ be a time scale with $\delta_0,\iota_{ij},\tau_{ij},\delta_{ij}\in\mathbb{T}$, $(i,j=1,\dots,n)$. Let $\nu_i,\varpi_i>1$, be constants and $\frac{1}{\nu_i}+\frac{1}{\varpi_i}=1$. Let $\vartheta_i(\tau_{1i},\dots,\tau_{ni})$ be real valued nth Δ^α-differentiable functions also that defined on $[\delta_0,\iota_{1i})_{\mathbb{T}}\times\cdots\times[\delta_0,\iota_{ni})_{\mathbb{T}}$, where $\delta_0\leqslant\iota_{ji}\leqslant\delta_{ji}$, $\delta_{ji}\in(0,\infty)$ and $i,j=1,\dots,n$. Suppose

$$\vartheta_i(\iota_{1i},\dots,\iota_{ni})=\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\frac{\partial^n}{\Delta^\alpha\tau_{1i}\dots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\dots,\tau_{ni})\Delta^\alpha\tau_{ni}\dots\Delta^\alpha\tau_{1i},$$

then

$$\int_{\delta_0}^{\delta_{11}}\cdots\int_{\delta_0}^{\delta_{n1}}\int_{\delta_0}^{\delta_{12}}\cdots\int_{\delta_0}^{\delta_{n2}}\cdots\int_{\delta_0}^{\delta_{1n}}\cdots\int_{\delta_0}^{\delta_{nn}} \tag{3.19}$$
$$\frac{\prod_{i=1}^n\Big(\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\Big|\frac{\partial^n}{\Delta^\alpha\tau_{1i}\dots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i}\dots\tau_{ni})\Big|^{\nu_i}\Delta^\alpha\tau_{ni}\dots\Delta^\alpha\tau_{1i}\Big)^{\frac{1}{\nu_i}}}{\Big(\sum_{i=1}^n\frac{\big[(\iota_{1i}-\delta_0)\dots(\iota_{ni}-\delta_0)\big]}{\varpi_i}\Big)^{\sum_{i=1}^n\frac{1}{\varpi_i}}}$$
$$\Delta^\alpha\iota_{11}\dots\Delta^\alpha\iota_{n1}\dots\Delta^\alpha\iota_{12}\dots\Delta^\alpha\iota_{n2}\dots\Delta^\alpha\iota_{1n}\dots\Delta^\alpha\iota_{nn}$$
$$\leqslant N\prod_{i=1}^n\Big(\int_{\delta_0}^{\delta_{1i}}\cdots\int_{\delta_0}^{\delta_{ni}}\prod_{j=1}^n(\sigma(\delta_{ji})-\iota_{ji})\Big|\frac{\partial^n}{\Delta^\alpha\iota_{1i}\dots\Delta^\alpha\iota_{ni}}\vartheta_i(\iota_{1i},\dots,\iota_{ni})\Big|^{\nu_i}\Delta^\alpha\iota_{1i}\dots\Delta^\alpha\iota_{ni}\Big)^{\frac{1}{\nu_i}},$$

where

$$N=N(\delta_{1i},\dots,\delta_{ni})\Big(n-\sum_{i=1}^n\frac{1}{\nu_i}\Big)^{\sum_{i=1}^n\frac{1}{\nu_i}-n}\prod_{i=1}^n\left[(\delta_{1i}-\delta_0)\dots(\delta_{ni}-\delta_0)\right]^{\frac{1}{\varpi_i}}.$$

Proof. From hypothesis of Theorem 3.3, we have

$$|\vartheta_i(\iota_{1i},\ldots,\iota_{ni})| \leqslant \int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i}\ldots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}. \tag{3.20}$$

On other hand, by using (3.3) and Hölder's dynamic inequality, we obtain

$$\begin{aligned}
&\prod_{i=1}^{n}|\vartheta_i(\iota_{1i},\ldots,\iota_{ni})|\\
&\leqslant \prod_{i=1}^{n}\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i}\ldots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}\\
&\leqslant \prod_{i=1}^{n}\left[(\iota_{1i}-\delta_0)\ldots(\iota_{ni}-\delta_0)\right]^{\frac{1}{\varpi_i}}\\
&\qquad\times\left(\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i},\ldots,\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|^{\nu_i}\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}\right)^{\frac{1}{\nu_i}}\\
&\leqslant \frac{\left(\sum_{i=1}^{n}\frac{\left[(\iota_{1i}-\delta_0)\ldots(\iota_{ni}-\delta_0)\right]}{\varpi_i}\right)^{\sum_{i=1}^{n}\frac{1}{\varpi_i}}}{\left(n-\sum_{i=1}^{n}\frac{1}{\nu_i}\right)^{n-\sum_{i=1}^{n}\frac{1}{\nu_i}}}\\
&\qquad\times\prod_{i=1}^{n}\left(\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i}\ldots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|^{\nu_i}\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}\right)^{\frac{1}{\nu_i}}.
\end{aligned} \tag{3.21}$$

Divide (3.21) by $\left(\sum_{i=1}^{n}\frac{\left[(\iota_{1i}-\delta_0)\ldots(\iota_{ni}-\delta_0)\right]}{\varpi_i}\right)^{\sum_{i=1}^{n}\frac{1}{\varpi_i}}$, then integrating over ι_{ji} from δ_0 to δ_{ji} $(i,j=1,\ldots,n)$, respectively, using dynamic Hölder's inequality and using the information $\sigma(n)\geqslant n$, we obtain

$$\begin{aligned}
&\int_{\delta_0}^{\delta_{11}}\cdots\int_{\delta_0}^{\delta_{n1}}\int_{\delta_0}^{\delta_{12}}\cdots\int_{\delta_0}^{\delta_{n2}}\cdots\int_{\delta_0}^{\delta_{1n}}\cdots\int_{\delta_0}^{\delta_{nn}}\\
&\frac{\prod_{i=1}^{n}\left(\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i}\ldots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|^{\nu_i}\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}\right)^{\frac{1}{\nu_i}}}{\left(\sum_{i=1}^{n}\frac{\left[(\iota_{1i}-\delta_0)\ldots(\iota_{ni}-\delta_0)\right]}{\varpi_i}\right)^{\sum_{i=1}^{n}\frac{1}{\varpi_i}}}\\
&\Delta^\alpha\iota_{11}\ldots\Delta^\alpha\iota_{n1}\Delta^\alpha\iota_{12}\ldots\Delta^\alpha\iota_{n2}\ldots\Delta^\alpha\iota_{1n}\ldots\Delta^\alpha\iota_{nn}\\
&\leqslant\left(n-\sum_{i=1}^{n}\frac{1}{\nu_i}\right)^{\sum_{i=1}^{n}\frac{1}{\nu_i}-n}\\
&\times\prod_{i=1}^{n}\int_{\delta_0}^{\delta_{1i}}\cdots\int_{\delta_0}^{\delta_{ni}}\left(\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i},\ldots,\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|^{\nu_i}\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}\right)^{\frac{1}{\nu_i}}\Delta^\alpha\iota_{ni}\ldots\Delta^\alpha\iota_{1i}\\
&\leqslant\left(n-\sum_{i=1}^{n}\frac{1}{\nu_i}\right)^{\sum_{i=1}^{n}\frac{1}{\nu_i}-n}\prod_{i=1}^{n}\left[(\delta_{1i}-\delta_0)\ldots(\delta_{ni}-\delta_0)\right]^{\frac{1}{\varpi_i}}\\
&\qquad\left(\int_{\delta_0}^{\delta_{1i}}\cdots\int_{\delta_0}^{\delta_{ni}}\left(\int_{\delta_0}^{\iota_{1i}}\cdots\int_{\delta_0}^{\iota_{ni}}\left|\frac{\partial^n}{\Delta^\alpha\tau_{1i}\ldots\Delta^\alpha\tau_{ni}}\vartheta_i(\tau_{1i},\ldots,\tau_{ni})\right|^{\nu_i}\Delta^\alpha\tau_{ni}\ldots\Delta^\alpha\tau_{1i}\right)\Delta^\alpha\iota_{ni}\ldots\Delta^\alpha\iota_{1i}\right)^{\frac{1}{\nu_i}}
\end{aligned}$$

$$= N\prod_{i=1}^{n}\left(\int_{\delta_0}^{\delta_{1i}}\cdots\int_{\delta_0}^{\delta_{ni}}\prod_{j=1}^{n}(\delta_{ji}-\iota_{ji})\left|\frac{\partial^n}{\Delta^\alpha\iota_{1i}\ldots\Delta^\alpha\iota_{ni}}\vartheta_i(\iota_{1i},\ldots,\iota_{ni})\right|^{\nu_i}\Delta^\alpha\iota_{ni}\ldots\Delta^\alpha\iota_{1i}\right)^{\frac{1}{\nu_i}}$$
$$\leqslant N\prod_{i=1}^{n}\left(\int_{\delta_0}^{\delta_{1i}}\cdots\int_{\delta_0}^{\delta_{ni}}\prod_{j=1}^{n}(\sigma(\delta_{ji})-\iota_{ji})\left|\frac{\partial^n}{\Delta^\alpha\iota_{1i}\ldots\Delta^\alpha\iota_{ni}}\vartheta_i(\iota_{1i},\ldots,\iota_{ni})\right|^{\nu_i}\Delta^\alpha\iota_{ni}\ldots\Delta^\alpha\iota_{1i}\right)^{\frac{1}{\nu_i}}.$$

This concludes the evidence. □

Remark 3.13. In Theorem 3.3, supposing $\mathbb{Z} = \mathbb{T}, andwith\alpha = 1$, obtains [4, Theorem 3.1].
Remark 3.14. In Theorem 3.3, supposing $\mathbb{R} = \mathbb{T}, andwith\alpha = 1$, obtains [4, Theorem 3.2].
Corollary 3.5. Let $\vartheta_i(\iota_{1i},\ldots,\iota_{ni})$ change to $\vartheta_i(\mathfrak{I}_i)$ in Theorem 3.3 and in view of $\vartheta_i(\delta_0) = 0$, $(i = 1,\ldots,n)$, then

$$\int_{\delta_0}^{\iota_1}\int_{\delta_0}^{\iota_2}\cdots\int_{\delta_0}^{\iota_n}\frac{\prod_{i=1}^{n}|\vartheta_i(\mathfrak{I}_i)|}{\left(\sum_{i=1}^{n}\frac{(\mathfrak{I}_i-\delta_0)}{\varpi_i}\right)^{\sum_{i=1}^{n}\frac{1}{\varpi_i}}}\Delta^\alpha\mathfrak{I}_n\Delta^\alpha\mathfrak{I}_{n-1}\ldots\Delta^\alpha\mathfrak{I}_1$$
$$\leqslant R\prod_{i=1}^{n}\left(\int_{\delta_0}^{\iota_i}(\sigma(\iota_i)-\sigma(\mathfrak{I}_i))\left|\vartheta_i^{\Delta^\alpha}(\mathfrak{I}_i)\right|^{\nu_i}\Delta^\alpha\tau_i\Delta^\alpha\mathfrak{I}_i\right)^{\frac{1}{\nu_i}}, \tag{3.22}$$

where

$$R = \left(n-\sum_{i=1}^{n}\frac{1}{\nu_i}\right)^{\sum_{i=1}^{n}\frac{1}{\nu_i}-n}\prod_{i=1}^{n}(\iota_i-\delta_0)^{\frac{1}{\varpi_i}}.$$

Remark 3.15. Taking $n = 2$, in Corollary 3.5, if ν_1, $\nu_2 > 1$ are such that $\frac{1}{\nu_1}+\frac{1}{\nu_2} \geqslant 1$ and $0 < \lambda = 2-\frac{1}{\nu_1}-\frac{1}{\nu_2} = \frac{1}{\varpi_1}+\frac{1}{\varpi_2} \leqslant 1$, inequality (3.22) reduces to inequality (3.6).

4. Conclusions

In this work, we used Holder's inequality to prove a number of Hilbert's inequalities on the time scale. Some integer and discrete inequalities were obtained as special cases of the results. This work builds on the multiple inequalities reported by Pachpatte in 1998 and his 2000 and by Handley et al. and by Zhao et al. in 2013. In the future, one can generalize the results proved here by using diamond alpha calculus and also by trying to get the inverse inequalities.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdul-Aziz University, Jeddah, KSA, under grant No. (DF-772-245-1441). The authors, therefore gratefully acknowledge DSR technical and financial support.

Conflict of interest

The authors declare that there is no competing interest.

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