Research Article

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On the convergence, stability and data dependence results of the JK iteration process in Banach spaces

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Abstract: This article analyzes the JK iteration process with the class of mappings that are essentially endowed with a condition called condition (E). The convergence of the iteration toward a fixed point of a specific mapping satisfying the condition (E) is obtained under some possible mild assumptions. It is worth mentioning that the iteration process JK converges better toward a fixed point compared to some prominent iteration processes in the literature. This fact is confirmed by a numerical example. Furthermore, it has been shown that the iterative scheme JK is stable in the setting of generalized contraction. The data dependence result is also established. Our results are new in the iteration theory and extend some recently announced results of the literature.

Keywords: JK- iteration, Garcia-Falset map, strong convergence, weak convergence, data dependence, stability, Banach space

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1 Introduction

Fixed point theory offers possibly alternative and some useful techniques and tools for solving those problems of applied sciences for which ordinary analytical methods are too much time consumer (or even they fails to solve them analytically). In fact, the known techniques of the fixed point theory suggests that a problem under consideration should be expressed in an operator equation form to obtain the fact that the solution set of the problem and the fixed point set of the operator that involved in the equation become same (for details, see [1–5]). In this way, the fixed point theory not only suggests the existence of a solution for such problems but also provides an algorithm to estimate the approximate value of such existed solutions. For example, the

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famous result due to Banach states that any contraction operator whose domain is essentially a closed subset of a Banach space will always admit a unique fixed point and the approximate value of this fixed point can be found by implementing the Picard iteration on the operator. This result become a very powerful tool for existence and approximation of the solutions of various problems in applied sciences. The Picard iteration is the simple iteration process and easy as a computation point of view. However, among the other aspects of Picard iteration, in [6], Krasnoselskii was the first who noticed the divergence of the Picard iterates in the case of nonexpansive operators. A well-known class of nonexpansive operators is a natural generalization of the class of contractions, and these operators are very extensively arise in many problems of science especially related to image processing. Furthermore, the speed at which the Picard iteration converges to a fixed point of any mappings is also very slow. Due to extensive use and slow rate of convergence of these operators suggest to use other iterative process like Mann iteration due to Mann [7], Ishikawa iteration due to Ishikawa [8], Noor iteration due to Noor [9], S iteration due to Agarwal et al. [10], Abbas iteration due to Abbas and Nazir [11], Thakur iteration due to Thakur et al. [12], M iteration due to Ullah and Arshad [13], and generalization of Thakur and Ishikawa by Saleem et al. [14] that converges very well in the class of nonexpansive operators and are more effective in the sense of rate of convergence as compared to the Picard iteration. It was proved in [12], that Thakur iteration is better than all the aforementioned iteration processes. Soon, Ullah and Arshad proved that M iteration is better than the leading Thakur iteration and hence all the aforementioned iterations. This article will show that there is an iteration process which is even better than all of these iterations.

In [15], Ahmad et al. suggested an up-to-date iteration, which they called JK (Junaid-Kifayat) iteration that generates a sequence of iterates using the following formula:

$$\begin{aligned} a_1 &= a \in K, \\ c_\xi &= (1-\beta_\xi)a_\xi + \beta_\xi \mathrm{F} a_\xi, \\ b_\xi &= \mathrm{F} c_\xi, \\ a_{\xi+1} &= \mathrm{F}((1-\alpha_\xi)b_\xi + \alpha_\xi \mathrm{F} b_\xi), \end{aligned} \tag{1.1}$$

where $\alpha_\xi, \beta_\xi \in (0,1)$.

They proved that speed of convergence of JK iteration process (1.1) is far better than the leading iteration processes due to Thakur et al. [12] and Agarwal et al. [10] in the case of Suzuki mappings (mappings with conditions (C)). In this article, we deeply studied this process as follows: first, we prove the convergence of this scheme in the class of mappings with condition (E), and in this way, we extend the previous outcome presented by Ahmad et al. [15] from mappings with condition (C) to mappings with condition (E). After this, we shall show it with the help of an example of the mapping with condition (E), which will show that the speed of the JK iteration is better than all of the Thakur et al. [12], Agarwal et al. [10], and M iteration of Ullah and Arshad [13]. Eventually, the stability and data dependence results of the iteration JK in the generalized setting of contractions are obtained.

Let K be a nonempty, closed, and convex subset of a Banach space S, a mapping $\mathrm{F} : K \to K$ is said to be contraction, if and only if for each $a, b \in K$, there exists a real number $\lambda \in [0,1)$, such that $\|\mathrm{F}a - \mathrm{F}b\| \le \lambda\|a - b\|$. Similarly, a mapping $\mathrm{F} : K \to K$ is said to be nonexpansive if $\lambda = 1$ in contraction mapping, that is, it satisfies the inequality $\|\mathrm{F}a - \mathrm{F}b\| \le \|a - b\|$, *for each* $a, b \in K$. Moreover, if $F_\Gamma \neq \emptyset$, where F_Γ represents the collection of all fixed points of F, and $\|\mathrm{F}a - p\| \le \|a - p\|$, *for each* $a \in K$ and $p \in F_\Gamma$, then F is called quasi-nonexpansive.

In 2010, Bosede and Rhoades [16] developed the concept of general class of mapping called contractive-like operators to prove the stability and convergence results for the Picard-Mann hybrid iterative process given below;

$$\|p - \mathrm{F}b\| \le \lambda\|p - b\|; \quad \forall b \in K, \quad \text{and} \quad p \in F_\Gamma.$$

Or,

$$\|\mathrm{F}b - p\| \le \lambda\|b - p\|; \quad \forall b \in K, \quad \text{and} \quad p \in F_\Gamma, \tag{1.2}$$

where $\lambda \in [0,1)$. We will regularly use (1.2) to establish the stability and strong convergence results for our proposed iteration.

The class of nonexpansive mappings plays a key role in the field of applied sciences; therefore, it is very important to study the extension of this mapping. In 2008, Suzuki [17] established the extension of nonexpansive mapping by weakening the inequality involved in the definition of nonexpansive mappings in the following manner; a self-map $\mathrm{F} : K \to K$ is said to satisfy condition (C), if for each $a, b \in K$, we have

$$\frac{1}{2}\|a - \mathrm{F}a\| \leq \|a - b\| \Rightarrow \|\mathrm{F}a - \mathrm{F}b\| \leq \|a - b\|. \tag{1.3}$$

Of course, every nonexpansive mapping F endowed with condition (C), but in [17], Suzuki also developed an example in which the converse is not valid in general. Hence, we can say that the class of Suzuki mappings is wider than the class of nonexpansive mappings. Keeping the concept of Suzuki (C) map in mind, Garcia-Falset et al. [18] established the concept of new map, which is more general than the Suzuki map and can be defined as follows: a self-map $\mathrm{F} : K \to K$ is called a Garcia-Falset map (or condition (E_μ)) if for every two points $a, b \in K$, we have

$$\|a - \mathrm{F}b\| \leq \mu\|a - \mathrm{F}a\| + \|a - b\|, \quad \text{for some } \mu \geq 1. \tag{1.4}$$

Garcia-Falset mentioned that any mapping F with condition (E_μ) is called simply a mapping with condition (E).

Since every Suzuki mapping F defined on a subset K of a Banach space S satisfies $\|a - \mathrm{F}b\| \leq 3\|a - \mathrm{F}a\| + \|a - b\|$, for each $a, b \in K$. From this, we can easily see that every Suzuki mapping is also Gracia-Falset mapping with real number $\mu = 3$. At the end of this article, we have constructed an example, which will be satisfies condition (E) but not condition (C). After the discovery of Garcia-Falset mappings, Bagherboum [19] was the first person who studied approximation of fixed points for these mappings under Ishikawa iteration in a certain distance space.

2 Preliminaries

We shall now provide some notions and results that are known in the literature, and we need here in our proposed work.

Notice that, a uniformly convex Banach space (UCBS) (see [20] for details) is simply a Banach space S having a property that for every chosen $\varepsilon \in (0, 2]$ and eventually a positive real number δ can be selected, so that one has:

$$a, b \in S, \|a\| \leq 1, \quad \|b\| \leq 1 \quad \text{and} \quad \|a - b\| \geq \varepsilon \Rightarrow \left\|\frac{a + b}{2}\right\| \leq 1 - \delta.$$

Also, if K is closed and convex in a UCBS S, and consider any bounded sequence $\{a_\xi\}$ in S, for a fixed element $a \in S$, we essentially define the asymptotic center of $\{a_\xi\}$ at the point a by the following formula:

$$r(a, \{a_\xi\}) = \limsup_{\xi\to\infty}\|a - a_\xi\|.$$

In this case, we define the asymptotic radius of $\{a_\xi\}$ corresponding to the set K by

$$r(K, \{a_\xi\}) = \inf\{r(a, \{a_\xi\}) : a \in K\},$$

and the asymptotic center of sequence $\{a_\xi\}$ relative to K is given by,

$$A(K, \{a_\xi\}) = \inf\{a \in K : r(a, \{a_\xi\}) = r(K, \{a_\xi\})\}.$$

In [21], Edelstein proved in this case that the set $A(K, \{a_\xi\})$ admits one and only one point.

Definition 2.1. [22] A UCBS S is said to be with Opial's condition if for all $\{a_\xi\} \subseteq S$ and suppose there is a point $l \in S$ such that $\{a_\xi\}$ is weakly convergent to l and the following holds:

$$\liminf_{\xi\to\infty}\|a_\xi - l\| < \liminf_{\xi\to\infty}\|a_\xi - m\|, \quad \text{for all } m \in S - \{l\}.$$

The following notion of condition (I) is essentially from Senter and Dotson [23], which provides an alternative tool for establishing a strong convergence of some iteration processes on noncompact domains.

Definition 2.2. A mapping F of a subset K in a UCBS is said to be with a condition (I) when one can set a function, namely, η such that $\eta(h) = 0$ if and only if $h = 0$, and $\eta(h) > 0$ for every choice of $h > 0$, and $\|a - \mathrm{F}a\| \geq \eta(d(a, F_\Gamma))$, for all $a \in K$, where $d(a, F_\Gamma)$ denotes distance of a from the set F_Γ.

Lemma 2.3. [24] *Assume that S is a UCBS and $0 < a \leq \gamma_\xi \leq b < 1$, $\forall\ \xi \in \mathbb{N}$. If $\{u_\xi\}$ and $\{v_\xi\}$ are two sequences in S with $\lim_{\xi\to\infty}\|u_\xi\| \leq d$, $\limsup_{\xi\to\infty}\|v_\xi\| \leq d$, and $\limsup_{\xi\to\infty}\|(1-\gamma_\xi)u_\xi + \gamma_\xi v_\xi\| = d$ holds for some $d \geq 0$. Then $\lim_{\xi\to\infty}\|u_\xi - v_\xi\| = 0$.*

Proposition 2.4. [18] *Assume that $K \neq \varnothing$ is a subset of a Banach space S, and F is a self-map on K. Then*
(*i*) *If F is a Suzuki mapping, then it is Garcia-Falset mapping.*
(*ii*) *If F is a Garcia-Falset mapping with $F_\Gamma \neq \varnothing$, then for every $a \in K$ and $p \in F_\Gamma$, we have $\|\mathrm{F}a - \mathrm{F}p\| \leq \|a - p\|$.*

Lemma 2.5. [18] *Assume that F is a self-map on a $K \neq \varnothing$ subset of a Banach space S enjoys the* Opial*'s property, and suppose that F also endowed with the condition (E). If $\{b_\xi\}$ is a sequence weakly converges to p and $\lim_{\xi\to\infty}\|\mathrm{F}a_\xi - a\| = 0$, then $\mathrm{F}p = p$. That is, $I - \mathrm{F}$ is demiclosed.*

3 Approximation results

Main purpose of this section is to suggest some theorems concerning the JK iteration (1.1) for mappings with condition (E). We start with a very key result. This result is valid in general Banach space setting.

Lemma 3.1. *Assume that S is UCBS and $\varnothing \neq K \subseteq S$, is closed as well as convex. Suppose $\mathrm{F} : K \to K$ is a mapping satisfying condition (E) with $F_\Gamma \neq \varnothing$. If $\{a_\xi\}$ is a sequence given in* (1.1). *Then $\limsup_{\xi\to\infty}\|a_\xi - p\|$ exists for all $p \in F_\Gamma$.*

Proof. To prove our objective, let $p \in F_\Gamma$, then by Proposition 2.4 (ii), we have,

$$\begin{aligned}\|c_\xi - p\| &= \|(1-\beta_\xi)a_\xi + \beta_\xi \mathrm{F}a_\xi - p\|\\ &\leq (1-\beta_\xi)\|a_\xi - p\| + \beta_\xi\|\mathrm{F}a_\xi - p\|\\ &\leq (1-\beta_\xi)\|a_\xi - p\| + \beta_\xi\|a_\xi - p\|\\ &= \|a_\xi - p\|.\end{aligned} \tag{3.1}$$

Now using (3.1), we have

$$\|b_\xi - p\| \leq \|\mathrm{F}c_\xi - p\| \leq \|c_\xi - p\|. \tag{3.2}$$

Similarly using (3.2), we obtain as follows:

$$\begin{aligned}\|a_{\xi+1} - p\| &= \|\mathrm{F}((1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi \mathrm{F}b_\xi) - p\|\\ &\leq \|(1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi\mathrm{F}b_\xi - p\|\\ &\leq (1-\alpha_\xi)\|\mathrm{F}c_\xi - p\| + \alpha_\xi\|\mathrm{F}b_\xi - p\|\\ &\leq (1-\alpha_\xi)\|c_\xi - p\| + \alpha_\xi\|b_\xi - p\|\\ &\leq (1-\alpha_\xi)\|c_\xi - p\| + \alpha_\xi\|c_\xi - p\|\\ &= \|c_\xi - p\|\\ &\leq \|a_\xi - p\|.\end{aligned} \tag{3.3}$$

From the aforementioned observation, we see that the sequence $\{\|a_\xi - p\|\}$ is bounded and non-increasing. Hence, $\lim_{\xi\to\infty}\|a_\xi - p\|$ exists for all $p \in F_\Gamma$. □

Theorem 3.2. *Suppose K, S, F and $\{a_\xi\}$ are as given in Lemma 3.1. Then, $F_\Gamma \neq \emptyset$ if and only if $\{a_\xi\}$ is bounded and $\lim_{\xi\to\infty}\|\mathrm{F}a_\xi - a_\xi\| = 0$.*

Proof. Let us first prove the direct statement. We assume that $\{a_\xi\}$ is bounded and $\lim_{\xi\to\infty}\|\mathrm{F}a_\xi - a\| = 0$. We need to prove that $F_\Gamma \neq \emptyset$. For this, we choose any $p \in A(K, \{a_\xi\})$, and we prove that $\mathrm{F}p = p$. Since F is Garcia-Falset mapping, so

$$\begin{aligned} r(\mathrm{F}p, a_\xi) &= \limsup_{\xi\to\infty}\|a_\xi - \mathrm{F}a_\xi\| \\ &\leq \limsup_{\xi\to\infty}(\mu\|a_\xi - \mathrm{F}a_\xi\| + \|a_\xi - p\|) \\ &= \|a_\xi - p\| \\ &= r(p, \{a_\xi\}). \end{aligned}$$

Clearly we see that $\mathrm{F}p \in A(K, \{a_\xi\})$, but we know that $A(K, \{a_\xi\})$ contains exactly one point. Therefore, we conclude that $\mathrm{F}p = p$. Hence, $p \in F_\Gamma$.

Let us prove the converse implication. Assume that $F_\Gamma \neq \emptyset$, we need to prove that $\{a_\xi\}$ is bounded and $\|\mathrm{F}a_\xi - a\| = 0$ for all $p \in F_\Gamma$. Boundedness of $\{a_\xi\}$ is obvious from the conclusion of Lemma 3.1. Choose any $p \in F_\Gamma$, since by Lemma 3.1, $\|a_\xi - p\|$ exists for every $p \in F_\Gamma$. Put

$$\lim_{\xi\to\infty}\|a_\xi - p\| = i.$$

From (3.1), of Lemma 3.1, we have proved that

$$\|c_\xi - p\| \leq \|a_\xi - p\|. \tag{3.4}$$

Applying limsup on both sides of (3.4), we obtain

$$\limsup_{\xi\to\infty}\|c_\xi - p\| \leq \limsup_{\xi\to\infty}\|a_\xi - p\| = i. \tag{3.5}$$

From Proposition 2.4 (ii), we have,

$$\limsup_{\xi\to\infty}\|\mathrm{F}a_\xi - p\| \leq \limsup_{\xi\to\infty}\|a_\xi - p\| = i. \tag{3.6}$$

Equation (3.3) of Lemma 3.1 also suggests that

$$\|a_{\xi+1} - p\| \leq \|c_\xi - p\| \Rightarrow i \leq \liminf_{\xi\to\infty}\|c_\xi - p\|. \tag{3.7}$$

Combining (3.5) and (3.7), we obtain

$$i = \lim_{\xi\to\infty}\|c_\xi - p\|. \tag{3.8}$$

From (1.1) and (3.8), we have

$$i = \lim_{\xi\to\infty}\|(1 - \beta_\xi)(a_\xi - p) + \beta_\xi(\mathrm{F}a_\xi - p)\|. \tag{3.9}$$

Now for $u_\xi = (a_\xi - p)$, $v_\xi = (\mathrm{F}a_\xi - p)$, and $\gamma_\xi = \beta_\xi$, then by Lemma 2.3, we obtain

$$\lim_{\xi\to\infty}\|a_\xi - \mathrm{F}a_\xi\| = 0.$$

Hence, proof is completed. □

Now, we can prove a weak convergence result for Garcia-Falset mappings as follows.

Theorem 3.3. *Suppose K, S, F, and $\{a_\xi\}$ are as given in Lemma 3.1. If F satisfies Opial's property, then $\{a_\xi\}$ converges weakly to a point of F_Γ.*

Proof. By Lemma 3.1, the strong limit $\lim_{\xi\to\infty}\|a_\xi - p\|$ exists even for all choices of $p \in F_\Gamma$. We need weak limit point of any subsequence of $\{a_\xi\}$ in F_Γ. Let's choose two weak limits say ρ_1 and ρ_2 of the subsequences $\{a_{\xi_j}\}$ and $\{a_{\xi_k}\}$ of $\{a_\xi\}$, respectively. By Theorem 3.2, $\lim_{\xi\to\infty}\|a_\xi - \mathrm{F}a_\xi\| = 0$, and by Lemma 2.5, $I - \mathrm{F}$ is demiclosed at zero. Therefore, $(I - \mathrm{F})\rho_1 = 0 \Rightarrow \rho_1 = \mathrm{F}\rho_1$, similarly $\mathrm{F}\rho_2 = \rho_2$.

Next we just show the uniqueness of weak limit point. For this, if $\rho_1 \neq \rho_2$, then by using Opial's property, we have

$$\begin{aligned}\lim_{\xi\to\infty}\|a_\xi - \rho_1\| &= \lim_{\xi_j\to\infty}\|a_{\xi_j} - \rho_1\| \\ &< \lim_{\xi_j\to\infty}\|a_{\xi_j} - \rho_2\| \\ &= \lim_{\xi\to\infty}\|a_\xi - \rho_2\| \\ &= \lim_{\xi_k\to\infty}\|a_{\xi_k} - \rho_2\| \\ &< \lim_{\xi_k\to\infty}\|a_{\xi_k} - \rho_1\| \\ &= \lim_{\xi\to\infty}\|a_\xi - \rho_1\|.\end{aligned}$$

Obviously this is a contradiction, so $\rho_1 = \rho_2$. This imply that $\{a_\xi\}$ weakly converges to some point of F_Γ. □

Next, we will prove some strong convergence results for self-mapping F.

Theorem 3.4. *Suppose K, S,* F, *and $\{a_\xi\}$ are as given in Lemma* 3.1. *If K is compact, then the sequence $\{a_\xi\}$ converges strongly to a fixed point of* F.

Proof. K is compact, and due to the compactness of K, one can find a subsequence, say, $\{a_{\xi_j}\}$ of $\{a_\xi\}$, having a limit say p_0 such that $\lim_{j\to\infty}\|a_{\xi_j} - p_0\| = 0$. Since F is endowed with the condition (E), therefore

$$\|a_{\xi_j} - \mathrm{F}p_0\| \leq \mu\|a_{\xi_j} - \mathrm{F}a_{\xi_j}\| + \|a_{\xi_j} - p_0\|. \tag{3.10}$$

Taking $\lim_{j\to\infty}$ on both sides of (3.10) and using Theorem 3.2, we obtain

$$\lim_{j\to\infty}\|a_{\xi_j} - \mathrm{F}p_0\| = 0, \quad \text{that is}, \quad a_{\xi_j} \to \mathrm{F}p_0.$$

Since $a_{\xi_j} \to \mathrm{F}p_0$ and $a_{\xi_j} \to p_0$, we prove that $\mathrm{F}p_0 = p_0$. For this purpose,

$$\|\mathrm{F}p_0 - p_0\| \leq \|\mathrm{F}p_0 - a_{\xi_j}\| + \|a_{\xi_j} - p_0\| = \|a_{\xi_j} - \mathrm{F}p_0\| + \|a_{\xi_j} - p_0\| = \longrightarrow 0 + 0 = 0.$$

Hence, we obtain $\|\mathrm{F}p_0 - p_0\| = 0$. It follows that $\mathrm{F}p_0 = p_0$, that is $p_0 \in F_\Gamma$. By Lemma 3.1, $\lim_{\xi\to\infty}\|a_\xi - p_0\|$ exists for all $p_0 \in F_\Gamma$. Hence, p_0 is the strong limit point of F. This completes the proof. □

Theorem 3.5. *Suppose K, S,* F, *and $\{a_\xi\}$ are as given in Lemma* 3.1. *If* $\liminf_{\xi\to\infty} d(a_\xi, F_\Gamma) = 0$. *Then $\{a_\xi\}$ converges strongly to a point of F_Γ.*

Proof. The proof of the aforementioned theorem is elementary, and therefore, we do not need to prove it. □

Next, we established strong convergence result for a mapping F, which satisfies condition (I).

Theorem 3.6. *Suppose K, S,* F, *and $\{a_\xi\}$ are as given in Lemma* 3.1. *If* F *is endowed with condition (I), then $\{a_\xi\}$ converges strongly to a point of F_Γ.*

Proof. To prove our objective, since proof of Lemma 3.1, we suggest that

$$\lim_{\xi\to\infty}\|a_\xi - \mathrm{F}a_\xi\| = 0. \tag{3.11}$$

By condition (I), we have

$$\|a - \mathrm{F}a\| \geq \eta(d(a, F_\Gamma)), \quad \forall a \in K. \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$0 \leq \lim_{\xi\to\infty}\eta(d(a_\xi, F_\Gamma)) \leq \lim_{\xi\to\infty}\|a_\xi - \mathrm{F}a_\xi\| = 0 \Rightarrow \lim_{\xi\to\infty}\eta(d(a_\xi, F_\Gamma)) = 0. \tag{3.13}$$

Since $\eta : [0, \infty) \to [0, \infty)$ is nondecreasing function satisfying $\eta(h) = 0$ if and only if $h = 0$, and $\eta(h) > 0$ for every choice of $h > 0$, so we have

$$\lim_{\xi\to\infty} d(a_\xi, F_\Gamma) = 0.$$

Hence, all the requirements of Theorem 3.4 are satisfied, and therefore, we conclude that $\{a_\xi\}$ strongly converges to some point of F_Γ. □

4 Stability and data dependency

In many cases, an iteration procedure is not stable when we implement it on a problem to find the approximate value of the requested solution. Notice that a fixed point scheme is said to be stable if the produced error arising between any two successive iterative steps do not affect the convergence of the scheme. First time, the concept of F-stability (or simply stability) was suggested by Harder and Hicks [25]. Some details and results about the stability are presented below (see, also [26,27] and others).

Definition 4.1. [25] Assuming that K is a nonempty subset of a Banach space and $\{l_\xi\}_{n=0}^{\infty}$ is any sequence in K. In this case, the iterative scheme $a_{\xi+1} = f(\mathrm{F}, a_\xi)$ that is convergent to a point a, is said to be F-stable (or simply stable), if for any sequence $\varepsilon_\xi \in [0, \infty)$ with $\varepsilon_\xi = \|l_{\xi+1} - f(\mathrm{F}, l_\xi)\|$, if one has

$$\lim_{\xi\to\infty} \varepsilon_\xi = 0 \;\Leftrightarrow\; \lim_{\xi\to\infty} l_\xi = a.$$

Definition 4.2. [28] Suppose that $(S, \|.\|)$ is a Banach space. If one have two mappings, namely, $\mathrm{F}, \tilde{\mathrm{F}} : S \to S$, then the mapping $\tilde{\mathrm{F}}$ is known as an approximate mapping for F if and only if for any $\varepsilon > 0$, the following holds:

$$\|\mathrm{F}a - \tilde{\mathrm{F}}a\| \le \varepsilon, \quad \forall a \in S.$$

Lemma 4.3. [29] *Supposes $\{\psi_\xi\}$ is a non-negative real sequence for which one assumes there exists $\xi_0 \in \mathbb{N}$, such that for all $\xi \ge \xi_0$, the following inequality holds*:

$$\psi_{\xi+1} \le (1 - \phi_\xi)\psi_\xi + \phi_\xi\varphi_\xi,$$

where ϕ_ξ contained in $(0, 1)$, for every choice of $\xi \in \mathbb{N}$, $\sum_{\xi=0}^{\infty}\phi_\xi = \infty$ and $\varphi_\xi \ge 0$, for all $\xi \in \mathbb{N}$

$$0 \le \limsup_{\xi\to\infty} \psi_\xi \le \limsup_{\xi\to\infty} \varphi_\xi.$$

Next, we will prove strong convergence theorem for iteration procedure (1.1) with respect to the contraction mapping and show that our proposed iteration is converging effectively compared to all listed iteration processes. We will also show that our proposed iteration process is F-stable by using inequality (1.2).

Theorem 4.4. *Suppose $\emptyset \ne K \subseteq S$, such that K is closed and convex, and $\mathrm{F} : S \to S$ is a contraction mapping. Suppose $\{a_\xi\}$ is the iterative sequence generated by* (1.1), *with $\alpha_\xi, \beta_\xi \in (0, 1)$, satisfying $\sum_{\xi=0}^{\infty}\alpha_\xi < \infty$, $\sum_{\xi=0}^{\infty}\beta_\xi < \infty$ and $\sum_{\xi=0}^{\infty}\alpha_\xi\beta_\xi < \infty$. Then, $\{a_\xi\}$ strongly converges to the unique fixed point of* F.

Proof. From Equation (1.1), let $p = \mathrm{F}p$.

$$\begin{aligned}\|c_\xi - p\| &= \|(1 - \beta_\xi)a_\xi + \beta_\xi\mathrm{F}a_\xi - p\| \\ &\le (1 - \beta_\xi)\|a_\xi - p\| + \beta_\xi\|\mathrm{F}a_\xi - p\| \\ &\le (1 - \beta_\xi)\|a_\xi - p\| + \beta_\xi\theta\|a_\xi - p\| \\ &= (1 - \beta_\xi(1 - \theta))\|a_\xi - p\|.\end{aligned} \tag{4.1}$$

Similarly using (4.1), we have

$$\|b_\xi - p\| = \|\mathrm{F}c_\xi - \mathrm{F}p\| \le \theta\|c_\xi - p\| \le \theta(1 - \beta_\xi(1-\theta))\|a_\xi - p\|. \tag{4.2}$$

Hence, using (4.2), we obtain

$$\begin{aligned}\|a_{\xi+1} - p\| &= \|\mathrm{F}((1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi \mathrm{F}b_\xi) - p\| \\ &\le \theta\|(1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi \mathrm{F}b_\xi - p\| \\ &\le \theta[(1-\alpha_\xi)\|\mathrm{F}c_\xi - p\| + \alpha_\xi\|\mathrm{F}b_\xi - p\|] \\ &\le \theta[(1-\alpha_\xi)\theta\|c_\xi - p\| + \alpha_\xi\theta\|b_\xi - p\|] \\ &\le \theta^2[(1-\alpha_\xi)(1-\beta_\xi(1-\theta))\|a_\xi - p\| + \alpha_\xi\theta(1-\beta_\xi(1-\theta))\|a_\xi - p\|] \\ &= \theta^2[(1-\alpha_\xi(1-\theta))(1-\beta_\xi(1-\theta))]\|a_\xi - p\|.\end{aligned} \tag{4.3}$$

Now, by repetition of the aforementioned procedure (4.3), we can easily see that

$$\begin{aligned}\|a_{\xi+1} - p\| &\le \theta^{2+2}\prod_{k=\xi-1}^{\xi}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))\|a_{k-1} - p\| \\ &\le \theta^{2+2+2}\prod_{k=\xi-2}^{\xi}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))\|a_{k-2} - p\| \\ &\;\vdots \\ &\le \theta^{2(\xi+1)}\prod_{k=0}^{\xi}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))\|a_0 - p\|.\end{aligned} \tag{4.4}$$

Since $\sum_{\xi=0}^{\infty}\alpha_\xi < \infty$, $\sum_{\xi=0}^{\infty}\beta_\xi < \infty$ and $\theta \in [0,1)$, so $\sum_{\xi=0}^{\infty}\alpha_\xi\beta_\xi < \infty$. Thus,

$$\lim_{\xi\to\infty}\theta^{2(\xi+1)}\prod_{k=0}^{\xi}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta)) = \lim_{\xi\to\infty}\theta^{2(\xi+1)}\lim_{\xi\to\infty}\prod_{k=0}^{\xi}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta)) = 0. \tag{4.5}$$

By using (4.5), in the inequality (4.4), we obtain

$$\lim_{\xi\to\infty}\|a_{\xi+1} - p\| \le 0. \tag{4.6}$$

Since $\|.\| \ge 0$, therefore from (4.6), we can write as follows:

$$\lim_{\xi\to\infty}\|a_{\xi+1} - p\| = 0.$$

Hence, $\{a_\xi\}$ strongly converges to $p \in F_\Gamma$, as required. □

Theorem 4.5. *Suppose* K, S, *and* F *are as in Theorem* 4.4. *Let's consider the iterative sequence* (1.1), *with* $\alpha_\xi, \beta_\xi \in (0,1)$, *satisfying* $\sum_{\xi=0}^{\infty}\alpha_\xi < \infty$, $\sum_{\xi=0}^{\infty}\beta_\xi < \infty$, *and* $\sum_{\xi=0}^{\infty}\alpha_\xi\beta_\xi < \infty$. *Then the iterative process* (1.1), *is* F-*stable.*

Proof. Since self-mapping F on K with a fixed point p satisfying (1.2), that is,

$$\|\mathrm{F}b - p\| \le \lambda\|b - p\|; \quad \forall b \in K, \quad \text{and} \quad p \in F_\Gamma. \tag{4.7}$$

Assume that $\{l_\xi\} \subset K$ be an arbitrary sequence. Let the sequence generated by (1.1), $a_{n+1} = f(\mathrm{F}, a_\xi)$, converging to a unique fixed point p (by Theorem 4.4, and $\varepsilon_\xi \in [0,1)$ such that $\varepsilon_\xi = \|l_{\xi+1} - f(\mathrm{F}, l_\xi)\|$. Let $\mathrm{F}_{l_\xi} = f(\mathrm{F}, l_\xi)$. Then $\varepsilon_\xi = \|l_{\xi+1} - \mathrm{F}_{l_\xi}\|$.

To show that the iteration process (1.1), is F − *stable*, it is suffice to prove that $\lim_{\xi\to\infty}\varepsilon_\xi = 0$ if and only if $\lim_{\xi\to\infty} l_\xi = p$. For this, suppose $\lim_{\xi\to\infty} l_\xi = p$, we have

$$\varepsilon_\xi = \|l_{\xi+1} - f(\mathrm{F}, l_\xi)\| \le \|l_{\xi+1} - p\| + \|p - f(\mathrm{F}, l_\xi)\| = \|l_{\xi+1} - p\| + \|\mathrm{F}_{l_\xi} - p\|. \tag{4.8}$$

So by using (1.2), in (4.8), we have

$$\varepsilon_\xi \le \|l_{\xi+1} - p\| + \theta\|l_\xi - p\|. \tag{4.9}$$

Taking $\xi \to \infty$ of (4.9), and using given supposition, we obtain

$$\lim_{\xi\to\infty}\varepsilon_\xi = 0.$$

Let us prove the converse implication, consider $\lim_{\xi\to\infty}\varepsilon_\xi = 0$. We need to prove that $\lim_{\xi\to\infty} l_\xi = p$.

$$\|l_{\xi+1} - p\| \le \|l_{\xi+1} - f(\mathrm{F}, l_\xi)\| + \|f(\mathrm{F}, l_\xi) - p\| = \varepsilon_\xi + \|\mathrm{F}_{l_\xi} - p\| \le \varepsilon_\xi + \theta\|l_\xi - p\|. \tag{4.10}$$

Since from (4.4), of Theorem 4.4, we have

$$\|l_{\xi+1} - p\| \le \varepsilon_\xi + \theta^{2(\xi+1)}\prod_{k=0}^{\xi}(1 - \alpha_k(1-\theta))(1 - \beta_k(1-\theta))\|l_0 - p\|.$$

Since $\lim_{\xi\to\infty}\varepsilon_\xi = 0$, $\sum_{\xi=0}^{\infty}\alpha_\xi < \infty$, $\sum_{\xi=0}^{\infty}\beta_\xi < \infty$, and $\theta \in [0,1)$, so $\lim_{\xi\to\infty}\|l_{\xi+1} - p\| = 0$, this imply that, $\lim_{\xi\to\infty} l_\xi = p$.

Hence, $\{a_\xi\}$ is F- stable. □

Theorem 4.6. *Suppose K, S, and F are as in Theorem 4.4. Consider $\tilde{\mathrm{F}}$ being an approximate operator for the contraction mapping F with possible numerical error $\varepsilon > 0$. Further, $\{a_\xi\}$ is the sequence (1.1), for F, and define approximate sequence $\{\tilde{a}_\xi\}$ for $\tilde{\mathrm{F}}$ as follows:*

$$\begin{aligned} \tilde{a}_1 &= \tilde{a} \in K \\ \tilde{c}_\xi &= (1-\beta_\xi)\tilde{a}_\xi + \beta_\xi\tilde{\mathrm{F}}\tilde{a}_\xi \\ \tilde{b}_\xi &= \tilde{\mathrm{F}}\tilde{c}_\xi \\ \tilde{a}_{\xi+1} &= \tilde{\mathrm{F}}((1-\alpha_\xi)\tilde{\mathrm{F}}\tilde{c}_\xi + \alpha_\xi\tilde{\mathrm{F}}\tilde{b}_\xi). \end{aligned} \tag{4.11}$$

with real sequence $\{\alpha_\xi\}, \{\beta_\xi\} \in (0,1)$ satisfying, $\frac{1}{2} \le \alpha_\xi\beta_\xi$, for all $\xi \in \mathbb{N}$.

If $\mathrm{F}p = p$ and $\tilde{\mathrm{F}}\tilde{p} = \tilde{p}$ such that $\lim_{\xi\to\infty}\tilde{a}_\xi = \tilde{p}$, then we have

$$\|p - \tilde{p}\| \le \frac{6\varepsilon}{(1-\theta)}.$$

Proof. Let us consider (1.1) and (4.11), we have

$$\begin{aligned} \|c_\xi - \tilde{c}_\xi\| &\le (1-\beta_\xi)\|a_\xi - \tilde{a}_\xi\| + \beta_\xi\|\mathrm{F}a_\xi - \tilde{\mathrm{F}}\tilde{a}_\xi\| \\ &\le (1-\beta_\xi)\|a_\xi - \tilde{a}_\xi\| + \beta_\xi\{\|\mathrm{F}a_\xi - \mathrm{F}\tilde{a}_\xi\| + \|\mathrm{F}\tilde{a}_\xi - \tilde{\mathrm{F}}\tilde{a}_\xi\|\} \\ &\le (1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + \beta_\xi\varepsilon. \end{aligned} \tag{4.12}$$

By using (4.12), we have

$$\begin{aligned} \|b_\xi - \tilde{b}_\xi\| &= \|\mathrm{F}c_\xi - \tilde{\mathrm{F}}\tilde{c}_\xi\| \\ &\le \|\mathrm{F}c_\xi - \mathrm{F}\tilde{c}_\xi\| + \|\mathrm{F}\tilde{c}_\xi - \tilde{\mathrm{F}}\tilde{c}_\xi\| \\ &\le \theta\|c_\xi - \tilde{c}_\xi\| + \varepsilon \\ &\le \theta(1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + \theta\beta_\xi\varepsilon + \varepsilon. \end{aligned} \tag{4.13}$$

$$\begin{aligned} \|a_{\xi+1} - \tilde{a}_{\xi+1}\| &= \|\mathrm{F}((1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi\mathrm{F}b_\xi) - \tilde{\mathrm{F}}((1-\alpha_\xi)\tilde{\mathrm{F}}\tilde{c}_\xi + \alpha_\xi\tilde{\mathrm{F}}\tilde{b}_\xi)\| \\ &\le \|\mathrm{F}((1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi\tilde{\mathrm{F}}b_\xi) - \tilde{\mathrm{F}}((1-\alpha_\xi)\tilde{\mathrm{F}}\tilde{c}_\xi + \alpha_\xi\tilde{\mathrm{F}}\tilde{b}_\xi)\| + \|\tilde{\mathrm{F}}((1-\alpha_\xi)\tilde{\mathrm{F}}\tilde{c}_\xi + \alpha_\xi\tilde{\mathrm{F}}\tilde{b}_\xi) \\ &\quad - \tilde{\mathrm{F}}((1-\alpha_\xi)\tilde{\mathrm{F}}\tilde{c}_\xi + \alpha_\xi\tilde{\mathrm{F}}\tilde{b}_\xi)\| \\ &\le \theta\|(1-\alpha_\xi)\mathrm{F}c_\xi + \alpha_\xi\mathrm{F}b_\xi - (1-\alpha_\xi)\tilde{\mathrm{F}}\tilde{c}_\xi + \alpha_\xi\tilde{\mathrm{F}}\tilde{b}_\xi\| + \varepsilon. \\ &\le \theta[(1-\alpha_\xi)\|\mathrm{F}c_\xi - \tilde{\mathrm{F}}\tilde{c}_\xi\| + \alpha_\xi\|\mathrm{F}b_\xi - \tilde{\mathrm{F}}\tilde{b}_\xi\|] + \varepsilon \\ &\le \theta\begin{bmatrix} (1-\alpha_\xi)\{\|\mathrm{F}c_\xi - \tilde{\mathrm{F}}\tilde{c}_\xi\| + \|\tilde{\mathrm{F}}\tilde{c}_\xi - \tilde{\mathrm{F}}\tilde{c}_\xi\|\}+ \\ \alpha_\xi\{\|\mathrm{F}b_\xi - \tilde{\mathrm{F}}\tilde{b}_\xi\| + \|\tilde{\mathrm{F}}\tilde{b}_\xi - \tilde{\mathrm{F}}\tilde{b}_\xi\|\} \end{bmatrix} + \varepsilon. \\ &\le \theta[(1-\alpha_\xi)\{\theta\|c_\xi - \tilde{c}_\xi\| + \varepsilon\} + \alpha_\xi\{\theta\|b_\xi - \tilde{b}_\xi\| + \varepsilon\}] + \varepsilon. \end{aligned}$$

By using (4.12) and (4.13), we obtain

$$\begin{aligned}\|a_{\xi+1} - \tilde{a}_{\xi+1}\| &\le \theta\begin{bmatrix}(1-\alpha_\xi)\theta\{(1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + \beta_\xi\varepsilon\}+\\ \alpha_\xi\theta\{\theta(1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + \theta\beta_\xi\varepsilon + \varepsilon\}+\\ (1-\alpha_\xi)\varepsilon + \alpha_\xi\theta\end{bmatrix} + \varepsilon.\\ &= \theta\begin{bmatrix}\{(1-\alpha_\xi)\theta(1-\beta_\xi(1-\theta)) + \alpha_\xi\theta^2(1-\beta_\xi(1-\theta))\}\|a_\xi - \tilde{a}_\xi\|\\ +\beta_\xi(1-\alpha_\xi)\theta\varepsilon + \alpha_\xi\beta_\xi\theta^2\varepsilon + \alpha_\xi\theta\varepsilon + (1-\alpha_\xi)\varepsilon + \alpha_\xi\varepsilon\end{bmatrix} + \varepsilon.\end{aligned} \tag{4.14}$$

Since $\theta \in (0, 1)$ so $\theta^2 < 1$, from (4.14), we obtain

$$\le \begin{bmatrix}\{(1-\alpha_\xi)(1-\beta_\xi(1-\theta)) + \alpha_\xi(1-\beta_\xi(1-\theta))\}\|a_\xi - \tilde{a}_\xi\|\\ +\beta_\xi(1-\alpha_\xi)\varepsilon + \alpha_\xi\beta_\xi\varepsilon + \alpha_\xi\varepsilon + (1-\alpha_\xi)\varepsilon + \alpha_\xi\varepsilon\end{bmatrix} + \varepsilon. \tag{4.15}$$

$$= \begin{bmatrix}\{(1-\alpha_\xi+\alpha_\xi)(1-\beta_\xi(1-\theta))\}\|a_\xi - \tilde{a}_\xi\|\\ +\{\beta_\xi(1-\alpha_\xi) + \alpha_\xi\beta_\xi + \alpha_\xi + (1-\alpha_\xi) + \alpha_\xi\}\varepsilon\end{bmatrix} + \varepsilon. \tag{4.16}$$

After simplifying (4.16), we obtain,

$$\begin{aligned}\|a_{\xi+1} - \tilde{a}_{\xi+1}\| &\le (1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + (1+\beta_\xi+\alpha_\xi)\varepsilon + \varepsilon\\ &= (1-\beta_\xi(1-\theta_\xi))\|a_\xi - \tilde{a}_\xi\| + (\alpha_\xi+\beta_\xi)\varepsilon + 2\varepsilon\\ &\le (1-\beta_\xi(1-\theta_\xi))\|a_\xi - \tilde{a}_\xi\| + (\alpha_\xi\beta_\xi+\alpha_\xi\beta_\xi)\varepsilon + 2\varepsilon\\ &= (1-\beta_\xi(1-\theta_\xi))\|a_\xi - \tilde{a}_\xi\| + 2\alpha_\xi\beta_\xi\varepsilon + 2\varepsilon\\ &\le (1-\beta_\xi(1-\theta_\xi))\|a_\xi - \tilde{a}_\xi\| + 2\alpha_\xi\beta_\xi\varepsilon + 2(1-\alpha_\xi\beta_\xi+\alpha_\xi\beta_\xi)\varepsilon.\end{aligned} \tag{4.17}$$

Since from assumption, i.e., $\frac{1}{2} \le \alpha_\xi\beta_\xi$, we have that $1 - \alpha_\xi\beta_\xi < \alpha_\xi\beta_\xi$. Hence, from (4.17), we obtain that

$$\begin{aligned}\|a_{\xi+1} - \tilde{a}_{\xi+1}\| &\le (1-\beta_\xi(1-\theta_\xi))\|a_\xi - \tilde{a}_\xi\| + 2\alpha_\xi\beta_\xi\varepsilon + 2(2\alpha_\xi\beta_\xi)\varepsilon\\ &\le (1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + 6\alpha_\xi\beta_\xi\varepsilon\\ &= (1-\beta_\xi(1-\theta))\|a_\xi - \tilde{a}_\xi\| + \alpha_\xi\beta_\xi(1-\theta)\frac{6\varepsilon}{(1-\theta)}.\end{aligned} \tag{4.18}$$

Let $\psi_\xi = \|a_\xi - \tilde{a}_\xi\|$, $\phi_\xi = \alpha_\xi\beta_\xi(1-\theta)$, $\varphi_\xi = \frac{6\varepsilon}{1-\theta}$. Then from Lemma 4.3 and (4.18), we obtain

$$0 \le \limsup_{\xi\to\infty}\|a_\xi - \tilde{a}_\xi\| \le \limsup_{\xi\to\infty}\frac{6\varepsilon}{1-\theta}. \tag{4.19}$$

By Theorem 4.4, we have $\lim_{\xi\to\infty} a_\xi = p$, and by assumption, we have $\lim_{\xi\to\infty}\tilde{a}_\xi = \tilde{p}$. By using these facts together with (4.19), we obtain

$$\|p - \tilde{p}\| \le \frac{6\varepsilon}{1-\theta},$$

as required. □

5 Example and comparative study

To support the objective of this article, we compute an example for a self-map $\mathrm{F} : K \to K$, endowed with condition (E), but without the condition (C). And will compare the efficiency of JK iterative scheme with other existing iterations.

Example 1. Consider $K = [2, 8]$, and a self-map $\mathrm{F} : K \to K$ define by

$$\mathrm{F}a = \begin{cases} \dfrac{a+6}{2}; & \text{if } a \in K_1 = [2, 8) \\ 6; & \text{if } a \in K_2 = \{8\}. \end{cases}$$

To observe that F satisfies condition (E), we will check $\|a - \mathrm{F}b\| \le \nu\|b - \mathrm{F}b\| + \|a - b\|$, for some $\nu \ge 1$, and $\forall a, b \in K$.

For this, we fix the value of $\nu = 4$ and discuss the following cases.

(M_1): Choose any $a, b \in K_2 \Rightarrow \mathrm{F}a = 6 = \mathrm{F}b$. We have,

$$\begin{aligned} \|a - \mathrm{F}b\| &= |a - \mathrm{F}b| \\ &= |a - 6| \\ &= |a - \mathrm{F}a| \\ &\le 4|a - \mathrm{F}a| + |a - b| \\ &= 4\|a - \mathrm{F}a\| + \|a - b\|. \end{aligned} \tag{5.1}$$

(M_2): Choose any $a, b \in K_1 \Rightarrow \mathrm{F}a = \frac{a+6}{2}$ and $\mathrm{F}b = \frac{b+6}{2}$. We have

$$\begin{aligned} \|a - \mathrm{F}b\| &= |a - \mathrm{F}b| \\ &\le |a - \mathrm{F}a| + |\mathrm{F}a - \mathrm{F}b| \\ &= |a - \mathrm{F}a| + \left|\frac{a+6}{2} - \frac{b+6}{2}\right| \\ &= |a - \mathrm{F}a| + \frac{1}{2}|a - b| \\ &\le |a - \mathrm{F}a| + |a - b| \\ &\le 4|a - \mathrm{F}a| + |a - b| \\ &= 4\|a - \mathrm{F}a\| + \|a - b\|. \end{aligned} \tag{5.2}$$

(M_3): Choose any $a \in K_1$ and $b \in K_2 \Rightarrow \mathrm{F}a = \frac{a+6}{2}$ and $\mathrm{F} = 6$.

We have

Table 1: Comparison of iterative values of various iterations

ξ	JK	M	Thakur	S
11	4.5	4.5	4.5	4.5
12	5.8544531200	5.7843700000	5.7285937000	5.4571870000
13	5.9858774000	5.9690039000	5.9508924300	5.8035697200
14	5.9986296600	5.9955443100	5.9911145900	5.9289167900
15	5.9998670300	5.9993594900	5.9983922900	5.9742767600
16	5.9999870900	5.9999079200	5.9997091000	5.9906914000
17	5.9999987400	5.9999867600	5.9999473600	5.9966314500
18	5.9999998700	5.9999980900	5.9999904700	5.9987810000
19	5.9999999800	5.9999997200	5.9999982700	5.9995588700
20	5.9999999990	5.9999999600	5.9999996800	5.9998403600
21	**6**	5.9999999900	5.9999999400	5.9999422300
22	:	5.9999999990	5.9999999900	5.9999790900
23	:	**6**	5.9999999990	5.9999924300
24	:	:	**6**	5.9999972600
25	:	:	:	5.9999990000
26	:	:	:	5.9999996400
27	:	:	:	5.9999998000
28	:	:	:	5.9999999500
29	:	:	:	5.9999999800
30	:	:	:	5.9999999990
30	:	:	:	**6**

$$
\begin{aligned}
\|a - \mathrm{F}b\| &= |a - 6| \\
&= 2\left|\frac{a-6}{2}\right| \\
&= 2\left|a - \frac{a-6}{2}\right| \\
&= 2|a - \mathrm{F}a| \\
&\leq 4|a - \mathrm{F}a| \\
&\leq 4|a - \mathrm{F}a| + |a - b| \\
&= 4\|a - \mathrm{F}a\| + \|a - b\|.
\end{aligned} \tag{5.3}
$$

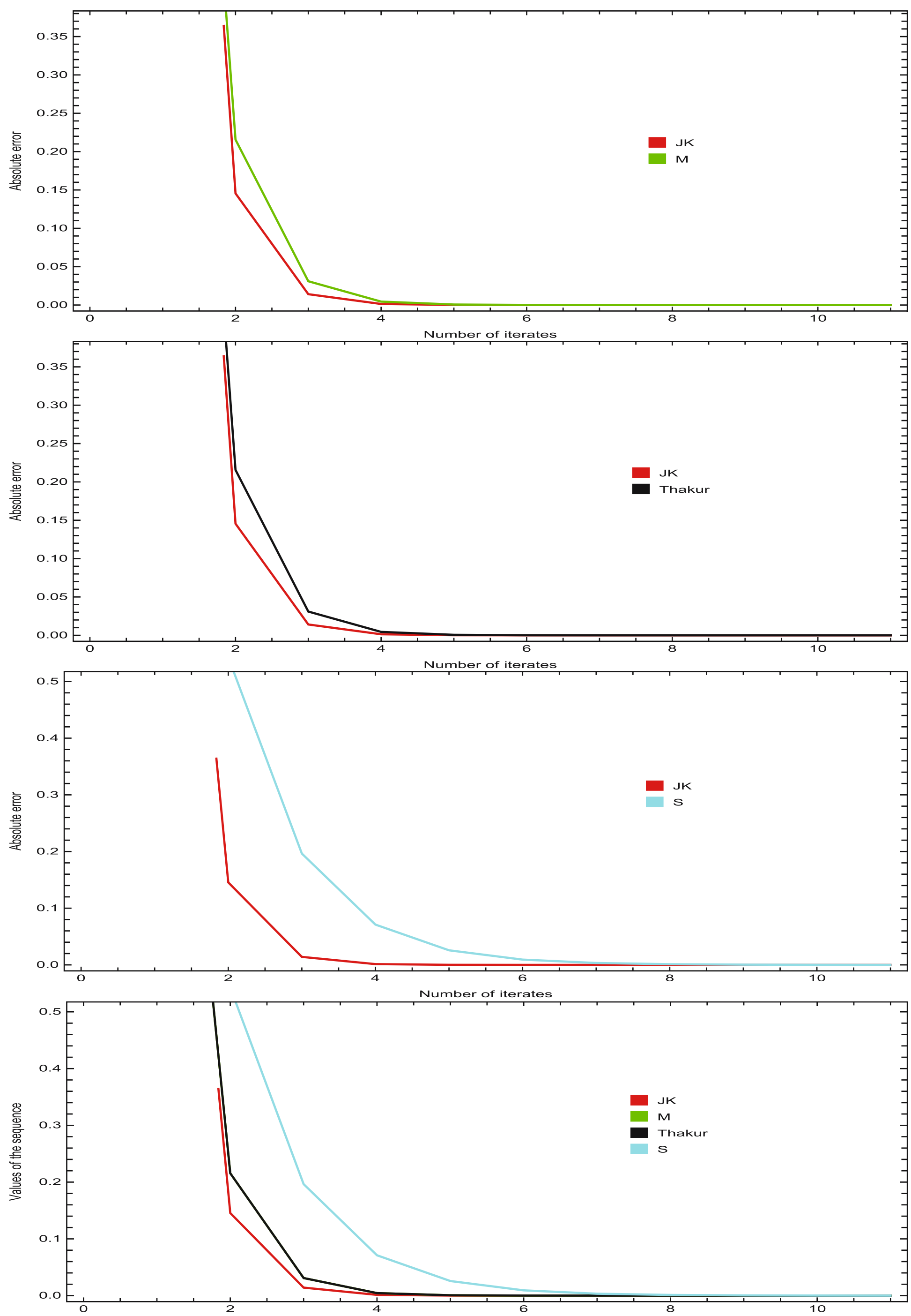

Figure 1: Error comparison of various iteration processes.

(M_4): Choose any $a \in K_2$ and $b \in K_1 \Rightarrow \mathrm{F}a = 6$ and $\mathrm{F}b = \frac{b+6}{2}$.
We have

$$\begin{aligned} \|a - \mathrm{F}b\| &= |a - \mathrm{F}b| \\ &= \left|a - \frac{b+6}{2}\right| \\ &= \left|\frac{2a - b + 6}{2}\right| \\ &\le \left|\frac{a-b}{2}\right| + \left|\frac{a-6}{2}\right| \\ &= \frac{1}{2}|a - \mathrm{F}a| + \frac{1}{2}|a - b| \\ &\le |a - \mathrm{F}a| + |a - b| \\ &\le 4|a - \mathrm{F}a| + |a - b|. \\ &= 4\|a - \mathrm{F}a\| + \|a - b\|. \end{aligned} \tag{5.4}$$

From above, it is obvious that in each case F satisfies condition (E).

Next, we show that for any value of $a, b \in K$, F does not endowed with Condition (C). For this choosing $a = 7.5$ and $b = 8$, it is easily can be seen that $\frac{1}{2}\|a - \mathrm{F}a\| = 0.375 \le \|a - b\| = 0.5$, but $\|\mathrm{F}a - \mathrm{F}b\| = 0.75 > \|a - b\| = 0.5$. Hence $\mathrm{F} : K \to K$ is not a Suzuki mapping.

The comparison of JK with other iterations can be found in Table 1, and graphical representation is shown in Figure 1.

6 Conclusion

The article successfully analyzed the JK iteration process with the so-called broad class of nonlinear mappings that are endowed with a condition (E). The results established under possible mild conditions and successfully supported by a numerical experiment. We have seen that the JK iteration process still suggests very accurate numerical results in the novel setting of mappings having a condition (E). All these results improved the previous results due to Ahmad et al. [15] from the setting of condition (C) to the novel setting of condition (E). We have also proved that JK iteration converges better than the M iteration of Ullah and Arshad [13]. We investigated some qualitative results like stability and data dependence results for the JK iteration process in the setting of generalized contractions. Our results thus at the same time improve the results presented by Abbas and Nazir [11], Thakur et al. [12], Ullah and Arshad [13] because their research areas were limited to the case of mappings with condition (C).

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