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$(\Delta\nabla)^{\nabla}-$Pachpatte Dynamic Inequalities Associated with Leibniz Integral Rule on Time Scales with Applications

Ahmed A. El-Deeb [1,*], Dumitru Baleanu [2,3,*] and Jan Awrejcewicz [4,*]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
[2] Institute of Space Science, 077125 Magurele, Romania
[3] Department of Mathematics, Cankaya University, Ankara 06530, Turkey
[4] Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski St., 90-924 Lodz, Poland
* Correspondence: ahmedeldeeb@azhar.edu.eg (A.A.E.-D.); dumitru@cankaya.edu.tr (D.B.); jan.awrejcewicz@p.lodz.pl (J.A.)

Abstract: We prove some new dynamic inequalities of the Gronwall–Bellman–Pachpatte type on time scales. Our results can be used in analyses as useful tools for some types of partial dynamic equations on time scales and in their applications in environmental phenomena and physical and engineering sciences that are described by partial differential equations.

Keywords: Gronwall's inequality; dynamic inequality; time scales; Leibniz integral rule on time scales

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1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Throughout the article, we assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for any $\tau \in \mathbb{T}$ by

$$\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ for any $\tau \in \mathbb{T}$ by

$$\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}.$$

In the preceding two definitions, we set $\inf\emptyset = \sup\mathbb{T}$ (i.e., if τ is the maximum of $\mathbb{T}$, then $\sigma(\tau) = \tau$) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., if τ is the minimum of $\mathbb{T}$, then $\rho(\tau) = \tau$), where $\emptyset$ denotes the empty set.

The set $\mathbb{T}^{\kappa}$is introduced as follows: If $\mathbb{T}$ has a left-scattered maximum ζ_1, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{\zeta_1\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

The interval $[\theta, \vartheta]$ in $\mathbb{T}$ is defined by

$$[\theta, \vartheta]_{\mathbb{T}} = \{\xi \in \mathbb{T} : \theta \leq \xi \leq \vartheta\}.$$

We define the open intervals and half-closed intervals similarly.

Assume $\chi : \mathbb{T} \to \mathbb{R}$ is a function and $\xi \in \mathbb{T}^{\kappa}$. Then, $\chi^{\Delta}(\xi) \in \mathbb{R}$ is said to be the delta derivative of χ at ξ if, for any $\varepsilon > 0$, there exists a neighborhood U of ξ such that, for every $s \in U$, we have

$$\big|[\chi(\sigma(\xi)) - \chi(s)] - \chi^{\Delta}(\xi)[\sigma(\xi) - s]\big| \leq \varepsilon|\sigma(\xi) - s|.$$

Moreover, χ is said to be delta-differentiable on $\mathbb{T}^{\kappa}$ if it is delta differentiable at every $\xi \in \mathbb{T}^{\kappa}$.

In what follows, we will need the set $\mathbb{T}_{\kappa}$, which is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$.

We introduce the nabla derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}_\kappa$ as follows.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}_\kappa$. We define $f^{\nabla}(t)$ as the real number (provided that it exists) with the property that, for any $\epsilon > 0$, there exists a neighborhood N of t (i.e., $N = (t-\delta, t+\delta)_{\mathbb{T}}$ for some $\delta > 0$) such that

$$\left|[f^{\rho}(t) - f(s)] - f^{\nabla}(t)[\rho(t) - s]\right| \leq \epsilon|\rho(t) - s| \quad for\ every \quad s \in N.$$

We say that $f^{\nabla}(t)$ is the nabla derivative of f at t.

Time scale calculus with the objective to unify discrete and continuous analysis was introduced by S. Hilger [1]. For additional subtleties on time scales, we refer the reader to the books by Bohner and Peterson [2,3].

Gronwall–Bellman-type inequalities, which have many applications in qualitative and quantitative behavior, have been developed by many mathematicians, and several refinements and extensions have been applied to the previous results; we refer the reader to the works [4–14]. For other types of dynamic inequalities on time scales, see [15–23].

Gronwall–Bellman's inequality [24] in the integral form stated the following. Let v and f be continuous and nonnegative functions defined on $[a, b]$, and let v_0 be a nonnegative constant. Then, the inequality

$$v(t) \leq v_0 + \int_a^t f(s)v(s)ds, \quad \text{for all} \quad t \in [a, b], \tag{1}$$

implies that

$$v(t) \leq v_0 \exp\left(\int_a^t f(s)ds\right), \quad \text{for all} \quad t \in [a, b].$$

Baburao G. Pachpatte [25] proved the discrete version of (1). In particular, he proved the following: if $v(n)$, $a(n)$, $\gamma(n)$ are nonnegative sequences defined for $n \in \mathbb{N}_0$ and $a(n)$ is non-decreasing for $n \in \mathbb{N}_0$, and if

$$v(n) \leq a(n) + \sum_{s=0}^{n-1} \gamma(n)v(n), n \in \mathbb{N}_0, \tag{2}$$

then

$$v(n) \leq a(n) \prod_{s=0}^{n-1} [1 + \gamma(n)], n \in \mathbb{N}_0.$$

Bohner and Peterson [2] unify the integral form (2) and the discrete form (1) by introducing a dynamic inequality on a time scale $\mathbb{T}$ as follows: if v, ζ are right-dense continuous functions and $\gamma \geq 0$ is a regressive and right-dense continuous function, then

$$v(t) \leq \zeta(t) + \int_{t_0}^t v(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all} \quad t \in \mathbb{T},$$

which implies

$$v(t) \leq \zeta(t) + \int_{t_0}^t e_{\gamma}(t, \sigma(\eta))\zeta(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all} \quad t \in \mathbb{T},$$

The authors [26] studied the following result:

$$\begin{aligned} \Xi(v(\ell, t)) \quad \leq \quad & a(\ell, t) + \int_0^{\theta(\ell)} \int_0^{\vartheta(t)} \Im_1(\varsigma, \eta)[f(\varsigma, \eta)\zeta(v(\varsigma, \eta))\varpi(v(\varsigma, \eta)) \\ & + \int_0^{\varsigma} \Im_2(\chi, \eta)\zeta(v(\chi, \eta))\varpi(v(\chi, \eta))d\chi\Big] d\eta d\varsigma \end{aligned}$$

where v, f, $\Im \in C(I_1 \times I_2, \mathbb{R}_+)$, $a \in C(\zeta, \mathbb{R}_+)$ are nondecreasing functions, $I_1, I_2 \in \mathbb{R}$, $\theta \in C^1(I_1, I_1), \vartheta \in C^1(I_2, I_2)$ are nondecreasing with $\theta(\ell) \leq \ell$ on I_1, $\vartheta(t) \leq t$ on I_2, $\Im_1$, $\Im_2 \in C(\zeta, \mathbb{R}_+)$, and Ξ, ζ, $\varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\{\Xi, \zeta, \varpi\}(v) > 0$ for $v > 0$, and $\lim_{v \to +\infty} \Xi(v) = +\infty$.

The following theorem was presented by Anderson [27].

$$\varphi(v(t,s)) \leq a(t,s) + c(t,s) \int_{t_0}^{t} \int_{s}^{\infty} \varphi'(v(\tau,\eta))[d(\tau,\eta)w(v(\tau,\eta)) + b(\tau,\eta)]\nabla\eta\Delta\tau, \tag{3}$$

where v, a, c, d are nonnegative continuous functions defined for $(t,s) \in \mathbb{T} \times \mathbb{T}$, and b is a nonnegative continuous function for $(t,s) \in [t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}$ and $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\varphi' > 0$ for $v > 0$.

Theorem 1 ([10]). (Leibniz Integral Rule on Time Scales) *In the following, by $\Upsilon^{\Delta}(r_1, r_2)$, we mean the delta derivative of $\Upsilon(r_1, r_2)$ with respect to r_1. Similarly, $\Upsilon^{\nabla}(r_1, r_2)$ is understood. If Υ, Υ^{Δ} and Υ^{∇} are continuous, and $u, h : \mathbb{T} \to \mathbb{T}$ are delta-differentiable functions, then the following formulas hold $\forall r_1 \in \mathbb{T}^{\kappa}$.*

(i) $\left[\int_{u(r_1)}^{h(r_1)} \Upsilon(r_1, r_2)\Delta r_2\right]^{\Delta} = \int_{u(r_1)}^{h(r_1)} \Upsilon^{\Delta}(r_1, r_2)\Delta r_2 + h^{\Delta}(r_1)\Upsilon(\sigma(r_1), h(r_1)) - u^{\Delta}(r_1) \Upsilon(\sigma(r_1), u(r_1))$;

(ii) $\left[\int_{u(r_1)}^{h(r_1)} \Upsilon(r_1, r_2)\Delta r_2\right]^{\nabla} = \int_{u(r_1)}^{h(r_1)} \Upsilon^{\nabla}(r_1, r_2)\Delta r_2 + h^{\nabla}(r_1)\Upsilon(\rho(r_1), h(r_1)) - u^{\nabla}(r_1) \Upsilon(\rho(r_1), u(r_1))$;

(iii) $\left[\int_{u(r_1)}^{h(r_1)} \Upsilon(r_1, r_2)\nabla r_2\right]^{\Delta} = \int_{u(r_1)}^{h(r_1)} \Upsilon^{\Delta}(r_1, r_2)\nabla r_2 + h^{\Delta}(r_1)\Upsilon(\sigma(r_1), h(r_1)) - u^{\Delta}(r_1) \Upsilon(\sigma(r_1), u(r_1))$;

(iv) $\left[\int_{u(r_1)}^{h(r_1)} \Upsilon(r_1, r_2)\nabla r_2\right]^{\nabla} = \int_{u(r_1)}^{h(r_1)} \Upsilon^{\nabla}(r_1, r_2)\nabla r_2 + h^{\nabla}(r_1)\Upsilon(\rho(r_1), h(r_1)) - u^{\nabla}(r_1) \Upsilon(\rho(r_1), u(r_1))$.

In this article, by employing the results of Theorems 1, we establish the delayed time scale case of the inequalities proven in [26]. Further, these results are proven here to extend some known results in [28–30].

2. Auxiliary Result

We prove the following fundamental lemma that will be needed in our main results.

Lemma 1. *Suppose $\mathbb{T}_1$, $\mathbb{T}_2$ are two times scales and $a \in C(\Omega = \mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R}_+)$ is nondecreasing with respect to $(\wp, t) \in \Omega$. Assume that $\Im$, F, $f \in C(\Omega, \mathbb{R}_+)$, $\ell_1 \in C^1(\mathbb{T}_1, \mathbb{T}_1)$ and $\ell_2 \in C^1(\mathbb{T}_2, \mathbb{T}_2)$ are nondecreasing functions with $\ell_1(\wp) \leq \wp$ on $\mathbb{T}_1$, $\ell_2(t) \leq t$ on $\mathbb{T}_2$. Furthermore, suppose that Ξ, $\zeta \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\Xi, \zeta\}(F) > 0$ for $F > 0$, and $\lim_{F \to +\infty} \Xi(F) = +\infty$. If $F(\wp, t)$ satisfies*

$$\Xi(F(\wp,t)) \leq a(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \zeta(F(\varsigma,\eta)) \Delta\eta \nabla\varsigma \tag{4}$$

for $(\wp, t) \in \Omega$, then

$$F(\wp,t) \leq \Xi^{-1}\left\{G^{-1}\left[G(a(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma\right]\right\} \tag{5}$$

for $0 \leq \wp \leq \wp_1, 0 \leq t \leq t_1$, *where*

$$G(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\zeta(\Xi^{-1}(\varsigma))}, v \geq v_0 > 0,\ G(+\infty) = \int_{v_0}^{+\infty} \frac{\nabla\varsigma}{\zeta(\Xi^{-1}(\varsigma))} = +\infty \tag{6}$$

and $(\wp_1, t_1) \in \Omega$ *is chosen so that*

$$\left(G(a(\wp, t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right) \in \mathrm{Dom}\left(G^{-1}\right).$$

Proof. Suppose that $a(\wp, t) > 0$. Fixing an arbitrary $(\wp_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\wp, t)$ by

$$\psi(\wp, t) = a(\wp_0, t_0) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im(\varsigma, \eta) f(\varsigma, \eta) \zeta(F(\varsigma, \eta)) \Delta\eta \nabla\varsigma, \tag{7}$$

for $0 \leq \wp \leq \wp_0 \leq \wp_1$, $0 \leq t \leq t_0 \leq t_1$, then $\psi(\wp_0, t) = \psi(\wp, t_0) = a(\wp_0, t_0)$ and

$$\Xi(F(\wp, t)) \leq \psi(\wp, t),$$

We obtain

$$F(\wp, t) \leq \Xi^{-1}(\psi(\wp, t)). \tag{8}$$

Taking the ∇-derivative for (7) while employing Theorem 1 (iv), we have

$$\begin{aligned}
\psi^{\nabla_\wp}(\wp, t) &= \ell_1^{\nabla}(\wp) \int_{t_0}^{\ell_2(t)} \Im(\ell_1(\wp), \eta) f(\ell_1(\wp), \eta) \zeta(F(\ell_1(\wp), \eta)) \Delta\eta \\
&\leq \ell_1^{\nabla}(\wp) \int_{t_0}^{\ell_2(t)} \Im(\ell_1(\wp), \eta) f(\ell_1(\wp), \eta) \zeta\left(\Xi^{-1}(\psi(\ell_1(\wp), \eta))\right) \Delta\eta \\
&\leq \zeta\left(\Xi^{-1}(\psi(\ell_1(\wp), \ell_2(t)))\right) \ell_1^{\nabla}(\wp) \int_{t_0}^{\ell_2(t)} \Im(\ell_1(\wp), \eta) f(\ell_1(\wp), \eta) \Delta\eta
\end{aligned} \tag{9}$$

Inequality (9) can be written in the form

$$\frac{\psi^{\nabla_\wp}(\wp, t)}{\zeta(\Xi^{-1}(\psi(\wp, t)))} \leq \ell_1^{\nabla}(\wp) \int_{t_0}^{\ell_2(t)} \Im(\ell_1(\wp), \eta) f(\ell_1(\wp), \eta) \Delta\eta. \tag{10}$$

Taking the ∇-integral for Inequality (10) obtains

$$\begin{aligned}
G(\psi(\wp, t)) &\leq G(\psi(\wp_0, t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \\
&\leq G(a(\wp_0, t_0)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma.
\end{aligned}$$

Since $(\wp_0, t_0) \in \Omega$ is chosen to be arbitrary,

$$\psi(\wp, t) \leq G^{-1}\left[G(a(\wp, t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right]. \tag{11}$$

From (8) and (11), we obtain the desired result (5). We carry out the above procedure with $\epsilon > 0$ instead of $a(\wp, t)$ when $a(\wp, t) = 0$ and subsequently let $\epsilon \to 0$. $\square$

Remark 1. *If we take* $\mathbb{T} = \mathbb{R}$, $\wp_0 = 0$ *and* $t_0 = 0$ *in Lemma 1, then Inequality (4) becomes the inequality obtained in [26] (Lemma 2.1).*

3. Main Results

In the following theorems, with the help of the Leibniz integral rule on time scales, Theorem 1 (item (iv)), and employing Lemma 1, we establish some new dynamics of the Gronwall–Bellman–Pachpatte type on time scales.

Theorem 2. *Let F, a, f, ℓ_1 and ℓ_2 be as in Lemma 1. Let $\Im_1, \Im_2 \in C(\Omega, \mathbb{R}_+)$. If $F(\wp, t)$ satisfies*

$$\begin{aligned}\Xi(F(\wp,t)) \quad \leq \quad & a(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(F(\varsigma,\eta)) \\ & + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(F(\chi,\eta))\Delta\chi\Big]\Delta\eta\nabla\varsigma \end{aligned} \tag{12}$$

for $(\wp, t) \in \Omega$, then

$$F(\wp,t) \leq \Xi^{-1}\left\{G^{-1}\left(p(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right)\right\} \tag{13}$$

for $0 \leq \wp \leq \wp_1, 0 \leq t \leq t_1$, where G is defined by (6) and

$$p(\wp,t) = G(a(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma \tag{14}$$

and $(\wp_1, t_1) \in \Omega$ is chosen so that

$$\left(p(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right) \in \mathrm{Dom}\left(G^{-1}\right).$$

Proof. By the same steps in the proof of Lemma 1, we can obtain (13), with suitable changes. □

Remark 2. *If we take $\Im_2(\wp, t) = 0$, then Theorem 2 reduces to Lemma 1.*

Corollary 1. *Let the functions F, f, $\Im_1$, $\Im_2$, a, ℓ_1 and ℓ_2 be as in Theorem 2. Further suppose that $q > p > 0$ are constants. If $F(\wp, t)$ satisfies*

$$\begin{aligned}F^q(\wp,t) \quad \leq \quad & a(\wp,t) + \frac{q}{q-p}\int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)F^p(\varsigma,\eta) \\ & + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)F^p(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma \end{aligned} \tag{15}$$

for $(\wp, t) \in \Omega$, then

$$F(\wp,t) \leq \left\{p(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right\}^{\frac{1}{q-p}} \tag{16}$$

where

$$p(\wp,t) = (a(\wp,t))^{\frac{q-p}{q}} + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma.$$

Proof. In Theorem 2, by letting $\Xi(F) = F^q, \zeta(F) = F^p$, we have

$$G(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\zeta(\Xi^{-1}(\varsigma))} = \int_{v_0}^{v} \frac{\nabla\varsigma}{\varsigma^{\frac{p}{q}}} \geq \frac{q}{q-p}\left(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}}\right), v \geq v_0 > 0$$

and

$$G^{-1}(v) \geq \left\{ v_0^{\frac{q-p}{q}} + \frac{q-p}{q} v \right\}^{\frac{1}{q-p}}$$

We obtain Inequality (16). □

Theorem 3. *Under the hypotheses of Theorem 2, suppose $\Xi, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\Xi, \zeta, \varpi\}(F) > 0$ for $F > 0$ and $F(\wp, t)$ satisfies*

$$\begin{aligned} \Xi(F(\wp,t)) \quad &\leq \quad a(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(F(\varsigma,\eta))\varpi(F(\varsigma,\eta)) \\ &\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(F(\chi,\eta))\Delta\chi \Big] \Delta\eta \nabla\varsigma \end{aligned} \tag{17}$$

for $(\wp, t) \in \Omega$, then

$$F(\wp,t) \leq \Xi^{-1}\left\{ G^{-1}\left(F^{-1}\left[F(p(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma \right] \right) \right\} \tag{18}$$

for $0 \leq \wp \leq \wp_1, 0 \leq t \leq t_1$, where G and p are as in (6) and (14), respectively, and

$$F(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\varpi(\Xi^{-1}(G^{-1}(\varsigma)))}, v \geq v_0 > 0, \qquad F(+\infty) = +\infty \tag{19}$$

and $(\wp_1, t_1) \in \Omega$ is chosen so that

$$\left[F(p(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma \right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\wp, t) > 0$. Fixing an arbitrary $(\wp_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\wp, t)$ by

$$\begin{aligned} \psi(\wp,t) \quad &= \quad a(\wp_0,t_0) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(F(\varsigma,\eta))\varpi(F(\varsigma,\eta)) \\ &\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(F(\chi,\eta))\Delta\chi \Big] \Delta\eta \nabla\varsigma, \end{aligned} \tag{20}$$

for $0 \leq \wp \leq \wp_0 \leq \wp_1, 0 \leq t \leq t_0 \leq t_1$, then $\psi(\wp_0, t) = \psi(\wp, t_0) = a(\wp_0, t_0)$ and

$$F(\wp,t) \leq \Xi^{-1}(\psi(\wp,t)). \tag{21}$$

Taking the ∇-derivative for (20) and employing Theorem 1 (iv) gives

$$\begin{aligned} \psi^{\nabla_\wp}(\wp,t) \quad &= \quad \ell_1^{\nabla}(\wp) \int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)[f(\ell_1(\wp),\eta)\zeta(F(\ell_1(\wp),\eta))\varpi(F(\ell_1(\wp),\eta)) \\ &\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\zeta(F(\chi,\eta))\Delta\chi \Big] \Delta\eta \\ &\leq \quad \ell_1^{\nabla}(\wp) \int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta) \Big[f(\ell_1(\wp),\eta)\zeta\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big)\varpi\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big) \\ &\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\zeta\Big(\Xi^{-1}(\psi(\chi,\eta))\Big)\Delta\chi \Big] \Delta\eta \\ &\leq \quad \ell_1^{\nabla}(\wp).\zeta\Big(\Xi^{-1}(\psi(\ell_1(\wp),\ell_2(t)))\Big) \times \\ &\quad \int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta) \Big[f(\ell_1(\wp),\eta)\varpi\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big) + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi \Big] \Delta\eta \end{aligned} \tag{22}$$

From (22), we have

$$\frac{\psi^{\nabla_\wp}(\wp,t)}{\zeta(\Xi^{-1}(\psi(\wp,t)))} \le \ell_1^{\nabla}(\wp)\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)\Big[f(\ell_1(\wp),\eta)\varpi\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big) + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta. \tag{23}$$

Taking the ∇-integral for (23) gives

$$\begin{aligned} G(\psi(\wp,t)) &\le G(\psi(\wp_0,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma \\ &\le G(a(\wp_0,t_0)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma. \end{aligned}$$

Since $(\wp_0,t_0)\in\Omega$ is chosen arbitrarily, the last inequality can be rewritten as

$$G(\psi(\wp,t)) \le p(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\Delta\eta\nabla\varsigma. \tag{24}$$

Since $p(\wp,t)$ is a nondecreasing function, an application of Lemma 1 to (24) gives us

$$\psi(\wp,t) \le G^{-1}\left(F^{-1}\left[F(p(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right). \tag{25}$$

From (21) and (25), we obtain the desired inequality (18).

Now, we take the case $a(\wp,t)=0$ for some $(\wp,t)\in\Omega$. Let $a_\epsilon(\wp,t)=a(\wp,t)+\epsilon$, for all $(\wp,t)\in\Omega$, where $\epsilon>0$ is arbitrary, and let $a_\epsilon(\wp,t)>0$ and $a_\epsilon(\wp,t)\in C(\Omega,\mathbb{R}_+)$ be nondecreasing with respect to $(\wp,t)\in\Omega$. We carry out the above procedure with $a_\epsilon(\wp,t)>0$ instead of $a(\wp,t)$, and we obtain

$$\digamma(\wp,t) \le \Xi^{-1}\left\{G^{-1}\left(F^{-1}\left[F(p_\epsilon(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right)\right\}$$

where

$$p_\epsilon(\wp,t) = G(a_\epsilon(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma.$$

Letting $\epsilon\to 0^+$, we obtain (18). The proof is complete. □

Remark 3. *If we take $\mathbb{T}=\mathbb{R}$, $\wp_0=0$ and $t_0=0$ in Theorem 3, then Inequality (17) becomes the inequality obtained in [26] (Theorem 2.2(A_2)).*

Corollary 2. *Let the functions $\digamma$, a, f, $\Im_1$, $\Im_2$, ℓ_1 and ℓ_2 be as in Theorem 2. Further suppose that q, p and r are constants with $p>0$, $r>0$ and $q>p+r$. If $\digamma(\wp,t)$ satisfies*

$$\digamma^q(\wp,t) \le a(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\digamma^p(\varsigma,\eta)\digamma^r(\varsigma,\eta) + \int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\digamma^p(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma \tag{26}$$

for $(\wp, t) \in \Omega$, *then*

$$F(\wp, t) \leq \left\{ [p(\wp, t)]^{\frac{q-p-r}{q-p}} + \frac{q-p-r}{q} \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right\}^{\frac{1}{q-p-r}} \quad (27)$$

where

$$p(\wp, t) = (a(\wp, t))^{\frac{q-p}{q}} + \frac{q-p}{q} \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \left(\int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) \Delta\eta \nabla\varsigma$$

Proof. An application of Theorem 3 with $\Xi(F) = F^q, \zeta(F) = F^p$, and $\varpi(F) = F^r$ yields the desired inequality (27). □

Theorem 4. *Under the hypotheses of Theorem 3, if* $F(\wp, t)$ *satisfies*

$$\begin{aligned} \Xi(F(\wp, t)) \quad &\leq \quad a(\wp, t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) [f(\varsigma, \eta) \zeta(F(\varsigma, \eta)) \varpi(F(\varsigma, \eta)) \\ &+ \int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \zeta(F(\chi, \eta)) \varpi(F(\chi, \eta)) \Delta\chi \Big] \Delta\eta \nabla\varsigma \end{aligned} \quad (28)$$

for $(\wp, t) \in \Omega$, *then*

$$F(\wp, t) \leq \Xi^{-1} \left\{ G^{-1} \left(F^{-1} \left[p_0(\wp, t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right] \right) \right\} \quad (29)$$

for $0 \leq \wp \leq \wp_1, 0 \leq t \leq t_1$ *where*

$$p_0(\wp, t) = F(G(a(\wp, t))) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \left(\int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) \Delta\eta \nabla\varsigma$$

and $(\wp_1, t_1) \in \Omega$ *is chosen so that*

$$\left[p_0(\wp, t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right] \in \mathrm{Dom}\left(F^{-1} \right).$$

Proof. Assume that $a(\wp, t) > 0$. Fixing an arbitrary $(\wp_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\wp, t)$ by

$$\begin{aligned} \psi(\wp, t) \quad &= \quad a(\wp_0, t_0) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) [f(\varsigma, \eta) \zeta(F(\varsigma, \eta)) \varpi(F(\varsigma, \eta)) \\ &+ \int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \zeta(F(\chi, \eta)) \varpi(F(\chi, \eta)) \Delta\chi \Big] \Delta\eta \nabla\varsigma \end{aligned}$$

for $0 \leq \wp \leq \wp_0 \leq \wp_1, 0 \leq t \leq t_0 \leq t_1$, then $\psi(\wp_0, t) = \psi(\wp, t_0) = a(\wp_0, t_0)$, and

$$F(\wp, t) \leq \Xi^{-1}(\psi(\wp, t)). \quad (30)$$

By the same steps as in the proof of Theorem 3, we obtain

$$\begin{aligned} \psi(\wp, t) \quad &\leq \quad G^{-1} \Big\{ G(a(\wp_0, t_0)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \Big[f(\varsigma, \eta) \varpi\left(\Xi^{-1}(\psi(\varsigma, \eta)) \right) \\ &+ \int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \varpi\left(\Xi^{-1}(\psi(\chi, \eta)) \right) \Delta\chi \Big] \Delta\eta \nabla\varsigma \Big\}. \end{aligned}$$

We define a nonnegative and nondecreasing function $v(\wp,t)$ by

$$\begin{aligned} v(\wp,t) &= G(a(\wp_0,t_0)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\Big] \\ &\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\varpi\Big(\Xi^{-1}(\psi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta\nabla\varsigma \end{aligned}$$

then $v(\wp_0,t) = v(\wp,t_0) = G(a(\wp_0,t_0))$,

$$\psi(\wp,t) \le G^{-1}[v(\wp,t)] \tag{31}$$

and then, employing Theorem 1 (iv), we have

$$\begin{aligned} v^{\nabla\wp}(\wp,t) &\le \ell_1^{\nabla}(\wp)\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)\Big[f(\ell_1(\wp),\eta)\varpi\Big(\Xi^{-1}\Big(G^{-1}(v(\ell_1(\wp),t))\Big)\Big) \\ &\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\varpi\Big(\Xi^{-1}\Big(G^{-1}(v(\chi,t))\Big)\Big)\Delta\chi\Big]\Delta\eta \\ &\le \ell_1^{\nabla}(\wp)\varpi\Big(\Xi^{-1}\Big(G^{-1}(v(\ell_1(\wp),\ell_2(t)))\Big)\Big)\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)[f(\ell_1(\wp),\eta) \\ &\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta \end{aligned}$$

or

$$\begin{aligned} \frac{v^{\nabla\wp}(\wp,t)}{\varpi(\Xi^{-1}(G^{-1}(v(\wp,t))))} &\le \ell_1^{\nabla}(\wp)\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)[f(\ell_1(\wp),\eta) \\ &\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta. \end{aligned}$$

Taking the ∇-integral for the above inequality gives

$$F(v(\wp,t)) \le F(v(\wp_0,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta) + \int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma$$

or

$$\begin{aligned} v(\wp,t) &\le F^{-1}\Big\{F(G(a(\wp_0,t_0))) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta) \\ &\quad + \int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma\Big\}. \end{aligned} \tag{32}$$

From (30)–(32), and since $(\wp_0,t_0) \in \Omega$ is chosen arbitrarily, we obtain the desired inequality (29). If $a(\wp,t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\wp,t)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Remark 4. *If we take $\mathbb{T} = \mathbb{R}$ and $\wp_0 = 0$ and $t_0 = 0$ in Theorem 4, then Inequality (28) becomes the inequality obtained in [26] (Theorem 2.2(A_3)).*

Corollary 3. *Under the hypotheses of Corollary 2, if $F(\wp,t)$ satisfies*

$$\begin{aligned} F^q(\wp,t) &\le a(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)F^p(\varsigma,\eta)F^r(\varsigma,\eta) \\ &\quad + \int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)F^p(\chi,\eta)F^r(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma \end{aligned} \tag{33}$$

for $(\wp, t) \in \Omega$, *then*

$$F(\wp, t) \leq \left\{ p_0(\wp, t) + \frac{q-p-r}{q} \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right\}^{\frac{1}{q-p-r}} \tag{34}$$

where

$$p_0(\wp, t) = (a(\wp, t))^{\frac{q-p-r}{q}} + \frac{q-p-r}{q} \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \left(\int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) \Delta\eta \nabla\varsigma$$

Proof. An application of Theorem 4 with $\Xi(F) = F^q, \zeta(F) = F^p$, and $\varpi(F) = F^r$ yields the desired inequality (34). $\square$

Theorem 5. *Under the hypotheses of Theorem 3, if* $F(\wp, t)$ *satisfies*

$$\begin{aligned} \Xi(F(\wp, t)) \quad \leq \quad & a(\wp, t) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \varpi(F(\varsigma, \eta)) \times \\ & \left[f(\varsigma, \eta) \zeta(F(\varsigma, \eta)) + \int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right] \Delta\eta \nabla\varsigma \end{aligned} \tag{35}$$

for $(\wp, t) \in \Omega$, *then*

$$F(\wp, t) \leq \Xi^{-1} \left\{ G_1^{-1} \left(F_1^{-1} \left[F_1(p_1(\wp, t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right] \right) \right\} \tag{36}$$

for $0 \leq \wp \leq \wp_2, 0 \leq t \leq t_2$, *where*

$$G_1(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\varpi(\Xi^{-1}(\varsigma))}, v \geq v_0 > 0, G_1(+\infty) = \int_{v_0}^{+\infty} \frac{\nabla\varsigma}{\varpi(\Xi^{-1}(\varsigma))} = +\infty \tag{37}$$

$$F_1(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\zeta\left[\Xi^{-1}\left(G_1^{-1}(\varsigma)\right)\right]}, v \geq v_0 > 0, F_1(+\infty) = +\infty \tag{38}$$

$$p_1(\wp, t) = G_1(a(\wp, t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \left(\int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \right) \Delta\eta \nabla\varsigma \tag{39}$$

and $(\wp_2, t_2) \in \Omega$ *is chosen so that*

$$\left[F_1(p_1(\wp, t)) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \nabla\varsigma \right] \in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. Suppose that $a(\wp, t) > 0$. Fixing an arbitrary $(\wp_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\wp, t)$ by

$$\begin{aligned} \psi(\wp, t) \quad = \quad & a(\wp_0, t_0) + \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \Im_1(\varsigma, \eta) \varpi(F(\varsigma, \eta)) [f(\varsigma, \eta) \zeta(F(\varsigma, \eta)) \\ & + \int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta) \Delta\chi \Big] \Delta\eta \nabla\varsigma \end{aligned}$$

for $0 \leq \wp \leq \wp_0 \leq \wp_2, 0 \leq t \leq t_0 \leq t_2$, then $\psi(\wp_0, t) = \psi(\wp, t_0) = a(\wp_0, t_0)$,

$$F(\wp, t) \leq \Xi^{-1}(\psi(\wp, t)). \tag{40}$$

Employing Theorem 1 (iv),

$$
\begin{aligned}
\psi^{\nabla_\wp}(\wp,t) &\leq \ell_1^{\nabla}(\wp)\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)\eta\Big[\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big]\Big[f(\ell_1(\wp),\eta)\zeta\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big) \\
&\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta \\
&\leq \ell_1^{\nabla}(\wp)\eta\Big[\Xi^{-1}(\psi(\ell_1(\wp),\ell_2(t)))\Big]\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)\Big[f(\ell_1(\wp),\eta)\zeta\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big) \\
&\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{\psi^{\nabla_\wp}(\wp,t)}{\eta[\Xi^{-1}(\psi(\wp,t))]} &\leq \ell_1^{\nabla}(\wp)\int_{t_0}^{\ell_2(t)} \Im_1(\ell_1(\wp),\eta)\Big[f(\ell_1(\wp),\eta)\zeta\Big(\Xi^{-1}(\psi(\ell_1(\wp),\eta))\Big) \\
&\quad + \int_{\wp_0}^{\ell_1(\wp)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta.
\end{aligned}
$$

Taking the ∇-integral for the above inequality gives

$$
\begin{aligned}
G_1(\psi(\wp,t)) &\leq G_1(\psi(0,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\
&\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma
\end{aligned}
$$

then

$$
\begin{aligned}
G_1(\psi(\wp,t)) &\leq G_1(a(\wp_0,t_0)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\
&\quad + \int_{\wp_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma.
\end{aligned}
$$

Since $(\wp_0,t_0)\in\Omega$ is chosen to be arbitrary, the last inequality can be restated as

$$
G_1(\psi(\wp,t)) \leq p_1(\wp,t) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\zeta\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\Delta\eta\nabla\varsigma \tag{41}
$$

It is easy to observe that $p_1(\wp,t)$ is a positive and nondecreasing function for all $(\wp,t)\in\Omega$, and an application of Lemma 1 to (41) yields the inequality

$$
\psi(\wp,t) \leq G_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\wp,t)) + \int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right). \tag{42}
$$

From (40) and (42), we obtain the desired inequality (36).

If $a(\wp,t)=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(\wp,t)$ and subsequently let $\epsilon\to 0$. The proof is complete. □

Remark 5. *If we take $\mathbb{T}=\mathbb{R}$ and $\wp_0=0$ and $t_0=0$ in Theorem 5, then Inequality (36) becomes the inequality obtained in [26] (Theorem 2.7).*

Theorem 6. *Under the hypotheses of Theorem 3 and letting p be a nonnegative constant, if $F(\wp,t)$ satisfies*

$$\begin{aligned}\Xi(F(\wp,t)) \leq\ & a(\wp,t)+\int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)}\Im_1(\varsigma,\eta)F^p(\varsigma,\eta)\times \\ & \left[f(\varsigma,\eta)\zeta(F(\varsigma,\eta))+\int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\nabla\varsigma \end{aligned} \tag{43}$$

for $(\wp,t)\in\Omega$, then

$$F(\wp,t)\leq \Xi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\wp,t))+\int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right)\right\} \tag{44}$$

for $0\leq\wp\leq\wp_2, 0\leq t\leq t_2$, where

$$G_1(v)=\int_{v_0}^{v}\frac{\nabla\varsigma}{[\Xi^{-1}(\varsigma)]^p}, v\geq v_0>0, G_1(+\infty)=\int_{v_0}^{+\infty}\frac{\nabla\varsigma}{[\Xi^{-1}(\varsigma)]^p}=+\infty \tag{45}$$

and F_1, p_1 are as in Theorem 5 and $(\wp_2,t_2)\in\Omega$ is chosen so that

$$\left[F_1(p_1(\wp,t))+\int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. An application of Theorem 5 with $\varpi(F)=F^p$ yields the desired inequality (44). □

Remark 6. *Taking $\mathbb{T}=\mathbb{R}$, the inequality established in Theorem 6 generalizes [30] (Theorem 1) (with $p=1$, $a(\wp,t)=b(\wp)+c(t)$, $\wp_0=0$, $t_0=0$, $\Im_1(\varsigma,\eta)f(\varsigma,\eta)=h(\varsigma,\eta)$, and $\Im_1(\varsigma,\eta)\left(\int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)=g(\varsigma,\eta)$).*

Corollary 4. *Under the hypotheses of Theorem 6 and $q>p>0$ being constants, if $F(\wp,t)$ satisfies*

$$\begin{aligned}F^q(\wp,t) \leq\ & a(\wp,t)+\frac{p}{p-q}\int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)}\Im_1(\varsigma,\eta)F^p(\varsigma,\eta)\times \\ & \left[f(\varsigma,\eta)\zeta(F(\varsigma,\eta))+\int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\nabla\varsigma \end{aligned} \tag{46}$$

for $(\wp,t)\in\Omega$, then

$$F(\wp,t)\leq\left\{F_1^{-1}\left[F_1(p_1(\wp,t))+\int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right\}^{\frac{1}{q-p}} \tag{47}$$

for $0\leq\wp\leq\wp_2$, $0\leq t\leq t_2$, where

$$p_1(\wp,t)=[a(\wp,t)]^{\frac{q-p}{q}}+\int_{\wp_0}^{\ell_1(\wp)}\int_{t_0}^{\ell_2(t)}\Im_1(\varsigma,\eta)\left(\int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma$$

and F_1 is defined in Theorem 6.

Proof. An application of Theorem 6 with $\Xi(F(\wp,t))=F^p$ to (46) yields Inequality (47); to save space, we omit the details. □

Remark 7. *Taking $\mathbb{T}=\mathbb{R}$, $\wp_0=0$, $t_0=0$, $a(\wp,t)=b(\wp)+c(t)$, $\Im_1(\varsigma,\eta)f(\varsigma,\eta)=h(\varsigma,\eta)$, and $\Im_1(\varsigma,\eta)\left(\int_{\wp_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)=g(\varsigma,\eta)$ in Corollary 4, we obtain [31] (Theorem 1).*

Remark 8. *Taking* $\mathbb{T} = \mathbb{R}$, $\wp_0 = 0$, $t_0 = 0$, $a(\wp, t) = c^{\frac{p}{p-q}}$, $\Im_1(\varsigma, \eta) f(\varsigma, \eta) = h(\eta)$, *and* $\Im_1(\varsigma, \eta)\left(\int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta)\Delta\chi\right) = g(\eta)$ *and keeping t fixed in Corollary 4, we obtain [32] (Theorem 2.1).*

4. Application

In the following, we discus the boundedness of the solutions of the initial boundary value problem for the partial delay dynamic equation of the form

$$(\psi^q)^{\nabla_\wp \Delta_t}(\wp, t) = A\left(\wp, t, \psi(\wp - h_1(\wp), t - h_2(t)), \int_{\wp_0}^{\wp} B(\varsigma, t, \psi(\varsigma - h_1(\varsigma), t))\Delta\varsigma\right) \tag{48}$$

$$\psi(\wp, t_0) = a_1(\wp), \psi(\wp_0, t) = a_2(t), a_1(\wp_0) = a_{t_0}(0) = 0$$

for $(\wp, t) \in \Omega$, where $\psi, b \in C(\Omega, \mathbb{R}_+), A \in C(\Omega \times R^2, R), B \in C(\zeta \times R, R)$ and $h_1 \in C^1(\mathbb{T}_1, \mathbb{R}_+), h_2 \in C^1(\mathbb{T}_2, \mathbb{R}_+)$ are nondecreasing functions such that $h_1(\wp) \le \wp$ on $\mathbb{T}_1$, $h_2(t) \le t$ on $\mathbb{T}_2$, and $h_1^{\nabla}(\wp) < 1, h_2^{\nabla}(t) < 1$.

Theorem 7. *Assume that the functions a_1, a_2, A, B in (48) satisfy the conditions*

$$|a_1(\wp) + a_2(t)| \le a(\wp, t) \tag{49}$$

$$|A(\varsigma, \eta, \psi, F)| \le \frac{q}{q-p}\Im_1(\varsigma, \eta)\left[f(\varsigma, \eta)|\psi|^p + |F|\right] \tag{50}$$

$$|B(\chi, \eta, \psi)| \le \Im_2(\chi, \eta)|\psi|^p \tag{51}$$

where $a(\wp, t), \Im_1(\varsigma, \eta), f(\varsigma, \eta)$, and $\Im_2(\chi, \eta)$ are as in Theorem 2, and $q > p > 0$ are constants. If $\psi(\wp, t)$ satisfies (48), then

$$|\psi(\wp, t)| \le \left\{p(\wp, t) + M_1 M_2 \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \bar{\Im}_1(\varsigma, \eta)\bar{f}(\varsigma, \eta)\Delta\eta\nabla\varsigma\right\}^{\frac{1}{q-p}} \tag{52}$$

where

$$\begin{aligned} p(\wp, t) &= (a(\wp, t))^{\frac{q-p}{q}} \\ &\quad + M_1 M_2 \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \bar{\Im}_1(\varsigma, \eta)\left(M_1 \int_{\wp_0}^{\varsigma} \bar{\Im}_2(\chi, \eta)\Delta\chi\right)\Delta\eta\nabla\varsigma \end{aligned}$$

and

$$M_1 = \underset{\wp \in I_1}{Max} \frac{1}{1 - h_1^{\nabla}(\wp)}, \qquad M_2 = \underset{t \in I_2}{Max} \frac{1}{1 - h_2^{\nabla}(t)}$$

and $\bar{\Im}_1(\gamma, \xi) = \Im_1(\gamma + h_1(\varsigma), \xi + h_2(\eta)), \bar{\Im}_2(\mu, \xi) = \Im_2(\mu, \xi + h_2(\eta)), \bar{f}(\gamma, \xi) = f(\gamma + h_1(\varsigma), \xi + h_2(\eta))$.

Proof. If $\psi(\wp, t)$ is any solution of (48), then

$$\psi^q(\wp, t) = a_1(\wp) + a_2(t)$$
$$+ \int_{\wp_0}^{\wp} \int_{t_0}^{t} A\left(\varsigma, \eta, \psi(\varsigma - h_1(\varsigma), \eta - h_2(\eta)), \int_{\wp_0}^{\varsigma} B(\chi, \eta, \psi(\chi - h_1(\chi), \eta))\Delta\chi\right)\Delta\eta\nabla\varsigma. \tag{53}$$

Using the conditions (49)–(51) in (53), we obtain

$$\begin{aligned} |\psi(\wp, t)|^q &\le a(\wp, t) + \frac{q-p}{q} \int_{\wp_0}^{\wp} \int_{t_0}^{t} \Im_1(\varsigma, \eta)\left[f(\varsigma, \eta)|\psi(\varsigma - h_1(\varsigma), \eta - h_2(\eta))|^p\right. \\ &\quad \left. + \int_{\wp_0}^{\varsigma} \Im_2(\chi, \eta)|\psi(\chi, \eta)|^p\Delta\chi\right]\Delta\eta\nabla\varsigma. \end{aligned} \tag{54}$$

Now, making a change of variables on the right side of (54), $\varsigma - h_1(\varsigma) = \gamma, \eta - h_2(\eta) = \xi, \wp - h_1(\wp) = \ell_1(\wp)$ for $\wp \in \mathbb{T}_1, t - h_2(t) = \ell_2(t)$ for $t \in \mathbb{T}_2$, we obtain the inequality

$$\begin{aligned} |\psi(\wp,t)|^q \quad &\leq \quad a(\wp,t) + \frac{q-p}{q} M_1 M_2 \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \bar{\Im}_1(\gamma,\xi) \Big[\bar{f}(\gamma,\xi) |\psi(\gamma,\xi)|^p \\ &+ M_1 \int_{\wp_0}^{\gamma} \bar{\Im}_2(\mu,\xi) |\psi(\mu,\eta)|^p \Delta\mu \Big] \Delta\xi \Delta\gamma. \end{aligned} \tag{55}$$

We can rewrite Inequality (55) as follows:

$$\begin{aligned} |\psi(\wp,t)|^q \quad &\leq \quad a(\wp,t) + \frac{q-p}{q} M_1 M_2 \int_{\wp_0}^{\ell_1(\wp)} \int_{t_0}^{\ell_2(t)} \bar{\Im}_1(\varsigma,\eta) \Big[\bar{f}(\varsigma,\eta) |\psi(\varsigma,\eta)|^p \\ &+ M_1 \int_{\wp_0}^{\varsigma} \bar{\Im}_2(\chi,\eta) |\psi(\chi,\eta)|^p \Delta\chi \Big] \Delta\eta \nabla\varsigma. \end{aligned} \tag{56}$$

As an application of Corollary 1 to (56) with $F(\wp,t) = |\psi(\wp,t)|$, we obtain the desired inequality (52). □

5. Conclusions

Using the Leibniz integral rule on time scales, we examined additional generalizations of the integral retarded inequality presented in [26,27] and generalized a few of these inequalities to a generic time scale. We also looked at the qualitative characteristics of various different dynamic equations' time scale solutions. As future work, we intend to generalize these results by using conformable calculus on time scales.

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