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ORIGINAL PAPER

A New (4 + 1)-Dimensional Burgers Equation: Its Bäcklund Transformation and Real and Complex *N*-Kink Solitons

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Abstract

Studying the dynamics of solitons in nonlinear evolution equations (NLEEs) has gained considerable interest in the last decades. Accordingly, the search for soliton solutions of NLEEs has been the main topic of many research studies. In the present paper, a new (4 + 1)-dimensional Burgers equation (n4D-BE) is introduced that describes specific dispersive waves in nonlinear sciences. Based on the truncated Painlevé expansion, the Bäcklund transformation of the n4D-BE is firstly extracted, then, its real and complex N-kink solitons are derived using the simplified Hirota method. Furthermore, several ansatz methods are formally adopted to obtain a group of other single-kink soliton solutions of the n4D-BE.

Keywords New (4 + 1)-dimensional Burgers equation · Specific dispersive waves · Bäcklund transformation · Simplified Hirota method · Real and complex N-kink solitons

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Introduction

Appearing nonlinear evolution equations in the vast areas of scientific disciplines including physics, engineering, etc. shows the importance of studying such equations. In recent decades, the search for soliton solutions of NLEEs has been the main topic of many research works. There are many effective methods such as exponential method [1, 2], Kudryashov method [3, 4], Jacobi method [5, 6], linear superposition method [7–9], and simplified Hirota method [10–14] to look for soliton solutions of NLEEs. Of these, the simplified Hirota method is capable of constructing multiple soliton solutions of NLEEs. For example, Wazwaz [10] used the simplified Hirota method to derive multiple soliton solutions of the Vakhnenko–Parkes equation. Wazwaz [11] also found multiple soliton solutions of the negative-order KdV equation using the simplified Hirota method. Hosseini et al. [12] employed the simplified Hirota method to acquire multiple soliton solutions of a new (3 + 1)-dimensional Hirota bilinear equation. Hosseini et al. [13] also extracted multiple soliton solutions of a generalized (2 + 1)-dimensional Hirota bilinear equation through the simplified Hirota method. Multiple soliton solutions of a new generalized Kadomtsev–Petviashvili equation were obtained by Hosseini et al. in [14] using the simplified Hirota method. The main goal of the present paper is introducing a new (4 + 1)-dimensional Burgers equation and obtaining its real and complex N-kink solitons with the use of the simplified Hirota method. The n4D-BE describing specific dispersive waves is expressed as

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\alpha_1 u\frac{\partial u}{\partial y}+\alpha_2 v\frac{\partial u}{\partial x}+\alpha_3 w\frac{\partial u}{\partial z}+\alpha_4\tau\frac{\partial u}{\partial s}+\alpha_5\frac{\partial^2 u}{\partial x^2}+\frac{\alpha_1\alpha_5}{\alpha_2}\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_3\alpha_5}{\alpha_2}\frac{\partial^2 u}{\partial z^2}\\
&\quad+\frac{\alpha_4\alpha_5}{\alpha_2}\frac{\partial^2 u}{\partial s^2}=0,\\
&\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0,\\
&\frac{\partial u}{\partial z}-\frac{\partial w}{\partial y}=0,\\
&\frac{\partial u}{\partial s}-\frac{\partial \tau}{\partial y}=0,
\end{aligned}
\tag{1}
$$

where u, v, w, and τ are functions in terms of x, y, z, s, and t, and α_i, $i = 1, 2, ..., 5$ are free constants. The (2 + 1)- and (3 + 1)-dimensional Burgers equations, namely [15–26]

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\alpha_1 u\frac{\partial u}{\partial y}+\alpha_2 v\frac{\partial u}{\partial x}+\alpha_3\frac{\partial^2 u}{\partial x^2}+\frac{\alpha_1\alpha_3}{\alpha_2}\frac{\partial^2 u}{\partial y^2}=0,\\
&\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\alpha_1 u\frac{\partial u}{\partial y}+\alpha_2 v\frac{\partial u}{\partial x}+\alpha_3 w\frac{\partial u}{\partial z}+\alpha_4\frac{\partial^2 u}{\partial x^2}+\frac{\alpha_1\alpha_4}{\alpha_2}\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_3\alpha_4}{\alpha_2}\frac{\partial^2 u}{\partial z^2}=0,\\
&\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0,\\
&\frac{\partial u}{\partial z}-\frac{\partial w}{\partial y}=0,
\end{aligned}
$$

have been investigated by many academic researchers. Very recently, Hosseini et al. [25] considered the (2 + 1)-dimensional Burgers equations and obtained its kinky breather-wave and lump solutions using the extended homoclinic and limit approaches. Gao and Wang [26] found lump-type and interaction solutions of (3 + 1)-dimensional Burgers equation through ansatz methods.

The simplified Hirota method considers the solution of a NLEE like $N(u, u_x, u_t, ...) = 0$ as

$$u = R(\ln\phi(x,t))_x,$$

where R is derived in the solution process.

1. For the single soliton, the auxiliary function $\phi(x,t)$ is as follows

$$\phi(x,t) = 1 + e^{a_1x - c_1t},$$

   where the dispersion relation c_1 is obtained in a systematic way.
2. For the double soliton, the auxiliary function $\phi(x,t)$ is as follows

$$\phi(x,t) = 1 + e^{a_1x-c_1t} + e^{a_2x-c_2t} + a_{12}e^{(a_1+a_2)x-(c_1+c_2)t},$$

where the dispersion relations $c_i, i = 1, 2$ and the phase shift a_{12} are derived in the solution process.

In the general case, the auxiliary function $\phi(x,t)$ (when phase shifts are zero) is considered as follows

$$\phi(x,t) = 1 + \sum_{i=1}^{N+1} e^{\theta_i}, \theta_i = a_ix - c_it.$$

This paper is organized as follows: In Sect. 2, based on the truncated Painlevé expansion, the Bäcklund transformation of the n4D-BE is extracted. In Sect. 3, real and complex N-kink solitons of the n4D-BE are derived using the simplified Hirota method. In Sect. 4, several ansatz methods are adopted to obtain a group of other single-kink soliton solutions of the n4D-BE. The Paper ends with a detailed review regarding the results in the last section.

New (4 + 1)-dimensional Burgers equation and its Bäcklund transformation

Based on the truncated Painlevé expansion [27–29], the following Bäcklund transformation to the n4D-BE can be considered

$$u = \frac{u_0}{\phi} + u_1, v = \frac{v_0}{\phi} + v_1, w = \frac{w_0}{\phi} + w_1, \tau = \frac{\tau_0}{\phi} + \tau_1, \tag{2}$$

where u_1, v_1, w_1, and τ_1 are arbitrary solutions of the n4D-BE, and u_0, v_0, w_0, and τ_0 are determined later.

Now, by setting the Bäcklund transformation (2) in Eq. (1) and equating the coefficients of ϕ^{-3} and ϕ^{-2} to zero, we find

$$-\alpha_1 u_0^2\frac{\partial\phi}{\partial y} - \alpha_2 v_0u_0\frac{\partial\phi}{\partial x} - \alpha_3 w_0u_0\frac{\partial\phi}{\partial z} - \alpha_4\tau_0u_0\frac{\partial\phi}{\partial s} + 2\alpha_5u_0\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{2\alpha_1\alpha_5}{\alpha_2}u_0\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{2\alpha_3\alpha_5}{\alpha_2}u_0\left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{2\alpha_4\alpha_5}{\alpha_2}u_0\left(\frac{\partial\phi}{\partial s}\right)^2 = 0,$$

$$-u_0\frac{\partial\phi}{\partial x} + v_0\frac{\partial\phi}{\partial y} = 0,$$

$$-u_0\frac{\partial\phi}{\partial z}+w_0\frac{\partial\phi}{\partial y}=0,$$
$$-u_0\frac{\partial\phi}{\partial s}+\tau_0\frac{\partial\phi}{\partial y}=0.$$

Solving the above system of nonlinear partial differential equations results in

$$u_0=\frac{2\alpha_5}{\alpha_2}\phi_y,\; v_0=\frac{2\alpha_5}{\alpha_2}\phi_x,\; w_0=\frac{2\alpha_5}{\alpha_2}\phi_z,\; \tau_0=\frac{2\alpha_5}{\alpha_2}\phi_s.$$

Inserting u_0, v_0, w_0, and τ_0 into (2) yields the following Bäcklund transformation to the n4D-BE

$$u=\frac{2\alpha_5}{\alpha_2}(\ln\phi)_y,\; v=\frac{2\alpha_5}{\alpha_2}(\ln\phi)_x,\; w=\frac{2\alpha_5}{\alpha_2}(\ln\phi)_z,\; \tau=\frac{2\alpha_5}{\alpha_2}(\ln\phi)_s,$$

which u_1, v_1, w_1, and τ_1 are considered to be zero.

Real and complex *N*-kink solitons of the new (4 + 1)-dimensional Burgers equation

In this section, the simplified Hirota method as a useful technique is adopted to extract real and complex N-kink solitons of the n4D-BE. For this aim, we consider the following terms

$$u=e^{\theta_i},\; v=e^{\theta_i},\; w=e^{\theta_i},\; \tau=e^{\theta_i},$$

where

$$\theta_i=a_ix+b_iy+p_iz+q_is-c_it.$$

Inserting $u=e^{a_ix+b_iy+p_iz+q_is-c_it}$ into the following linear PDE

$$\frac{\partial u}{\partial t}+\alpha_5\frac{\partial^2 u}{\partial x^2}+\frac{\alpha_1\alpha_5}{\alpha_2}\frac{\partial^2 u}{\partial y^2}+\frac{\alpha_3\alpha_5}{\alpha_2}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_4\alpha_5}{\alpha_2}\frac{\partial^2 u}{\partial s^2}=0,$$

results in

$$\left(-c_i+\alpha_5a_i^2+\frac{\alpha_1\alpha_5b_i^2}{\alpha_2}+\frac{\alpha_3\alpha_5p_i^2}{\alpha_2}+\frac{\alpha_4\alpha_5q_i^2}{\alpha_2}\right)e^{a_ix+b_iy+p_iz+q_is-c_it}=0.$$

By equating the coefficient of $e^{a_ix+b_iy+p_iz+q_is-c_it}$ to zero and solving the resulting equation, the following dispersion relation is obtained

$$c_i=\frac{\alpha_5\left(\alpha_2a_i^2+\alpha_1b_i^2+\alpha_3p_i^2+\alpha_4q_i^2\right)}{\alpha_2},$$

and hence

$$\theta_i=a_ix+b_iy+p_iz+q_is-\frac{\alpha_5\left(\alpha_2a_i^2+\alpha_1b_i^2+\alpha_3p_i^2+\alpha_4q_i^2\right)}{\alpha_2}t,$$

where θ_i, $i=1,2,...,N+1$ are phase variables. By substituting the logarithmic transformations

$$u=R(\ln\phi)_y,\; v=R(\ln\phi)_x,\; w=R(\ln\phi)_z,\; \tau=R(\ln\phi)_s,$$

into (1) where the auxiliary function ϕ is defined as

$$\phi = 1 + e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t},$$

we obtain a nonlinear algebraic equation as

$$Ra_1^2\alpha_2^2+R\alpha_1\alpha_2 b_1^2+R\alpha_2\alpha_3 p_1^2+R\alpha_2\alpha_4 q_1^2-2\alpha_2\alpha_5 a_1^2-2\alpha_1\alpha_5 b_1^2-2\alpha_3\alpha_5 p_1^2-2\alpha_4\alpha_5 q_1^2 = 0. \tag{3}$$

The solution of Eq. (3) is

$$R = \frac{2\alpha_5}{\alpha_2}.$$

Based on the above results, a single-kink soliton to the n4D-BE can be gained as follows

$$u(x,y,z,s,t) = \frac{2\alpha_5 b_1 e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}}{\alpha_2\left(1+e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}\right)},$$

$$v(x,y,z,s,t) = \frac{2\alpha_5 a_1 e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}}{\alpha_2\left(1+e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}\right)},$$

$$w(x,y,z,s,t) = \frac{2\alpha_5 p_1 e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}}{\alpha_2\left(1+e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}\right)},$$

$$\tau(x,y,z,s,t) = \frac{2\alpha_5 q_1 e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}}{\alpha_2\left(1+e^{a_1x+b_1y+p_1z+q_1s-\frac{\alpha_5(\alpha_2 a_1^2+\alpha_1 b_1^2+\alpha_3 p_1^2+\alpha_4 q_1^2)}{\alpha_2}t}\right)}.$$

Now, to get the double-kink soliton, we consider the following auxiliary function

$$\phi = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2},$$

where the phase variables θ_1 and θ_2 are as follows

$$\theta_1 = a_1x + b_1y + p_1z + q_1s - \frac{\alpha_5\left(\alpha_2 a_1^2 + \alpha_1 b_1^2 + \alpha_3 p_1^2 + \alpha_4 q_1^2\right)}{\alpha_2}t,$$
$$\theta_2 = a_2x + b_2y + p_2z + q_2s - \frac{\alpha_5\left(\alpha_2 a_2^2 + \alpha_1 b_2^2 + \alpha_3 p_2^2 + \alpha_4 q_2^2\right)}{\alpha_2}t.$$

After some operations, a double-kink soliton to the n4D-BE is extracted as

$$u = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_y,$$
$$v = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_x,$$
$$w = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_z,$$

$$\tau = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_s,$$

which the phase shift a_{12} is zero.

In a similar way, the real N-kink soliton to the n4D-BE is derived as

$$u = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_y,$$
$$v = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_x,$$
$$w = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_z,$$
$$\tau = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_s,$$

where

$$\phi = 1 + \sum_{i=1}^{N+1} e^{a_i x + b_i y + p_i z + q_i s - \frac{\alpha_5\left(\alpha_2 a_i^2 + \alpha_1 b_i^2 + \alpha_3 p_i^2 + \alpha_4 q_i^2\right)}{\alpha_2} t}.$$

It is noteworthy that the following complex N-kink soliton to the n4D-BE can be derived

$$u = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_y,\ v = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_x,\ w = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_z,\ \tau = \frac{2\alpha_5}{\alpha_2}(\ln\phi)_s,$$

where

$$\phi = I + \sum_{i=1}^{N+1} e^{a_i x + b_i y + p_i z + q_i s - \frac{\alpha_5\left(\alpha_2 a_i^2 + \alpha_1 b_i^2 + \alpha_3 p_i^2 + \alpha_4 q_i^2\right)}{\alpha_2} t}.$$

The three-dimensional plots of single-, double-, and triple-kink solitons have been presented in Figs. 1, Fig. 2 and Fig. 3 for a series of the involved parameters. Obviously, Fig. 1 signifies a kink soliton wave, Fig. 2 shows the interaction of two kink soliton waves, and Fig. 3 demonstrates the interaction of three kink soliton waves.

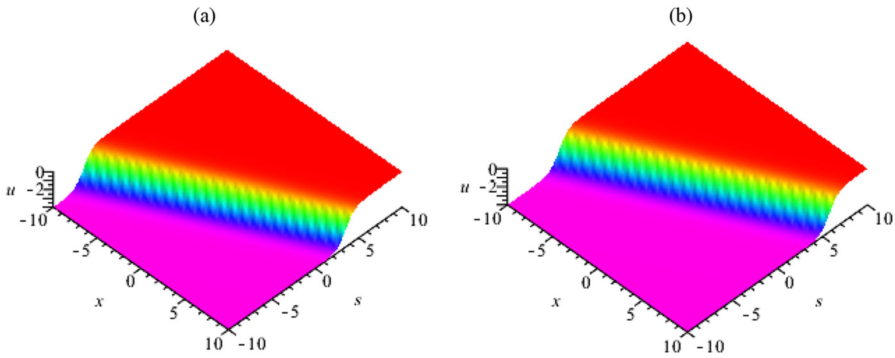

Fig. 1 Single-kink soliton on the x-s plane when $a_1 = 1$, $b_1 = -1$, $p_1 = 2$, $q_1 = -2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_4 = -2$, $\alpha_5 = 4$, $y = 1$, $z = -1$, and **a** $t = 0$; **b** $t = 0.25$

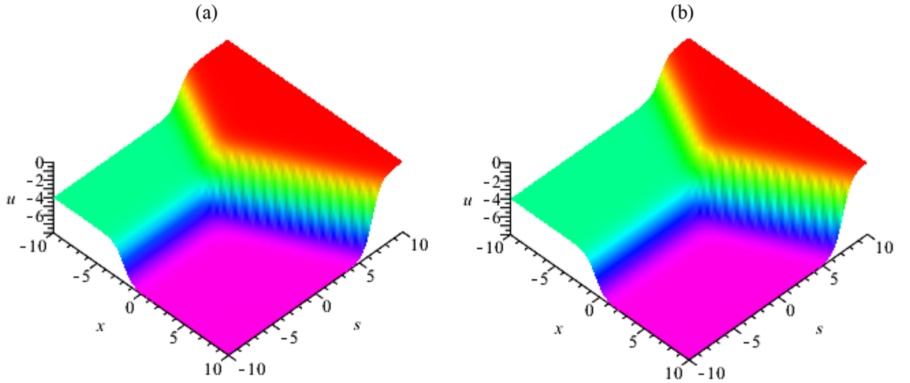

Fig. 2 Double-kink soliton on the x-s plane when $a_1 = 1$, $b_1 = 2$, $p_1 = -1$, $q_1 = -2$, $a_2 = -1$, $b_2 = 1$, $p_2 = 2$, $q_2 = -2$, $\alpha_1 = -1$, $\alpha_2 = -2$, $\alpha_3 = 1$, $\alpha_4 = 2$, $\alpha_5 = 4$, $y = 1$, $z = -1$, and **a** $t = 0$; **b** $t = 0.25$

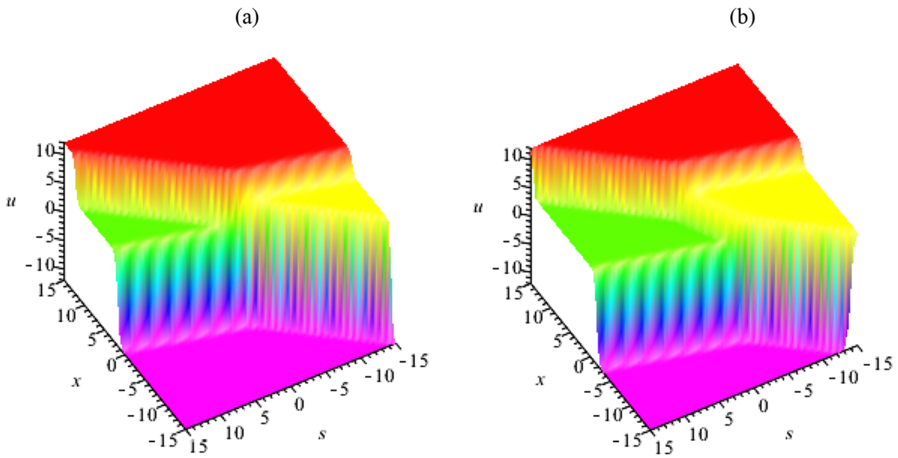

Fig. 3 Triple-kink soliton on the x-s plane when $a_1 = 1$, $b_1 = 2$, $p_1 = -1$, $q_1 = -2$, $a_2 = -1$, $b_2 = 1$, $p_2 = -2$, $q_2 = 2$, $a_3 = 5$, $b_3 = 3$, $p_3 = -2$, $q_3 = -3$, $a_4 = -5$, $b_4 = -3$, $p_4 = 2$, $q_4 = 3$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_4 = 2$, $\alpha_5 = 4$, $y = 1$, $z = -1$, and **a** $t = 0$; **b** $t = 0.1$

Other single-kink solitons of the new (4 + 1)-dimensional Burgers equation

To seek other single-kink solitons of the n4D-BE, one can consider the following ansatz

$$u(x, y, z, s, t) = A + B\tanh(ax + by + pz + qs - ct),$$

$$v(x, y, z, s, t) = \frac{a}{b}(A + B\tanh(ax + by + pz + qs - ct)),$$

$$w(x, y, z, s, t) = \frac{p}{b}(A + B\tanh(ax + by + pz + qs - ct)),$$

$$\tau(x, y, z, s, t) = \frac{q}{b}(A + B\tanh(ax + by + pz + qs - ct)),$$

where A, B, a, b, p, q, and c are found later. After substituting the above ansatz into Eq. (1), the following system of nonlinear algebraic equations is obtained

$$\begin{aligned}&Ba^2\alpha_2^2 + Bb^2\alpha_1\alpha_2 + Bp^2\alpha_2\alpha_3 + Bq^2\alpha_2\alpha_4 - 2a^2b\alpha_2\alpha_5\\&- 2b^3\alpha_1\alpha_5 - 2bp^2\alpha_3\alpha_5 - 2bq^2\alpha_4\alpha_5 = 0,\end{aligned}$$

$$Aa^2\alpha_2^2 + Ab^2\alpha_1\alpha_2 + Ap^2\alpha_2\alpha_3 + Aq^2\alpha_2\alpha_4 - bc\alpha_2 = 0.$$

The solution of the above system is

$$B = \frac{2b\alpha_5}{\alpha_2}, c = \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}.$$

Therefore, the following single-kink soliton to the n4D-BE is derived

$$u(x, y, z, s, t) = A + \frac{2b\alpha_5}{\alpha_2}\tanh\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right),$$

$$v(x, y, z, s, t) = \frac{a}{b}\left(A + \frac{2b\alpha_5}{\alpha_2}\tanh\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right)\right),$$

$$w(x, y, z, s, t) = \frac{p}{b}\left(A + \frac{2b\alpha_5}{\alpha_2}\tanh\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right)\right),$$

$$\tau(x, y, z, s, t) = \frac{q}{b}\left(A + \frac{2b\alpha_5}{\alpha_2}\tanh\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right)\right).$$

Similarly, one can find another single-kink soliton to the n4D-BE as

$$u(x, y, z, s, t) = A + \frac{2b\alpha_5}{\alpha_2}\coth\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right),$$

$$v(x, y, z, s, t) = \frac{a}{b}\left(A + \frac{2b\alpha_5}{\alpha_2}\coth\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right)\right),$$

$$w(x, y, z, s, t) = \frac{p}{b}\left(A + \frac{2b\alpha_5}{\alpha_2}\coth\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right)\right),$$

$$\tau(x, y, z, s, t) = \frac{q}{b}\left(A + \frac{2b\alpha_5}{\alpha_2}\coth\left(ax + by + pz + qs - \frac{A\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)}{b}t\right)\right).$$

It is noteworthy that one can adopt another ansatz as follows

$$\begin{aligned} u(x, y, z, s, t) &= \frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}, \\ v(x, y, z, s, t) &= \frac{a}{b}\left(\frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}\right), \\ w(x, y, z, s, t) &= \frac{p}{b}\left(\frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}\right), \\ \tau(x, y, z, s, t) &= \frac{q}{b}\left(\frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}\right), \end{aligned}$$

where A, B, a, b, p, q, and c are determined later. By inserting the above ansatz into Eq. (1), one arrives at a system of nonlinear algebraic equations as

$$\begin{aligned} &2Aa^2b\alpha_2\alpha_5 + 2Ab^3\alpha_1\alpha_5 + 2Abp^2\alpha_3\alpha_5 + 2Abq^2\alpha_4\alpha_5 \\ &\quad + Bbc\alpha_2 - a^2\alpha_2^2 - b^2\alpha_1\alpha_2 - p^2\alpha_2\alpha_3 - q^2\alpha_2\alpha_4 = 0, \end{aligned}$$

$$2Ba^2b\alpha_2\alpha_5 + 2Bb^3\alpha_1\alpha_5 + 2Bbp^2\alpha_3\alpha_5 + 2Bbq^2\alpha_4\alpha_5 + Abc\alpha_2 = 0.$$

By solving the above system, the following single-kink soliton to the n4D-BE is obtained

$$\begin{aligned} u(x, y, z, s, t) &= \frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}, \\ v(x, y, z, s, t) &= \frac{a}{b}\left(\frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}\right), \\ w(x, y, z, s, t) &= \frac{p}{b}\left(\frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}\right), \\ \tau(x, y, z, s, t) &= \frac{q}{b}\left(\frac{\tanh(ax + by + pz + qs - ct)}{A + B\tanh(ax + by + pz + qs - ct)}\right), \end{aligned}$$

where

$$\begin{aligned} B &= \frac{1}{2}\sqrt{-\frac{-4A^2b\alpha_5 + 2A\alpha_2}{b\alpha_5}}, \\ c &= -\frac{\alpha_5\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)\sqrt{-\frac{-4A^2b\alpha_5+2A\alpha_2}{b\alpha_5}}}{A\alpha_2}. \end{aligned}$$

In a similar manner, one can construct the following single-kink soliton to the n4D-BE

$$\begin{aligned} u(x, y, z, s, t) &= \frac{\coth(ax + by + pz + qs - ct)}{A + B\coth(ax + by + pz + qs - ct)}, \\ v(x, y, z, s, t) &= \frac{a}{b}\left(\frac{\coth(ax + by + pz + qs - ct)}{A + B\coth(ax + by + pz + qs - ct)}\right), \\ w(x, y, z, s, t) &= \frac{p}{b}\left(\frac{\coth(ax + by + pz + qs - ct)}{A + B\coth(ax + by + pz + qs - ct)}\right), \\ \tau(x, y, z, s, t) &= \frac{q}{b}\left(\frac{\coth(ax + by + pz + qs - ct)}{A + B\coth(ax + by + pz + qs - ct)}\right), \end{aligned}$$

where

$$B = \frac{1}{2}\sqrt{-\frac{-4A^2b\alpha_5 + 2A\alpha_2}{b\alpha_5}},$$

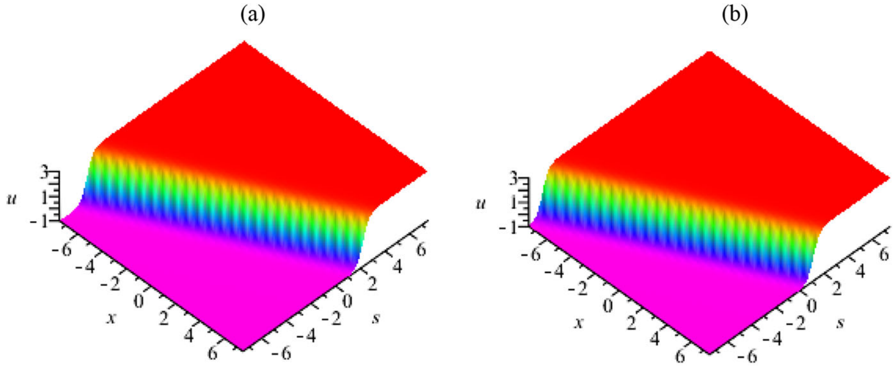

Fig. 4 The first single-kink soliton on the x-s plane when $A = 1$, $a = 1$, $b = -1$, $p = 2$, $q = -2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_4 = -2$, $\alpha_5 = 2$, $y = 1$, $z = -1$, and **a** $t = 0$; **b** $t = 0.25$

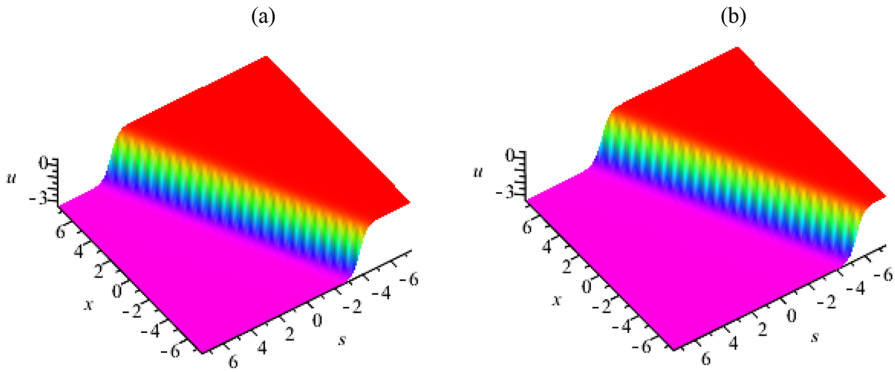

Fig. 5 The third single-kink soliton on the x-s plane when $A = 1$, $a = 1$, $b = 1$, $p = 2$, $q = -2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = -1$, $\alpha_4 = -2$, $\alpha_5 = 2$, $y = 1$, $z = -1$, and **a** $t = 0$; **b** $t = 0.25$

$$c = -\frac{\alpha_5\left(a^2\alpha_2 + b^2\alpha_1 + p^2\alpha_3 + q^2\alpha_4\right)\sqrt{-\frac{-4A^2b\alpha_5+2A\alpha_2}{b\alpha_5}}}{A\alpha_2}.$$

The three-dimensional plots of the first and third single-kink solitons have been shown in Figs. 4 and 5 for a series of the involved parameters. Clearly, these figures signify the single-kink soliton waves.

Conclusion

A new $(4+1)$-dimensional Burgers equation describing specific dispersive waves in nonlinear sciences was introduced and studied in the current paper. In this respect, first, the Bäcklund transformation of the n4D-BE was extracted through the use of the truncated Painlevé expansion. Through applying the simplified Hirota method, real and complex N-kink solitons of the n4D-BE were then retrieved. Moreover, several ansatz methods were used to acquire a number of other single-kink soliton solutions of the n4D-BE. As future works, the authors are interested in applying other methods [30–35] to derive other exact solutions of the n4D-BE.

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Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

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