Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 9389–9404.
DOI: 10.3934/math.2022521
Received: 08 December 2021
Revised: 21 February 2022
Accepted: 06 March 2022
Published: 11 March 2022

Research article

A hybrid analytical technique for solving nonlinear fractional order PDEs of power law kernel: Application to KdV and Fornberg-Witham equations

Shabir Ahmad[1,*], **Aman Ullah**[1], **Ali Akgül**[2] **and Fahd Jarad**[3,4,5,*]

[1] Department of Mathematics, University of Malakand, Chakdara, Dir Lower, Khyber Pakhtunkhwa, Pakistan
[2] Art and Science Faculty, Department of Mathematics, Siirt University, TR-56100 Siirt, Turkey
[3] Department of Mathematics, Cankaya University, Etimesgut 06790, Ankara, Turkey
[4] King Abdulaziz University Jeddah, Saudi Arabia
[5] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: shabirahmad2232@gmail.com, fahd@cankaya.edu.tr.

Abstract: It is important to deal with the exact solution of nonlinear PDEs of non-integer orders. Integral transforms play a vital role in solving differential equations of integer and fractional orders. To obtain analytical solutions to integer and fractional-order DEs, a few transforms, such as Laplace transforms, Sumudu transforms, and Elzaki transforms, have been widely used by researchers. We propose the Yang transform homotopy perturbation (YTHP) technique in this paper. We present the relation of Yang transform (YT) with the Laplace transform. We find a formula for the YT of fractional derivative in Caputo sense. We deduce a procedure for computing the solution of fractional-order nonlinear PDEs involving the power-law kernel. We show the convergence and error estimate of the suggested method. We give some examples to illustrate the novel method. We provide a comparison between the approximate solution and exact solution through tables and graphs.

Keywords: Yang transform; homotopy perturbation method; power law kernel
Mathematics Subject Classification: 35R11

1. Introduction

Non-integer calculus is a prominent branch of mathematics that seeks to describe real-world events using fractional-order operators. Non-integer-order derivatives are used for differentiation and integration in this discipline. When the order is zero, we obtain the main function, and when the order is one, we get the ordinary derivative [1]. Fractional derivatives have two advantages: The memory effect and preserved demonstrative physical characteristics. Implementing these sorts of operators, more accurate and up-to-date research has been presented over time. In this way, fractional calculus (FC) is becoming increasingly popular around the world. Because of the memory effect, fractional-order models include all prior knowledge from the past, making it easier to anticipate and evaluate dynamical models. Due to its effective features, non-integer order calculus has a lot of implications in different fields. For instance, Alqahtani et al. used the Caputo operator to investigate the model for the production of bioethanol [2]. Ozarslan et al. used fractional operator to study the projectile motion with wind influence of an object [3]. Ahmad et al. have used nonsingular fractional operator to analyze the Ambartsumian equation [4]. FC has many applications in finance, engineering, mathematical biology, mathematical physics, and many applied sciences [5–8].

Fractional order partial differential equations (PDEs) have been used in the analysis of various physical problems [9–11]. Solving fractional order (PDEs) is notoriously difficult, and obtaining an accurate solution is much more challenging. Approximate and exact solutions to this sort of equation are required in the disciplines of physical science and engineering. As a consequence, many approaches for finding analytical solutions that are almost precise have been designed. DEs were often solved using integral transforms. In differential and integral equations, integral transformations are important for solving IVPs and BVPs. Some researchers used different sorts of integral transformations and looked at how they influenced various kinds of DEs. The Laplace transform is the most often used integral transform [15]. Watugala [12] introduced the Sumudu transform to solve DEs and control engineering issues in 1998. In 2011, Elzaki presented the Elzaki Transform, a novel Integral transform that is widely utilized in the solution of partial DEs [13]. Aboodh proposes the Aboodh transform in 2013 and uses it to solve partial DEs [14].

He formulated the homotopy perturbation method (HPM) [16] in 1999, which is a combination of the homotopy method and the classical perturbation technique and has been widely applied on both linear and nonlinear problems like nonlinear wave equation [17], fractional diffusion equation [18], fractional Burger's equation [19] and many more. The importance of HPM is that it does not need a small parameter in the equation, so it reduces the drawbacks of traditional perturbation methods. The major goal of this manuscript is to use hybrid HPM to compute semianalytical solution to nonlinear partial DEs of fractional-order in Caputo sense. The hybrid HPM is combination of Yang transform (YT) [20] and HPM. The suggested approach is used to solve two well-known nonlinear PDEs. In the context of a fast converging series, we acquire a power series solution, and just a few iterations are required to produce extremely effective outcomes. There is no need for discretization or linearization for the nonlinear problem, and only a few iterations can yield a result that can be simply approximated using these approaches. The paper is organized as follows: Section 2 provides definition of Caputo operators and Yang transform. Section 3 gives the general method for solution of fractional PDEs via proposed method, results of convergence and error estimate, and test problem for validity and efficiency of the suggested method. Section 4 gives the conclusion of the paper.

2. Preliminaries

This section is devoted to the fundamental definitions of fractional calculus.

Definition 2.1. *[1] Let* $\mathbb{O}$ *be a continuous function on* $[0, T]$. *The Caputo fractional derivative may be* *expressed as:*

$$ {}^{C}\mathscr{D}_{\tau}^{\alpha}\mathbb{O}(\tau) = \frac{1}{\Gamma(j-\alpha)}\left[\int_0^{\tau}(t-\delta)^{j-\alpha-1}\frac{d^j}{d\delta^j}\mathbb{O}(\delta)\,d\delta\right], $$

where $j = \lfloor\alpha\rfloor + 1$ *and* $\lfloor\alpha\rfloor$ *represents the bracket function of* α.

Definition 2.2. *[1] Let* $\mathbb{O}$ *be a continous function on* $L^1([0,T],\mathbb{R})$*, a fractional integral in Riemann-Liouvilli sense of order* $0 < \alpha \leq 1$ *corresponding to* τ *is defined as:*

$$ I^{\alpha}\mathbb{O}(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(t-\delta)^{\alpha-1}\mathbb{O}(\delta)d\delta. $$

Definition 2.3. *[1] For the nonlocal Caputo operator of* $\mathbb{O}(\tau)$*, the Laplace transform is given by*

$$ \mathcal{L}\left[{}^{C}\mathscr{D}_{\tau}^{\alpha}\mathbb{O}(\tau)\right] = \mu^{\alpha}\mathcal{L}[\mathbb{O}(\tau)] - \mu^{\alpha-1}\mathbb{O}(0), $$

where μ *is the transform variable.*

Definition 2.4. *[20] The YT of* $\mathbb{O}(\tau)$ *is given by:*

$$ Y[\mathbb{O}(\tau)] = \chi(\mu) = \int_0^{\infty}\mathbb{O}(\tau)e^{-\frac{\tau}{\mu}}d\tau,\ \ \tau > 0, \tag{2.1} $$

where μ *represents transform variable.*

Remark 2.5. *YT of some basic function is given by:*

$$ \begin{aligned} Y[1] &= \mu, \\ Y[\tau] &= \mu^2, \\ Y[\tau^n] &= \Gamma(n+1)\mu^{n+1}. \end{aligned} $$

3. Main work

In this part, we establish the relation between the YT and LT. We deduce a formula for YT of fractional operator in Caputo sense. We establish a general procedure for computing the solution of nonlinear fractional partial DEs in Caputo sense. We prove the convergence and estimate results of the suggested method. We provide two test problems to verify the efficiency, validity and accuracy of the proposed method.

Theorem 3.1 (Laplace-Yang relationship)**.** *[19] Suppose that* $F(\mu)$ *be the LT of* $\mathbb{O}(\tau)$*, then* $\chi(\mu) = F(\frac{1}{\mu})$.

Proof. From Eq (2.1), on can get another form of the YT by putting $\frac{-\tau}{u} = x$ as

$$ \mathrm{Y}[\mathbb{O}(\tau)] = \chi(\mu) = \mu\int_0^{\infty}\mathbb{O}(\mu x)\odot e^{-x}dx,\ \ x > 0 \tag{3.1} $$

Since $\mathcal{L}[\mathbb{O}(\tau)] = \mathrm{F}(\mu)$, this implies that

$$ \mathrm{F}(\mu) = \mathcal{L}[\mathbb{O}(\tau)] = \int_0^{\infty}e^{-\mu\tau}\mathbb{O}(\tau)d\tau. \tag{3.2} $$

Put $\tau = \frac{x}{\mu}$ in (3.2), we have

$$\mathrm{F}(\mu) = \frac{1}{\mu}\int_0^\infty e^{-x}\mathbb{O}(\frac{x}{\mu})dx. \tag{3.3}$$

Thus, from Eq (3.1), we get

$$\mathrm{F}(\mu) = \chi(\frac{1}{\mu}) \tag{3.4}$$

Also from Eqs (2.1) and (3.2), we get

$$\mathrm{F}(\frac{1}{\mu}) = \chi(\mu). \tag{3.5}$$

The Eqs (3.4) and (3.5) give the relation bewteen LT and YT, which is called Laplace-Yang duality relation. □

Theorem 3.2. *For a continuous function $\mathbb{O}(\tau)$, the YT of Caputo derivative of $\mathbb{O}(\tau)$ is given by*

$$Y[{}^C\mathscr{D}_\tau^\alpha\mathbb{O}(\tau)] = \frac{Y[\mathbb{O}(\tau)]}{\mu^\alpha} - \frac{\mathbb{O}(0)}{\mu^{\alpha-1}}.$$

Proof. Since, for Caputo fractional derivative, the Laplace transform is given by

$$\mathcal{L}[{}^C\mathscr{D}_\tau^\alpha\mathbb{O}(\tau)] = \mu^\alpha\mathcal{L}[\mathbb{O}(\tau)] - \mu^{\alpha-1}\mathbb{O}(0). \tag{3.6}$$

Due to LT-YT duality relation, i.e., $\chi(\mu) = \mathrm{F}(\frac{1}{\mu})$, replace μ by $\frac{1}{\mu}$ in Eq (3.6), we obtain

$$\mathrm{Y}[{}^C\mathscr{D}_\tau^\alpha\mathbb{O}(\tau)] = \frac{\mathrm{Y}[\mathbb{O}(\tau)]}{\mu^\alpha} - \frac{\mathbb{O}(0)}{\mu^{\alpha-1}}.$$

This ends the proof. □

3.1. General method for finding solution

Here, we proposed a general method for finding the solution to a general nonlinear Caputo fractional DEs through YHPM. In this section, for the sake of simplicity, we will use $\boldsymbol{F}$ instead of $\boldsymbol{F}(x,\tau)$. Consider a general nonlinear Caputo fractional PDE with nonlinear term $\mathrm{N}(\boldsymbol{F}(x,\tau))$ and linear term $\mathrm{L}(\boldsymbol{F})$ as

$$\begin{cases} {}^C\mathsf{D}_\tau^\alpha\boldsymbol{F} + \mathrm{L}(\boldsymbol{F}) + \mathrm{N}(\boldsymbol{F}) = \mathsf{g}(x,\tau), \\ \boldsymbol{F}(x,0) = \mathsf{h}(x). \end{cases} \tag{3.7}$$

where the term $\mathsf{g}(x,\tau)$ represents the source term. Implement Yang transform to Eq (3.7), we get

$$\begin{aligned} \frac{1}{\mu^\alpha}\mathrm{Y}[\boldsymbol{F}] &= \frac{\boldsymbol{F}(x,0)}{\mu^{\alpha-1}} - \mathrm{Y}[\mathrm{L}(\boldsymbol{F}) + \mathrm{N}(\boldsymbol{F})] + \mathrm{Y}[\mathsf{g}] \\ \mathrm{Y}[\boldsymbol{F}] &= \mu\mathsf{h}(x) - \mu^\alpha[\mathrm{Y}[\mathrm{L}(\boldsymbol{F}) + \mathrm{N}(\boldsymbol{F})] + \mathrm{Y}[\mathsf{g}]], \end{aligned}$$

applying inverse of Yang transform, we get

$$\boldsymbol{F} = \mathcal{G}(x,\tau) - \mathrm{Y}^{-1}[\mu^\alpha\mathrm{Y}[\mathrm{L}(\boldsymbol{F}) + \mathrm{N}(\boldsymbol{F})]], \tag{3.8}$$

where the term $\mathcal{G}(x,\tau)$ arises from the source term and the stated initial condition. Now HPM is given by:

$$\boldsymbol{F} = \sum_{q=0}^{\infty} \rho^q \boldsymbol{F}_q. \tag{3.9}$$

The nonlinear term $\mathrm{N}(\boldsymbol{F})$ can be written as

$$\mathrm{N}(\boldsymbol{F}) = \sum_{q=0}^{\infty} \rho^q \boldsymbol{H}_q(\boldsymbol{F}), \tag{3.10}$$

where $\boldsymbol{H}_q(\boldsymbol{F})$ denotes the He's polynomial and is calculated by:

$$\boldsymbol{H}_q(\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_q) = \frac{1}{\Gamma(q+1)} \frac{\partial^q}{\partial \rho^q} \left[\mathrm{N}\left(\sum_{i=0}^{\infty} \rho^i \boldsymbol{F}_i \right) \right]_{\rho=0}, \quad q = 0, 1, \cdots \tag{3.11}$$

Substituting Eqs (3.9) and (3.10) in Eq (3.8), we achieve

$$\sum_{q=0}^{\infty} \rho^q \boldsymbol{F}_q = \mathcal{G} - \rho \left(\mathrm{Y}^{-1} \left[\mu^\alpha \mathrm{Y} \left[\mathrm{L} \sum_{q=0}^{\infty} \rho^q \boldsymbol{F}_q + \sum_{q=0}^{\infty} \rho^q \boldsymbol{H}_q(\boldsymbol{F}) \right] \right] \right). \tag{3.12}$$

One can achieve the approximations by comparing the coefficients of ρ on both sides of Eq (3.12) as:

$$\begin{aligned}
\rho^0 : \boldsymbol{F}_0 &= \mathcal{G}(x,\tau), \\
\rho^1 : \boldsymbol{F}_1 &= \mathrm{Y}^{-1} \left[\mu^\alpha \mathrm{Y} \left[\mathrm{L}\left(\boldsymbol{F}_0\right) + \boldsymbol{H}_0(\boldsymbol{F}) \right] \right], \\
\rho^2 : \boldsymbol{F}_2 &= \mathrm{Y}^{-1} \left[\mu^\alpha \mathrm{Y} \left[\mathrm{L}\left(\boldsymbol{F}_1\right) + \boldsymbol{H}_1(\boldsymbol{F}) \right] \right], \\
\rho^3 : \boldsymbol{F}_3 &= \mathrm{Y}^{-1} \left[\mu^\alpha \mathrm{Y} \left[\mathrm{L}\left(\boldsymbol{F}_2\right) + \boldsymbol{H}_2(\boldsymbol{F}) \right] \right], \\
&\vdots \\
\rho^q : \boldsymbol{F}_q &= \mathrm{Y}^{-1} \left[\mu^\alpha \mathrm{Y} \left[\mathrm{L}\left(\boldsymbol{F}_{q-1}(x,\tau) \right) + \boldsymbol{H}_{q-1}(\boldsymbol{F}) \right] \right].
\end{aligned}$$

Thus the required approximate solution of the Eq (3.7) is given by

$$\boldsymbol{F} = \boldsymbol{F}_0 + \boldsymbol{F}_1 + \cdots$$

3.2. Convergence analysis and error estimate

Here, we give the convergence of the proposed scheme and provide the result for error estimate.

Theorem 3.3. *Let $\mathcal{H}$ be a Banach space. Let $\boldsymbol{F}$ be the exact solution of (3.7) and let $\boldsymbol{F}_q, \boldsymbol{F}_n \in \mathcal{H}$ and $0 < \sigma < 1$. Then the approximate solution $\sum_{q=0}^{\infty} \boldsymbol{F}_q$ converges to $\boldsymbol{F}$ whenever $\boldsymbol{F}_q \leq \sigma \boldsymbol{F}_{q-1}$ for all $q > \mathbb{A}$, i.e., for each $\omega > 0$ $\exists$ $\mathbb{A} > 0$ with the relation $\left\| \boldsymbol{F}_{q+n} \right\| \leq \beta$, $\forall m, n \in \mathcal{N}$.*

Proof. Consider a sequence $\sum_{q=0}^{\infty} \boldsymbol{F}_q$ as:

$$\mathsf{C}_0 = \boldsymbol{F}_0,$$

$$\begin{aligned}
\mathsf{C}_1 &= \boldsymbol{F}_0 + \boldsymbol{F}_1, \\
\mathsf{C}_2 &= \boldsymbol{F}_0 + \boldsymbol{F}_1 + \boldsymbol{F}_2, \\
\mathsf{C}_3 &= \boldsymbol{F}_0 + \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3, \\
&\vdots \\
\mathsf{C}_q &= \boldsymbol{F}_0 + \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots + \boldsymbol{F}_q.
\end{aligned}$$

To prove the theorem, we have to show that $\mathsf{C}_q(x,\tau)$ form a "Cauchy sequence". Further, let us take

$$\begin{aligned}
\left\|\mathsf{C}_{q+1} - \mathsf{C}_q\right\| &= \left\|\boldsymbol{F}_{q+1}\right\| \\
&\leq \sigma \left\|\boldsymbol{F}_q\right\| \\
&\leq \sigma^2 \left\|\boldsymbol{F}_{q-1}\right\| \\
&\leq \sigma^3 \left\|\boldsymbol{F}_{q-2}\right\| \\
&\vdots \\
&\leq \sigma^{q+1} \left\|\boldsymbol{F}_0\right\|.
\end{aligned}$$

Now for any $q, n \in \mathbb{N}$, we have

$$\begin{aligned}
\left\|\mathsf{C}_q - \mathsf{C}_n\right\| &= \left\|\boldsymbol{F}_{q+n}\right\| \\
&= \Big\|\left(\mathsf{C}_q - \mathsf{C}_{q-1}\right) + \left(\mathsf{C}_{q-1} - \mathsf{C}_{q-2}\right) \\
&\quad + \left(\mathsf{C}_{q-2} - \mathsf{C}_{q-3}\right) + \cdots + \left(\mathsf{C}_{n+1} - \mathsf{C}_n\right)\Big\| \\
&\leq \left\|\mathsf{C}_q - \mathsf{C}_{q-1}\right\| + \left\|\mathsf{C}_{q-1} - \mathsf{C}_{q-2}\right\| \\
&\quad + \cdots + \left\|\mathsf{C}_{n+1} - \mathsf{C}_n\right\|. \\
&\leq \sigma^q \left\|\boldsymbol{F}_0\right\| + \sigma^{q-1} \left\|\boldsymbol{F}_0\right\| + \cdots + \sigma^{q+1} \left\|\boldsymbol{F}_0\right\| \\
&= \left\|\boldsymbol{F}_0\right\| \left(\sigma^q + \sigma^{q-1} + \cdots + \sigma^{q+1}\right) \\
&= \left\|\boldsymbol{F}_0\right\| \frac{1 - \sigma^{q-n}}{1 - \sigma} \sigma^{n+1}.
\end{aligned}$$

Let us take $\beta = \frac{1-\sigma}{(1-\sigma^{q-n})\sigma^{n+1}\|\boldsymbol{F}_0\|} < \infty$, one gets

$$\left\|\boldsymbol{F}_{q+n}\right\| \leq \beta,\ \forall q, n \in \mathbb{N}.$$

Thus, $\left\{\boldsymbol{F}_q\right\}_{q=0}^{\infty}$ form a "Cauchy sequence" in $\mathcal{H}$. Consequently, the sequence $\left\{\boldsymbol{F}_q\right\}_{q=0}^{\infty}$ is a convergent sequence with the limit $\lim\limits_{q\to\infty} \boldsymbol{F}_q = \boldsymbol{F}$ for $\exists\, \boldsymbol{F} \in \mathcal{H}$. □

Theorem 3.4. *Suppose that* $\sum_{h=0}^{k} \boldsymbol{F}_h < \infty$ *and* $\boldsymbol{F}$ *denotes the obtained solution. Let* $\wp > 0$ *such that* $\|\boldsymbol{F}_{h+1}\| \leq \wp \|\boldsymbol{F}_h\|$, *then maximum absolute error is given by*

$$\left\|\boldsymbol{F} - \sum_{h=0}^{k} \boldsymbol{F}_h\right\| < \frac{\wp^{k+1}}{1 - \wp} \left\|\boldsymbol{F}_0\right\|.$$

Proof. Since $\sum_{h=0}^{k} \boldsymbol{F}_h$ is finite. This implies that $\sum_{h=0}^{k} \boldsymbol{F}_h < \infty$. Consider

$$\begin{aligned}
\left\| \boldsymbol{F} - \sum_{h=0}^{k} \boldsymbol{F}_h \right\| &= \left\| \sum_{h=k+1}^{\infty} \boldsymbol{F}_h \right\| \\
&\leq \sum_{h=k+1}^{\infty} \|\boldsymbol{F}_h\| \\
&\leq \sum_{h=k+1}^{\infty} \wp^{k} \|\boldsymbol{F}_0\| \\
&\leq \wp^{k+1}(1 + \wp + \wp^2 + \cdots) \|\boldsymbol{F}_0\| \\
&\leq \frac{\wp^{k+1}}{1-\wp} \|\boldsymbol{F}_0\| .
\end{aligned}$$

□

3.3. Examples

In this portion, the YT homotopy perturbation approach is implemented to solve well-known nonlinear fractional PDEs, proving its simplicity and validity.

Example 3.5. *Consider the nonlinear KdV equation as:*

$$^{C}\mathsf{D}_{\tau}^{\alpha}\boldsymbol{v}(x,\tau) = -\boldsymbol{v}\boldsymbol{v}_x - \boldsymbol{v}\boldsymbol{v}_{xxx}, \quad 0 < \alpha \leq 1 \tag{3.13}$$

subjected to I.C $\boldsymbol{v}(x,0) = x$.

Solution 3.6. *Implementing the Yang transform to Eq (3.13), we achieve*

$$Y[\boldsymbol{v}(x,\tau)] = \mu\boldsymbol{v}(x,0) - \mu^{\alpha} Y[\boldsymbol{v}\boldsymbol{v}_x + \boldsymbol{v}\boldsymbol{v}_{xxx}] .$$

Applying Yang inverse transform, we achieve

$$\boldsymbol{v}(x,\tau) = \boldsymbol{v}(x,0) - Y^{-1}\left[\mu^{\alpha} Y[\boldsymbol{v}\boldsymbol{v}_x + \boldsymbol{v}\boldsymbol{v}_{xxx}]\right] . \tag{3.14}$$

The solution through HPT is represented by:

$$\boldsymbol{v}(x,\tau) = \sum_{q=0}^{\infty} \rho^{q} \boldsymbol{v}_q(x,\tau), \tag{3.15}$$

thus, Eq (3.14) becomes

$$\sum_{q=0}^{\infty} \rho^{q} \boldsymbol{v}_q(x,\tau) = x - \rho Y^{-1}\left[\mu^{\alpha} Y\left[\sum_{q=0}^{\infty} \rho^{q} \boldsymbol{H}_q(\boldsymbol{v})\right]\right], \tag{3.16}$$

where $\boldsymbol{H}_q(\boldsymbol{v})$ *are He's polynomial which represent the nonlinear term* $\boldsymbol{v}\boldsymbol{v}_x + \boldsymbol{v}\boldsymbol{v}_{xxx}$. *We give the first three terms as*

$$\boldsymbol{H}_0(\boldsymbol{v}) = \boldsymbol{v}_0 \frac{\partial}{\partial x}\boldsymbol{v}_0 + \boldsymbol{v}_0 \frac{\partial^3}{\partial x^3}\boldsymbol{v}_0,$$

$$\begin{aligned}
\boldsymbol{H}_1(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial}{\partial x}\boldsymbol{v}_1+\boldsymbol{v}_1\frac{\partial}{\partial x}\boldsymbol{v}_0+\boldsymbol{v}_0\frac{\partial^3}{\partial x^3}\boldsymbol{v}_1+\boldsymbol{v}_1\frac{\partial^3}{\partial x^3}\boldsymbol{v}_0,\\
\boldsymbol{H}_2(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial}{\partial x}\boldsymbol{v}_2+\boldsymbol{v}_1\frac{\partial}{\partial x}\boldsymbol{v}_1+\boldsymbol{v}_2\frac{\partial}{\partial x}\boldsymbol{v}_0\\
&\quad+\boldsymbol{v}_0\frac{\partial^3}{\partial x^3}\boldsymbol{v}_2+\boldsymbol{v}_1\frac{\partial^3}{\partial x^3}\boldsymbol{v}_1+\boldsymbol{v}_2\frac{\partial^3}{\partial x^3}\boldsymbol{v}_0,\\
&\vdots
\end{aligned}$$

comparing the like powers of ρ, one can get

$$\rho^0:\boldsymbol{v}_0(x,\tau)=x.$$

Now, via He's polynomials, one can obtain $\boldsymbol{H}_0(\boldsymbol{v})=x$. Further,

$$\begin{aligned}
\rho^1:\boldsymbol{v}_1(x,\tau) &= -Y^{-1}\left[\mu^\alpha Y[\boldsymbol{H}_0(\boldsymbol{v})]\right]\\
&= -Y^{-1}\left[\mu^\alpha Y[x]\right]\\
&= -Y^{-1}\left[x\mu^{\alpha+1}\right]\\
\rho^1:\boldsymbol{v}_1(x,\tau) &= -x\frac{\tau^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

The third approximation is given

$$\rho^1:\boldsymbol{v}_1(x,\tau)=-Y^{-1}\left[\mu^\alpha Y[\boldsymbol{H}_1(\boldsymbol{v})]\right]. \tag{3.17}$$

The second term $\boldsymbol{H}_1(\boldsymbol{v})$ of He's polynomial is calculated as

$$\begin{aligned}
\boldsymbol{H}_1(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial}{\partial x}\boldsymbol{v}_1+\boldsymbol{v}_1\frac{\partial}{\partial x}\boldsymbol{v}_0+\boldsymbol{v}_0\frac{\partial^3}{\partial x^3}\boldsymbol{v}_1+\boldsymbol{v}_1\frac{\partial^3}{\partial x^3}\boldsymbol{v}_0\\
&= -x\frac{\partial}{\partial x}x\frac{\tau^\alpha}{\Gamma(\alpha+1)}-x\frac{\tau^\alpha}{\Gamma(\alpha+1)}\frac{\partial}{\partial x}x\\
&\quad -x\frac{\partial^3}{\partial x^3}x\frac{\tau^\alpha}{\Gamma(\alpha+1)}-x\frac{\tau^\alpha}{\Gamma(\alpha+1)}\frac{\partial^3}{\partial x^3}x,
\end{aligned}$$

after simple calculation, we get

$$\boldsymbol{H}_1(\boldsymbol{v})=-2x\frac{\tau^\alpha}{\Gamma(\alpha+1)}. \tag{3.18}$$

Now, putting Eq (3.18) into Eq (3.17), we obtain

$$\begin{aligned}
\rho^2:\boldsymbol{v}_2(x,\tau) &= -Y^{-1}\left[\mu^\alpha Y\left[-2x\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right]\\
&= \frac{2x}{\Gamma(\alpha+1)}Y^{-1}\left[\mu^\alpha Y[\tau^\alpha]\right]\\
&= 2xY^{-1}\left[\mu^{2\alpha+1}\right]\\
&= 2x\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

Likewise, the other terms may be computed. The desired solution is given by

$$\boldsymbol{v}(x,\tau) = x - x\frac{\tau^{\alpha}}{\Gamma(\alpha+1)} + 2x\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \tag{3.19}$$

Remark 3.7. *The suggested approach is more computationally efficient and precise since the resulting solution quickly converges to the classical exact solution when $\alpha = 1$ is substituted in Eq (3.19), i.e.,*

$$\begin{aligned}
\boldsymbol{v}(x,\tau) &= x\left(1 - \frac{\tau^{1}}{\Gamma(1+1)} + 2\frac{\tau^{2}}{\Gamma(2+1)} + \cdots\right)\\
&= x\left(1 - \tau + \tau^{2} + \cdots\right)\\
&= x\sum_{i=0}^{\infty}(-1)^{i}\tau^{i}\\
\boldsymbol{v}(x,\tau) &= \frac{x}{1+\tau}.
\end{aligned} \tag{3.20}$$

Equation (3.20) denotes the exact solution of the Eq (3.13).

Figures 1 and 2 shows the graphical representation of the solution of the KdV for $\alpha = 0.7$, 0.8 and $\alpha = 0.9$ and 0.1, respectively. We observe that the solution of the KdV equation converges to the classical solution, when $\alpha = 1$. The validity and convergence of the suggested method is illustrated in the Figure 3. It show the comparison of the obtained and classical solution of KdV equation. The graph demonstrates that the approximate and the exact solution are in good agreement. Table 1 shows the numeric data of absolute error between approximate and exact solutions for $\alpha = 1$. Table 2 also shows the absolute errors for $\alpha = 1$ and $\alpha = 0.9$. The approximate solutions for $\alpha = 1$ and 0.9 are respectively denoted by $\boldsymbol{v}_{\text{approx}}$ and $\boldsymbol{v}'_{\text{approx}}$.

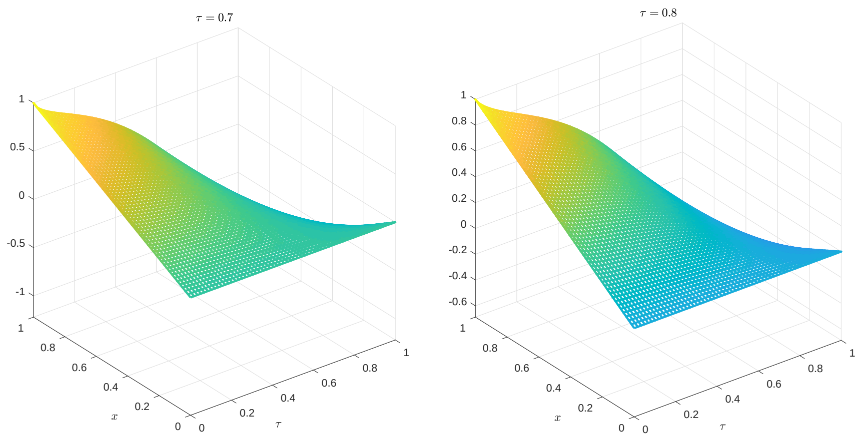

Figure 1. Graphical illustration of $\boldsymbol{v}(x,\tau)$ for $\alpha = 0.7$ *and* 0.8.

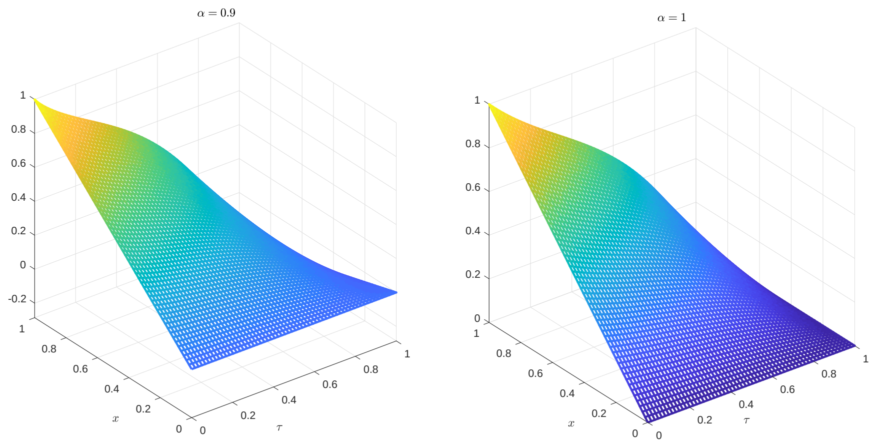

Figure 2. Graphical illustration of $\boldsymbol{v}(x, \tau)$ for $\alpha = 0.7$ *and* 0.8.

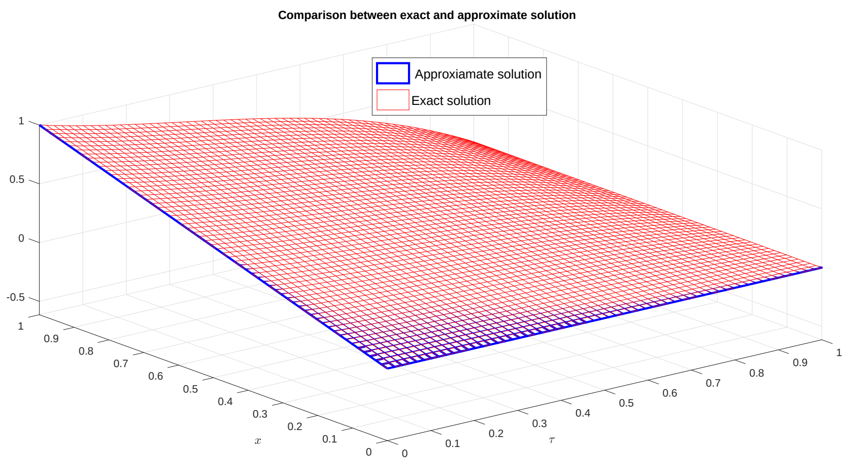

Figure 3. Comparison between exact and approximate solution of Problem 1.

Table 1. Numeric data for absolute errors between exact solution and obtained solution of Problem 4.3 at $\alpha = 1$ and $\tau = 0.1$.

x	$\boldsymbol{v}_{\text{exact}}(x, \tau)$	$\boldsymbol{v}_{\text{approx}}(x, \tau)$	$\boldsymbol{v}_{\text{exact}} - \boldsymbol{v}_{\text{approx}}$
0	0	0	0
0.1	0.0667	0.0625	0.0042
0.2	0.1333	0.1250	0.0083
0.3	0.2000	0.1875	0.0125
0.4	0.2667	0.2500	0.0167
0.5	0.3333	0.3125	0.0208

Table 2. Numeric data of error between approximate solution of Problem 4.3 at $\alpha = 1$ and $\alpha = 0.95$.

x	$\boldsymbol{v}_{\text{approx}}(x,\tau)$	$\boldsymbol{v}'_{\text{approx}}(x,\tau)$	$\boldsymbol{v}_{\text{approx}} - \boldsymbol{v}'_{\text{approx}}$
0	0	0	0
0.1	0.0909	0.0898	0.0011
0.2	0.1818	0.1795	0.0023
0.3	0.2727	0.2693	0.0034
0.4	0.3636	0.3590	0.0046
0.5	0.4545	0.4488	0.0057

Example 3.8. *Let us take nonlinear time fractional order Fornberg-Witham equation as*

$$\begin{cases} {}^{C}\mathrm{D}_{\tau}^{\alpha}\boldsymbol{v}(x,\tau) & = \frac{\partial^3 \boldsymbol{v}}{\partial x^2 \partial \tau} - \frac{\partial \boldsymbol{v}}{\partial x} + \boldsymbol{v}\frac{\partial^3 \boldsymbol{v}}{\partial x^3} - \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial x} \\ & +3\frac{\partial \boldsymbol{v}}{\partial x}\frac{\partial^3 \boldsymbol{v}}{\partial x^3}, \ \tau > 0, x \in \mathbb{R}, \ \alpha \in (0,1]. \end{cases} \tag{3.21}$$

subjected to the initial condition $\boldsymbol{v}(x,0) = \exp(\frac{x}{2})$.

Solution 3.9. *Implementing Yang transform to Eq (3.21), we achieve*

$$Y[\boldsymbol{v}(x,\tau)] = \mu\boldsymbol{v}(x,0) + \mu^{\alpha}Y\left[\frac{\partial^3 \boldsymbol{v}}{\partial x^2 \partial \tau} - \frac{\partial \boldsymbol{v}}{\partial x} + \boldsymbol{v}\frac{\partial^3 \boldsymbol{v}}{\partial x^3} - \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial x} + 3\frac{\partial \boldsymbol{v}}{\partial x}\frac{\partial^3 \boldsymbol{v}}{\partial x^3}\right].$$

Applying inverse Yang transform, we get

$$\begin{cases} \boldsymbol{v}(x,\tau) & = \exp(\frac{x}{2}) + Y^{-1}\left[\mu^{\alpha}Y\left[\frac{\partial^3 \boldsymbol{v}}{\partial x^2 \partial \tau} - \frac{\partial \boldsymbol{v}}{\partial x} + \boldsymbol{v}\frac{\partial^3 \boldsymbol{v}}{\partial x^3} - \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial x} \right.\right. \\ & \left.\left. +3\frac{\partial \boldsymbol{v}}{\partial x}\frac{\partial^3 \boldsymbol{v}}{\partial x^3}\right]\right]. \end{cases} \tag{3.22}$$

Using HPT method, the approximate solution is

$$\sum_{q=0}^{\infty}\rho^q \boldsymbol{v}_q(x,\tau) = \exp(\frac{x}{2}) + \rho Y^{-1}\left[\mu^{\alpha}Y\left[\sum_{q=0}^{\infty}\rho^q(\boldsymbol{v}_q)_{xx\tau} - \sum_{q=0}^{\infty}\rho^q(\boldsymbol{v}_q)_x + \sum_{q=0}^{\infty}\rho^q \boldsymbol{H}_q(\boldsymbol{v})\right]\right], \tag{3.23}$$

where the He's polynomials $\boldsymbol{H}_q(\boldsymbol{v})$ *represents the nonlinear terms* $\boldsymbol{v}\frac{\partial^3 \boldsymbol{v}}{\partial x^3} - \boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial x} + 3\frac{\partial \boldsymbol{v}}{\partial x}\frac{\partial^3 \boldsymbol{v}}{\partial x^3}$. *The first three terms of* $\boldsymbol{H}_q(\boldsymbol{v})$ *are given by*

$$\begin{aligned} \boldsymbol{H}_0(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial^3 \boldsymbol{v}_0}{\partial x^3} - \boldsymbol{v}_0\frac{\partial \boldsymbol{v}_0}{\partial x} + 3\frac{\partial \boldsymbol{v}_0}{\partial x}\frac{\partial^2 \boldsymbol{v}_0}{\partial x^2}, \\ \boldsymbol{H}_1(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial^3 \boldsymbol{v}_1}{\partial x^3} + \boldsymbol{v}_1\frac{\partial^3 \boldsymbol{v}_0}{\partial x^3} - \boldsymbol{v}_0\frac{\partial \boldsymbol{v}_1}{\partial x} \\ & \quad -\boldsymbol{v}_1\frac{\partial \boldsymbol{v}_0}{\partial x} + 3\frac{\partial \boldsymbol{v}_0}{\partial x}\frac{\partial^2 \boldsymbol{v}_1}{\partial x^2} + 3\frac{\partial \boldsymbol{v}_1}{\partial x}\frac{\partial^2 \boldsymbol{v}_0}{\partial x^2}, \\ \boldsymbol{H}_2(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial^3 \boldsymbol{v}_2}{\partial x^3} + \boldsymbol{v}_1\frac{\partial^3 \boldsymbol{v}_1}{\partial x^3} + \boldsymbol{v}_2\frac{\partial^3 \boldsymbol{v}_0}{\partial x^3} - \boldsymbol{v}_0\frac{\partial \boldsymbol{v}_2}{\partial x} \\ & \quad -\boldsymbol{v}_1\frac{\partial \boldsymbol{v}_1}{\partial x} - \boldsymbol{v}_2\frac{\partial \boldsymbol{v}_0}{\partial x} + 3\frac{\partial \boldsymbol{v}_2}{\partial x}\frac{\partial^2 \boldsymbol{v}_0}{\partial x^2} \end{aligned}$$

$$+3\frac{\partial \boldsymbol{v}_1}{\partial x}\frac{\partial^2 \boldsymbol{v}_1}{\partial x^2}+3\frac{\partial \boldsymbol{v}_0}{\partial x}\frac{\partial^2 \boldsymbol{v}_2}{\partial x^2}.$$

Similarly other terms can be calculated. We compare like powers of ρ, and obtain

$$\begin{aligned}\rho^0:\boldsymbol{v}_0(x,\tau) &= \exp(\frac{x}{2}),\\ \rho^1:\boldsymbol{v}_1(x,\tau) &= Y^{-1}\left[\mu^\alpha Y\left[(\boldsymbol{v}_0)_{xx\tau}-(\boldsymbol{v}_0)_x+\boldsymbol{H}_0(\boldsymbol{v})\right]\right],\end{aligned}\tag{3.24}$$

now we compute $\boldsymbol{H}_0(\boldsymbol{v})$ as:

$$\begin{aligned}\boldsymbol{H}_0(\boldsymbol{v}) &= \boldsymbol{v}_0\frac{\partial^3\boldsymbol{v}_0}{\partial x^3}-\boldsymbol{v}_0\frac{\partial\boldsymbol{v}_0}{\partial x}+3\frac{\partial\boldsymbol{v}_0}{\partial x}\frac{\partial^2\boldsymbol{v}_0}{\partial x^2}\\ &= \exp(\frac{x}{2})\frac{\partial^3\exp(\frac{x}{2})}{\partial x^3}-\exp(\frac{x}{2})\frac{\partial\exp(\frac{x}{2})}{\partial x}+3\frac{\partial\exp(\frac{x}{2})}{\partial x}\frac{\partial^2\exp(\frac{x}{2})}{\partial x^2}\\ &= \frac{1}{8}\exp(\frac{x}{2})\exp(\frac{x}{2})-\frac{1}{2}\exp(\frac{x}{2})\exp(\frac{x}{2})+\frac{3}{8}\exp(\frac{x}{2})\exp(\frac{x}{2})\\ &= 0,\end{aligned}$$

thus, Eq (3.24) becomes

$$\begin{aligned}\rho^1:\boldsymbol{v}_1(x,\tau) &= Y^{-1}\left[\mu^\alpha Y\left[0-\frac{1}{2}\exp(\frac{x}{2})+0\right]\right]\\ &= -\frac{1}{2}\exp(\frac{x}{2})Y^{-1}\left[\mu^{\alpha+1}\right]\\ \rho^1:\boldsymbol{v}_1(x,\tau) &= -\frac{1}{2}\exp(\frac{x}{2})\frac{\tau^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

on the same fashion the 3rd approximation is given as

$$\rho^2:\boldsymbol{v}_2(x,\tau)=-\frac{1}{8}\exp(\frac{x}{2})\frac{\tau^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{4}\exp(\frac{x}{2})\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)},$$

and so on. The obtained solution in series form is written as

$$\boldsymbol{v}(x,\tau)=\exp(\frac{x}{2})-\frac{1}{2}\exp(\frac{x}{2})\frac{\tau^\alpha}{\Gamma(\alpha+1)}-\frac{1}{8}\exp(\frac{x}{2})\frac{\tau^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{1}{4}\exp(\frac{x}{2})\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\tag{3.25}$$

Remark 3.10. *The suggested approach is more computationally efficient and precise since the resulting solution quickly converges to the classical exact solution when $\alpha=1$ is substituted in Eq (3.25), i.e.,*

$$\begin{aligned}\boldsymbol{v}(x,\tau) &= \exp(\frac{x}{2})-\frac{1}{2}\exp(\frac{x}{2})\frac{\tau}{\Gamma(1+1)}-\frac{1}{8}\exp(\frac{x}{2})\frac{\tau}{\Gamma(2)}+\frac{1}{4}\exp(\frac{x}{2})\frac{\tau^2}{\Gamma(2+1)}+\cdots\\ &= \exp(\frac{x}{2})-\frac{1}{2}\exp(\frac{x}{2})\tau-\frac{1}{8}\exp(\frac{x}{2})\tau+\frac{1}{4}\exp(\frac{x}{2})\tau^2+\cdots\\ \boldsymbol{v}(x,\tau) &= \exp(\frac{x}{2}-\frac{4\tau}{6}).\end{aligned}\tag{3.26}$$

Equation (3.26) denotes the exact solution of the Eq (22).

Figures 4 and 5 shows the graphical representation of the solution of the Fornberg-Witham equation for $\alpha = 0.7$, 0.8 and $\alpha = 0.9$ and 0.1, respectively. We observe that the solution of the Fornberg-Witham equation converges to the classical solution, when $\alpha = 1$. The validity and convergence of the suggested method is illustrated in the Figure 6. It show the comparison of the obtained and classical solution of Fornberg-Witham equation. The graph demonstrates that the approximate solution and the exact solution are in good agreement. Table 3 shows the numeric data of absolute error between approximate and exact solutions for $\alpha = 1$. Table 4 also shows the absolute errors for $\alpha = 1$ and $\alpha = 0.9$. The approximate solutions for $\alpha = 1$ and 0.9 are respectively denoted by $\boldsymbol{v}_{\text{approx}}$ and $\boldsymbol{v}'_{\text{approx}}$.

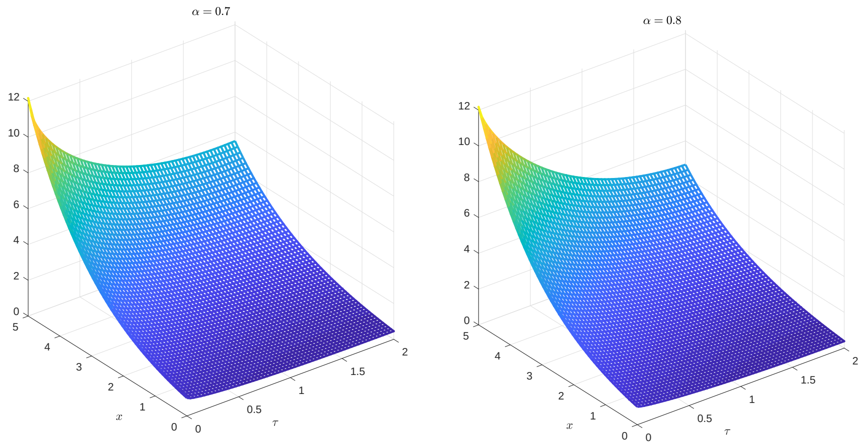

Figure 4. Graphical illustration of $\boldsymbol{v}(x, \tau)$ for $\alpha = 0.7$ and 0.8.

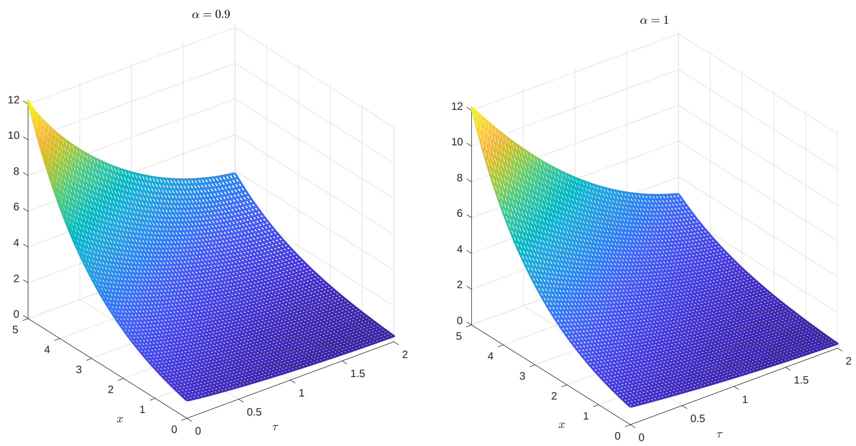

Figure 5. Graphical illustration of $\boldsymbol{v}(x, \tau)$ for $\alpha = 0.9$ and 1.

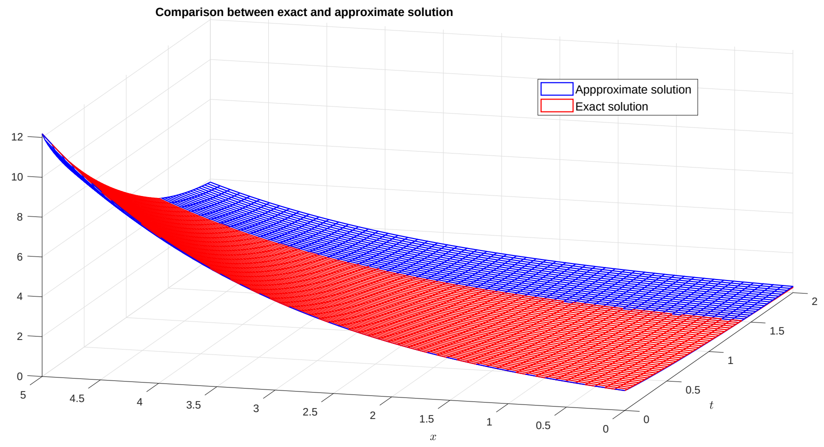

Figure 6. Graphical comparison exact and approximate solution of Problem 4.6.

Table 3. Numeric data for absolute errors between exact and obtained solution of Problem 4.6 at $\alpha = 1$ and $\tau = 0.5$.

x	$\boldsymbol{v}_{\text{exact}}(x, \tau)$	$\boldsymbol{v}_{\text{approx}}(x, \tau)$	$\boldsymbol{v}_{\text{exact}} - \boldsymbol{v}_{\text{approx}}$
0.1	0.7553	0.7556	0.0023
0.2	0.7919	0.7943	0.0025
0.3	0.8325	0.8351	0.0026
0.4	0.8752	0.8779	0.0027
0.5	0.9200	0.9229	0.0028

Table 4. Absolute errors between approximate solution of Problem 4.6 at $\alpha = 0.95$ and 1.

τ	$\boldsymbol{v}_{\text{approx}}(x, \tau)$	$\boldsymbol{v}'_{\text{approx}}(x, \tau)$	$\boldsymbol{v}_{\text{approx}} - \boldsymbol{v}'_{\text{approx}}$
0.1	1.2054	1.2002	0.0052
0.2	1.1299	1.1226	0.0073
0.3	1.0577	1.0494	0.0083
0.4	0.9887	0.9802	0.0085
0.5	0.9229	0.9147	0.0082

4. Conclusions and future work

In this article, we have utilized a novel hybrid method to compute the approximate solution of nonlinear fractional order PDEs in the Caputo sense. The novel method is the combination of the YT and HPM, which we called the Yang homotopy perturbation transform method (YHPTM). We have utilized the HPM to decompose the nonlinear term into He's polynomial. We have deduced a general method to find a solution of nonlinear PDEs defined by Caputo derivative. We have shown the convergence of the approximate solution and presented a result for the absolute error estimate. To verify the simplicity and validity of the suggested method, we have solved well-known nonlinear PDEs; the KdV equation and the Fornberg-Witham equation. We have achieved the required solution in series form, which rapidly converges to the exact solution of the KdV and Fornberg-Witham equations by substituting $\alpha = 1$, as given in the remarks after the solution of KdV and Fornberg-Witham equation. We have calculated the numeric data of the absolute errors between the approximate solution and exact solution, which shows an agreement of the exact solution with the approximate solution. We have presented the comparison between the exact solution and achieved solution through 3D graphs, which also present the high accuracy and fast convergence of the suggested technique. Another plus point of the method is that it does not require linearization, discretization, and extra memory. Thus, we have concluded that the suggested hybrid method is efficient, accurate, and less computational. In near future, we will use this method for solving the non-singular fractional-order PDEs.

Conflict of interest

In this research, there are no conflicts of interest.

References

1. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional calculus: Models and numerical methods*, World Scientific, 2012.
2. R. T. Alqahtani, S. Ahmad, A. Akgül, Dynamical analysis of Bio-Ethanol production model under generalized nonlocal operator in Caputo Sense, *Mathematics*, **9** (2021), 2370. https://doi.org/10.3390/math9192370
3. R. Ozarslan, E. Bas, D. Baleanu, B. Acay, Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel, *AIMS Mathematics*, **5** (2020), 467–481. https://doi.org/10.3934/math.2020031
4. S. Ahmad, A. Ullah, A. Akgül, M. De la Sen, A study of fractional order Ambartsumian equation involving exponential decay kernel, *AIMS Mathematics*, **6** (2021), 9981–9997. https://doi.org/10.3934/math.2021580
5. B. Acay, E. Bas, T. Abdeljawad, Fractional economic models based on market equilibrium in the frame of different type kernels, *Chaos Soliton. Fract.*, **130** (2020), 109438. https://doi.org/10.1016/j.chaos.2019.109438
6. S. Ahmad, A. Ullah, K. Shah, A. Akgül, Computational analysis of the third order dispersive fractional PDE under exponential-decay and Mittag-Leffler type kernels, *Numer. Meth. Part. Differ. Equ.*, 2020. https://doi.org/10.1002/num.22627
7. S. Ahmad, A. Ullah, A. Akgül, D. Baleanu, Analysis of the fractional tumour-immune-vitamins model with Mittag–Leffler kernel, *Res. Phys.*, **19** (2020), 103559. https://doi.org/10.1016/j.rinp.2020.103559
8. Gulalai, S. Ahmad, F. A. Rihan, A. Ullah, Q. M. Al-Mdallal, A. Akgül, Nonlinear analysis of a nonlinear modified KdV equation under Atangana Baleanu Caputo derivative, *AIMS Mathematics*, **7** (2022), 7847–7865. https://doi.org/10.3934/math.2022439
9. S. Saifullah, A. Ali, M. Irfan, K. Shah, Time-fractional Klein–Gordon equation with solitary/shock waves solutions, *Math. Probl. Eng.*, **2021** (2021), 6858592. https://doi.org/10.1155/2021/6858592

10. F. Rahman, A. Ali, S. Saifullah, Analysis of time-fractional Φ^4 equation with singular and non-singular Kernels, *Int. J. Appl. Comput. Math.*, **7** (2021), 192. https://doi.org/10.1007/s40819-021-01128-w

11. S. Saifullah, A. Ali, Z. A. Khan, Analysis of nonlinear time-fractional Klein-Gordon equation with power law kernel, *AIMS Mathematics,* **7** (2022), 5275–5290. https://doi.org/10.3934/math.2022293

12. G. K. Watugala, Sumudu transform-a new integral transform to solve differential equations and control engineering problems, *Math. Eng. Ind.*, **6** (1998), 319–329.

13. T. M. Elzaki, S. M. Elzaki, Application of new integral transform Elzaki transform to partial differential equations, *Glob. J. Pure Appl. Math.*, **7** (2011), 65–70.

14. K. S. Aboodh, Application of new integral transform "Aboodh Transform" to partial differential equations, *Glob. J. Pure Appl. Math.*, **10** (2014), 249–254.

15. J. L. Schiff, *Laplace transform: Theory and applications*, New York: Springer, 1999. https://doi.org/10.1007/978-0-387-22757-3

16. J. H. He, Homotopy perturbation technique, *Comput. Meth. Appl. Mech. Eng.*, **178** (1999), 257–262.

17. J. H. He, Application of homotopy perturbation method to nonlinear wave equations, *Chaos Soliton. Fract.*, **26** (2005), 695–700. https://doi.org/10.1016/j.chaos.2005.03.006

18. S. Das, P. K. Gupta, An approximate analytical solution of the fractional diusion equation with absorbent term and external force by homotopy perturbation method, *Zeitschrift für Naturforschung A*, **65** (2014), 182–190. https://doi.org/10.1515/zna-2010-0305

19. S. Ahmad, A. Ullah, A. Akgül, M. De la Sen, A novel homotopy perturbation method with applications to nonlinear fractional order KdV and Burger equation with exponential-decay kernel, *J. Funct. Spaces*, **2021** (2021), 8770488. https://doi.org/10.1155/2021/8770488

20. X. J. Yang, A new integral transform method for solving steady heat-transfer problem, *Therm. Sci.*, **20** (2016), S639–S642.

AIMS Press