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A numerical study of dengue internal transmission model with fractional piecewise derivative

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ABSTRACT

The goal of this paper is to study the dynamics of the dengue internal transmission model using a novel piecewise derivative approach in the sense of singular and non-singular kernels. The singular kernel operator is in the sense of Caputo, whereas the non-singular kernel operator is the Atangana–Baleanu Caputo operator. The existence and uniqueness of a solution with piecewise derivative is presented for the considered problem by using fixed point theorems. The suggested problem's approximate solution is demonstrated using the piecewise numerical iterative Newton polynomial approach. A numerical scheme for piecewise derivatives is established in terms of singular and non-singular kernels. The numerical simulation for the piecewise derivable problem under consideration is depicted using data for various fractional orders. This work makes the idea of piecewise derivatives and the dynamics of the crossover problem much clearer.

Introduction

In the second part of the twentieth century, medical research successes in terms of immunisation, antibiotics, and improved living circumstances led to the expectation that infectious illnesses would be eradicated. As a result, in industrialised countries, efforts have been focused on diseases such as cancer. Infectious illnesses, however, continue to cause pain and death in underdeveloped nations around the turn of the century. Malaria, yellow fever, AIDS, Ebola, and other diseases will live on in the collective memory of mankind.

Among these diseases, Dengue fever, which is most prevalent in Southeast Asia, is spreading the globe, affecting nations with tropical and warm climates. It is spread to humans by the Aedes mosquito, and there are two types of dengue fever: basic dengue and Dengue Haemorrhagic Fever (DHF), which can progress to an extreme condition called Dengue Shock Syndrome (DSS). The fact that dengue is caused by four different serotypes classified as DEN1, DEN2, DEN3, and DEN4 is a serious issue. A person who has been attacked by one of the four serotypes will never be infected by that serotype again (homologous immunity), but he will lose resistance to the other three serotypes in around 12 weeks (heterologous immunity), making him more susceptible to dengue haemorrhagic fever.

Dengue (Breakbone) fever is a mosquito-borne viral infection that has been rapidly spreading over the world. Dengue virus is the name given to the virus that causes dengue fever (DENV). A severe case of dengue causes significant sickness and death, although many cases of DENV generate relatively minor symptoms. Dengue fever has been linked to a variety of symptoms. If a person has a high fever (40 °C/104 °F) and two of the symptoms/indications (severe headache, discomfort behind the eyes, muscle and joint aches, nausea, vomiting, swollen glands, and rash) during the febrile phase, dengue may be

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considered. For the time being, we must combat the illness by limiting vector transmission [1,2]. It is important to remember that this break bone fever is caused by a virus, with transmission occurring through the bite of female mosquitoes. In particular Aedes albopictus and Aedes aegypti, [3,4]. The broadcast has taken place when an infected human comes into contact with mosquitoes and becomes infected, the mosquito bites the sick person, infecting them and keeping them infected until death. In contrast to mosquitoes, infected people heal from their infections within a short period of time, and these healed people are unable to transfer the virus again to mosquitoes, allowing them to keep their immunity against transmission [4–7].

Furthermore, environmental degradation, climate changes, filthy habitat, poverty, and uncontrolled urbanisation are all favourable conditions for the spread of infectious diseases in general, and dengue fever in particular. The global frequency of dengue fever has risen considerably in recent decades. The illness has already spread to over 100 African and Latin American countries. The virus is wreaking havoc on Southeast Asia and the Western Pacific.

During dengue fever outbreaks, the infection rate among susceptibles is usually between 40 and 50 percent, but it can reach 80–90 percent under favourable geographic and environmental circumstances. Each year, around 500, 000 cases of dengue hemorrhagic fever necessitate hospitalisation. Dengue fever is a viral illness spread by mosquitoes of the genus Aedes. It is caused by the contact of susceptible people with any of the four serotypes. Aedes aegypti and Aedes albopictus are the two species of vectors that transmit dengue fever. The first is extremely anthropophilic, thriving in densely populated places and biting largely during the day, whilst the second is less anthropophilic and prefers to live in rural settings. As a result, dengue is important in two ways: (i) Even in the absence of deadly forms, the illness causes enormous economic and social costs due to its global dissemination and various serotypes (absenteeism, immobilisation debilitation, medication). (ii) The danger of the disease evolving into a hemorrhagic form and dengue shock syndrome, both of which have large economic implications and can cause death.

Mathematical modelling has shown to be a useful approach for gaining a better understanding of certain diseases and developing treatment plans. The model's formulation and the feasibility of a simulation with parameter estimates enable sensitivity testing and conjuncture comparisons [8]. In the case of dengue fever, the mathematical models we identified in the literature for dengue illness offer compartmental dynamics with Susceptible, Exposed, Infectious, and Removed compartments (immunised). SEIRS models [9,10] with only one virus or two viruses working concurrently [11] were examined in particular.

The bulk of integer order calculus-based dynamical frameworks have recently been converted to non-integer order domain. The notion of fractional calculus has become an alternate mathematical way to explain models with non-local behaviour because of the additional interest variable and flexibility that may be used to firmly suit the test information far better than anything in integer order modelling. The historical memory and global knowledge of physical issues are stored in these models, which are represented by fractional differential equations. Fractional order models describe the dynamics of models with greater clarity than integer models. There are several applications of nonlocal operators in the literature. Nonlocal operators have been used in the study of problems that occurs in mathematical physics [12, 13], biomathematics [14,15], engineering [16,17] and other areas of applied sciences [18–20]. Indeed, the fractional technique generalises the dengue model's classical models. The goal of this change is to improve our knowledge and prediction of epidemic trends as well as intervention strategies. The fractional order models are considered to be helpful in identifying specific patterns in patients' illness development and may give better data fit. Clinicians can utilise the data from the universal fractional order system to create novel therapies for each individual by fitting their data to the most appropriate fixed index. As a result, the fractional differential equation can provide natural solutions for this system. Choosing a meaningful fractional index based on available real data yields a more trustworthy model. In [21–24], the authors introduced a newly established fractional order dengue mathematical model that is more reliable than previous models They look into the dynamical behaviour and describe the solution to the fractional dengue epidemic model Zain et al. [25] studied the following non-integer order dengue internal transmission model

$$\begin{aligned}
&{}_0D_t^{\sigma}(\mathrm{S_H}(t)) = \mathcal{B} - \gamma \mathrm{S_H}(t)\mathrm{R_H}(t) - \hbar \mathrm{S_H}(t),\\
&{}_0D_t^{\sigma}(\mathrm{I_H}(t)) = \gamma \mathrm{S_H}(t)\mathrm{R_H}(t) - \lambda \mathrm{I_H}(t),\\
&{}_0D_t^{\sigma}(\mathrm{R_H}(t)) = \beta_n \mathrm{I_H}(t) - (\nu_1+\nu_2)\mathrm{R_H}(t) - \gamma \mathrm{S_H}(t)\mathrm{R_H}(t),\\
&\mathrm{S_H}(0) = \mathrm{S_{H0}} \geq 0,\ \mathrm{I_H}(0) = \mathrm{I_{H0}} \geq 0,\ \mathrm{R_H}(0) = \mathrm{R_{H0}} \geq 0,\\
&\mathcal{S}_{\mathrm{M}}(0) = \mathcal{S}_{\mathrm{M0}} \geq 0,\ \mathcal{I}_{\mathcal{M}}(0) = \mathcal{I}_{\mathcal{M}0} \geq 0,\\
&0 < \sigma \leq 1,\ t \in [0,T].
\end{aligned} \tag{1}$$

We utilised the supposition that all of the parameters are positive constants in this model. The most environmentally friendly hypothesis was that sensitive cells are generated at a constant rate $\mathcal{B}$ and die at a rate $\hbar \mathrm{S_H}(t)$. The product of their plenitudes $\gamma \mathrm{S_H}(t)\mathrm{R_H}(t)$ infects vulnerable cells at a rate equal to the product of their plenitude $\gamma \mathrm{S_H}(t)\mathrm{R_H}(t)$. The steady rate γ declares the process' viability, including the rate and possibility of successful septicity. Free infection particles are expelled from the system at a rate $(nu_1 + nu_2)\mathrm{R_H}(t)$, where ν_1 is the natural demise rate of virus and ν_2 is the death rate of virus by T-cells, and septic cells yield free virus at a rate proportional to their plenitude $\beta_n \mathrm{I_H}(t)$, with n being the multiplication rate. As $\gamma \mathrm{S_H}(t)\mathrm{R_H}(t)$, the free virus also goes to the vulnerable cells compartment. At a rate of $\lambda \mathrm{I_H}(t)$, infected cells bite the dust.

Different operators, such as fractal derivative, non-integer order derivative with kernel of singularity and non-singularity, fractal-fractional operator, and other derivative operators, have been presented for the study of crossover issues [26–30]. Although the incorporation of randomness in the form of a stochastic equation results in more realistic results, the crossover dynamics remain unsolved. Many infectious disease models, heat movement, fluid flow, and many complicated advection issues all have this trait [31,32]. The exponential and Mittag-Lefler mappings in fractional calculus are unable to determine the timing of crossovers. As a result, one of the novel approaches of piecewise differentiation and integration has been developed in [33] to address such difficulties. They discussed the classical and global piecewise derivatives, as well as several applicable examples. Piecewise operators have bee used to study mathematical models in epidemiology [34–36].

In this study, we reinterpret the model (1) for qualitative analysis as well as numerical iterative analysis in the sense of Caputo and Atangana–Baleanu piecewise derivative. The rest of the work is arranged as follows: In Section "Model derivation in piecewise derivative", we present the model (1) in piecewise derivative in sense of singular and non-singular kernel. The preliminaries are covered in Section "Preliminaries", while the existence results is presented in Section "Qualitative analysis". Section "Numerical Scheme for piecewise model ((2)) with fractional order" present the model analysis with the numerical approaches while simulations and discussion of the model presented in Section "Simulations and Discussion". Section "Conclusion" concludes the with a few concluding observations.

Model derivation in piecewise derivative

The (1) can be written in piecewise derivative in sense of singular and non-singular kernel as follows:

$$\begin{aligned}
&{}_0^{CABC}D_t^{\sigma}(\mathrm{S_H}(t)) = \mathcal{B} - \gamma \mathrm{S_H}(t)\mathrm{R_H}(t) - \hbar \mathrm{S_H}(t),\\
&{}_0^{CABC}D_t^{\sigma}(\mathrm{I_H}(t)) = \gamma \mathrm{S_H}(t)\mathrm{R_H}(t) - \lambda \mathrm{I_H}(t),\\
&{}_0^{CABC}D_t^{\sigma}(\mathrm{R_H}(t)) = \beta_n \mathrm{I_H}(t) - (\nu_1+\nu_2)\mathrm{R_H}(t) - \gamma \mathrm{S_H}(t)\mathrm{R_H}(t)\\
&\mathrm{S_H}(0) = \mathrm{S_{H0}} \geq 0,\ \mathrm{I_H}(0) = \mathrm{I_{H0}} \geq 0,\ \mathrm{R_H}(0) = \mathrm{R_{H0}} \geq 0,\\
&\mathcal{S}_{\mathrm{M}}(0) = \mathcal{S}_{\mathrm{M0}} \geq 0,\ \mathcal{I}_{\mathcal{M}}(0) = \mathcal{I}_{\mathcal{M}0} \geq 0,\\
&0 < \sigma \leq 1,\ t \in [0,T].
\end{aligned} \tag{2}$$

In more detail, we can write Eq. (2) as

$$
\begin{aligned}
&{}_0^{CABC}D_t^{\sigma}(\mathrm{S_H}(t)) = \begin{cases} {}_0^{C}D_t^{\sigma}(\mathrm{S_H}(t)) = {}^{C}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t), & 0<t\le t_1,\\ {}_0^{ABC}D_t^{\sigma}(\mathrm{S_H}(t)) = {}^{ABC}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t), & t_1<t\le T,\end{cases}\\
&{}_0^{CABC}D_t^{\sigma}(\mathrm{I_H}(t)) = \begin{cases} {}_0^{C}D_t^{\sigma}(\mathrm{I_H}(t)) = {}^{C}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t), & 0<t\le t_1,\\ {}_0^{ABC}D_t^{\sigma}(\mathrm{I_H}(t)) = {}^{ABC}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t), & t_1<t\le T,\end{cases}\\
&{}_0^{CABC}D_t^{\sigma}(\mathrm{R_H}(t)) = \begin{cases} {}_0^{C}D_t^{\sigma}(\mathrm{R_H}(t)) = {}^{C}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t), & 0<t\le t_1,\\ {}_0^{ABC}D_t^{\sigma}(\mathrm{R_H}(t)) = {}^{ABC}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t), & t_1<t\le T,\end{cases}
\end{aligned}
\tag{3}
$$

where ${}_0^{C}D_t^{\sigma}$ and ${}_0^{C}D_t^{\sigma}$ are Caputo and ABC derivative respectively. and

$$
\begin{aligned}
&{}^{C}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) = {}^{ABC}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) = \mathcal{B} - \gamma \mathrm{S_H}(t)\mathrm{R_H}(t) - \hbar \mathrm{S_H}(t),\\
&{}^{C}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) = {}^{ABC}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) = \gamma \mathrm{S_H}(t)\mathrm{R_H}(t) - \lambda \mathrm{I_H}(t),\\
&{}^{C}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) = {}^{ABC}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t)\\
&= \beta_n \mathrm{I_H}(t) - (\nu_1+\nu_2)\mathrm{R_H}(t) - \gamma \mathrm{S_H}(t)\mathrm{R_H}(t).
\end{aligned}
$$

Preliminaries

In this section we will give some preliminaries definition of Caputo and ABC fractional derivative and integrals.

Definition 1. The ABC derivative of a function $U(t)$ under the condition $U(t) \in \mathcal{H}^1(0,\tau)$ is defined as follows:

$$
{}^{ABC}{}_0D_t^{\sigma}(U(t)) = \frac{ABC(\sigma)}{1-\sigma}\int_0^t \frac{d}{d\zeta}U(\zeta)E_{\sigma}\left[\frac{-\sigma}{1-\sigma}\left(t-\zeta\right)^{\sigma}\right]d\zeta. \tag{4}
$$

Replace $E_{\sigma}\left[\frac{-\sigma}{1-\sigma}\left(t-\zeta\right)^{\zeta}\right]$ by $E_1 = \exp\left[\frac{-\sigma}{1-\sigma}\left(t-\zeta\right)\right]$, in (4), to get the Caputo–Fabrizio differential operator. Next, it is notified that

$$
{}^{ABC}{}_0D_t^{\sigma}[constant] = 0.
$$

Here, $\mathcal{ABC}(\varrho)$ is called normalisation operator which is formulated as $ABC(0) = ABC(1) = 1$. Also E_{σ} represents the special function known as Mittag-Leffler function, which is the generalisation of the exponential function.

Definition 2. Let $U(t) \in L[0,T]$, then the fractional integral in $\mathbb{ABC}$ sense as:

$$
{}^{ABC}{}_0I_t^{\sigma}U(t) = \frac{1-\sigma}{ABC(\sigma)}U(t) + \frac{\sigma}{ABC(\sigma)\Gamma(\sigma)}\int_0^t (t-\zeta)^{\sigma-1}U(\zeta)d\zeta. \tag{5}
$$

Definition 3. Consider $U(t)$, for the definition of arbitrary order derivative in Caputo sense w.r.t t as

$$
{}_0^{C}D_t^{\sigma}U(t) = \frac{1}{\Gamma(1-\sigma)}\int_0^t (t-\zeta)^{n-\sigma-1}[U'(\zeta)]d\zeta.
$$

Definition 4. Consider $U(t)$ for the definition Caputo integration w.r.t t as

$$
{}_0^{C}I_t^{\sigma}U(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-\zeta)^{\sigma-1}d\zeta,\ \sigma>0,
$$

having converging integral.

Definition 5. Consider $U(t)$ differentiable and $g(t)$ is increasing function then for the definition of classical piecewise derivative [33] as

$$
{}_0^{PG}D_tU(t) = \begin{cases} U(t), & 0<t\le t_1,\\ \dfrac{U'(t)}{g'(t)} & t_1<t\le T,\end{cases}
$$

here ${}_0^{PG}D_tU(t)$ is for classical derivative for $0 < t \le t_1$ and global derivative for $t_1 < t \le T$.

Definition 6. Consider $U(t)$ differentiable and $g(t)$ is increasing function then for the definition of classical piecewise integration [33] as

$$
{}_0^{PG}I_tU(t) = \begin{cases} \displaystyle\int_0^t U(\tau)d\tau, & 0<t\le t_1,\\ \displaystyle\int_{t_1}^t U(\tau)g'(\tau)d(\tau) & t_1<t\le T,\end{cases}
$$

here ${}_0^{PG}I_tU(t)$ is for classical integration for $0 < t \le t_1$ and global integration for $t_1 < t \le T$.

Definition 7. Consider $U(t)$ differentiable then for the definition of classical and fractional piecewise derivative [33] as

$$
{}_0^{PC}D_t^{\sigma}U(t) = \begin{cases} U'(t), & 0<t\le t_1,\\ {}_0^{C}D_t^{\sigma}U(t) & t_1<t\le T,\end{cases}
$$

here ${}_0^{PC}D_t^{\sigma}U(t)$ is classical derivative for $0 < t \le t_1$ and fractional derivative for $t_1 < t \le T$.

Definition 8. Consider $U(t)$ differentiable then for the definition of classical and fractional piecewise integration [33] as

$$
{}_0^{PC}I_tU(t) = \begin{cases} \displaystyle\int_0^t U(\tau)d\tau, & 0<t\le t_1,\\ \dfrac{1}{\Gamma\sigma}\displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1}U(\zeta)d(\zeta) & t_1<t\le T,\end{cases}
$$

here ${}_0^{PC}I_tU(t)$ is for classical integration for $0 < t \le t_1$ and Caputo integration for $t_1 < t \le T$.

Definition 9. Consider $U(t)$ differentiable then for the definition of Caputo and ABC fractional piecewise derivative [33] as

$$
{}_0^{PCABC}D_t^{\sigma}U(t) = \begin{cases} {}_0^{C}D_t^{\sigma}U(t), & 0<t\le t_1,\\ {}_0^{ABC}D_t^{\sigma}U(t) & t_1<t\le T,\end{cases}
$$

here ${}_0^{PCABC}D_t^{\sigma}U(t)$ is Caputo derivative for $0 < t \le t_1$ and fractional ABC derivative for $t_1 < t \le T$.

Definition 10. Consider $U(t)$ differentiable then for the definition of fractional Caputo and fractional ABC piecewise integration [33] as

$$
{}_0^{PCABC}I_tU(t) = \begin{cases} \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1}U(\zeta)d(\zeta), & 0<t\le t_1,\\ \dfrac{1-\sigma}{ABC\sigma}U(t) + \dfrac{\sigma}{ABC\sigma\Gamma\sigma}\\ \times\displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1}U(\zeta)d(\zeta) & t_1<t\le T,\end{cases}
$$

here ${}_0^{PCABC}I_tU(t)$ is for Caputo singular kernel integration for $0 < t \le t_1$ and ABC integration for $t_1 < t \le T$.

Lemma 1. *The solution of piecewise derivable equation*

$$
{}_0^{PCABC}D_t^{\sigma}\mathrm{R_H}(t) = H(t,\mathrm{R_H}(t)),\quad 0<\sigma\le 1,
$$

is

$$
\mathrm{R_H}(t) = \begin{cases} \mathrm{R_{H0}} + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t H(\zeta,\mathrm{R_H}(\zeta))(t-\zeta)^{\sigma-1}d\zeta, & 0<t\le t_1\\ \mathrm{R_H}(t_1) + \dfrac{1-\sigma}{ABC(\sigma)}H(t,\mathrm{R_H}(t)) + \dfrac{\sigma}{ABC\sigma\Gamma(\sigma)}\\ \times\displaystyle\int_{t_1}^t (t-\zeta)^{\sigma-1}H(\zeta\mathrm{R_H}(\zeta))d(\zeta) & t_1<t\le T.\end{cases}
$$

Qualitative analysis

In this section we find the existence as well as the uniqueness of the proposed model (3) in piecewise concept. Now we will find the existence of solution along with unique solution property of the

considered piecewise derivable function. For this we can write the system (3) as given in Lemma 1 and by further description we write as follows

$$ {}_{0}^{PC}{}^{ABC}D_{t}^{\rho}\mathrm{Z}(t) = \mathcal{F}(t,\mathrm{Z}(t)), \quad 0<\rho\leq 1, $$

is

$$ \mathrm{Z}(t) = \begin{cases} \mathrm{Z}_0 + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^t \mathcal{F}(\mu,\mathrm{Z}(\mu,))(t-\mu,)^{\mu,-1}d\mu, \ 0<t\leq t_1, \\ \mathrm{Z}(t_1) + \dfrac{1-\sigma}{ABC(\sigma)}\mathcal{F}(t,\mathrm{Z}(t)) + \dfrac{\sigma}{ABC(\sigma)\Gamma(\sigma)} \\ \times \displaystyle\int_{t_1}^t (t-\mu)^{\mu-1}\mathcal{F}(\mu,\mathrm{Z}(\mu,))d\mu, \quad t_1<t\leq T, \end{cases} \tag{6} $$

where

$$ \mathrm{Z}(t) = \begin{cases} \mathrm{S_H}(t) \\ \mathrm{I_H}(t) \\ \mathrm{R_H}(t) \end{cases}, \quad \mathrm{Z}_0 = \begin{cases} \mathrm{S_{H0}} \\ \mathrm{I_{H0}} \\ \mathrm{R_{H0}} \end{cases}, \quad \mathrm{Z}(t_1) = \begin{cases} \mathrm{S_H}(t_1) \\ \mathrm{I_H}(t_1) \\ \mathrm{R_H}(t_1) \end{cases}, \tag{7} $$

$$ \mathcal{F}(t,\mathrm{Z}(t)) = \begin{cases} \mathcal{F}_1 = \begin{cases} {}^{C}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) \\ {}^{ABC}\mathcal{F}_1(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) \end{cases}, \\ \mathcal{F}_2 = \begin{cases} {}^{C}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) \\ {}^{ABC}\mathcal{F}_2(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) \end{cases}, \\ \mathcal{F}_3 = \begin{cases} {}^{C}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t) \\ {}^{ABC}\mathcal{F}_3(\mathrm{S_H},\mathrm{I_H},\mathrm{R_H},t). \end{cases} \end{cases} \tag{8} $$

We take $0<t\leq T<\infty$ with a Banach space define as $E_1 = C[0,T]$ with a norm

$$ \|\mathrm{Z}\| = \max_{t\in[0,T]}|\mathrm{Z}(t)|. $$

To obtain our result, we suppose growth condition on a non-linear operator as:

(C1) $\exists\ \mathcal{L}_{\mathrm{Z}}>0;\ \forall\ \mathcal{F},\ \bar{\mathrm{Z}}\in E$ we have

$$ |\mathcal{F}(t,\mathrm{Z})-\mathcal{F}(t,\bar{\mathrm{Z}})|\leq\mathcal{L}_{\mathcal{F}}|\mathrm{Z}-\bar{\mathrm{Z}}|, $$

(C2) $\exists\ C_{\mathcal{F}}>0\ \&\ M_{\mathcal{F}}>0$,;

$$ |\mathcal{F}(t,\mathrm{Z}(t))|\leq C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}. $$

If $\mathcal{F}$ be piecewise continuous on sub interval $0<t\leq t_1$ and $t_1<t\leq T$ on $[0,T]$, also obeying $(C2)$, then piecewise problem (3) has ≥ 1 solution on each sub interval.

Proof. Let us define a closed subset in both subintervals of $0,T$ as $\mathbb{B}$ and E as E using the Schauder fixed point theorem.

$$ B = \{\mathrm{Z}\in E:\ \|\mathrm{Z}\|\leq R_{1,2},\ R>0\}. $$

Next consider an operator $\mathcal{T}:\mathbb{B}\to\mathbb{B}$ and applying (6) as

$$ \mathcal{T}(\mathrm{Z}) = \begin{cases} \mathrm{Z}_0 + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}\mathcal{F}(\mu,\mathbb{U}(\mu))(t-\mu)^{\mu-1}d\mu, \ 0<t\leq t_1, \\ \mathrm{Z}(t_1) + \dfrac{1-\rho}{ABC(\rho)}\mathcal{F}(t,\mathrm{Z}(t)) + \dfrac{\rho}{ABC(\rho)\Gamma(\rho)} \\ \times \displaystyle\int_{t_1}^t (t-\mu)^{\rho-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d(\mu), \quad t_1<t\leq T. \end{cases} \tag{9} $$

On any $\mathrm{Z}\in B$, we get

$$ |\mathcal{T}(\mathrm{Z})(t)| \leq \begin{cases} |\mathrm{Z}_0| + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\mu)^{\sigma-1}|\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu, \\ |\mathrm{Z}(t_1)| + \dfrac{1-\sigma}{ABC(\sigma)}|\mathcal{F}(t,\mathrm{Z}(t))| + \dfrac{\sigma}{ABC(\sigma)\Gamma(\sigma)} \\ \times \displaystyle\int_{t_1}^t (t-\mu)^{\sigma-1}|\mathcal{F}(\mu\mathrm{Z}(\mu))|d(\mu), \end{cases} $$

$$ \leq \begin{cases} |\mathrm{Z}_0| + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_0^{t_1}(t-\mu)^{\sigma-1}[C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}]d\mu, \\ |\mathrm{Z}(t_1)| + \dfrac{1-\sigma}{ABC(\sigma)}[C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}] + \dfrac{\sigma}{ABC(\sigma)\Gamma(\sigma)} \\ \times \displaystyle\int_{t_1}^t (t-\mu)^{\sigma-1}[C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}]d(\mu), \end{cases} $$

$$ \leq \begin{cases} |\mathrm{Z}_0| + \dfrac{\mathbf{T}^{\sigma}}{\Gamma(\sigma+1)}[C_H|\mathbb{U}|+M_{\mathcal{F}}] = R_1, \ 0<t\leq t_1, \\ |\mathrm{Z}(t_1)| + \dfrac{1-\sigma}{ABC(\sigma)}[C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}] \\ + \dfrac{\sigma(T-\mathbf{T})^{\sigma}}{ABC(\sigma)\Gamma\sigma+1}[C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}]d(\mu) = R_2, \ t_1<t\leq T, \end{cases} $$

$$ \leq \begin{cases} R_1, \ 0<t\leq t_1, \\ R_2, \ t_1<t\leq T. \end{cases} $$

From the above equation, as $\mathrm{Z}\in\mathbf{B}$. Thus $\mathcal{T}(\mathbf{B})\subset\mathbf{B}$. Hence it proves that $\mathcal{T}$ is close and complete. Further to show the complete continuity we can write as We take $t_i<t_j\in[0,t_1]$ as first interval in Caputo sense, consider

$$ \begin{aligned} |\mathcal{T}(\mathrm{Z})(t_j)-\mathcal{T}(\mathrm{Z})(t_i)| &= \left|\frac{1}{\Gamma(\sigma)}\int_0^{t_j}(t_j-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu,\right. \\ &\quad \left. -\frac{1}{\Gamma(\sigma)}\int_0^{t_i}(t_i-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu\right| \\ &\leq \frac{1}{\Gamma(\sigma)}\int_0^{t_i}[(t_i-\mu)^{\sigma-1}-(t_j-\mu)^{\sigma-1}]|\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\quad + \frac{1}{\Gamma(\sigma)}\int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}|\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\leq \frac{1}{\Gamma(\sigma)}\left[\int_0^{t_i}[(t_i-\mu)^{\sigma-1}-(t_j-\mu)^{\sigma-1}]d\mu\right. \\ &\quad \left. + \int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}d\mu\right](C_H|\mathrm{Z}|+M_{\mathcal{F}}) \\ &\leq \frac{(C_{\mathcal{F}}\mathrm{Z}+M_{\mathcal{F}})}{\Gamma(\sigma+1)}[t_j^{\sigma}-t_i^{\sigma}+2(t_j-t_i)^{\sigma}]. \end{aligned} \tag{10} $$

Next (10), we obtain $t_i\to t_j$, then

$$ |\mathcal{T}(\mathrm{Z})(t_j)-\mathcal{T}(\mathrm{Z})(t_i)|\to 0, \ \text{as}\ t_i\to t_j. $$

So $\mathcal{T}$ is equi-continuous in $[0,t_1]$ interval. Next we consider other interval $t_i,t_j\in[t_1,T]$ in the ABC sense as

$$ \begin{aligned} |\mathcal{T}(\mathrm{Z})(t_j)-\mathcal{T}(\mathrm{Z})(t_i)| &= \left|\frac{1-\sigma}{ABC(\sigma)}\mathcal{F}(t,\mathrm{Z}(t)) + \frac{\sigma}{ABC(\sigma)\Gamma(\sigma)}\right. \\ &\quad \times \int_{t_1}^{t_j}(t_j-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu, \\ &\quad - \frac{1-\sigma}{ABC(\sigma)}\mathcal{F}(t,\mathrm{Z}(t)) + \frac{(\sigma)}{ABC(\sigma)\Gamma(\sigma)} \\ &\quad \left.\times \int_{t_1}^{t_i}(t_i-\mu)^{\sigma-1}\mathcal{F}(\mu,\mathrm{Z}(\mu))d\mu\right| \\ &\leq \frac{\sigma}{ABC(\sigma)\Gamma(\sigma)}\int_{t_1}^{t_i}[(t_i-\mu)^{\sigma-1}-(t_j-\mu)^{\sigma-1}] \\ &\quad \times |\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\quad + \frac{\sigma}{ABC(\sigma)\Gamma(\sigma)}\int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}|\mathcal{F}(\mu,\mathrm{Z}(\mu))|d\mu \\ &\leq \frac{\sigma}{ABC(\sigma)\Gamma(\sigma)}\left[\int_{t_1}^{t_i}[(t_i-\mu)^{\sigma-1}-(t_j-\mu)^{\sigma-1}]d\mu\right. \\ &\quad \left. + \int_{t_i}^{t_j}(t_j-\mu)^{\sigma-1}d\mu\right](C_{\mathcal{F}}|\mathrm{Z}|+M_{\mathcal{F}}) \\ &\leq \frac{\sigma(C_{\mathcal{F}}\mathrm{Z}+M_{\mathcal{F}})}{ABC(\sigma)\Gamma(\sigma+1)}[t_j^{\sigma}-t_i^{\sigma}+2(t_j-t_i)^{\sigma}]. \end{aligned} \tag{11} $$

Next as (11), we obtain $t_i\to t_j$, then

$$ |\mathcal{T}(\mathrm{Z})(t_j)-\mathcal{T}(\mathrm{Z})(t_i)|\to 0, \ \text{as}\ t_i\to t_j. $$

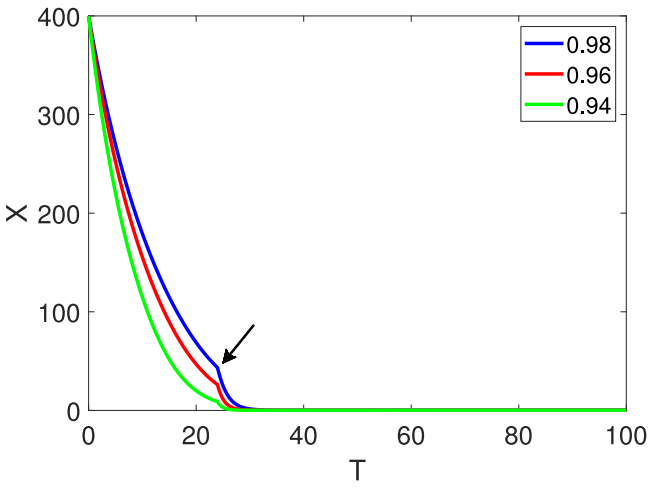

Fig. 1. The population dynamics of susceptible population $S_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 23$.

So $\mathcal{T}$ show the equi-continuity in $[t_1, T]$ interval. Therefore, $\mathcal{T}$ is equi-continuous mapping. The operator $\mathcal{T}$ is completely continuous, uniform continuous, and has bounds, according to the Arzel'a–Ascoli theorem. As a result, the piecewise derivable problem (3) has a ≥ 1 solution on each sub interval, according to Schauder's fixed point Theorem. □

With $(C1)$, the proposed model has unique root if $\mathcal{T}$ be a contraction operator.

Proof. As we have taken an operator $\mathcal{T} : \mathbf{B} \to \mathbf{B}$ piecewise continuous, take Z and $\bar{Z} \in B$ on $[0, t_1]$ in Caputo sense as

$$\begin{aligned}\|\mathcal{T}(Z) - \mathcal{T}(\bar{Z})\| &= \max_{t\in[0,t_1]} \left| \frac{1}{\Gamma(\sigma)} \int_0^t (t-\mu)^{\sigma-1} \mathcal{F}(\mu, Z(\mu)) d\mu \right. \\ &\quad \left. - \frac{1}{\Gamma(\sigma)} \int_0^t (t-\mu)^{\sigma-1} \mathcal{F}(\mu, \bar{Z}(\mu)) d\mu \right| \\ &\leq \frac{\mathbf{T}^\sigma}{\Gamma(\sigma+1)} L_{\mathcal{F}} \|Z - \bar{Z}\|. \end{aligned} \tag{12}$$

From (12), we have

$$\|\mathcal{T}(Z) - \mathcal{T}(\bar{Z})\| \leq \frac{\mathbf{T}^\sigma}{\Gamma(\sigma+1)} L_{\mathcal{F}} \|Z - \bar{Z}\|. \tag{13}$$

So $\mathcal{T}$ is contracted. Therefore, finally in sense of Banach contraction theorem the considered problem has unique solution in given sub interval. Next for the other interval $t \in [t_1, T]$ in the sense of ABC derivative as

$$\|\mathcal{T}(Z) - \mathcal{T}(\bar{Z})\| \leq \frac{1-\sigma}{ABC(\sigma)} L_{\mathcal{F}} \|Z - \bar{Z}\| + \frac{\sigma(\mathbf{T} - T^\sigma)}{ABC(\sigma)\Gamma(\sigma+1)} L_{\mathcal{F}} \|Z - \bar{Z}\|. \tag{14}$$

or

$$\|\mathcal{T}(Z) - \mathcal{T}(\bar{Z})\| \leq L_{\mathcal{F}} \left[\frac{1-\sigma}{ABC(\sigma)} + \frac{\sigma(\mathbf{T} - T^\sigma)}{ABC(\sigma)\Gamma(\sigma+1)} \right] \|Z - \bar{Z}\|. \tag{15}$$

Therefore, $\mathcal{T}$ is contracted. Therefore, finally in sense of Banach contraction theorem the considered problem has unique solution in given sub interval. So by Eqs. (13) and (15) the piecewise derivable problem have unique solution on each sub-intervals. □

Numerical scheme for piecewise model (2) with fractional order

Next we will establish a numerical scheme for the proposed piecewise differentiable problem (3). We will developed a numerical scheme for the two subinterval of $[0, T]$, in Caputo and ABC sense. We will take help from the piecewise derivative integer order numerical scheme as in [33]. Applying the piecewise integration to Eq. (3) for Caputo and ABC format as follows

$$\begin{aligned}
S_H(t) &= \begin{cases} S_H(0) + \frac{1}{\Gamma(\sigma)} \int_0^{t_1} (t-\tau)^{\sigma-1} {}^C\mathcal{F}_1(t, S_H, I_H, R_H) d\tau & 0 < t \leq t_1, \\ S_H(t_1) + \frac{1-\sigma}{AB(\sigma)} \mathcal{F}_1(t, S_H, I_H, R_H) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_{t_1}^t (t-\tau)^{\sigma-1} \mathcal{F}_1(t, S_H, I_H, R_H) d\tau & t_1 < t \leq T, \end{cases} \\
I_H(t) &= \begin{cases} I_H(0) + \frac{1}{\Gamma(\sigma)} \int_0^{t_1} (t-\tau)^{\sigma-1} {}^C\mathcal{F}_2(t, S_H, I_H, R_H) d\tau & 0 < t \leq t_1, \\ I_H(t_1) + \frac{1-\sigma}{AB(\sigma)} \mathcal{F}_2(t, S_H, I_H, R_H) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_{t_1}^t (t-\tau)^{\sigma-1} \mathcal{F}_2(t, S_H, I_H, R_H) d\tau & t_1 < t \leq T, \end{cases} \\
R_H(t) &= \begin{cases} R_H(0) + \frac{1}{\Gamma(\sigma)} \int_0^{t_1} (t-\tau)^{\sigma-1} {}^C\mathcal{F}_3(t, S_H, I_H, R_H) d\tau & 0 < t \leq t_1, \\ R_H(t_1) + \frac{1-\sigma}{AB(\sigma)} \mathcal{F}_3(t, S_H, I_H, R_H) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_{t_1}^t (t-\tau)^{\sigma-1} \mathcal{F}_3(t, S_H, I_H, R_H) d\tau & t_1 < t \leq T, \end{cases}
\end{aligned} \tag{16}$$

where ${}^C\mathcal{F}_i(t) = {}^C_i\mathcal{F}(S_H, I_H, R_H, t)$ and ${}^{ABC}\mathcal{F}_i(t) = {}^{ABC}_i\mathcal{F}(S_H, I_H, R_H, t)$ are the left hand side of Eq. (16) for $i = 1, 2, 3$, also given in Eq. (3). We will derive the scheme for system (16) and the same procedure will be for the rest of the compartments.

At $t = t_{n+1}$

$$S_H(t_{n+1}) = \begin{cases} S_{H0} + \frac{1}{\Gamma(\sigma)} \int_0^{t_1} (t-\zeta)^{\sigma-1} {}^C\mathcal{F}_1(S_H, I_H, R_H, \zeta) d\zeta, \\ S_H(t_1) + \frac{1-\sigma}{ABC(\sigma)} {}^{ABC}\mathcal{F}_1(S_H, I_H, R_H, t_n) \\ + \frac{\sigma}{ABC(\sigma)\Gamma(\sigma)} \int_{t_1}^{t_{n+1}} (t-\zeta)^{\sigma-1} {}^{ABC}\mathcal{F}_1(\zeta) d\zeta, \quad t_1 < t \leq T. \end{cases} \tag{17}$$

Writing Eq. (17) in the Newton interpolation approximation given in [33] as follows

$$S_H(t_{n+1}) = \begin{cases} S_{H0} + \begin{cases} \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{K=2}^{i} \left[{}^C\mathcal{F}_1(S_H{}^{K-2}, I_H{}^{K-2}, R_H{}^{K-2}, t_{K-2}) \right] \Pi \\ + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{K=2}^{i} \left[{}^C\mathcal{F}_1(S_H{}^{K-1}, I_H{}^{K-1}, R_H{}^{K-1}, t_{K-1}) \right. \\ \left. - {}^C\mathcal{F}_1(S_H{}^{K-2}, I_H{}^{K-2}, R_H{}^{K-2}, S_M{}^{K-2}, I_M{}^{K-2}, t_{K-2}) \right] \sum \\ + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{K=2}^{i} \left[{}^C\mathcal{F}_1(S_H{}^K, I_H{}^K, R_H{}^K, t_K) \right. \\ - 2\,{}^C\mathcal{F}_1(S_H{}^{K-1}, I_H{}^{K-1}, R_H{}^{K-1}, t_{K-1}) \\ \left. + {}^C\mathcal{F}_1(S_H{}^{K-2}, I_H{}^{K-2}, R_H{}^{K-2}, t_{K-2}) \right] \Delta \end{cases}, \\ S_H(t_1) + \begin{cases} \frac{1-\sigma}{ABC(\sigma)} {}^{ABC}\mathcal{F}_1(S_H{}^n, I_H{}^n, R_H{}^n, t_n) \\ + \frac{\sigma}{ABC(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{K=i+3}^{n} \left[{}^{ABC}\mathcal{F}_1(S_H{}^{K-2}, I_H{}^{K-2}, R_H{}^{K-2}, t_{K-2}) \right] \Pi \\ + \frac{\sigma}{ABC(\sigma)} \frac{(vt)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{K=i+3}^{n} \left[{}^{ABC}\mathcal{F}_1(S_H{}^{K-1}, I_H{}^{K-1}, R_H{}^{K-1}, t_{K-1}) \right. \\ \left. + ABC\mathcal{F}_1(S_H{}^{K-2}, I_H{}^{K-2}, R_H{}^{K-2}, t_{K-2}) \right] \sum \\ + \frac{\sigma}{ABC(\sigma)} \frac{\sigma(vt)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{K=i+3}^{n} \left[{}^{ABC}\mathcal{F}_1(S_H{}^K, I_H{}^K, R_H{}^K, t_K) \right. \\ - 2\,{}^{ABC}\mathcal{F}_1(S_H{}^{K-1}, I_H{}^{K-1}, R_H{}^{K-1}, t_{K-1}) \\ \left. + {}^{ABC}\mathcal{F}_1(S_H{}^{K-2}, I_H{}^{K-2}, R_H{}^{K-2}, t_{K-2}) \right] \Delta. \end{cases} \end{cases}$$

For the remaining three compartment we can write the Newton interpolation approximation as follows

$$
\mathrm{I_H}(t_{n+1}) = \begin{cases} \mathrm{I_{H0}} + \begin{cases} \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi \\ + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\ \left. - {}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \sum \\ + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, \mathcal{S_M}^{\mathrm{K}}, \mathcal{I_M}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\ - 2\,{}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ \left. + {}^{C}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta \end{cases}, \\ \mathrm{I_H}(t_1) + \begin{cases} \frac{1-\sigma}{ABC(\sigma)} {}^{ABC}\mathcal{F}_2(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, t_n) \\ + \frac{\sigma}{ABC(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi \\ + \frac{\sigma}{ABC(\sigma)} \frac{(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\ \left. + ABC\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \sum \\ + \frac{\sigma}{ABC(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\ - 2\,{}^{ABC}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ \left. + {}^{ABC}\mathcal{F}_2(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta. \end{cases} \end{cases}.
$$

$$
\mathrm{R_H}(t_{n+1}) = \begin{cases} \mathrm{R_{H0}} + \begin{cases} \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi \\ + \frac{(\Delta t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\ \left. - {}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, \mathcal{S_M}^{\mathrm{K}-2}, \mathcal{I_M}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \sum \\ + \frac{\sigma(\Delta t)^{\sigma-1}}{2\Gamma(\sigma+3)} \sum_{\mathrm{K}=2}^{i} \left[{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\ - 2\,{}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ \left. + {}^{C}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta \end{cases}, \\ \mathrm{R_H}(t_1) + \begin{cases} \frac{1-\sigma}{ABC(\sigma)} {}^{ABC}\mathcal{F}_3(\mathrm{S_H}^{n}, \mathrm{I_H}^{n}, \mathrm{R_H}^{n}, t_n) \\ + \frac{\sigma}{ABC(\sigma)} \frac{(\delta t)^{\sigma-1}}{\Gamma(\sigma+1)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Pi \\ + \frac{\sigma}{ABC(\sigma)} \frac{(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+2)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \right. \\ \left. + ABC\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \sum \\ + \frac{\sigma}{ABC(\sigma)} \frac{\sigma(\upsilon t)^{\sigma-1}}{\Gamma(\sigma+3)} \sum_{\mathrm{K}=i+3}^{n} \left[{}^{ABC}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}}, \mathrm{I_H}^{\mathrm{K}}, \mathrm{R_H}^{\mathrm{K}}, t_{\mathrm{K}}) \right. \\ - 2\,{}^{ABC}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-1}, \mathrm{I_H}^{\mathrm{K}-1}, \mathrm{R_H}^{\mathrm{K}-1}, \mathcal{S_M}^{\mathrm{K}-1}, \mathcal{I_M}^{\mathrm{K}-1}, t_{\mathrm{K}-1}) \\ \left. + {}^{ABC}\mathcal{F}_3(\mathrm{S_H}^{\mathrm{K}-2}, \mathrm{I_H}^{\mathrm{K}-2}, \mathrm{R_H}^{\mathrm{K}-2}, t_{\mathrm{K}-2}) \right] \Delta. \end{cases} \end{cases}.
$$

Here

$$
\Delta = \begin{bmatrix} (1+n-\mathrm{K})^{\sigma} \left(2(n-\mathrm{K})^2 + (3\sigma+10)(n-\mathrm{K}) + 2\sigma^2 + 9\sigma + 12 \right) \\ - (n-\mathrm{K}) \left(2(n-\mathrm{K})^2 + (5\sigma+10)(-\mathrm{K}+n) + 6\sigma^2 + 18\sigma + 12 \right) \end{bmatrix},
$$

$$
\sum = \begin{bmatrix} (1+n-\mathrm{K})^{\sigma} \left(3 + 2\sigma - \mathrm{K} + n \right) \\ - (n-\mathrm{K}) \left(n - \mathrm{K} + 3\sigma + 3 \right) \end{bmatrix},
$$

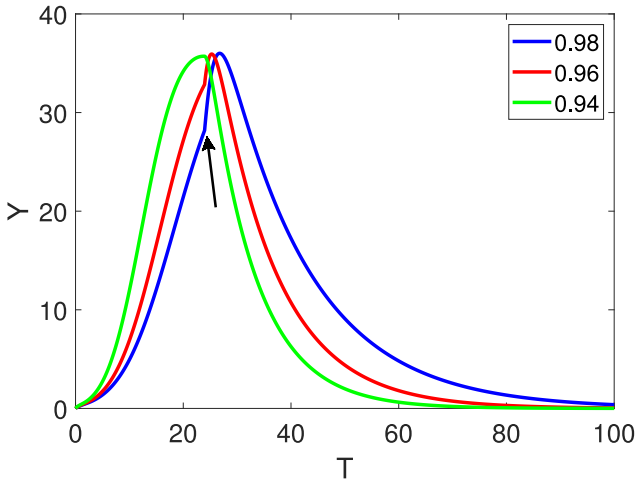

Fig. 2. The population dynamics of the infected population $\mathrm{I_H}(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 23$.

$$
\Delta = \left[(1+n-\mathrm{K})^{\sigma} - (n-\mathrm{K})^{\sigma} \quad \right].
$$

Simulations and discussion

The purpose of this section is to present the simulations of model (2), which is considered in Caputo and ABC piece-wise derivative. For Figs. 1–6, we have assumed the parameter values to be $\mathcal{B} = 0.56$, $\gamma = 0.009$, $\hbar = 2.5$, $\lambda = 0.5$, $\beta_n = 175$, $\nu_1 = 4$, and $\nu_2 = 25$. The initial values are assumed to be $\mathcal{X}(0) = 400$, $\mathcal{Y}(0) = 0$, $\mathcal{Z}(0) = 4$. We divide the whole interval into two sub intervals which are $[0, t_1] = [0, 23]$ and $[t_1, T] = [23, 100]$.

The arrow presented in the figures points towards t_1. The population dynamics of the susceptible class $\mathcal{X}$ with different fractional order σ is demonstrated in Figs. 1 and 4. Similarly in Figs. 2 and 5 the population behaviour of the infected individuals is depicted with various values of σ, while the affect of the fractional piecewise operator on the free virus class of the proposed model is projected in Figs. 3 and 6. From the simulations it is observed that the susceptible population decreases with time and becomes zero soon after we advances to the second sub-interval. Similarly, from the infected population we see that the infected individuals increases and reach its peak value around $t = 30$, where after this time a decline in the population of infected individuals is observed. Finally, the recovered or the free virus population shows increase in the recovered individuals which decreases gradually after $t = 33$, and becomes stable at $t = 100$. However, it is observed that the recovered population at lower fractional orders become stable soon as compared to the high values of σ.

For Figs. 7–12, we have assumed the parameter values to be $\mathcal{B} = 0.56$, $\gamma = 0.001$, $\hbar = 0.1313$, $\lambda = 0.5$, $\beta_n = 156$, $\nu_1 = 4$, and $\nu_2 = 25$. The initial values are assumed to be $\mathcal{X}(0) = 400$, $\mathcal{Y}(0) = 0$, $\mathcal{Z}(0) = 4$. Here we split the whole interval into two sub intervals which are $[0, t_1] = [0, 40]$ and $[t_1, T] = [40, 100]$.

The arrow presented in the figures points towards t_1, where the second interval starts and the operator shifts to ABC. The dynamics of the susceptible class $\mathcal{X}$ with different fractional order σ is demonstrated in Figs. 7 and 10. Similarly in Figs. 8 and 11 the population behaviour of the infected individuals is demonstrated with various fractional orders σ, while the affect of the fractional piecewise operator on the free virus class of the proposed model with the second set of parameter values are projected in Figs. 9 and 12. From the graphs one can see that the susceptible population decreases with time and becomes zero gradually. Similarly, from the infected population we see that the infected individuals increases and reach its peak value at different

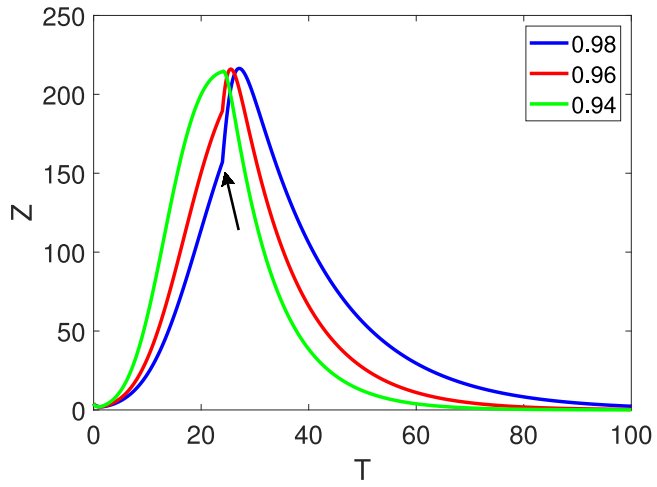

Fig. 3. The dynamics of recovered population $R_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 23$.

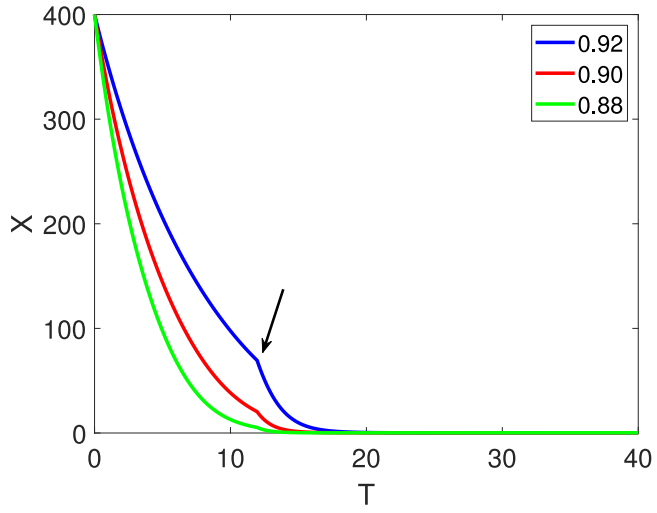

Fig. 4. The population dynamics of susceptible population $S_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 13$.

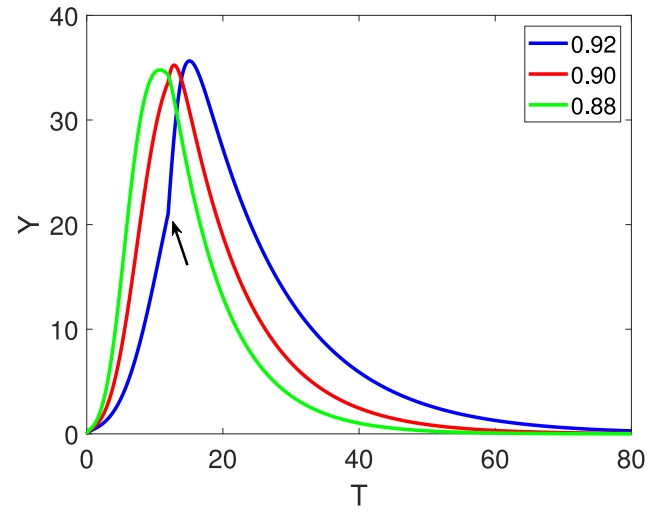

Fig. 5. The population dynamics of the infected population $I_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 13$.

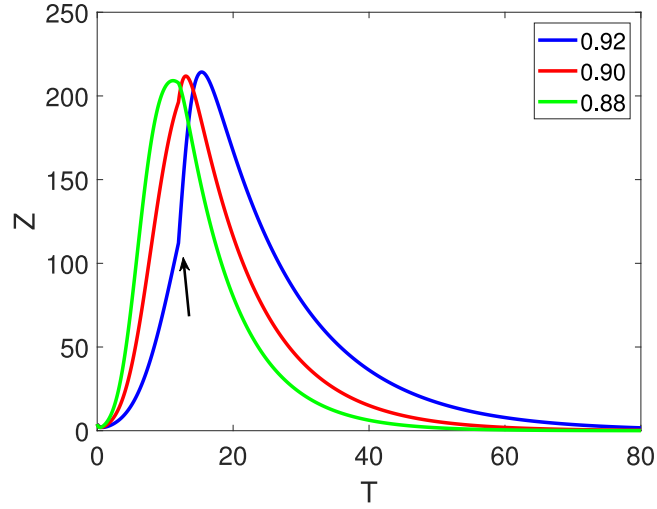

Fig. 6. The dynamics of recovered population $R_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 13$.

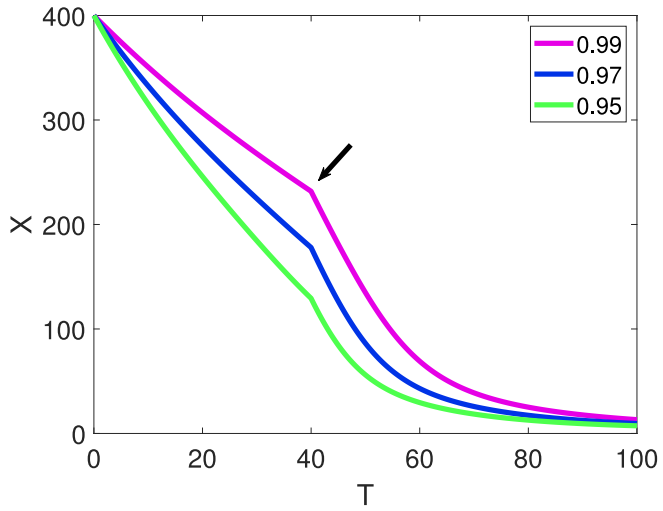

Fig. 7. The population dynamics of susceptible population $S_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 40$.

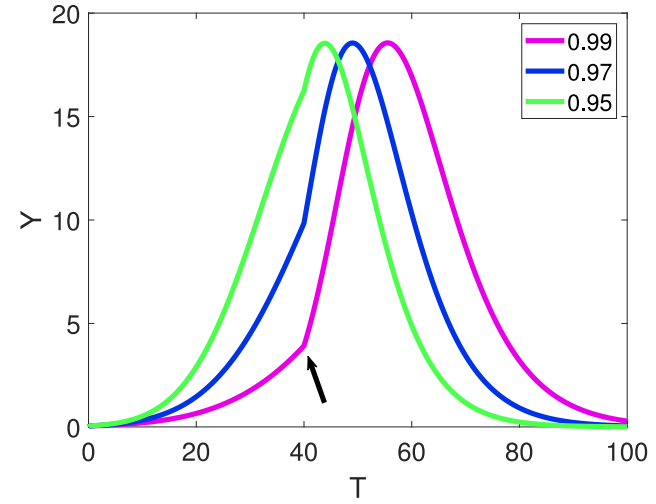

Fig. 8. The population dynamics of the infected population $I_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 40$.

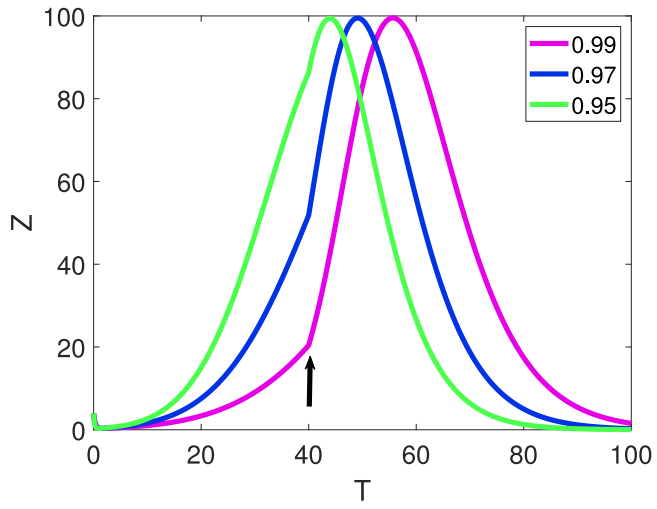

Fig. 9. The dynamics of recovered population $R_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 40$.

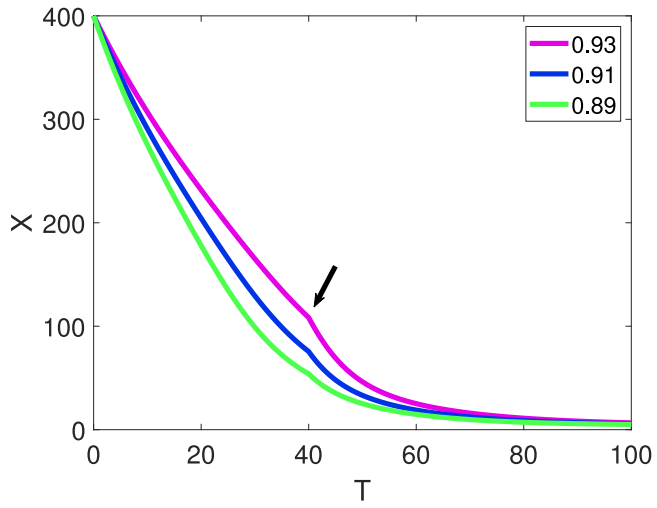

Fig. 10. The population dynamics of susceptible population $S_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 40$.

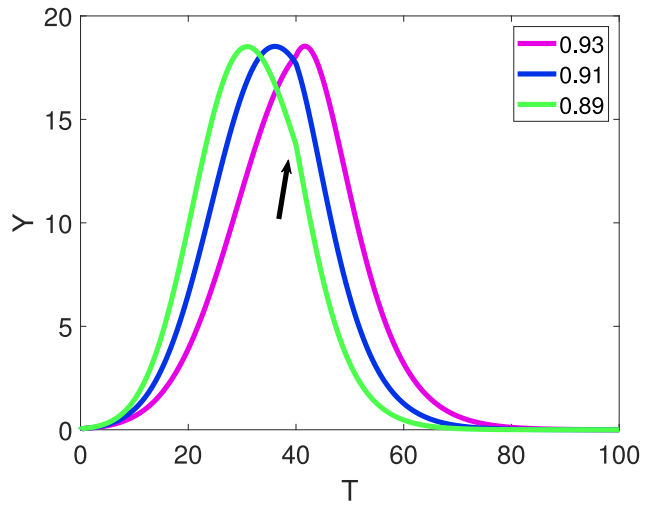

Fig. 11. The population dynamics of the infected population $I_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 40$.

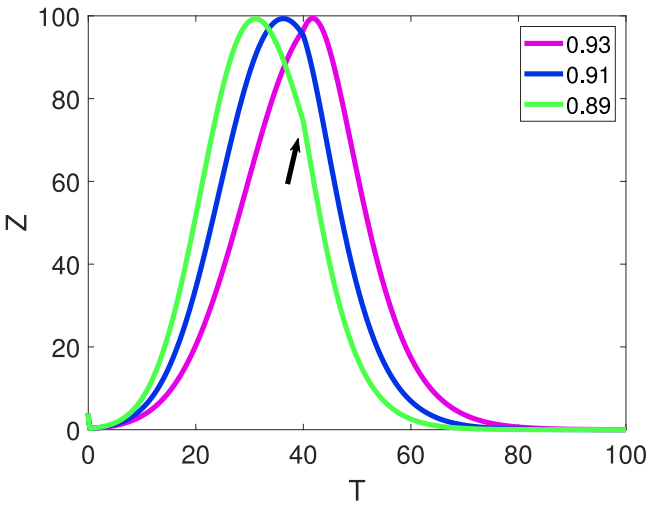

Fig. 12. The dynamics of recovered population $R_H(t)$ in the Caputo-ABC piecewise model (2) with $t_1 = 40$.

values of t for different fractional orders, after which a decline in the population of infected individuals is can be seen.

Further we see that the recovered or the free virus population shows increase in the recovered individuals which decreases gradually after $t = 60$, and becomes stable at $t = 100$ when $\sigma = 0.99$. However, it is observed that the recovered population at lower fractional orders become stable soon as compared to the high values of σ. From the lower fractional orders, one can see that the model's state variables becomes stable soon as compared to the high values of σ.

Conclusion

In this article, we have analysed the dynamics of the dengue epidemic model with a novel piecewise derivative in the sense of the Caputo and Atangana–Baleanu Caputo operators. The existence and uniqueness of a solution with piecewise derivative has been examined for the aforesaid disease model. The suggested problem's approximate solution was obtained using the piecewise approach Newton polynomial approach. A numerical scheme for piecewise derivatives has been established in terms of singular and non-singular kernels. The numerical simulation for the piecewise dengue model was presented for various fractional orders. We have observed that piecewise operators present better dynamics of the models as compared to the classical ones. This work advances the idea of the piecewise derivatives and presents the dynamics of the crossover behaviour in a more clear way. In the future, we will study the proposed model with optimal control theory and piece-wise operators.

CRediT authorship contribution statement

Shabir Ahmad: Methodology, Writing – original draft. **Mansour F. Yassen:** Methodology, Formal analysis, Investigation. **Mohammad Mahtab Alam:** Methodology, Validation. **Soliman Alkhati:** Validation. **Fahd Jarad:** Conceptualization. **Muhammad Bilal Riaz:** Conceptualization, Formal analysis, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Caminade C, Medlock JM, Ducheyne E, McIntyre KM, Leach S, Baylis M, Morse AP. Suitability of European climate for the Asian tiger mosquito aedes albopictus: recent trends and future scenarios. J R Soc Interface 2012;9(7):2708–17.
[2] Altaf KM, Atangana A. Dynamics of ebola disease in the framework of different fractional derivatives. Entropy 2019;21(3):303.
[3] Zhang M, Lin Z. A reaction–diffusion-advection model for aedes aegypti mosquitoes in a time-periodic environment. Nonlinear Anal-Real 2019;46:219–37.
[4] Nishiura H. Mathematical and statistical analyses of the spread of dengue. Dengue Bull 2006;30:51–67.
[5] Acay B, Inc M, Khan A, et al. Fractional methicillin-resistant staphylococcus aureus infection model under Caputo operator. J Appl Math Comput 2021. http://dx.doi.org/10.1007/s12190-021-01502-3.
[6] Amin Jajarmi, Sadia Arshad, Dumitru Baleanu. A new fractional modelling and control strategy for the outbreak of dengue fever. Phys A 2019;535:122524.
[7] Hopp MJ, Foley JA. Global-scale relationships between climate and the dengue fever vector. Aedes Aegypt Clim Change 2001;48:441–63.
[8] Hethcote HW. The mathematics of infectious diseases. SIAM Rev 2000;42:599–653.
[9] Newton EA, Reiter P. A model of the transmission of dengue fever with an evaluation of the impact of ultra-low volume (ULV) Insecticide applications on dengue epidemics. Am J Trop Med Hyg 1992;47:709–20.
[10] Esteva L, Vargas C. Analysis of a dengue disease transmission model. Math Biosci 1998;150:131–51. http://dx.doi.org/10.1016/S0025-5564(98)10003-2.
[11] Feng Z, Hernàndez V. Competitive exclusion in a vector-host model for the dengue fever. J Math Biol 1997;35:523–44.
[12] Ahmad S, Ullah A, Akgül A, De la Sen M. A study of fractional order Ambartsumian equation involving exponential decay kernel. AIMS Math 2021;6(9):9981–97.
[13] Saifullah S, Ali A, Khan Z. Analysis of nonlinear time-fractional Klein–Gordon equation with power law kernel. AIMS Math 7(4):5275-5290.
[14] Ahmad S, Ullah A, Partohaghighi M, Saifullah S, Akgül A, Jarad F. Oscillatory and complex behaviour of Caputo–Fabrizio fractional order HIV-1 infection model. AIMS Math 2021;7(3):4778–92.
[15] Xu C, Farman M, Hassan A, Akgül A, Zakarya M, Albalawi W, Park C. Lyapunov stability and wave analysis of Covid-19 omicron variant of real data with fractional operator. Alex Eng J 2022;61(12):11787–802.
[16] Alqahtani RT, Ahmad S, Akgül A. Dynamical analysis of bio-ethanol production model under generalized nonlocal operator in Caputo sense. Mathematics 2021;9(19):2370.
[17] Li P, Li Y, Gao R, Xu C, Shang Y. New exploration on bifurcation in fractional-order genetic regulatory networks incorporating both type delays. Eur Phys J Plus 2022;137:598.
[18] Xu C, Liu Z, Liao M, Yao L. Theoretical analysis and computer simulations of a fractional order bank data model incorporating two unequal time delays. Expert Syst Appl 2022;199:116859.
[19] Rahman M, Ahmad S, Arfan M, Akgül A, Jarad F. Fractional order mathematical model of serial killing with different choices of control strategy. Fractal Fract 2022;6(3):162.
[20] Xu C, Liao M, Li P, Yuan S. Impact of leakage delay on bifurcation in fractional-order complex-valued neural networks. Chaos Solitons Fractals 2021;142:110535.
[21] Pooseh S, Rodrigues HS. Fractional derivatives in dengue epidemics. 2011, http://dx.doi.org/10.1063/1.3636838.
[22] Jan R, Jan A. MSGDTM for solution of fractional order dengue disease model. Int J Sci Res 2017;6(3):1140–4.
[23] Al-Sulami H, El-Shahed JJ, Nieto W, Shammakh M. On fractional order dengue epidemic model. Math Probl Eng 2014. http://dx.doi.org/10.1155/2014/456537.
[24] Faiz Muhammad, Ullah Khan Zia, Lv Yu-Pei, Yusuf Abdullahi, Din Anwarud. Investigating of fractional order dengue epidemic model with ABC operator. Results Phys 2021;24.
[25] Zafar ZUA, Mushtaq M, Rehan K. A non-integer order dengue internal transmission model. Adv Differ Equ 2018 2018;23.
[26] Atangana A, Baleanu D. New fractional derivatives with non-local and non-singular kernel. Theory Appl Heat Transf Model Therm Sci 2016;20(2):763–9.
[27] Goufo EFD. Application of the Caputo–Fabrizio fractional derivative without singular kernel to Korteweg–de Vries-Burgers equation. Math Model Anal 2016;21(2):188–98.
[28] Goufo EFD. A bio mathematical view on the fractional dynamics of cellulose degradation. Fract Calc Appl Anal 2015;18(3):554–64.
[29] Atangana A. Extension of rate of change concept: from local to nonlocal operators with applications. Res Phys 2020.
[30] Atangana A, Araz SI. Nonlinear equations with global differential and integral operators:existence, uniqueness with application to epidemiology. Res Phys 2020.
[31] Kabunga SK, Goufo EFD, Tuong VH. Analysis and simulation of a mathematical model of tuberculosis transmission in democratic Republic of the Congo. Adv Differential Equations 2020;(1):1–19.
[32] Atangana A, Araz SI. Mathematical model of COVID-19 spread in Turkey and South Africa: theory, methods and applications. Adv Differ Equ 2020;2020:659.
[33] Atangana A, Araz SI. New concept in calculus: Piecewise differential and integral operators. Chaos Solitons Fractals 2021;145:110638.
[34] Xu C, Alhejaili W, Saifullah S, Khan A, Khan J, El-Shorbagy MA. Analysis of huanglongbing disease model with a novel fractional piecewise approach. Chaos Solitons Fractals 2022;161:112316.
[35] Cao Y, Alamri S, Rajhi AA, Anqi AE, Riaz MB, Elagan SK, Jawa TM. A novel piece-wise approach to modeling interactions in a food web model. Results Phys 2021;31:104951.
[36] Zeb A, Atangana A, Khan ZA, Djillali S. A robust study of a piecewise fractional order COVID-19 mathematical model. Alex Eng J 2022;61(7):5649–65.