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Research article

Bennett-Leindler nabla type inequalities via conformable fractional derivatives on time scales

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Abstract: In this work, we prove several new (γ, a)-nabla Bennett and Leindler dynamic inequalities on time scales. The results proved here generalize some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. Our results will be proved by using integration by parts, chain rule and Hölder inequality for the (γ, a)-nabla-fractional derivative on time scales.

Keywords: Steffensen's inequality; dynamic inequality; dynamic integral; time scales
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1. Introduction

In [13], Hardy presented the discrete version.

Theorem 1.1. If $\{r(n)\}_{n=0}^{\infty}$ is a nonnegative real sequence and $l > 1$, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{m=1}^{n} r(m)\right)^{l} \leq \left(\frac{l}{l-1}\right)^{l}\sum_{n=1}^{\infty} r^{l}(n). \tag{1.1}$$

Also, Hardy [14] gave the continuous version of (1.1).

Theorem 1.2. Let $r \geqslant 0$ be a continuous over $[0, \infty)$ and $l > 1$. Then

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x r(s)ds\Big)^l dx \leq \Big(\frac{l}{l-1}\Big)^l \int_0^\infty r^l(x)dx. \tag{1.2}$$

Copson [6] obtained a new version of inequality (1.1) by replacing the arithmetic mean of a sequence by a weighted arithmetic mean in the following manner: Let $b(m) \geqslant 0$, $\eta(m) \geqslant 0$ for all m. If $l > 1$, $c > 1$, then

$$\sum_{m=1}^{\infty} \frac{\eta(m)}{[\bar{\xi}(m)]^c}\Big(\sum_{j=1}^{m} b(j)\eta(j)\Big)^l \leqslant \Big(\frac{l}{c-1}\Big)^l \sum_{m=1}^{\infty} \eta(m)[\bar{\xi}(m)]^{l-c} b^l(m), \tag{1.3}$$

where $\bar{\xi}(m) = \sum_{j=1}^{m} \eta(j)$, and if $0 \leqslant c < 1 < l$, then

$$\sum_{m=1}^{\infty} \frac{\eta(m)}{[\bar{\xi}(m)]^c}\Big(\sum_{j=1}^{\infty} b(j)\eta(j)\Big)^l \leqslant \Big(\frac{l}{1-c}\Big)^l \sum_{m=1}^{\infty} \eta(m)[\bar{\xi}(m)]^{l-c} b^l(m). \tag{1.4}$$

The reverse versions of the inequalities (1.3) and (1.4), which have been derived by Bennett and Leindler [4, 17], can be deduced for $\bar{\xi}(m) \to \infty$, $b(m) \geqslant 0$ and $\eta(m) \geqslant 0$ for all m that if $0 < l < 1 < c$, then

$$\sum_{m=1}^{\infty} \frac{\eta(m)}{[\bar{\xi}(m)]^c}\Big(\sum_{j=1}^{m} b(j)\eta(j)\Big)^l \geqslant \Big(\frac{lL}{c-1}\Big)^l \sum_{m=1}^{\infty} \eta(m)[\bar{\xi}(m)]^{l-c} b^l(m), \tag{1.5}$$

where $L = \inf \frac{\bar{\xi}(m)}{\bar{\xi}(m+1)}$, and if $c \leqslant 0 < l < 1$, then

$$\sum_{m=1}^{\infty} \frac{\eta(m)}{[\bar{\xi}(m)]^c}\Big(\sum_{j=1}^{\infty} b(j)\eta(j)\Big)^l \geqslant \Big(\frac{l}{1-c}\Big)^l \sum_{m=1}^{\infty} \eta(m)[\bar{\xi}(m)]^{l-c} b^l(m), \tag{1.6}$$

respectively. Copson [7] gave the continuous version of the inequalities (1.5) and (1.6), respectively, as follows: Let η and f be nonnegative functions and $\bar{\xi}(\wp) = \int_0^\wp \eta(\varrho)d\varrho$, $B(\wp) = \int_0^\wp \eta(\varrho)f(\varrho)d\varrho$, $\overline{B}(\wp) = \int_\wp^\infty \eta(\varrho)f(\varrho)d\varrho$. If $0 < l \leqslant 1 < c$, $a > 0$ then

$$\int_a^\infty \frac{\eta(\wp)}{[\bar{\xi}(\wp)]^c}[B(\wp)]^l dt \geqslant \Big(\frac{p}{c-1}\Big)^l \int_a^\infty \eta(\wp)[\bar{\xi}(\wp)]^{l-c} f^l(\wp)dt, \tag{1.7}$$

and if $0 < l < 1$, $c < 1$ then

$$\int_a^\infty \frac{\eta(\wp)}{[\bar{\xi}(\wp)]^c}[\overline{B}(\wp)]^l dt \geqslant \Big(\frac{l}{1-c}\Big)^l \int_a^\infty \eta(\wp)[\bar{\xi}(\wp)]^{l-c} f^l(\wp)dt. \tag{1.8}$$

For further results on Hardy inequalities and other types see [1–3,5,8–10,12,15,18–21,23–27]. In [20] the author proved the time scales version of (1.1) and (1.2).

$$\int_a^\infty \Big(\frac{\int_a^{\sigma(\wp)} \eta(\varrho)\Delta\varrho}{\sigma(\wp)-a}\Big)^l \Delta\wp < \Big(\frac{l}{l-1}\Big)^l \int_a^\infty \eta^l(\wp)\Delta\wp, \tag{1.9}$$

unless $\eta \equiv 0$.

In [11] El-Deeb et al. extended (1.9)

$$\int_a^\infty \frac{\tilde{\lambda}(\varsigma)\breve{\Psi}^p(\varsigma)}{\tilde{\Lambda}^{\hat{\gamma}}(\varsigma)}\Delta\varsigma \geq \frac{p}{\hat{\gamma}-1}\int_a^\infty \tilde{\lambda}(\varsigma)\tilde{\Lambda}^{p-\hat{\gamma}}(\varsigma)\upsilon^p(\varsigma)\Delta\varsigma, \tag{1.10}$$

where

$$\breve{\Psi}(\varsigma) = \int_a^\varsigma \tilde{\lambda}(\upsilon)\upsilon(\varrho)\Delta\upsilon, \qquad \text{and} \qquad \tilde{\Lambda}(\varsigma) = \int_a^\varsigma \tilde{\lambda}(\upsilon)\Delta\upsilon.$$

In 2021 Kayar et al. [16], established the time scale version unification of discrete and continuous Bennett-Leindler inequalities (1.5) and (1.7) as following theorem.

Theorem 1.3. Let λ, f be nonnegative, ld-continuous, ∇-differentiable and ∇-integrable functions on $[a,\infty)_{\mathbb{T}}$ where $a \in [0,\infty)_{\mathbb{T}}$. Define

$$\bar{\xi}(\wp) = \int_a^\wp \lambda(\varrho)\nabla\varrho \qquad B(\wp) = \int_a^\wp \lambda(\varrho)f(\varrho)\nabla\varrho.$$

If $L = \inf_{\wp\in\mathbb{T}} \frac{\bar{\xi}^\rho(\wp)}{\bar{\xi}(\wp)} > 0$, $0 < p < 1$ and $c \geqslant 1$, then

$$\int_a^\infty \frac{\lambda(\wp)}{[\bar{\xi}^c(\wp)]}[B(\wp)]^p\nabla\wp \geqslant \left(\frac{pL^c}{c-1}\right)^p \int_a^\infty \frac{\lambda(\wp)f^p(\wp)}{(\bar{\xi}(\wp))^{c-p}}\nabla\wp. \tag{1.11}$$

Lately, Zakarya et al. proved an α-conformable version of Hardy inequalities [22].

Theorem 1.4. Assume that $\mathbb{T}$ is a time scale with $\omega \in (0,\infty)_{\mathbb{T}}$. Let λ and ξ be rd-continuous and α-fractional differentiable functions on $[\omega,\infty)_{\mathbb{T}}$. Define

$$\chi(\wp) = \int_t^\infty \lambda(\varrho)\Delta_\alpha\varrho \ \textit{ and } \ \Theta(\wp) = \int_\omega^\wp \lambda(\varrho)\xi(\varrho)\Delta_\alpha\varrho.$$

Then, for $k \leqslant 0 < h < 1$, and $\alpha \in (0,1]$, we have that

$$\int_\omega^\infty \frac{\lambda(\wp)}{\chi^{k-\alpha+1}(\wp)}(\Theta^\sigma(\wp))^h\Delta_\alpha\wp \geqslant \left(\frac{h}{\alpha-k}\right)^h \int_\omega^\infty \lambda(\wp)\xi^h(\wp)\chi^{h-k+\alpha-1}(\wp)\Delta_\alpha\wp.$$

We will need the following chain rule for γ-nabla derivative, integration by parts for γ-nabla derivative [29] and generalized γ-nabla Hölder fractional inequality on timescales [28] respectively

$$\nabla_a^\gamma(\varpi\circ\xi)(\wp) = \varpi'(\xi(c))\nabla_a^\gamma(\xi(\wp)). \tag{1.12}$$

$$\int_d^b \varpi(\wp)[\nabla_a^\gamma\xi(\wp)]\nabla_a^\gamma\wp = \Big[\varpi(\wp)\xi(\wp)\Big]_d^b - \int_d^b [\nabla_a^\gamma\varpi(\wp)]\xi^\rho(\wp)\nabla_a^\gamma\wp. \tag{1.13}$$

$$\int_d^b |\varpi(\wp)\xi(\wp)|\nabla_a^\gamma\wp \leq \left(\int_d^b |\varpi(\wp)|^p\nabla_a^\gamma\wp\right)^{1/p}\left(\int_d^b |\xi(\wp)|^q\nabla_a^\gamma\wp\right)^{1/q}, \tag{1.14}$$

where $\frac{1}{p}+\frac{1}{q} = 1$ and $0 < \gamma \leq 1$. Now, we start to state our main results.

2. Main results

We focus in this section, on investigating corresponding results for γ-nabla conformable time scales.
Theorem 2.1. Let $\mathbb{T}$ be a time scale with $r \in [0,\infty)_{\mathbb{T}}$, $\gamma \in (0,1]$ and $\wp \geqslant a$. In addition, let $\daleth$ and λ be nonnegative ld-continuous and (γ, a)-nabla fractional differentiable functions on $[r,\infty)_{\mathbb{T}}$ where

$$\Omega(\wp) = \int_{\wp}^{\infty} \lambda(\varrho)\nabla_a^\gamma \varrho \quad \textit{and} \quad \Psi(\wp) = \int_r^{\wp} \lambda(\varrho)\daleth(\varrho)\nabla_a^\gamma \varrho, \quad \wp \in [r,\infty)_{\mathbb{T}}.$$

If $0 < p < \gamma$ and $c \leqslant \gamma - 1$, then

$$\int_r^\infty \frac{\lambda(\wp)}{[\Omega^\rho(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^\gamma \wp \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\daleth^p(\wp)[\Psi(\wp)]^{1-\gamma}}{(\Omega^\rho(\wp))^{c-p-\gamma+1}}\nabla_a^\gamma \wp. \tag{2.1}$$

Proof. Using (1.13), with

$$\nabla_a^\gamma \eta(\wp) = \lambda(\wp)/[\Omega^\rho(\wp)]^{c-\gamma+1}, \qquad \xi(\wp) = [\Psi(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^\infty \frac{\lambda(\wp)}{[\Omega^\rho(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\alpha+1}\nabla_a^\gamma \wp = [\eta(\wp)\Psi^{p-\alpha+1}(\wp)]_r^\infty + \int_r^\infty (-\eta^\rho(\wp))\nabla_a^\gamma(\Psi^{p-\gamma+1}(\wp))\nabla_a^\gamma \wp,$$

where we assumed that

$$\eta(\wp) = -\int_\wp^\infty \lambda(\wp)/\Omega^{c-\gamma+1}(\wp)\nabla_a^\gamma \varrho. \tag{2.2}$$

Using $\Psi(r) = 0$ and $\eta(\infty) = 0$, we get

$$\int_r^\infty \frac{\lambda(\wp)}{[\Omega^\rho(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^\gamma \wp = \int_r^\infty -\eta^\rho(\wp)\nabla_a^\gamma(\Psi^{p-\gamma+1}(\wp))\nabla_a^\gamma \wp. \tag{2.3}$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$\nabla_a^\gamma(\Psi^{p-\gamma+1}(\wp)) = \frac{p-\gamma+1}{\Psi^{\gamma-p}(d)}\nabla_a^\gamma \Psi(\wp) \geqslant \frac{(p-\gamma+1)\lambda(\wp)\daleth(\wp)}{\Psi^{\gamma-p}(\wp)}. \tag{2.4}$$

Next note $\nabla_a^\gamma \Omega(\wp) = -\lambda(\wp) \leqslant 0$. By chain rule, we see that

$$\begin{aligned}
\nabla_a^\gamma(\Omega(\wp))^{\gamma-c} &= (\gamma-c)\int_0^1 \frac{\nabla_a^\gamma \Omega(\wp)dh}{[h\Omega(\wp)+(1-h)\Omega^\rho(\wp)]^{c-\alpha+1}} \\
&= -(\gamma-c)\int_0^1 \frac{\lambda(\wp)dh}{[h\Omega(\wp)+(1-h)\Omega^\rho(\wp)]^{c-\gamma+1}} \\
&\geqslant -(\gamma-c)\int_0^1 \frac{\lambda(\wp)dh}{[h\Omega^\rho(\wp)+(1-h)\Omega^\rho(\wp)]^{c-\gamma+1}} \\
&= -(\gamma-c)\frac{\lambda(\wp)}{[\Omega^\rho(\wp)]^{c-\gamma+1}}.
\end{aligned}$$

This implies that

$$\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} \geqslant \frac{-1}{\gamma-c}\nabla_a^{\gamma}(\Omega(\wp))^{\gamma-c}, \tag{2.5}$$

and then, we have that

$$\begin{aligned} -\eta^{\rho}(\wp) = \int_{\rho(\wp)}^{\infty} \frac{\lambda(\varrho)}{[\Omega^{\rho}(\varrho)]^{c-\gamma+1}}\nabla_a^{\gamma}\varrho &\geqslant \frac{-1}{\gamma-c}\int_{\rho(\wp)}^{\infty} \nabla_a^{\gamma}(\Omega(\varrho))^{\gamma-c}\nabla_a^{\gamma}\varrho \\ &= \frac{1}{(\gamma-c)(\Omega^{\rho}(\wp))^{c-\gamma}}. \end{aligned} \tag{2.6}$$

Using (2.4) and (2.6) in (2.3) yields

$$\left(\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\right)^p \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p\left(\int_r^{\infty}\left[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\Omega^{\rho}(\wp))^{p(c-\gamma)}}\right]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\right)^p. \tag{2.7}$$

Applying Hölder inequality on the term

$$\left(\int_r^{\infty}\left[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\Omega^{\rho}(\wp))^{p(\gamma-c)}}\right]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\right)^p,$$

with indices $1/p$ and $1/1-p$ and

$$F(\wp) = \frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\Omega^{\rho}(\wp))^{p(c-\gamma)}} \quad \textit{and} \quad \beth(\wp) = \left(\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}\right)^{1-p}[\Psi(\wp)]^{(1-p)(p-\gamma+1)}$$

$$\begin{aligned} &\left(\int_r^{\infty} F^{1/p}(\wp)\nabla_a^{\gamma}\wp\right)^p = \left(\int_r^{\infty}\left[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\Omega^{\rho}(\wp))^{p(c-\gamma)}}\right]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\right)^p \\ &\geqslant \frac{\int_r^{\infty} F(\wp)\beth(\wp)\nabla_a^{\gamma}\wp}{\left(\int_r^{\infty}\beth^{\frac{1}{1-p}}(\wp)\nabla_a^{\gamma}\wp\right)^{p-1}} = \left[\int_r^{\infty}\frac{\lambda^p(\wp)(\lambda(\wp)[\Omega^{\rho}(\wp)]^{\gamma-c-1})^{1-p}\beth^p(\wp)[\Psi(\wp)]^{(1-p)(p-\gamma+1)}}{(\Psi(\wp))^{p(\alpha-p)}(\Omega^{\rho}(\wp))^{p(c-\alpha)}}\nabla_a^{\gamma}\wp\right] \\ &\quad\times\left(\int_r^{\infty}\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\alpha+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\right)^{p-1} \\ &= \left(\int_a^{\infty}\frac{\lambda(\wp)\beth^p(\wp)[\Psi(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp\right)\left(\int_r^{\infty}\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\right)^{p-1}. \end{aligned} \tag{2.8}$$

From (2.8) and (2.7) yields

$$\begin{aligned} &\left(\int_r^{\infty}\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\right)^p \\ &\geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p\left(\int_r^{\infty}\frac{\lambda(\wp)\beth^p(\wp)[\Psi(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp\right)\left(\int_r^{\infty}\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\right)^{p-1}. \end{aligned} \tag{2.9}$$

Therefore,

$$\int_r^{\infty}\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p\int_r^{\infty}\frac{\lambda(\wp)\beth^p(\wp)[\Psi(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp.$$

□

Remark 2.1. In Theorem 2, if we take $\gamma = 1$ then inequality (2.1) reduces to

$$\int_r^\infty \frac{\lambda(\wp)\Psi^p(\wp)}{(\Omega^\rho(\wp))^c}\nabla\wp \geqslant \left(\frac{p}{1-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\daleth^p(\wp)}{(\Omega^\rho(\wp))^c}\nabla\wp,$$

where

$$\Psi(\wp) = \int_r^\wp \lambda(\varrho)\daleth(\varrho)\nabla\varrho \qquad \textit{and} \qquad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)\nabla\varrho,$$

which is Theorem 3.1 in [16].

Corollary 2.1. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.1) obtains

$$\int_r^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{(\Omega(\wp))^{c-\gamma+1}}(\wp-a)^{\gamma-1}d\wp \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\daleth^p(\wp)\Psi^{1-\gamma}(\wp)}{(\Omega(\wp))^{c-\gamma+1}}(\wp-a)^{\gamma-1}dt,$$

where

$$\Psi(\wp) = \int_r^\wp \lambda(\varrho)\daleth(\varrho)(\wp-a)^{\gamma-1}d\varrho \qquad \textit{and} \qquad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)(\wp-a)^{\gamma-1}ds.$$

Corollary 2.2. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.1) get

$$\sum_{\wp=\frac{r}{h}}^{\infty} \frac{\lambda(h\wp)\Psi^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(h\wp-h)}(\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)} \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp=\frac{r}{h}}^{\infty} \frac{\lambda(h\wp)\daleth^p(h\wp)\Psi^{1-\gamma}(\wp)}{\Omega^{c-\gamma+1}(h\wp-h)}(\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)},$$

where

$$\Psi(\wp) = h\sum_{\varrho=\frac{\wp}{h}}^{\wp} \lambda(h\varrho)\daleth(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)} \qquad \textit{and} \qquad \Omega(\wp) = h\sum_{\varrho=\frac{\wp}{h}}^{\infty} \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)}.$$

Corollary 2.3. From Corollary 2, assumer $\mathbb{T} = \mathbb{Z}$ and $h = 1$, then (2.1) obtains

$$\sum_{\wp=r}^{\infty} \frac{\lambda(\wp)\Psi^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(\wp-1)}(\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)} \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp=r}^{\infty} \frac{\lambda(\wp)\daleth^p(\wp)\Psi^{1-\gamma}(\wp)}{\Omega^{c-\gamma+1}(\wp-1)}(\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)},$$

where

$$\Psi(\wp) = \sum_{\varrho=r}^{\wp} \lambda(\varrho)\daleth(\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)} \qquad \textit{and} \qquad \Omega(\wp) = h\sum_{\varrho=\wp}^{\infty} \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)}.$$

Corollary 2.4. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.10) obtains

$$\sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{\Omega^{c-\gamma+1}(\rho(\wp))} \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\daleth^p(\wp)\Psi^{1-\gamma}(\wp)}{\Omega^{c-\gamma+1}(\rho(\wp))},$$

where

$$\Psi(\wp) = (\tilde{q}-1)\sum_{\varrho\in(r,\wp)} \varrho\lambda(\varrho)\daleth(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)} \qquad \textit{and} \qquad \Omega(\wp) = (\tilde{q}-1)\sum_{\varrho\in(\wp,\infty)} \varrho\lambda(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)}.$$

Theorem 2.2. Let $\mathbb{T}$ be a time scale with $r \in [0,\infty)_{\mathbb{T}}$, $\gamma \in (0,1]$ and $\wp \geqslant a$. In addition, let $\gimel$ and λ be nonnegative ld-continuous and (γ, a)-nabla fractional differentiable functions on $[r,\infty)_{\mathbb{T}}$ where

$$\Omega(\wp) = \int_{\wp}^{\infty} \lambda(\varrho)\nabla_a^{\gamma}\varrho \quad \textit{and} \quad \overline{\Psi}(\wp) = \int_{\wp}^{\infty} \lambda(\varrho)\gimel(\varrho)\nabla_a^{\gamma}\varrho, \quad \wp \in [r,\infty)_{\mathbb{T}}.$$

If $0 < p < \gamma$ and $c \geqslant \gamma$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^{p} \int_r^{\infty} \frac{\lambda(\wp)\gimel^{p}(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp. \tag{2.10}$$

Proof. Using (1.13), with

$$\nabla_a^{\gamma}\eta(\wp) = \lambda(\wp)/[\Omega^{\rho}(\wp)]^{c-\gamma+1}, \qquad \xi(\wp) = [\overline{\Psi}(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\overline{\Psi}^{\rho}(\wp)]^{p-\alpha+1}\nabla_a^{\gamma}\wp = [\eta(\wp)\overline{\Psi}^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty} (\eta(\wp))\nabla_a^{\gamma}(-\overline{\Psi}^{p-\gamma+1}(\wp))\nabla_a^{\gamma}\wp,$$

where we assumed that

$$\eta(\wp) = \int_r^{\wp} \lambda(\wp)/[\Omega^{\rho}(\wp)]^{c-\gamma+1}\nabla_a^{\gamma}\varrho.$$

Using $\overline{\Psi}(\infty) = 0$ and $\eta(r) = 0$, we get

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp = \int_r^{\infty} \eta(\wp)\nabla_a^{\gamma}(-\overline{\Psi}^{p-\gamma+1}(\wp))\nabla_a^{\gamma}\wp. \tag{2.11}$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$\nabla_a^{\gamma}(-\overline{\Psi}^{p-\gamma+1}(\wp)) \;=\; -\frac{p-\gamma+1}{\Psi^{\gamma-p}(d)}\nabla_a^{\gamma}\Psi(\wp) \geqslant \frac{(p-\gamma+1)\lambda(\wp)\gimel(\wp)}{\overline{\Psi^{\rho}}^{\gamma-p}(\wp)}. \tag{2.12}$$

Next note $\nabla_a^{\gamma}\Omega(\wp) = -\lambda(\wp) \leqslant 0$. By chain rule, we see that

$$\begin{aligned}
\nabla_a^{\gamma}(\Omega(\wp))^{\gamma-c} &= (\gamma-c)\int_0^1 \frac{\nabla_a^{\gamma}\Omega(\wp)dh}{[h\Omega(\wp)+(1-h)\Omega^{\rho}(\wp)]^{c-\alpha+1}} \\
&= (c-\gamma)\int_0^1 \frac{\lambda(\wp)dh}{[h\Omega(\wp)+(1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\
&\geqslant (c-\gamma)\int_0^1 \frac{\lambda(\wp)dh}{[h\Omega^{\rho}(\wp)+(1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\
&= (c-\gamma)\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}}.
\end{aligned}$$

This implies that

$$\frac{-\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} \geqslant \frac{-1}{c-\gamma}\nabla_a^{\gamma}(\Omega(\wp))^{\gamma-c}, \tag{2.13}$$

and thus, we get

$$\eta(\wp) = \int_r^{\wp} \frac{\lambda(\varrho)}{[\Omega^{\rho}(\varrho)]^{c-\gamma+1}} \nabla_a^{\gamma}\varrho = \int_r^{\infty} \frac{\lambda(\varrho)}{[\Omega^{\rho}(\varrho)]^{c-\gamma+1}} \nabla_a^{\gamma}\varrho - \int_{\wp}^{\infty} \frac{\lambda(\varrho)}{[\Omega^{\rho}(\varrho)]^{c-\gamma+1}} \nabla_a^{\gamma}\varrho \geqslant - \int_{\wp}^{\infty} \frac{\lambda(\varrho)}{[\Omega^{\rho}(\varrho)]^{c-\gamma+1}} \nabla_a^{\gamma}\varrho$$
$$= \frac{-1}{c-\gamma} \int_{\wp}^{\infty} \nabla_a^{\gamma}(\Omega(\varrho))^{\gamma-c} \nabla_a^{\gamma}\varrho = \frac{1}{(c-\gamma)(\Omega^{\rho}(\wp))^{\gamma-c}}. \tag{2.14}$$

Substituting (2.13), (2.14) into (2.11) obtains

$$\left(\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma}\wp \right)^p \geqslant \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^{\infty} \left[\frac{\lambda^p(\wp)\gimel^p(\wp)}{(\overline{\Psi^{\rho}}(\wp))^{p(\gamma-p)} (\Omega^{\rho}(\wp))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^{\gamma}\wp \right)^p. \tag{2.15}$$

Applying Hölder inequality on the term

$$\left(\int_r^{\infty} \left[\frac{\lambda^p(\wp)\gimel^p(\wp)}{(\overline{\Psi^{\rho}}(\wp))^{p(\gamma-p)} (\Omega^{\rho}(\wp))^{p(\gamma-c)}} \right]^{\frac{1}{p}} \nabla_a^{\gamma}\wp \right)^p,$$

with indices $1/p$ and $1/1-p$ and

$$F(\wp) = \frac{\lambda^p(\wp)\gimel^p(\wp)}{(\overline{\Psi^{\rho}}(\wp))^{p(\gamma-p)} (\Omega^{\rho}(\wp))^{p(c-\gamma)}} \quad \textit{and} \quad \gimel(\wp) = \left(\frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} \right)^{1-p} [\overline{\Psi^{\rho}}(\wp)]^{(1-p)(p-\gamma+1)}$$

$$\left(\int_r^{\infty} F^{1/p}(\wp) \nabla_a^{\gamma}\wp \right)^p = \left(\int_r^{\infty} \left[\frac{\lambda^p(\wp)\gimel^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)} (\Omega^{\rho}(\wp))^{p(c-\gamma)}} \right]^{\frac{1}{p}} \nabla_a^{\gamma}\wp \right)^p$$
$$\geqslant \frac{\int_r^{\infty} F(\wp)\gimel(\wp)\nabla_a^{\gamma}\wp}{\left(\int_r^{\infty} \gimel^{\frac{1}{1-p}}(\wp)\nabla_a^{\gamma}\wp \right)^{p-1}} = \left[\int_r^{\infty} \frac{\lambda^p(\wp)\left(\lambda(\wp)[\Omega^{\rho}(\wp)]^{\gamma-c-1}\right)^{1-p} \gimel^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{(1-p)(p-\gamma+1)}}{(\overline{\Psi^{\rho}}(\wp))^{p(\alpha-p)} (\Omega^{\rho}(\wp))^{p(c-\alpha)}} \nabla_a^{\gamma}\wp \right]$$
$$\times \left(\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\alpha+1}} [\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma}\wp \right)^{p-1}$$
$$= \left(\int_r^{\infty} \frac{\lambda(\wp)\gimel^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}} \nabla_a^{\gamma}\wp \right) \left(\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma}\wp \right)^{p-1}. \tag{2.16}$$

Substituting (2.16) into (2.15) yields

$$\left(\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma}\wp \right)^p \tag{2.17}$$
$$\geqslant \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^{\infty} \frac{\lambda(\wp)\gimel^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}} \nabla_a^{\gamma}\wp \right) \left(\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma}\wp \right)^{p-1}.$$

Therefore,

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\Omega^{\rho}(\wp)]^{c-\gamma+1}} [\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1} \nabla_a^{\gamma}\wp \geqslant \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \int_r^{\infty} \frac{\lambda(\wp)\gimel^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\Omega^{\rho}(\wp))^{c-p-\gamma+1}} \nabla_a^{\gamma}\wp.$$

□

Remark 2.2. In Theorem 2, if we take $\gamma = 1$ then inequality (2.10) reduces to

$$\int_r^\infty \frac{\lambda(\wp)\overline{\Psi}^p(\wp)}{(\Omega^\rho(\wp))^c}\nabla\wp \geqslant \left(\frac{p}{c-1}\right)^p \int_r^\infty \frac{\lambda(\wp)\gimel^p(\wp)}{(\Omega^\rho(\wp))^{c-p}}\nabla\wp,$$

where

$$\overline{\Psi}(\wp) = \int_\wp^\infty \lambda(\varrho)\gimel(\varrho)\nabla\varrho \qquad \textit{and} \qquad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)\nabla\varrho,$$

which is Theorem 3.4 in [16].

Corollary 2.5. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.10) gets

$$\int_r^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{(\Omega(\wp))^{c-\gamma+1}}(\wp-a)^{\gamma-1}d\wp \geqslant \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \int_r^\infty \frac{\lambda(\wp)\gimel^p(\wp)\Psi^{1-\gamma}(\wp)}{(\Omega(\wp))^{c-p-\gamma+1}}(\wp-a)^{\gamma-1}dt,$$

where

$$\overline{\Psi}(\wp) = \int_\wp^\infty \lambda(\varrho)\gimel(\varrho)(\wp-a)^{\gamma-1}d\varrho \qquad \textit{and} \qquad \Omega(\wp) = \int_\wp^\infty \lambda(\varrho)(\wp-a)^{\gamma-1}ds.$$

Corollary 2.6. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.10) gets

$$\sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\overline{\Psi}^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(h\wp-h)}(\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)} \geqslant \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\gimel^p(h\wp)\overline{\Psi}^{1-\gamma}(\wp)}{\Omega^{c-p-\gamma+1}(h\wp-h)}(\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)},$$

where

$$\overline{\Psi}(\wp) = h\sum_{\varrho=\frac{\wp}{h}}^\infty \lambda(h\varrho)\gimel(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)} \qquad \textit{and} \qquad \Omega(\wp) = h\sum_{\varrho=\frac{\wp}{h}}^\infty \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)}.$$

Corollary 2.7. From Corollary 2, assume $\mathbb{T} = \mathbb{Z}$, and $h = 1$, then (2.10) obtains

$$\sum_{\wp=r}^\infty \frac{\lambda(\wp)\overline{\Psi}^{p-\gamma+1}(h\wp)}{\Omega^{c-p-\gamma+1}(\wp-1)}(\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)} \geqslant \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \sum_{\wp=r}^\infty \frac{\lambda(\wp)\gimel^p(\wp)\overline{\Psi}^{1-\gamma}(\wp)}{\Omega^{c-p-\gamma+1}(\wp-1)}(\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)},$$

where

$$\overline{\Psi}(\wp) = \sum_{\varrho=\wp}^\infty \lambda(\varrho)\gimel(\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)} \qquad \textit{and} \qquad \Omega(\wp) = h\sum_{\varrho=\wp}^\infty \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)}.$$

Corollary 2.8. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.10) obtains

$$\sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\overline{\Psi}^{p-\gamma+1}(\wp)}{\Omega^{c-\gamma+1}(\rho(\wp))} \geqslant \left(\frac{p-\gamma+1}{c-\gamma}\right)^p \sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\gimel^p(\wp)\overline{\Psi}^{1-\gamma}(\wp)}{\Omega^{c-p-\gamma+1}(\rho(\wp))},$$

where

$$\overline{\Psi}(\wp) = (\tilde{q}-1)\sum_{\varrho\in(\wp,\infty)} \varrho\lambda(\varrho)\gimel(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)} \qquad \textit{and} \qquad \Omega(\wp) = (\tilde{q}-1)\sum_{\varrho\in(\wp,\infty)} \varrho\lambda(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)}.$$

Theorem 2.3. Let $\mathbb{T}$ be a time scale with $r \in [0,\infty)_{\mathbb{T}}$, $\gamma \in (0,1]$ and $\wp \geqslant a$. In addition, let $\gimel$ and λ be nonnegative ld-continuous and (γ, a)-nabla fractional differentiable functions on $[r,\infty)_{\mathbb{T}}$ where

$$\overline{\Omega}(\wp) = \int_r^{\wp} \lambda(\varrho)\nabla_a^{\gamma}\varrho \quad \textit{and} \quad \overline{\Psi}(\wp) = \int_{\wp}^{\infty} \lambda(\varrho)\gimel(\varrho)\nabla_a^{\gamma}\varrho, \quad \wp \in [r,\infty)_{\mathbb{T}}.$$

If $0 < p < \gamma$ and $c \leqslant \gamma - 1$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^{p}\int_r^{\infty}\frac{\lambda(\wp)\gimel^{p}(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp. \quad (2.18)$$

Proof. Using (1.13), with

$$\nabla_a^{\gamma}\eta(\wp) = \lambda(\wp)/[\overline{\Omega}(\wp)]^{c-\gamma+1}, \qquad \xi(\wp) = [\overline{\Psi}(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi}^{\rho}(\wp)]^{p-\alpha+1}\nabla_a^{\gamma}\wp = [\eta(\wp)\overline{\Psi}^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty}(-\eta(\wp))\nabla_a^{\gamma}(\overline{\Psi}^{p-\gamma+1}(\wp))\nabla_a^{\gamma}\wp,$$

where we assumed that

$$\eta(\wp) = \int_r^{\wp} \lambda(\wp)/[\overline{\Omega}(\wp)]^{c-\gamma+1}\nabla_a^{\gamma}\varrho.$$

Using $\overline{\Psi}(\infty) = 0$ and $\eta(r) = 0$, we have that

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp = \int_r^{\infty} -\eta(\wp)\nabla_a^{\gamma}(\overline{\Psi}^{p-\gamma+1}(\wp))\nabla_a^{\gamma}\wp. \quad (2.19)$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$-\nabla_a^{\gamma}(\overline{\Psi}^{p-\gamma+1}(\wp)) \;=\; -\frac{p-\gamma+1}{\overline{\Psi}^{\gamma-p}(d)}\nabla_a^{\gamma}\overline{\Psi}(\wp) \geqslant \frac{(p-\gamma+1)\lambda(\wp)\gimel(\wp)}{\overline{\Psi^{\rho}}^{\gamma-p}(\wp)}. \quad (2.20)$$

Next note $\nabla_a^{\gamma}\Omega(\wp) = \lambda(\wp) \leqslant 0$. By chain rule, we see that

$$\begin{aligned}
\nabla_a^{\gamma}(\Omega(\wp))^{\gamma-c} &= (\gamma-c)\int_0^1 \frac{\nabla_a^{\gamma}\overline{\Omega}(\wp)dh}{[h\overline{\Omega}(\wp)+(1-h)\overline{\Omega}^{\rho}(\wp)]^{c-\gamma+1}} \\
&= (\gamma-c)\int_0^1 \frac{\lambda(\wp)dh}{[h\Omega(\wp)+(1-h)\Omega^{\rho}(\wp)]^{c-\gamma+1}} \\
&\leqslant (\gamma-c)\int_0^1 \frac{\lambda(\wp)dh}{[h\overline{\Omega}(\wp)+(1-h)\overline{\Omega}(\wp)]^{c-\gamma+1}} \\
&= (\gamma-c)\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}.
\end{aligned}$$

This implies that

$$\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} \geqslant \frac{-1}{\gamma-c}\nabla_a^{\gamma}(\overline{\Omega}(\wp))^{\gamma-c}, \quad (2.21)$$

and then, we have that

$$\eta(\wp)=\int_r^{\wp}\frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}}\nabla_a^{\gamma}\varrho = \int_r^{\infty}\frac{\lambda(\varrho)}{[\Omega^{\rho}(\varrho)]^{c-\gamma+1}}\nabla_a^{\gamma}\varrho-\int_{\wp}^{\infty}\frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}}\nabla_a^{\gamma}\varrho \geqslant -\int_{\wp}^{\infty}\frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}}\nabla_a^{\gamma}\varrho$$
$$= \frac{1}{\gamma-c}\int_{\wp}^{\infty}\nabla_a^{\gamma}(\Omega(\varrho))^{\gamma-c}\nabla_a^{\gamma}\varrho=\frac{1}{(\gamma-c)(\overline{\Omega}(\wp))^{c-\gamma}}. \tag{2.22}$$

Substituting (2.21), (2.22) into (2.19) yields

$$\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^p \geqslant \Big(\frac{p-\gamma+1}{c-\gamma}\Big)^p\Big(\int_r^{\infty}\Big[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\overline{\Psi^{\rho}}(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(c-\gamma)}}\Big]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\Big)^p. \tag{2.23}$$

Applying Hölder inequality on the term

$$\Big(\int_r^{\infty}\Big[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\overline{\Psi^{\rho}}(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(\gamma-c)}}\Big]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\Big)^p,$$

with indices $1/p$ and $1/1-p$ and

$$F(\wp)=\frac{\lambda^p(\wp)\beth^p(\wp)}{(\overline{\Psi^{\rho}}(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(c-\gamma)}} \quad \textit{and} \quad \beth(\wp)=\Big(\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}\Big)^{1-p}[\overline{\Psi^{\rho}}(\wp)]^{(1-p)(p-\gamma+1)}$$

$$\Big(\int_r^{\infty}F^{1/p}(\wp)\nabla_a^{\gamma}\wp\Big)^p=\Big(\int_r^{\infty}\Big[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\overline{\Psi}^{\rho}(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(c-\gamma)}}\Big]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\Big)^p$$
$$\geqslant \frac{\int_r^{\infty}F(\wp)\beth(\wp)\nabla_a^{\gamma}\wp}{(\int_r^{\infty}\beth^{\frac{1}{1-p}}(\wp)\nabla_a^{\gamma}\wp)^{p-1}}=\Big[\int_r^{\infty}\frac{\lambda^p(\wp)(\lambda(\wp)[\overline{\Omega}(\wp)]^{\gamma-c-1})^{1-p}\beth^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{(1-p)(p-\gamma+1)}}{(\overline{\Psi^{\rho}}(\wp))^{p(\alpha-p)}(\overline{\Omega}(\wp))^{p(c-\alpha)}}\nabla_a^{\gamma}\wp\Big]$$
$$\times\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\alpha+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^{p-1}$$
$$=\Big(\int_r^{\infty}\frac{\lambda(\wp)\beth^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp\Big)\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^{p-1}. \tag{2.24}$$

From (2.24) and (2.23) gets

$$\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^p \tag{2.25}$$
$$\geqslant\Big(\frac{p-\gamma+1}{c-\gamma}\Big)^p\Big(\int_r^{\infty}\frac{\lambda(\wp)\beth^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp\Big)\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^{p-1}.$$

Therefore,

$$\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\overline{\Psi^{\rho}}(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\geqslant\Big(\frac{p-\gamma+1}{\gamma-c}\Big)^p\int_r^{\infty}\frac{\lambda(\wp)\beth^p(\wp)[\overline{\Psi^{\rho}}(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp.$$

□

Remark 2.3. In Theorem 2, if we take $\gamma = 1$ then inequality (2.18) reduces to

$$\int_r^\infty \frac{\lambda(\wp)[\overline{\Psi^\rho}(\wp)]^p}{(\overline{\Omega}(\wp))^c}\nabla\wp \geqslant \left(\frac{p}{1-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\beth^p(\wp)}{(\overline{\Omega}(\wp))^{c-p}}\nabla\wp,$$

where

$$\overline{\Psi}(\wp) = \int_\wp^\infty \lambda(\varrho)\beth(\varrho)\nabla\varrho \qquad \textit{and} \qquad \overline{\Omega}(\wp) = \int_r^\wp \lambda(\varrho)\nabla\varrho,$$

which is Theorem 3.9 in [16].

Corollary 2.9. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.18) gets

$$\int_r^\infty \frac{\lambda(\wp)\overline{\Psi}^{p-\gamma+1}(\wp)}{(\overline{\Omega}(\wp))^{c-\gamma+1}}(\wp-a)^{\gamma-1}d\wp \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \int_r^\infty \frac{\lambda(\wp)\beth^p(\wp)\overline{\Psi}^{1-\gamma}(\wp)}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}}(\wp-a)^{\gamma-1}dt,$$

where

$$\overline{\Psi}(\wp) = \int_\wp^\infty \lambda(\varrho)\beth(\varrho)(\wp-a)^{\gamma-1}d\varrho \qquad \textit{and} \qquad \overline{\Omega}(\wp) = \int_r^\wp \lambda(\varrho)(\wp-a)^{\gamma-1}ds.$$

Remark 2.4. In Corollary 2, if we take $\gamma = 1$ yields discrete Bennett-Leindler type inequality (1.8).

Corollary 2.10. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.18) gets

$$\sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\overline{\Psi}^{p-\gamma+1}(h\wp-h)}{\overline{\Omega}^{c-\gamma+1}(h\wp)}(\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)} \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\beth^p(h\wp)\overline{\Psi}^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(h\wp)}(\rho^{\gamma-1}(h\wp)-a)_h^{(\gamma-1)},$$

where

$$\overline{\Psi}(\wp) = h\sum_{\varrho=\frac{\wp}{h}}^\infty \lambda(h\varrho)\beth(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)} \qquad \textit{and} \qquad \overline{\Omega}(\wp) = h\sum_{\varrho=\frac{r}{h}}^{\wp} \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)_h^{(\gamma-1)}.$$

Corollary 2.11. For $\mathbb{T} = \mathbb{Z}$, we take $h = 1$ in Corollary 2. In this case, inequality (2.18) reduces to

$$\sum_{\wp=r}^\infty \frac{\lambda(\wp)\overline{\Psi}^{p-\gamma+1}(\wp-1)}{\overline{\Omega}^{c-\gamma+1}(\wp)}(\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)} \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp=r}^\infty \frac{\lambda(\wp)\beth^p(\wp)\overline{\Psi}^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)}(\rho^{\gamma-1}(\wp)-a)^{(\gamma-1)},$$

where

$$\overline{\Psi}(\wp) = \sum_{\varrho=\wp}^\infty \lambda(\varrho)\beth(\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)} \qquad \textit{and} \qquad \overline{\Omega}(\wp) = h\sum_{\varrho=r}^{\wp} \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho)-a)^{(\gamma-1)}.$$

Remark 2.5. In Corollary 2, if we take $\gamma = 1$ and $r = 1$, yields discrete Bennett-Leindler type inequality (1.6), which is the converse of Copson inequality (1.4).

Corollary 2.12. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.18) gets

$$\sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\overline{\Psi}^{p-\gamma+1}(\rho(\wp))}{\overline{\Omega}^{c-\gamma+1}(\wp)} \geqslant \left(\frac{p-\gamma+1}{\gamma-c}\right)^p \sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp)-a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\beth^p(\wp)\overline{\Psi}^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)},$$

where

$$\overline{\Psi}(\wp) = (\tilde{q}-1)\sum_{\varrho\in(\wp,\infty)} \varrho\lambda(\varrho)\beth(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)} \qquad \textit{and} \qquad \overline{\Omega}(\wp) = (\tilde{q}-1)\sum_{\varrho\in(r,\wp)} \varrho\lambda(\varrho)(\rho^{\gamma-1}(\varrho)-a)_{\tilde{q}}^{(\gamma-1)}.$$

Theorem 2.4. Let $\mathbb{T}$ be a time scale with $r \in [0,\infty)_{\mathbb{T}}$, $\gamma \in (0,1]$ and $\wp \geqslant a$. In addition, let $\beth$ and λ be nonnegative ld-continuous and (γ, a)-nabla fractional differentiable functions on $[r,\infty)_{\mathbb{T}}$ where

$$\overline{\Omega}(\wp) = \int_r^{\wp} \lambda(\varrho)\nabla_a^{\gamma}\varrho, \quad \overline{\Omega}(\infty) = \infty, \quad \Psi(\wp) = \int_r^{\wp} \lambda(\varrho)\beth(\varrho)\nabla_a^{\gamma}\varrho, \quad \wp \in [r,\infty)_{\mathbb{T}}.$$

If $L = \inf_{\wp\in\mathbb{T}} \frac{\overline{\Omega}^{\rho}(\wp)}{\overline{\Omega}(\wp)} > 0$, $0 < p < \gamma$ and $c \geqslant \gamma$, then

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp \geqslant \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma}\right)^p \int_r^{\infty} \frac{\lambda(\wp)\beth^p(\wp)[\Psi(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}}\nabla_a^{\gamma}\wp. \tag{2.26}$$

Proof. Using (1.13), with

$$\nabla_a^{\gamma}\eta(\wp) = \lambda(\wp)/[\overline{\Omega}(\wp)]^{c-\gamma+1}, \qquad \xi(\wp) = [\Psi(\wp)]^{p-\gamma+1},$$

we have

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\alpha+1}\nabla_a^{\gamma}\wp = [\eta(\wp)\Psi^{p-\alpha+1}(\wp)]_r^{\infty} + \int_r^{\infty}(-\eta^{\rho}(\wp))\nabla_a^{\gamma}(\Psi^{p-\gamma+1}(\wp))\nabla_a^{\gamma}\wp,$$

where we assumed that

$$\eta(\wp) = -\int_{\wp}^{\infty} \lambda(\wp)/[\overline{\Omega}(\wp)]^{c-\gamma+1}\nabla_a^{\gamma}\varrho.$$

Using $\Psi(r) = 0$ and $\eta(\infty) = 0$, we have that

$$\int_r^{\infty} \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp = \int_r^{\infty} -\eta^{\rho}(\wp)\nabla_a^{\gamma}(\Psi^{p-\gamma+1}(\wp))\nabla_a^{\gamma}\wp. \tag{2.27}$$

Applying (1.12), then there exists $d \in [\wp, \rho(\wp)]$ such that

$$\nabla_a^{\gamma}(\Psi^{p-\gamma+1}(\wp)) \;=\; \frac{p-\gamma+1}{\Psi^{\gamma-p}(d)}\nabla_a^{\gamma}\Psi(\wp) \geqslant \frac{(p-\gamma+1)\lambda(\wp)\beth(\wp)}{\Psi^{\gamma-p}(\wp)}. \tag{2.28}$$

Next note $\nabla_a^{\gamma}\overline{\Omega}(\wp) = \lambda(\wp) \leqslant 0$. By chain rule, we see that

$$\nabla_a^{\gamma}(\overline{\Omega}(\wp))^{\gamma-c} \;=\; (\gamma-c)\int_0^1 \frac{\nabla_a^{\gamma}\overline{\Omega}(\wp)dh}{[h\overline{\Omega}(\wp)+(1-h)\overline{\Omega}^{\rho}(\wp)]^{c-\gamma+1}}$$

$$
\begin{aligned}
&= -(c-\gamma)\int_0^1 \frac{\lambda(\wp)dh}{[h\overline{\Omega}(\wp)+(1-h)\overline{\Omega}^{\rho}(\wp)]^{c-\gamma+1}} \\
&\geqslant -(c-\gamma)\int_0^1 \frac{\lambda(\wp)dh}{[h\overline{\Omega}^{\rho}(\wp)+(1-h)\overline{\Omega}^{\rho}(\wp)]^{c-\gamma+1}} \\
&= -(c-\gamma)\frac{\lambda(\wp)}{[\overline{\Omega}^{\rho}(\wp)]^{c-\gamma+1}} = -(\gamma-c)\frac{\lambda(\wp)}{[\overline{\Omega}^{\rho}(\wp)]^{c-\gamma+1}}\frac{[\overline{\Omega}(\wp)]^{c-\gamma+1}}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} \\
&\geqslant -(c-\gamma)\frac{\lambda(\wp)}{L^{c-\gamma+1}[\overline{\Omega}(\wp)]^{c-\gamma+1}}.
\end{aligned}
$$

This implies that

$$
\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} \geqslant \frac{-L^{c-\gamma+1}}{c-\gamma}\nabla_a^{\gamma}(\overline{\Omega}(\wp))^{\gamma-c} \tag{2.29}
$$

and then, we have that

$$
\begin{aligned}
-\eta^{\rho}(\wp) = \int_{\rho(\wp)}^{\infty}\frac{\lambda(\varrho)}{[\overline{\Omega}(\varrho)]^{c-\gamma+1}}\nabla_a^{\gamma}\varrho &\geqslant -\int_{\rho(\wp)}^{\infty}\frac{L^{c-\gamma+1}}{c-\gamma}\nabla_a^{\gamma}(\overline{\Omega}(\wp))^{\gamma-c}\nabla_a^{\gamma}\varrho = \frac{L^{c-\gamma+1}}{c-\gamma}\Big\{(\overline{\Omega}^{\rho}(\wp))^{\gamma-c}-(\overline{\Omega}(\infty))^{\gamma-c}\Big\} \\
&= \frac{L^{c-\gamma+1}}{c-\gamma}(\overline{\Omega}^{\rho}(\wp))^{\gamma-c} \geqslant \frac{L^{c-\gamma+1}}{c-\gamma}(\overline{\Omega}(\wp))^{\gamma-c}.
\end{aligned} \tag{2.30}
$$

Substituting (2.30), (2.28) into (2.27) yields

$$
\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^p \geqslant \Big(\frac{p-\gamma+1)L^{c-\gamma+1}}{c-\gamma}\Big)^p\Big(\int_r^{\infty}\Big[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(c-\gamma)}}\Big]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\Big)^p. \tag{2.31}
$$

Applying Hölder inequality on the term

$$
\Big(\int_r^{\infty}\Big[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(\gamma-c)}}\Big]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\Big)^p,
$$

with indices $1/p$ and $1/1-p$ and

$$
F(\wp) = \frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(c-\gamma)}} \quad \textit{and} \quad \beth(\wp) = \Big(\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}}\Big)^{1-p}[\Psi(\wp)]^{(1-p)(p-\gamma+1)}
$$

$$
\begin{aligned}
&\Big(\int_r^{\infty}F^{1/p}(\wp)\nabla_a^{\gamma}\wp\Big)^p = \Big(\int_r^{\infty}\Big[\frac{\lambda^p(\wp)\beth^p(\wp)}{(\Psi(\wp))^{p(\gamma-p)}(\overline{\Omega}(\wp))^{p(c-\gamma)}}\Big]^{\frac{1}{p}}\nabla_a^{\gamma}\wp\Big)^p \\
&\geqslant \frac{\int_r^{\infty}F(\wp)\beth(\wp)\nabla_a^{\gamma}\wp}{(\int_r^{\infty}\beth^{\frac{1}{1-p}}(\wp)\nabla_a^{\gamma}\wp)^{p-1}} = \Big[\int_r^{\infty}\frac{\lambda^p(\wp)(\lambda(\wp)[\overline{\Omega}(\wp)]^{\gamma-c-1})^{1-p}\beth^p(\wp)[\Psi(\wp)]^{(1-p)(p-\gamma+1)}}{(\Psi(\wp))^{p(\alpha-p)}(\overline{\Omega}(\wp))^{p(c-\alpha)}}\nabla_a^{\gamma}\wp\Big] \\
&\quad\times\Big(\int_r^{\infty}\frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\alpha+1}}[\Psi(\wp)]^{p-\gamma+1}\nabla_a^{\gamma}\wp\Big)^{p-1}
\end{aligned}
$$

$$= \left(\int_r^\infty \frac{\lambda(\wp)\beth^p(\wp)[\overline{\Psi^\rho}(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp \right) \left(\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \right)^{p-1}. \tag{2.32}$$

From (2.32) and (2.31) gets

$$\left(\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \right)^p \tag{2.33}$$
$$\geqslant \left(\frac{p-\gamma+1}{c-\gamma} \right)^p \left(\int_r^\infty \frac{\lambda(\wp)\beth^p(\wp)[\overline{\Psi^\rho}(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp \right) \left(\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \right)^{p-1}.$$

Therefore,

$$\int_r^\infty \frac{\lambda(\wp)}{[\overline{\Omega}(\wp)]^{c-\gamma+1}} [\Psi(\wp)]^{p-\gamma+1} \nabla_a^\gamma \wp \geqslant \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \int_r^\infty \frac{\lambda(\wp)\beth^p(\wp)[\Psi(\wp)]^{1-\gamma}}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} \nabla_a^\gamma \wp.$$

□

Remark 2.6. In Theorem 2, if we take $\gamma = 1$ then we get Theorem 1.

Corollary 2.13. From Theorem 2, assume $\mathbb{T} = \mathbb{R}$, then (2.26) gets

$$\int_r^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{(\overline{\Omega}(\wp))^{c-\gamma+1}} (\wp - a)^{\gamma-1} d\wp \geqslant \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \int_r^\infty \frac{\lambda(\wp)\beth^p(\wp)\Psi^{1-\gamma}(\wp)}{(\overline{\Omega}(\wp))^{c-p-\gamma+1}} (\wp - a)^{\gamma-1} dt,$$

where

$$\Psi(\wp) = \int_r^\wp \lambda(\varrho)\beth(\varrho)(\wp - a)^{\gamma-1} d\varrho \qquad \textit{and} \qquad \overline{\Omega}(\wp) = \int_r^\wp \lambda(\varrho)(\wp - a)^{\gamma-1} ds.$$

Remark 2.7. In Corollary 2, if we take $L = \gamma = 1$ yields continuous variant of Bennett-Leindler type inequality (1.7).

Corollary 2.14. From Theorem 2, assume $\mathbb{T} = h\mathbb{Z}$, then (2.26) gets

$$\sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\Psi^{p-\gamma+1}(h\wp)}{\Omega^{c-\gamma+1}(h\wp)} (\rho^{\gamma-1}(h\wp) - a)_h^{(\gamma-1)} \geqslant \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \sum_{\wp=\frac{r}{h}}^\infty \frac{\lambda(h\wp)\beth^p(h\wp)\Psi^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(h\wp)} (\rho^{\gamma-1}(h\wp) - a)_h^{(\gamma-1)},$$

where

$$\Psi(\wp) = h \sum_{\varrho=\frac{r}{h}}^{\wp} \lambda(h\varrho)\beth(h\varrho)(\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)} \qquad \textit{and} \qquad \overline{\Omega}(\wp) = h \sum_{\varrho=\frac{r}{h}}^{\wp} \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho) - a)_h^{(\gamma-1)}.$$

Corollary 2.15. For $\mathbb{T} = \mathbb{Z}$, we take $h = 1$ in Corollary 2. In this case, inequality (2.26) reduces to

$$\sum_{\wp=r}^\infty \frac{\lambda(\wp)\Psi^{p-\gamma+1}(h\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)} (\rho^{\gamma-1}(\wp) - a)^{(\gamma-1)} \geqslant \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma} \right)^p \sum_{\wp=r}^\infty \frac{\lambda(\wp)\beth^p(\wp)\Psi^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)} (\rho^{\gamma-1}(\wp) - a)^{(\gamma-1)},$$

where

$$\Psi(\wp) = \sum_{\varrho=r}^{\wp} \lambda(\varrho)\gimel(\varrho)(\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)} \qquad \textit{and} \qquad \overline{\Omega}(\wp) = h\sum_{\varrho=r}^{\wp} \lambda(h\varrho)(\rho^{\gamma-1}(h\varrho) - a)^{(\gamma-1)}.$$

Remark 2.8. In Corollary 2, if we take $\gamma = 1$ and $r = 1$, yields discrete Bennett-Leindler type inequality (1.5), which is the converse of Copson inequality (1.3).

Corollary 2.16. From Theorem 2, assume $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.26) gets

$$\sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp) - a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\Psi^{p-\gamma+1}(\wp)}{\overline{\Omega}^{c-\gamma+1}(\rho(\wp))} \geqslant \left(\frac{(p-\gamma+1)L^{c-\gamma+1}}{c-\gamma}\right)^{p} \sum_{\wp\in(r,\infty)} \frac{\wp(\rho^{\gamma-1}(\wp) - a)_{\tilde{q}}^{(\gamma-1)}\lambda(\wp)\gimel^{p}(\wp)\Psi^{1-\gamma}(\wp)}{\overline{\Omega}^{c-p-\gamma+1}(\wp)},$$

where

$$\Psi(\wp) = (\tilde{q}-1)\sum_{\varrho\in(r,\wp)} \varrho\lambda(\varrho)\gimel(\varrho)(\rho^{\gamma-1}(\varrho) - a)_{\tilde{q}}^{(\gamma-1)} \qquad \textit{and} \qquad \overline{\Omega}(\wp) = (\tilde{q}-1)\sum_{\varrho\in(r,\wp)} \varrho\lambda(\varrho)(\rho^{\gamma-1}(\varrho) - a)_{\tilde{q}}^{(\gamma-1)}.$$

3. Conclusions

In this paper, with the help of a simple consequence of Keller's chain rule and Hölder inequality for the (γ, a)-nabla-fractional derivative on time scales, we generalized a number of Bennett and Leindler Hardy-type inequalities to a general time scale. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete and continuous calculus. In order to illustrate the theorems for each type of inequality applied to various time scales such as $\mathbb{R}$, $h\mathbb{Z}$, $\overline{q^{\mathbb{Z}}}$ and $\mathbb{Z}$ as a sub case of $h\mathbb{Z}$. For future studies researchers may obtain some different generalizations for dynamic Hardy inequality and its companion inequalities by using the results presented in this paper.

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Conflict of interest

The authors declare that there is no competing interest.

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