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A reduction technique to solve the generalized nonlinear dispersive mK(m,n) equation with new local derivative

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ABSTRACT

In this work, we consider the generalized nonlinear dispersive mK(m,n) equation with a recently defined local derivative in the temporal direction. Different types of exact solutions are extracted by Nucci's reduction technique. Combinations of the exponential, trigonometric, hyperbolic, and logarithmic functions constitute the exact solutions especially of the soliton and Kink-type soliton solutions. The influence of the derivative order α, for the obtained results, is graphically investigated. In some cases, exact solutions are achieved for arbitrary values of n and m, which can be interesting from the mathematical point of view. We provided 2-D and 3-D figures to illustrate the reported solutions. Computational results indicate that the reduction technique is superior to some other methods used in the literature to solve the same equations. To the best of the author's knowledge, this method is not applied for differential equations with the recently hyperbolic local derivative.

Introduction

Nonlinear partial differential equations play significant role in almost all branches of science and technology. Solutions of these problems can describe many natural phenomena in engineering, chemistry, and physics and so on. Therefore, exact solutions of Nonlinear partial differential equations is interesting field of many researchers and there are various types of methods to find exact solutions of these problems. Soliton's theory is one of the most desirable branches of researchers in science and engineering. This useful theory appears in different aspects of life. Soliton type solutions are well-known in some branches of physics and engineering such as optics, surface wave propagation and fluid dynamics. In the current work we try to extract some soliton type solutions for considered equation.

Many studies have been done in recent years to find the new solutions of these equations with various techniques. For example, the Lie symmetry method [1–4], invariant subspace method [5,6], the exponential rational function method [7,8], the modified simple equation method [9–12], the Exp function method [13,14], the modified extended tanh-function method [15,16], the Kudryashov method [17,18].

One of the interesting NPDEs which firstly devoted by Rosenau and Hyman [19] is the K(m,n) equation:

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3. \tag{1}$$

Indeed, this equation is the Korteweg–de Vries-like equation with nonlinear dispersion. The role of nonlinear dispersion in the formation of patterns in liquid drops (nuclear physics) is interpreted by the mentioned K(m,n) equation. Very closed behave and stability of solitary waves with compact support (compactons) to completely integrable systems are founded.

A natural generalization of the K(m,n) equation is the generalized nonlinear dispersive mK(m,n) equations: [20,21]:

$$u^{n-1}u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{2}$$

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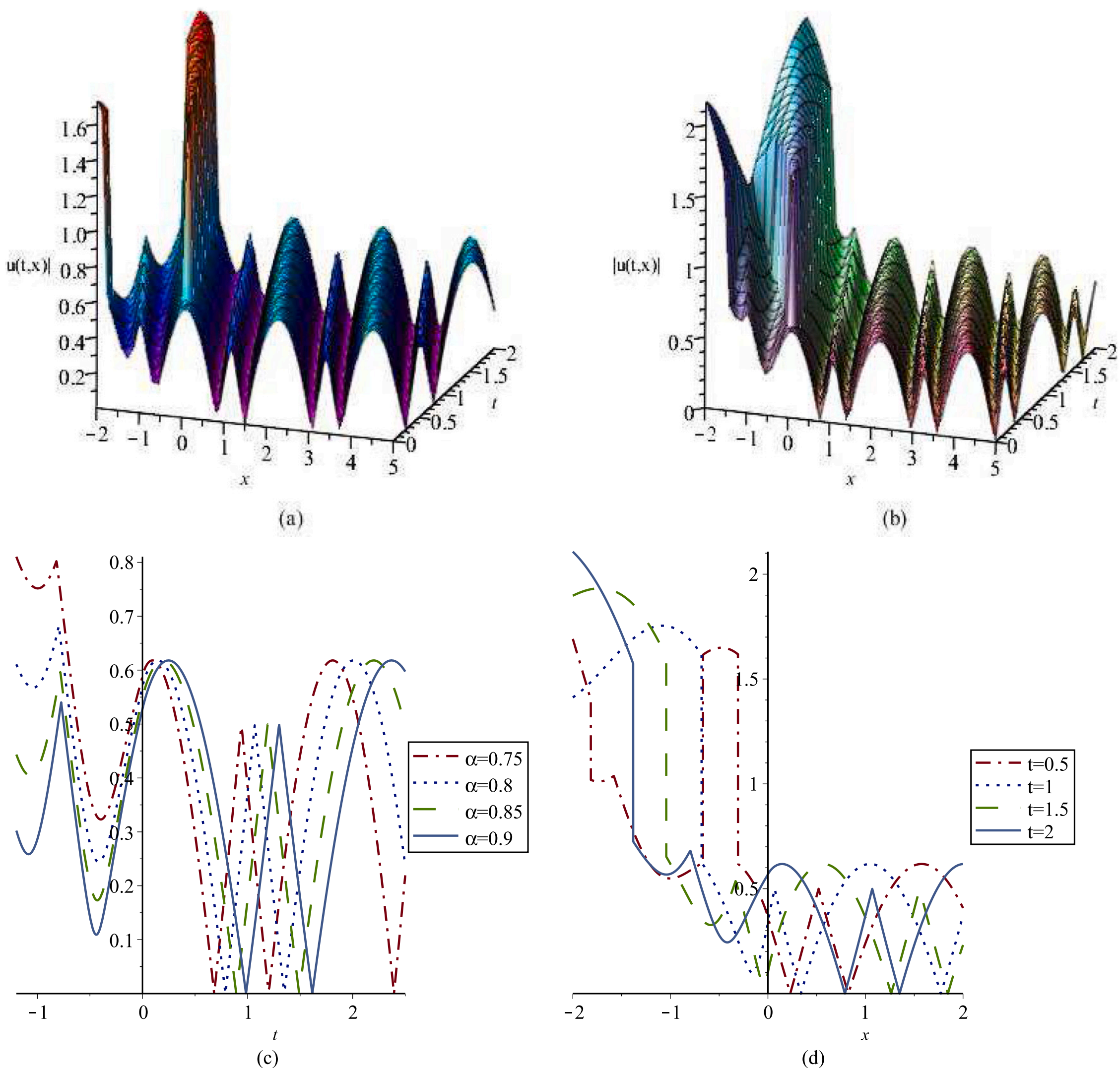

Fig. 1. Exact solution of (15) with $R_1 = R_2 = 0$, $R_3 = -1$, $\chi = 1$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $x = -5$, and various α, (d) $\alpha = 0.8$, and various t.

where in which a, m, n are constants and $m, n \geq 1$. In [22], the bifurcation behaviour of travelling wave solutions of Eq. (2) along with all possible exact explicit parametric representations for periodic travelling wave solutions, solitary wave solutions, kink and anti-kink wave solutions and periodic cusp wave solutions are investigated. Moreover, a new version of Eq. (2), that is the modified K(m,n,k), is discussed in [23]. Some compacton solutions and solitary pattern solutions of mK(m,n, k) equations are reported in this paper.

The concept of fractional differential operators in local and non-local senses, has captured minds of many scientist in the recent years due to the operators' wider applicability to almost all fields of science, engineering, and technology [24–28]. These operators play significant role in the modelling of complex real-world problems. Fractional derivatives and integrals is utilized by researchers for modelling of physical problems more precises than the integer ones. In these physical models, the results offered by fractional differential operators in both local and non-local cases, were in good agreement of experimental data. This issue, motivates us to consider the generalized nonlinear dispersive mK(m,n) equation with fractional derivative.

In this work, we investigate analytical solutions of the generalized nonlinear dispersive mK(m,n) equation with a recently defined local derivative [29]:

$$u^{n-1}\mathfrak{N}^{\alpha}_{hyp,t}u + a(u^m)_x + (u^n)_{xxx} = 0. \tag{3}$$

The plan of the paper is organized as follows.

In section "Preliminaries", we give some preliminaries and discussions about definitions and basic properties of the utilized local derivative. The section "Nucci's reduction method", which contains the main body of this research, deals with the exact solutions of the mK(m,n) equation with local derivative in temporal direction by a novel reduction method. Finally in "Conclusion" we draw our conclusions.

Preliminaries

Recently, the local fractional-order derivatives absorbed attention of many researched in science and technology. The concept of local fractional calculus which also is known as fractal calculus, firstly proposed in [30,31]. Indeed, the proposed fractals defined based on the Riemann–Liouville fractional derivative [32–34], was utilized to deal with non-differentiable equations raised from science and engineering [35–38].

Recently, a new type of local fractional derivatives is defined as follows:

Definition 1 (*[29]*). Let $\alpha \in (0, 1)$ and $t > 0$. Then

$$\mathfrak{N}^{\alpha}_{h}\varpi(t) = \lim_{\varepsilon\to 0}\frac{\varpi\left(t + \varepsilon\, t^{\frac{1-\alpha}{2}}\,\mathrm{sech}((1-\alpha)t^{\frac{1+\alpha}{2}})\right) - \varpi(t)}{\varepsilon}.$$

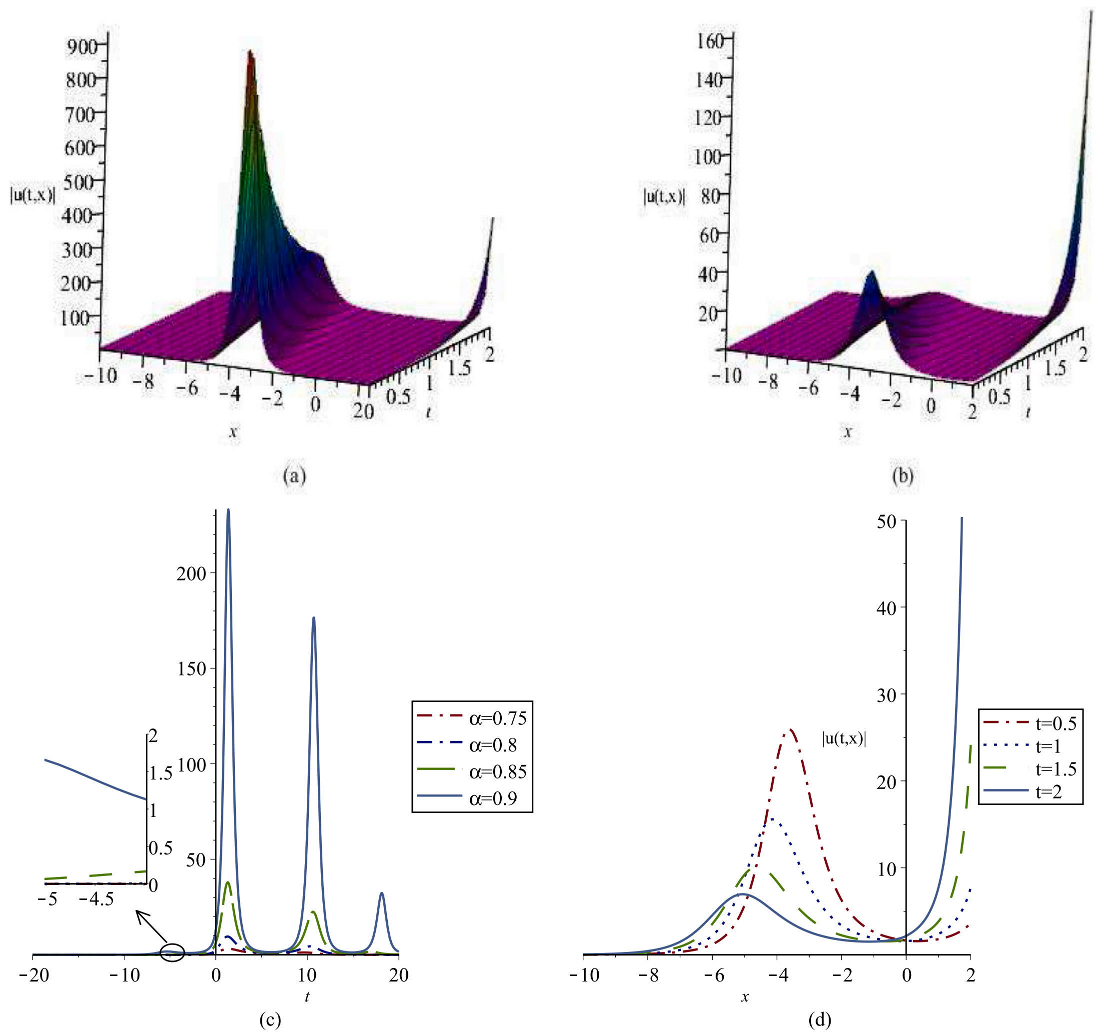

Fig. 2. Exact solution of (16) with $R_1 = R_2 = 0$, $R_3 = -1$, $\chi = 1$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $x = -5$, and various α, (d) $\alpha = 0.8$, and various t.

Indeed, this derivative is not fractional, but it is a natural extension of the classical derivative. It is clear that physical interpretation of the above derivative is a modification of classical velocity in direction and magnitude. That is, it depends on not only the time direction but also the real value order α. It is easily seen from the above definition that for every $\varpi \in C^1$, we have

$$\begin{aligned}\lim_{\alpha\to 1} \mathfrak{N}_h^\alpha \varpi(t) &= \lim_{\alpha\to 1}\lim_{\varepsilon\to 0} \frac{\varpi\left(t + \varepsilon\, t^{\frac{1-\alpha}{2}} \operatorname{sech}((1-\alpha)t^{\frac{1+\alpha}{2}})\right) - \varpi(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\lim_{\alpha\to 1} \varpi\left(t + \varepsilon\, t^{\frac{1-\alpha}{2}} \operatorname{sech}((1-\alpha)t^{\frac{1+\alpha}{2}})\right) - \varpi(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\varpi\left(\lim_{\alpha\to 1}\left[t + \varepsilon\, t^{\frac{1-\alpha}{2}} \operatorname{sech}((1-\alpha)t^{\frac{1+\alpha}{2}})\right]\right) - \varpi(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\varpi(t+\varepsilon) - \varpi(t)}{\varepsilon} = \varpi'(t). \end{aligned} \tag{4}$$

Hence, the considered local derivative degenerate to the usual first-order derivative when fractional order equals one. It is notable that a real function f defined on $[x_0, x_f]$ is said α-differentiable if

$$\lim_{t\to x_0^+} \mathfrak{N}_h^\alpha \varpi(t) = \mathfrak{N}_h^\alpha \varpi(x_0^+),$$

provided that $\lim_{t\to x_0^+} \mathfrak{N}_h^\alpha \varpi(t)$ exists.

From

$$\mathfrak{N}_h^\alpha\left[\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)t^{\frac{1+\alpha}{2}}\right)\right] = 1,$$

one can find that

$$\mathfrak{N}_h^\alpha \varpi(t) = t^{\frac{1-\alpha}{2}} \operatorname{sech}\left((1-\alpha)t^{\frac{1+\alpha}{2}}\right)\varpi'(t).$$

Moreover, this property is consistent with (4), whenever $\alpha \to 1$. One important result for the new fractional local derivative is

$$\mathfrak{N}_h^\alpha \varpi(\zeta) = \chi\varpi'(t), \quad \zeta = \frac{2}{1-\alpha^2}\sinh((1-\alpha)\chi t^{\frac{1+\alpha}{2}}), \tag{5}$$

for the constant χ.

Moreover, some other properties of this derivative is gathered in the following theorem.

Theorem 1 ([29]). *Let f_1 and f_2 be α-differentiable at t and $0 < \alpha \le 1$. Then*

- $\mathfrak{N}_h^\alpha(a_1 f_1 + a_2 f_2)(t) = a_1\mathfrak{N}_h^\alpha(f_1)(t) + a_2\mathfrak{N}_h^\alpha(f_2)(t), \quad a_1, a_2 \in \mathbb{R},$
- $\mathfrak{N}_h^\alpha(t^\mu) = \mu t^{\frac{2\mu-\alpha-1}{2}} \operatorname{sech}\left((1-\alpha)t^{\frac{1+\alpha}{2}}\right), \quad \mu \in \mathbb{R},$
- $\mathfrak{N}_h^\alpha(C) = 0, \quad C \in \mathbb{R},$
- $\mathfrak{N}_h^\alpha(f_1 f_2)(t) = f_1\mathfrak{N}_h^\alpha(f_2)(t) + f_2\mathfrak{N}_h^\alpha(f_1)(t),$
- $\mathfrak{N}_h^\alpha\left(\frac{f_1}{f_2}\right)(t) = \frac{f_2(t)\mathfrak{N}_h^\alpha(f_1)(t) - f_1(t)\mathfrak{N}_h^\alpha(f_2)(t)}{f_2^2(t)}.$

In this work, we investigate analytical solutions of the mK(m,n) equation with the local derivative

$$u^{n-1}\mathfrak{N}_{h,t}^\alpha u + a(u^m)_x + (u^n)_{xxx} = 0, \tag{6}$$

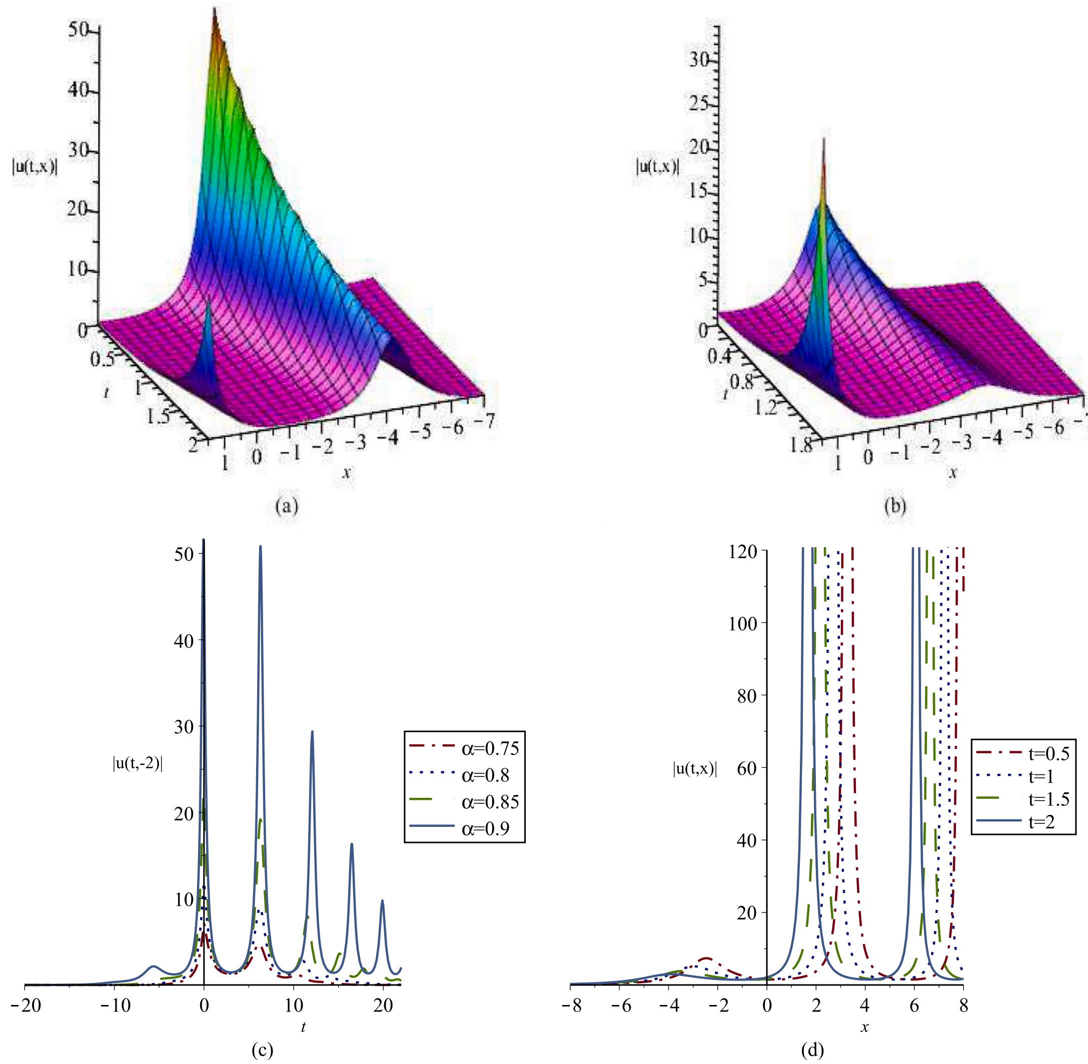

Fig. 3. Exact solution of (17) with $R_1 = R_2 = 0$, $R_3 = -1$, $\chi = 1$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $x = -2$, and various α, (d) $\alpha = 0.8$, and various t.

where

$$\mathfrak{N}^{\alpha}_{h,t}u(t,x) = \lim_{\varepsilon\to 0} \frac{u\left(t+\varepsilon\, t^{\frac{1-\alpha}{2}}\operatorname{sech}((1-\alpha)t^{\frac{1+\alpha}{2}}),x\right)-u(t,x)}{\varepsilon}.$$

Nucci's reduction method

In this section, we consider the nonlinear mK(m,n) equation with mentioned temporal local derivative. The transformation (5) can convert this equation into a nonlinear ordinary differential equation. Then by the Nucci's reduction technique, different types of exact solution can be extracted. All computations are accomplished by the Maple software. To the best of authors knowledge, this is first development of reduction technique to a differential equation with recently defined local derivative.

Let us assume the mK(m,n) Eq. (6) with new local derivative and corresponding transformation

$$\mathcal{U}(\zeta) = u(t,x),\quad \zeta = \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right). \tag{7}$$

Applying transformation (5), we get the following nonlinear third-order ODE *w.r.t.* n and m:

$$\chi U^{n-1}(\zeta)U'(\zeta) + a(U^m(\zeta))' + (U^n(\zeta))''' = 0. \tag{8}$$

Let us assume the change of variables [32,39–41]:

$$\varphi_1(\zeta) = \mathcal{U}(\zeta),\quad \varphi_2(\zeta) = \mathcal{U}'(\zeta),\quad \varphi_3(\zeta) = \mathcal{U}''(\zeta).$$

So, the Eq. (8) reduces into the following autonomous system of equations:

$$\begin{cases} \dfrac{d\varphi_1}{d\zeta} = \varphi_2, \\ \dfrac{d\varphi_2}{d\zeta} = \varphi_3, \\ \dfrac{d\varphi_3}{d\zeta} = -\dfrac{\chi\varphi_1^{n-1}\varphi_2 + m\varphi_1^{m-1}\varphi_2 + 3n(n-1)\varphi_1^{n-2}\varphi_2\varphi_3 + n(n-1)(n-2)\varphi_2^3\varphi_1^{n-3}}{n\varphi_1^{n-1}}. \end{cases} \tag{9}$$

Selecting φ_1 as a new independent variable, converts the system (9) into

$$\begin{cases} \dfrac{d\varphi_2}{d\varphi_1} = \dfrac{\varphi_3}{\varphi_2}, \\ \dfrac{d\varphi_3}{d\varphi_1} = -\dfrac{\chi\varphi_1^{n-1} + m\varphi_1^{m-1} + 3n(n-1)\varphi_1^{n-2}\varphi_3 + n(n-1)(n-2)\varphi_2^2\varphi_1^{n-3}}{n\varphi_1^{n-1}}. \end{cases} \tag{10}$$

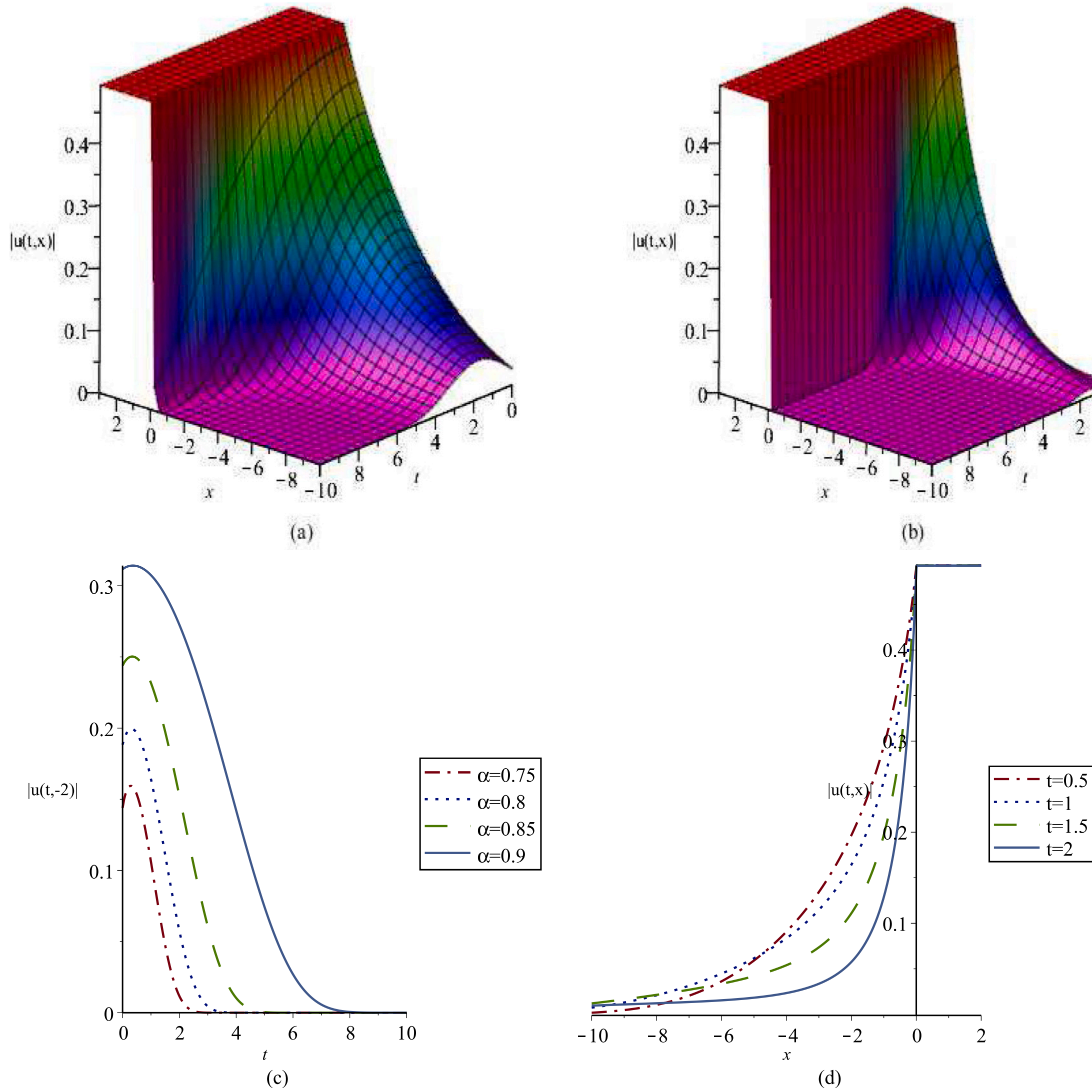

Fig. 4. Exact solution of (18) with $R_1 = R_2 = 0,\ R_3 = 1,\ \chi = 5$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $x = -2$, and various α, (d) $\alpha = 0.8$, and various t.

From the first equation in (10) we have

$$\varphi_3 = \varphi_2 \frac{d\varphi_2}{d\varphi_1}. \tag{11}$$

Therefore, second equation of (10) can be written as:

$$\left(\frac{d\varphi_2}{d\varphi_1}\right)^2 + \varphi_2 \frac{d^2\varphi_2}{d\varphi_1^2} = -\frac{\chi\varphi_1^{n-1} + m\varphi_1^{m-1} + 3n(n-1)\varphi_1^{n-2}\varphi_2\frac{d\varphi_2}{d\varphi_1} + n(n-1)(n-2)\varphi_2^2\varphi_1^{n-3}}{n\varphi_1^{n-1}}. \tag{12}$$

General solution of Eq. (12) for arbitrary values of m and n is inaccessible. So, we try to find exact solutions of some special cases:

- **Case 1:** $n = 1$

In this case, solving Eq. (12) concludes

$$\varphi_2(\varphi_1) = \pm\frac{\sqrt{-(m+1)\left(\chi\,{\varphi_1}^2(m+1) + 2R_1(m+1)\varphi_1 - 2R_2(m+1) + 2{\varphi_1}^{m+1}\right)}}{m+1}, \tag{13}$$

with R_1 and R_2 arbitrary constants. Now after assuming that φ_1 is a dependent variable *w.r.t.* ζ, we substitute (13) into the first equation of (9) which yields the following first order ODE:

$$\varphi_1'(\zeta) = \pm\frac{\sqrt{-(m+1)\left(\chi\,{\varphi_1}^2(m+1) + 2R_1(m+1)\varphi_1 - 2R_2(m+1) + 2{\varphi_1}^{m+1}\right)}}{m+1}.$$

This equation is a separable ODE, and corresponding implicit solution is

$$\zeta \mp \int \frac{(m+1)d\varphi_1}{\sqrt{-(m+1)\left(\chi\,{\varphi_1}^2(m+1) + 2R_1(m+1)\varphi_1 - 2R_2(m+1) + 2{\varphi_1}^{m+1}\right)}} + R_3 = 0, \tag{14}$$

where R_3 is an arbitrary constant. Explicit solutions can be extracted by assuming some special values of m.

- **Case 1.1.** $m = 1$

By using this assumption, from Eq. (14) we obtain

$$\zeta \mp \frac{1}{\sqrt{\chi+1}}\arctan \times \left(\frac{\sqrt{\chi+1}}{\sqrt{(-\chi-1)\,{\varphi_1}^2 - 2R_1\varphi_1 + 2R_2}}\left(\varphi_1 + \frac{R_1}{\chi+1}\right)\right) + R_3 = 0.$$

Hence, solving the obtained equation *w.r.t.* the variable φ_1 concludes

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = \pm\frac{1}{\chi+1}$$

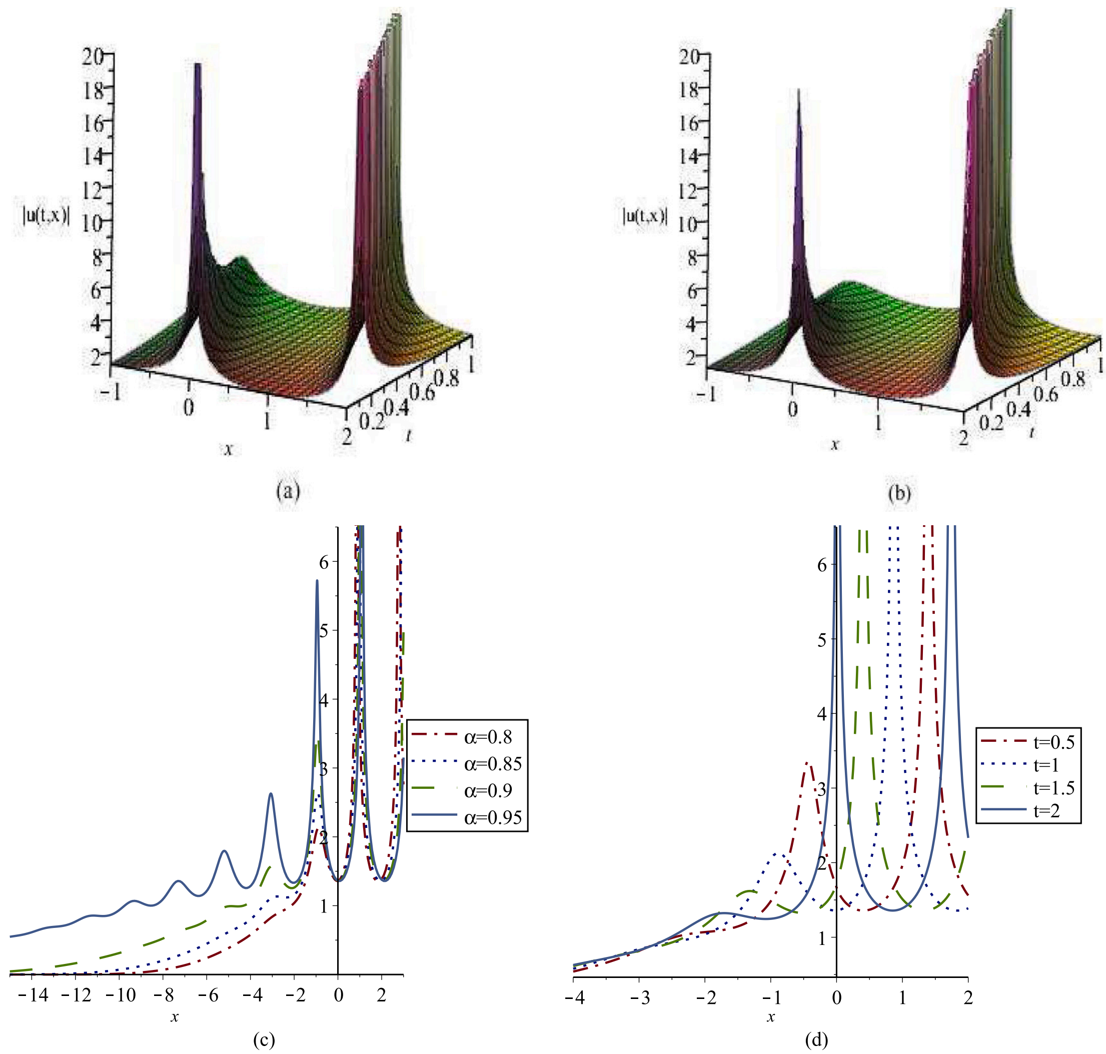

Fig. 5. Exact solution of (19) with $R_1 = 0,\ R_2 = 0,\ R_3 = \chi = 1$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $t = 1$, and various α, (d) $\alpha = 0.8$, and various t.

$$\times\left(\sqrt{-\frac{\left(\left(\cos\left(R_3\sqrt{\chi+1}+\zeta\sqrt{\chi+1}\right)\right)^2-1\right)\left(R_1^2+2R_2\chi+2R_2\right)}{\cos^4\left(R_3\sqrt{\chi+1}+\zeta\sqrt{\chi+1}\right)}}\right.$$
$$\left.\times\cos^2\left(R_3\sqrt{\chi+1}+\zeta\sqrt{\chi+1}\right)-R_1\right).$$

Finally, from the obtained solution and transformation (7) we get:

$$u(t,x) = \pm\frac{1}{\chi+1}$$
$$\times\left(\sqrt{-\frac{\left(\left(\cos\left(R_3\sqrt{\chi+1}+\left(\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\sqrt{\chi+1}\right)\right)^2-1\right)\vartheta}{\cos^4\left(R_3\sqrt{\chi+1}+\left(\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\sqrt{\chi+1}\right)}}\right.$$
$$\left.\times\cos^2\left(R_3\sqrt{\chi+1}+\left(\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\sqrt{\chi+1}\right)-R_1\right), \quad (15)$$

where $\vartheta = R_1^2 + 2R_2\chi + 2R_2$. Some plots corresponding to the (15) is represented in Fig. 1 with various selected parameters and order values. 3-D and 2-D periodic W-shaped soliton solutions shown in this figure, demonstrate the effects of fractional order into the final results.

- **Case 1.2.** $m = \frac{3}{2}$

By using this assumption and $R_1 = R_2 = 0$, from Eq. (14) we obtain

$$\zeta \mp 4\frac{\varphi_1\sqrt{-20\sqrt{\varphi_1}-25\chi}}{\sqrt{-25\chi\,\varphi_1{}^2-20\varphi_1^{5/2}}\sqrt{\chi}}\arctan\left(\frac{\sqrt{-20\sqrt{\varphi_1}-25\,\chi}}{5\sqrt{\chi}}\right)+R_3=0,$$

which yields its explicit solution

$$\mathcal{U}(\zeta)=\varphi_1(\zeta)=\frac{25\,\chi^2\left(\tan^2\left(\pm\frac{\sqrt{2}}{4}\left(R_3+\zeta\right)\right)+1\right)^2}{16}.$$

Therefore, transformation (7) concludes the following final solution:

$$u(t,x)=\frac{25\,\chi^2\left(\tan^2\left(\pm\frac{\sqrt{2}}{4}\left(R_3+\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right)+1\right)^2}{16}. \quad (16)$$

Fig. 2. shows some periodic bright soliton solutions with different values of derivative order and temporal t.

- **Case 1.3.** $m = 2$

Assuming $m = 2$ and $R_1 = R_2 = 0$, from Eq. (14) we get the following implicit solution

$$\zeta \mp 2\frac{\varphi_1\sqrt{-6\varphi_1-9\chi}}{\sqrt{-6\varphi_1{}^3-9\chi\,\varphi_1{}^2}\sqrt{\chi}}\arctan\left(\frac{\sqrt{-6\varphi_1-9\chi}}{3\sqrt{\chi}}\right)+R_3=0,$$

where R_3 is an arbitrary constant and solving the obtained implicit solution with respect to φ_1 concludes

$$\mathcal{U}(\zeta)=\varphi_1(\zeta)=-\frac{3\chi}{2\cos^2\left(\pm\frac{1}{2}\sqrt{\chi}\left(R_3+\zeta\right)\right)}.$$

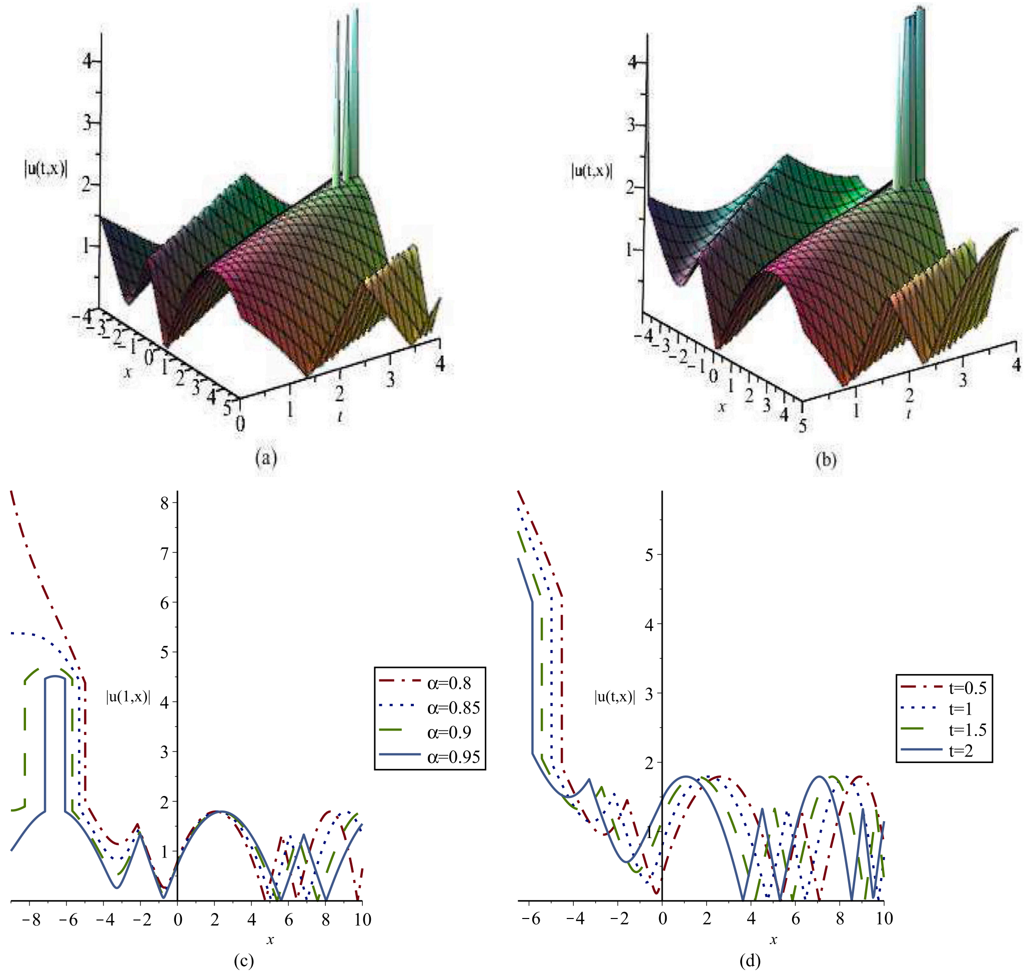

Fig. 6. Exact solution of (22) with $R_1 = 0$, $R_2 = -1$, $R_3 = \chi = 1$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $t = 1$, and various α, (d) $\alpha = 0.8$, and various t.

Hence

$$u(t,x) = -\frac{3\chi}{2\cos^2\left(\pm\frac{1}{2}\sqrt{\chi}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + \chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right)}. \tag{17}$$

Periodic bright soliton solutions are plotted in Fig. 3. Effects of differential order are plotted in this figure.

• **Case 1.4.** $m = 3$

Eq. (14) with supposing $R_1 = R_2 = 0$ and $m = 3$ yields the following implicit solution:

$$\zeta + \frac{\varphi_1\sqrt{-2\varphi_1^2 - 4\chi}}{\sqrt{-2\varphi_1^4 - 4\chi\varphi_1^2}\sqrt{-\chi}}\ln\left(4\,\frac{\sqrt{-\chi}\sqrt{-2\varphi_1^2 - 4\chi} - 2\chi}{\varphi_1}\right) + R_3 = 0,$$

or, equivalently

$$\mathcal{V}(\zeta) = \varphi_1(\zeta) = \frac{-16e^{\pm i(R_3+\zeta)}}{e^{\pm 2i(R_3+\zeta)} - 32},$$

and

$$\mathcal{V}(\zeta) = \varphi_1(\zeta) = \frac{-32e^{\pm i\sqrt{2}(R_3+\zeta)}}{e^{\pm 2i\sqrt{2}(R_3+\zeta)} - 64},$$

for $\chi = 1$ and $\chi = 2$, respectively. Therefore, transformation (7) concludes the following final solutions:

$$u(t,x) = \frac{-16e^{\pm i\left(R_3+\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)}}{e^{\pm 2i\left(R_3+\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)} - 32},$$

and

$$u(t,x) = \frac{-32e^{\pm i\sqrt{2}\left(R_3+\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)}}{e^{\pm 2i\sqrt{2}\left(R_3+\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)} - 64}, \tag{18}$$

where $i^2 = -1$.

Profile of exact solution (18) in 2 and 3 dimensions with different α and t are plotted in Fig. 4.

• **Case 1.5.** $m = 4$

Similarly, in this case for $\chi = 1$, we get

$$\mathcal{V}(\zeta) = \varphi_1(\zeta) = \frac{\sqrt[3]{20}}{2}\sqrt[3]{-\cos^{-2}\left(\frac{3}{2}(R_3+\zeta)\right)},$$

Therefore, transformation (7) concludes the following final solutions:

$$u(t,x) = \frac{\sqrt[3]{20}}{2}\sqrt[3]{-\cos^{-2}\left(\frac{3}{2}(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + t^{\frac{1+\alpha}{2}}\right)\right))\right)}. \tag{19}$$

Smooth–cuspon bright soliton of (19), is demonstrated in Fig. 5. Different behave of solution in negative and positive values of space direction can be seen from the plotted figures.

Now, let us consider the second case of non-linearity power n.

• **Case 2:** $n = 2$

In this case, solving Eq. (12) concludes

$\varphi_2(\varphi_1)$

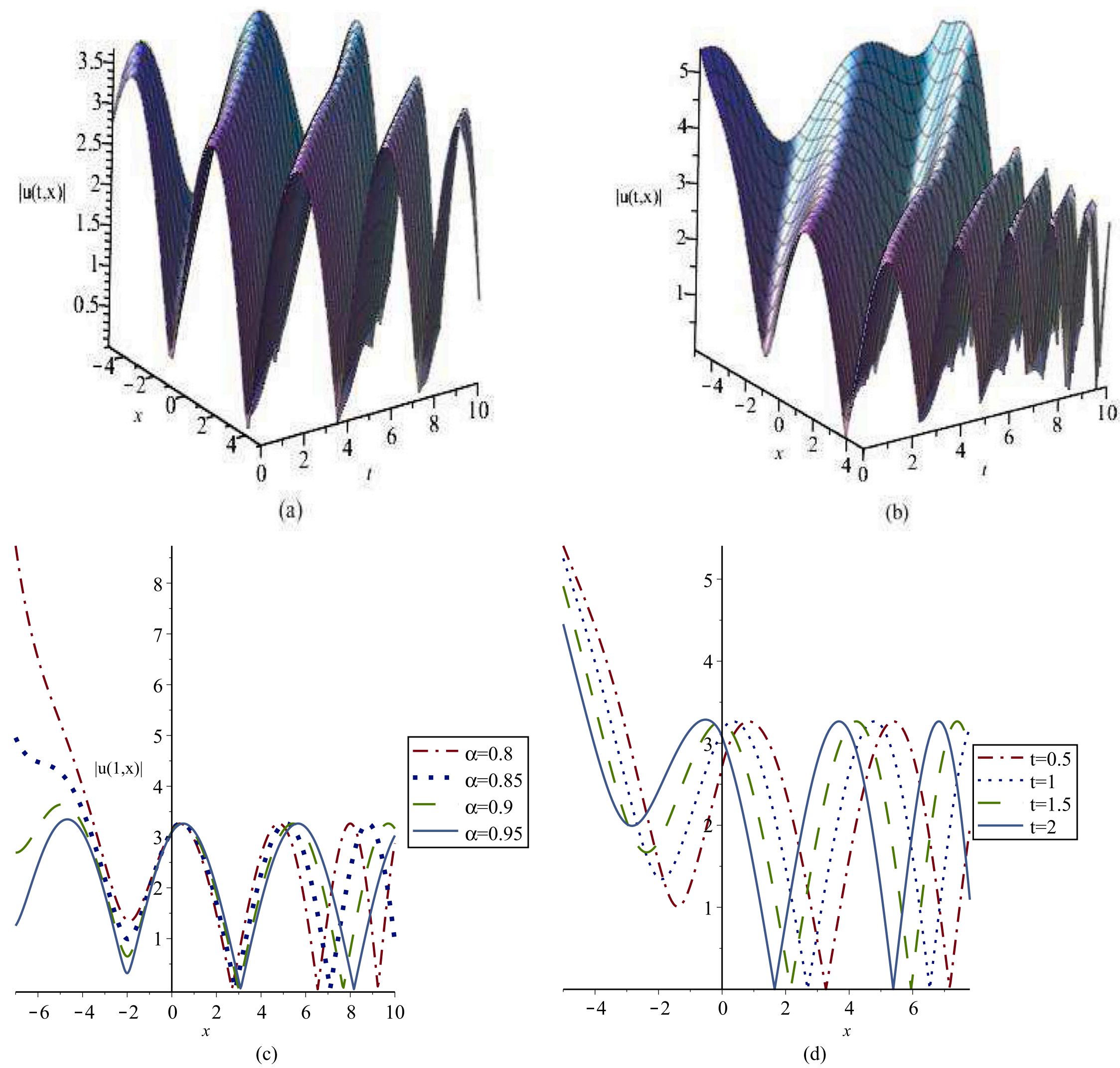

Fig. 7. Exact solution of (23) with $R_1 = 0,\ R_2 = -1,\ R_3 = \chi = 1$, and (a) $\alpha = 0.9$, (b) $\alpha = 0.8$, (c) $t = 1$, and various α, (d) $\alpha = 0.8$, and various t.

$$= \pm \frac{\sqrt{-(2m+4)\left(\chi \varphi_1{}^4 m + 2\chi \varphi_1{}^4 + 8R_2 \varphi_1{}^2 m + 16R_2 \varphi_1{}^2 - 8R_1 m + 8\varphi_1{}^{m+2} - 16R_1\right)}}{4(m+2)\varphi_1}, \tag{20}$$

with R_1 and R_2 arbitrary constants. Lastly, we substitute (20) into the first equation of (9) which concludes the following single ODE:

$$\varphi_1'(\zeta) = \pm \frac{\sqrt{-(2m+4)\left(\chi \varphi_1{}^4 m + 2\chi \varphi_1{}^4 + 8R_2 \varphi_1{}^2 m + 16R_2 \varphi_1{}^2 - 8R_1 m + 8\varphi_1{}^{m+2} - 16R_1\right)}}{4(m+2)\varphi_1}.$$

Corresponding implicit solution is

$$\zeta \mp \int \frac{4(m+2)\varphi_1 d\varphi_1}{\sqrt{-(m+2)\left(\chi\varphi_1^4 m + 2\chi\varphi_1^4 + 8R_1 m\varphi_1^2 + 16R_1\varphi_1^2 - 8R_2 m + 8\varphi_1^{m+2} - 16R_2\right)}} + R_3 = 0, \tag{21}$$

where R_3 is an arbitrary constant. In order to extract explicit solutions from the obtained implicit one, we consider the following cases for values of m.

- **Case 2.1.** $m = 1$

In order to solve the Eq. (21), assuming $R_1 = 0$ and $m = 1$ yields

$$\zeta \mp 2\frac{\sqrt{2}}{\sqrt{\chi}} \arctan\left(\frac{\sqrt{2}\,(3\varphi_1\chi + 4)}{\sqrt{\chi}\sqrt{-18\chi\varphi_1{}^2 - 144R_2 - 48\varphi_1}}\right) + R_3 = 0,$$

or equivalently

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = \frac{2}{3\chi}\left(\sqrt{2}\sqrt{\frac{\left(\cos^2\left(\frac{\sqrt{2\chi}}{4}(R_3+\zeta)\right)-1\right)(9\chi R_2 - 2)}{\cos^4\left(\frac{\sqrt{2\chi}}{4}(R_3+\zeta)\right)}} \times \cos^2\left(\frac{\sqrt{2\chi}}{4}(R_3+\zeta)\right) - 2\right).$$

Therefore, substituting transformation (7) concludes the following final solution:

$$u(t,x) = \frac{2}{3\chi}\left(\sqrt{2} \times \sqrt{\frac{\left(\cos^2\left(\frac{\sqrt{2\chi}}{4}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + \chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right)-1\right)(9\chi R_2 - 2)}{\cos^4\left(\frac{\sqrt{2\chi}}{4}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + \chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right)}} \times \cos^2\left(\frac{\sqrt{2\chi}}{4}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + \chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right) - 2\right). \tag{22}$$

By choosing $R_2 = -1$ and $R_3 = \chi = 1$, corresponding W-shaped soliton solutions with different values of α and t are plotted in Fig. 6. However, different behaviour of solution can be seen in the negative part of space direction.

- **Case 2.2.** $m = 2$

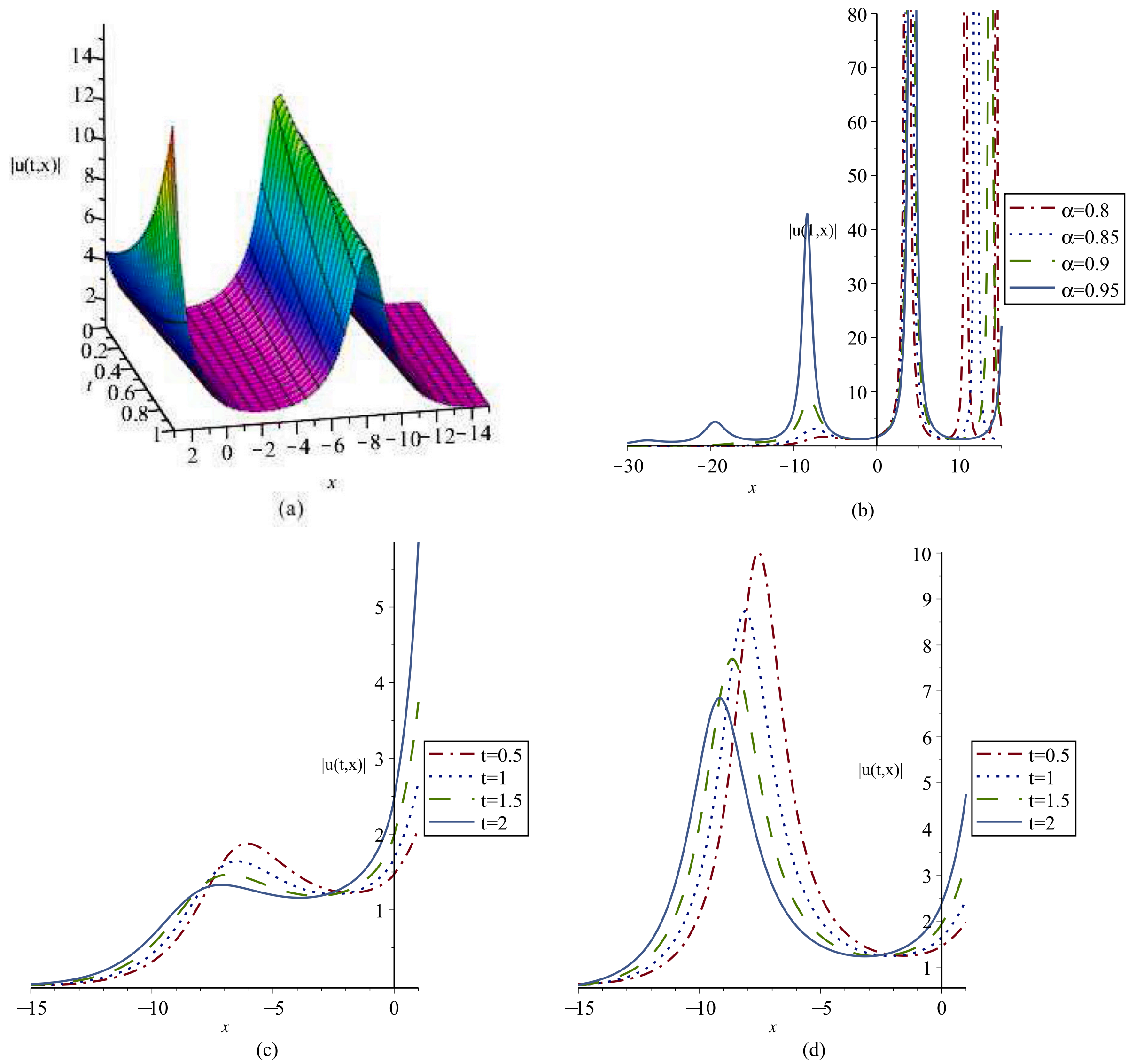

Fig. 8. Exact solution of (24) with $R_1 = R_2 = 0$, $R_3 = \chi = 1$, and (a) $\alpha = 0.9$, (b) $t = 1$ and various α, (c) $\alpha = 0.8$, and various t, (d) $\alpha = 0.9$, and various t.

By using this assumption and $R_1 = 0$, from Eq. (21) we obtain

$$\zeta \mp \frac{4}{\sqrt{2\chi+4}} \arctan\left(\frac{\sqrt{2\chi+4}\,\varphi_1}{\sqrt{-2\chi\varphi_1^2 - 4\varphi_1^2 - 16R_2}}\right) + R_3 = 0.$$

Hence,

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = \frac{2\sqrt{2}\sin\left(\frac{\sqrt{2\chi+4}}{2}(R_3+\zeta)\right)}{\chi+2}\sqrt{-\frac{(\chi+2)R_2}{\cos^2\left(\frac{\sqrt{2\chi+4}}{4}(R_3+\zeta)\right)}},$$

which concludes

$$u(t,x) = \frac{2\sqrt{2}\sin\left(\frac{\sqrt{2\chi+4}}{2}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + \chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right)}{\chi+2}$$
$$\times\sqrt{-\frac{(\chi+2)R_2}{\cos^2\left(\frac{\sqrt{2\chi+4}}{4}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + \chi t^{\frac{1+\alpha}{2}}\right)\right)\right)\right)}}. \qquad (23)$$

In Fig. 7, the periodic wave solutions of (23) with different values of differential order α and temporal values t, are plotted. From Fig. 7(b)–(c) we find that the order of differential operator causes different behave of solution whenever α goes far away from the integer one.

• **Case 2.3.** $m = 3$

By using this assumption and $R_1 = R_2 = 0$, from Eq. (21) we get

$$\zeta \mp 4\sqrt{\frac{2}{\chi}}\arctan\left(\frac{\sqrt{-2(80\varphi_1 + 50\chi)}}{10\sqrt{\chi}}\right) + R_3 = 0,$$

in other words

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = -\frac{5}{8}\tan^2\left(\frac{\sqrt{2}}{8}(R_3+\zeta)\right) - \frac{5}{8},$$

and

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = -\frac{5}{4}\tan^2\left(\frac{1}{4}(R_3+\zeta)\right) - \frac{5}{4},$$

with respect to $\chi = 1$ and $\chi = 2$, respectively. Therefore, transformation (7) concludes the following final solutions:

$$u(t,x) = -\frac{5}{8}\tan^2\left(\frac{\sqrt{2}}{8}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + t^{\frac{1+\alpha}{2}}\right)\right)\right)\right) - \frac{5}{8},$$

and

$$u(t,x) = -\frac{5}{4}\tan^2\left(\frac{1}{4}\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} + 2t^{\frac{1+\alpha}{2}}\right)\right)\right)\right) - \frac{5}{4}. \qquad (24)$$

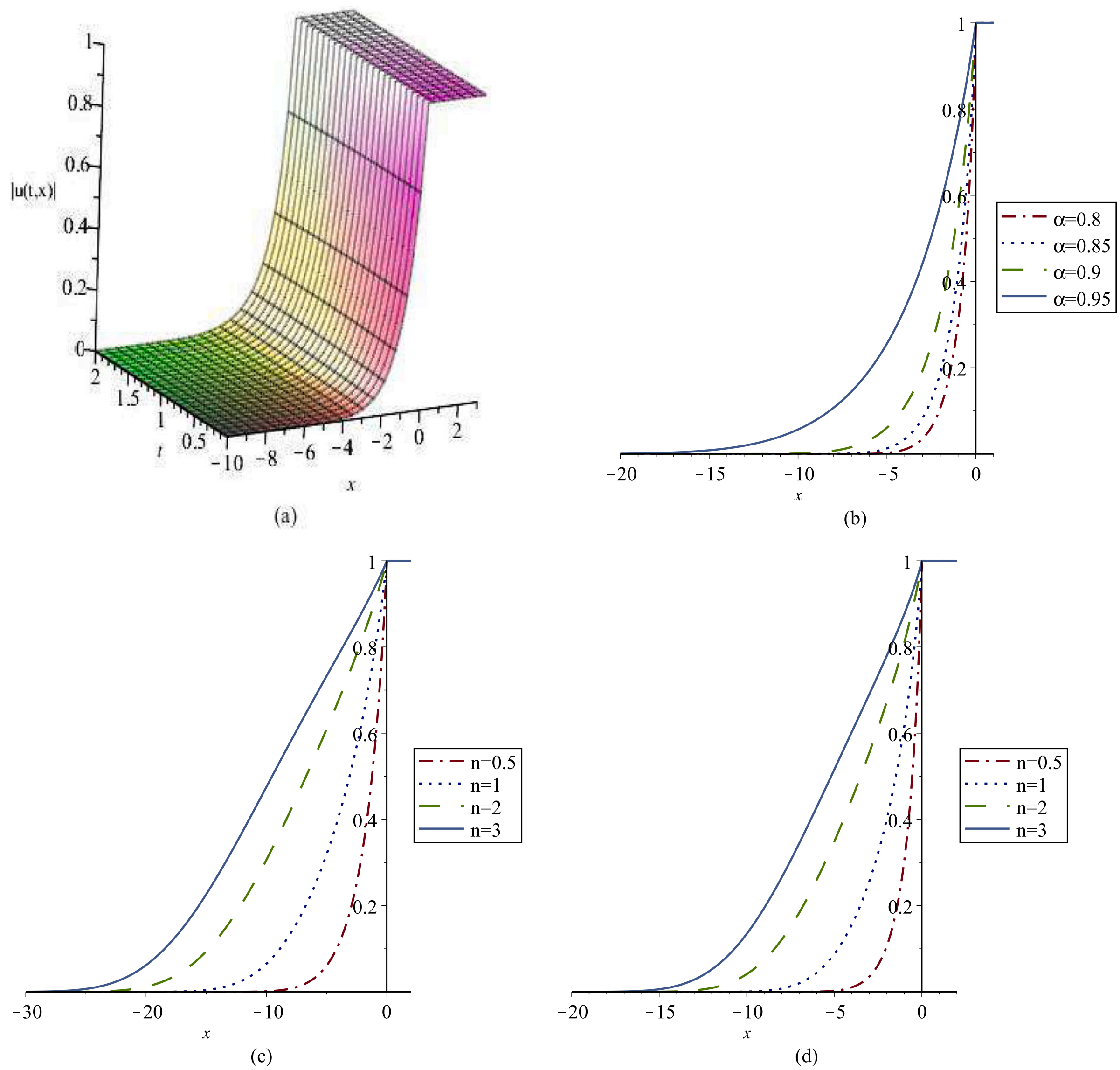

Fig. 9. Exact solution of (28) with $R_1 = R_2 = 0$, $R_3 = \chi = 1$, and (a) $n = 0.5$, $\alpha = 0.8$, (b) $n = 0.5$, $t = 1$ and various α, (c) $t = 1$, $\alpha = 0.9$, and various n, (d) $t = 1$, $\alpha = 0.8$, and various n.

Smooth-smooth bright soliton of (24) is plotted in Fig. 8.

• **Case 3:** $m = n$ In this case that Eq. (6) coincide with the mK(n,n) equation, we find exact solution for an arbitrary value of n. By solving Eq. (12) we get

$$\varphi_2(\varphi_1) = \pm \frac{{\varphi_1}^{-n+2}}{n^2}\sqrt{-\frac{n\left(-2R_1 n^2 + 2\varphi_1^n R_2 n^2 + \chi\varphi_1^{2n} + \varphi_1^{2n} n\right)}{\varphi_1^2}}, \tag{25}$$

with R_1 and R_2 arbitrary constants. Lastly, we substitute (25) into the first equation of (9) which concludes the following single ODE:

$$\varphi_1'(\zeta) = \pm \frac{{\varphi_1}^{-n+2}}{n^2}\sqrt{-\frac{n\left(-2R_1 n^2 + 2\varphi_1^n R_2 n^2 + \chi\varphi_1^{2n} + \varphi_1^{2n} n\right)}{\varphi_1^2}}. \tag{26}$$

Corresponding implicit solution by assuming $R_1 = R_2 = 0$, is

$$\zeta \mp \frac{n^2 {\varphi_1}^{n-1}\ln\left(\varphi_1\right)}{\sqrt{-n\left(\chi+n\right){\varphi_1}^{2n-2}}} + R_3 = 0, \tag{27}$$

where R_3 is an arbitrary constant. Solving this equation concludes

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = \mathrm{e}^{\pm\frac{\sqrt{-n(\chi+n)}\left(R_3+\zeta\right)}{n^2}},$$

or

$$u(t,x) = \mathrm{e}^{\pm\frac{\sqrt{-n(\chi+n)}\left(R_3+\frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}}+\chi t^{\frac{1+\alpha}{2}}\right)\right)\right)}{n^2}}. \tag{28}$$

King shape wave solution of (28) is presented in Fig. 9 with different values of the non-linearity power n and order of fractional derivative α.

In addition, for the nonzero values of R_1 and R_2, the implicit solution of Eq. (26) by choosing $\chi = -n$, can be written as

$$\zeta \mp \frac{n\sqrt{2}\left({\varphi_1}^n R_2 - R_1\right)}{\varphi_1 R_2\sqrt{-\frac{n^3\left(\varphi_1^n R_2 - R_1\right)}{\varphi_1^2}}} + R_3 = 0,$$

or

$$\mathcal{U}(\zeta) = \varphi_1(\zeta) = \mathrm{e}^{\frac{1}{n}\ln\left(\frac{2R_1 - nR_2^2(R_3+\zeta)^2}{2R_2}\right)},$$

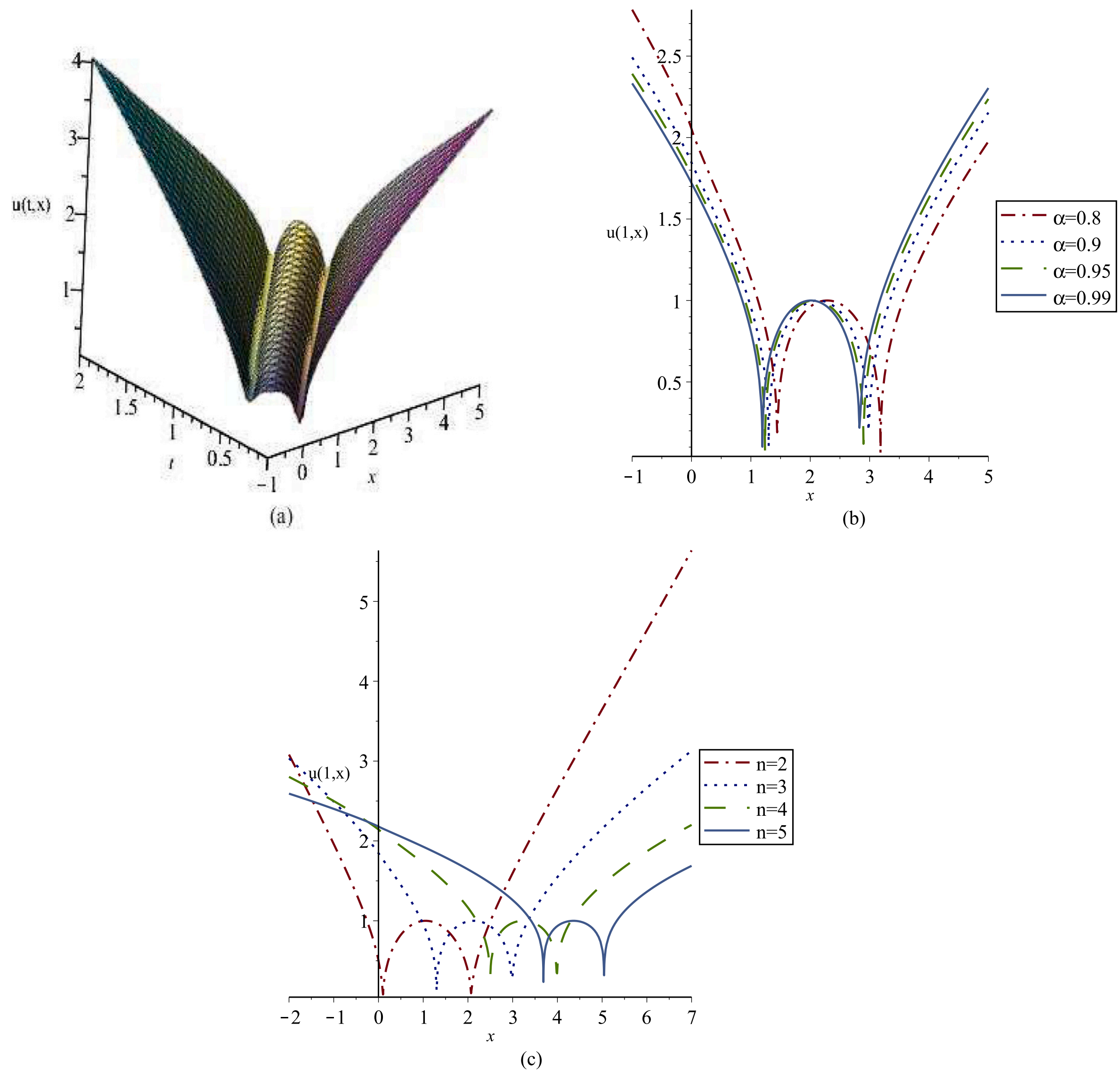

Fig. 10. Exact solution of (29) with $R_1 = R_2 = R_3 = 1$, $\chi = -n$ and (a) $n = 3$, $\alpha = 0.9$, (b) $n = 3$, $t = 1$ and various α, (c) $t = 1$, $\alpha = 0.9$, and various n.

and therefore

$$u(t,x) = e^{\frac{1}{n}\ln\left(\frac{2R_1 - nR_2^2\left(R_3 + \frac{2}{1-\alpha^2}\sinh\left((1-\alpha)\left(x^{\frac{1+\alpha}{2}} - nt^{\frac{1+\alpha}{2}}\right)\right)\right)^2}{2R_2}\right)}. \tag{29}$$

Fig. 10, shows the W-shaped solution of (29) with respect to different values of n and α. As a summary of the results and discussing about the novelties of current work, we can list the following items:

- This is the first work to consider the generalized nonlinear dispersive mK(m,n) equation with fractional local derivative.
- The Nucci's reduction method is novel for the differential equations with local derivatives.
- Considered Eq. (6) and reduction method are novel. So the reported exact solutions in this section are novel.

Conclusion

Consideration of differential equations with new local or nonlocal derivative operators and finding corresponding exact solutions is a major study field of many researchers. In this paper, an important differential equation, namely, the generalized nonlinear dispersive mK(m,n) equation is considered with different values of m and n. The supposed derivative in temporal direction is a recently defined local derivative. Different types of soliton, wave and W-shape solutions are extracted by a reduction method. The super deformed nuclei, preformation of cluster in hydrodynamic models, the fission of liquid drops (nuclear physics), inertial fusion and others are some scientific applications of compactons (compact support solitons). Therefore, compactons of an important kind of applicable KdV equation in physics, is investigated. To the best of authors knowledge, this paper is the only work which is developed to the differential equations with this type of local derivative and therefore, the obtained exact solutions and methodology are novel. It is notable that our obtained results for the generalized mK(m,n) with integer order, are not reachable. However, when our developed model by fractional operator cover the integer order model when $\alpha = 1$. It is easily deducible from the Eq. (4).

CRediT authorship contribution statement

Fang-Li Xia: Investigation, Software, Formulation, Review and checking results, Conceptualization. **Fahd Jarad:** Software, Visualization, Supervision, Formal analysis, Writing – review & editing, Conceptualization. **Mir Sajjad Hashemi:** Data curation, Data anaysis, Project administration, Final checking, Validation, Writing. **Muhammad Bilal Riaz:** Investigation, Methodology, Initial writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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