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Additive Trinomial Fréchet distribution with practical application

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A R T I C L E I N F O

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A B S T R A C T

This article presents an innovative model called Additive Trinomial Fré chet (ATF) distribution using six parameters. The indicated model is worthy of modeling survival data with a non-monotonic hazard rate. The statistical characteristics of ATF model such as probability generating function, Renyi, Shannon, Tsallis and Mathai–Houbold entropy, quantile function, order statistics, maximum likelihood estimation, factorial and characteristic function, moment generating function, Stress-Strength analysis are thoroughly discussed. The effectiveness of suggested model is demonstrated by the use of a data set from real life. The suggested model has demonstrated better performance and fits the data used superior than other significant counterparts.

Introduction

A very well-known approach in the literature is the procedure of extending models by adding flexibility or to form covariate models. In several applied sciences such as engineering, medicine, and finance, the modeling and interpretation of survival data is significant. Several models of lifetime were utilized to model these types of data. The accuracy of the methods for use of a statistical study is highly based on the conjecture probability distributions. Due to this, significant effort was put into developing broad classes of standard models of probability along by means of appropriate statistical methods. Nevertheless, there are some big complications where data does not fit either the classical or normal probability distributions. In theory of extreme value, Fréchet distribution is one of main models used for the data on rainfall, sea wave dynamics, wind speed, earthquakes, queues, floods, horse racing and track race records. More information about the Fréchet model and its usages were discussed in [1].

Many modifications of the Fréchet model have recently been examined. The exponentiated Fréchet [2], beta Fréchet [3,4], transmuted Fr échet [5], gamma extended Fréchet [6], Marshall–Olkin Fréchet [7], Kumaraswamy Fréchet [8], transmuted Marshall–Olkin Fréchet (TMOF) [9], Transmuted exponentiated generalized Fréchet [10], Weibull Fr échet [11], transmuted exponentiated Fréchet [12], Bur X Fréchet [13] and beta exponential Fréchet [14] models.

The amount of uncertainty corresponding to a random variable is described by entropy. Theory of knowledge has mathematical roots in entropy-related notion of statistical mechanics and thermodynamic. In [15], Hartley introduced concepts of information theory in engineering and to this purpose the theory of information was taken as a branch of theory of communication. Shannon [16] then introduced entropy's mathematical concept and succeeded in evaluating channel's capacity to transmit numerical knowledge utilizing sources of information. To strengthen his effort on theory of information, other researchers reached at the entropy in Shannon by looking at different characteristics. For example, one may study that in Refs. [16–22].

Various modifications of Shannon entropy, like Renyi [23], Harvda Charvat [24], and Tsallis [25] entropy, involving special case (Shannon entropy (SE)), were introduced after 1948. Campbell [26] and Koski and Persson [27] introduced the other measures known as exponential entropy and generalized exponential entropy (GEE). Renyi and Tsallis entropies converge on Shannon entropy. GEE can be correlated with a generalized variant of RE (Renyi entropy), GRE (generalized Renyi entropy), and GEE converging into SE.

Motivated by the above literature, here we adding a new improvement to the Fr échet model by describing the Fréchet model cumulative distribution with three Fréchet model combination. The new proposed Fréchet model is named as "Additive Trinomial Frechet (ATF) distribution." The suggested six parameters model is more flexible than the Fréchet model. Here we calculate the uncertainty measures that are entropies such as Shannon, Renyi, Tsallis and Mathai–Houbold entropy according to our proposed model ATF. Furthermore, hazard rate function (HRF) its characterization and cumulative hazard rate function (CHRF) are calculated. Besides this conditional moments and

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mean deviations is also discussed. Residual life function with a certain measure of reliability is also derived. Also we calculate some other statistical attributes of the ATF model such as probability generating function, quantile function, order statistics, factorial and characteristics functions, moment generating function, Stress-Strength analysis, Stochastic Ordering, Non-central moments, generating function (GF) and maximum likelihood estimation,. The representation of pdf, cdf, cumulative cdf, quantile, median, skewness and kurtosis are examined graphically.

We implement one real-life data set for comparisons to demonstrate that model being proposed offers a better fit than other models including additive binomial Frechet distribution and Frechet distribution.

The additive trinomial Fréchet distribution

The Fréchet model is an unique type of extended extreme value model, a family of continuous models which takes into account the Weibull, Fré chet and Gumbel models often recognized as extreme value models. The cumulative distribution (CDF) function is presented by

$$\mathcal{F}(y|\xi,\psi)=\exp\{-\xi y^{-\psi}\}\quad y,\ \xi,\ \psi>0$$

Extrapolating distributions is an common method, which has often been found as important in statistical field. The current era of distribution theory emphasizes the problem solving faced by practical investigators to develop a variety of distributions so that data that are available in various areas of life can be properly analyzed and explored. In certain ways, there is an evident need to create practical models to explore the real-life occurrences properly. Lai et al. [21] efficiently characterized the modified Weibull distribution by three parameters by taking reasonable limits on the integrated beta models. It is therefore achieved by expanding Weibull model by using added term $\exp(-\upsilon y)$ to reduce the survival function more efficiently. In this study, we introduce a new update to the Frechet distribution by describing the Frechet cumulative distribution based on a combination of three Frechet distributions. We research the ATF distribution by strengthening the CDF of Frechet model to the format

$$\begin{aligned}\mathcal{F}\left(y|\xi_1,\xi_2,\xi_3,\psi_1,\psi_2,\psi_3\right) &= \exp\left\{-\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)\right\},\\ \mathcal{F}(y|\boldsymbol{\xi},\boldsymbol{\psi}) &= \exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right),\xi_i,\ \psi_i>0,\\ & i=1,2,3. \end{aligned}\tag{1}$$

Where the ψ_i and ξ_i, $i=1,2,3$, are parameters for shape and scale. It is straightforward and obvious to see that $\mathcal{F}(y)$ differentiable and rises strictly in 0 to ∞ and $\lim_{y\to 0}\mathcal{F}(y|\boldsymbol{\xi},\boldsymbol{\psi})=0$ and $\lim_{y\to\infty}\mathcal{F}(y|\boldsymbol{\xi},\boldsymbol{\psi})=1$. The respective ATF density is thus to the form

$$\begin{aligned} f(y|\boldsymbol{\xi},\boldsymbol{\psi}) &= \left\{\xi_1\psi_1 y^{-(\psi_1+1)}+\xi_2\psi_2 y^{-(\psi_2+1)}+\xi_3\psi_3 y^{-(\psi_3+1)}\right\}\\ &\quad \exp\left\{-\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)\right\},\\ f(y|\boldsymbol{\xi},\boldsymbol{\psi}) &= \sum_{i=1}^{3}\psi_i\xi_i y^{-(\psi_i+1)}\exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right),\xi_i,\ \psi_i,\ >0,\\ &\quad i=1,2,3. \end{aligned}\tag{2}$$

Notice that the model of ATF ($Y\sim\text{ATF}(\xi_i,\psi_i)$, $i=1,2,3$) is a particular case of F ($Y\sim\text{ATF}(\xi_1,\ \psi_1)$) if ξ_2,ξ_3,ψ_2, and $\psi_3=0$.

Reliability function

In general, the reliability paradigm is concerned with assessing a system's probability of longevity or failure. The $\mathcal{R}(y)=P(Y>y)$ indicates the survival function Y. The $\mathcal{R}(y)$ of the ATF is defined by

$$\mathcal{R}(y|\boldsymbol{\xi},\boldsymbol{\psi})=1-\exp\left\{-\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)\right\},$$

$$\mathcal{R}(y|\boldsymbol{\xi},\boldsymbol{\psi})=1-\exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right),\xi_1,\xi_2,\xi_3,\psi_1,\psi_2,\psi_3>0.\tag{3}$$

Hazard rate function

The hrf $\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})=f(y|\boldsymbol{\xi},\boldsymbol{\psi})/[1-\mathcal{F}(y|\boldsymbol{\xi},\boldsymbol{\psi})]$, in lifespan analysis, it is a highly useful tool. The $\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})$ of ATF model is

$$\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})=\frac{\left\{\xi_1\psi_1 y^{-(\psi_1+1)}+\xi_2\psi_2 y^{-(\psi_2+1)}+\xi_3\psi_3 y^{-(\psi_3+1)}\right\}\times\exp\left\{-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right\}}{1-\exp\left\{-\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)\right\}},$$
$$\xi_i,\ \psi_i>0,\ i=1,2,3.$$

After little simplification it takes the form

$$\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})=\frac{\sum_{i=1}^{3}\psi_i\xi_i y^{-(\psi_i+1)}}{\exp\left(\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right)-1},\xi_i,\ \psi_i>0,\ i=1,2,3.\tag{4}$$

Cumulative Hazard rate function

The CHRF is defined as

$$H(y)=\int_0^y \hbar(t|\boldsymbol{\xi},\boldsymbol{\psi})dt,\tag{5}$$

Therefore,

$$\begin{aligned} H(y) &= -\log\left\{1-\exp\left\{-\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)\right\}\right\},\\ &= \log\left\{1-\exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right)\right\}^{-1}. \end{aligned}\tag{6}$$

where $\hbar_t(t)$ is defined in (3) and (4).

Eqs. (1)–(6) can be easily investigated numerically employing computer software like MATLAB, Mathematica, Maple, Minitab, and R. For selected values of parameter, the plots of (2) and (3) are given in Figs. 1 and 2. Fig. 1 presents how well the parameters ψ_i, $i=1,2,3$ influence the density of ATF, as well as the flexibility of the pdf (2) structures, which can be used to measure skewness, modality, high tails, and little symmetry. These curves exhibit ATF model's adaptability. Fig. 2 shows the increasing-decreasing and inverted U pattern of hrfs. Survival and cumulative hrf curves are shown in Figs. 3 and 4. We notice that for modeling positive data, this model is quite flexible.

Characterization based on hazard function

We state the below definition, just for clarity.

Definition 1. Let $F(y|\boldsymbol{\xi},\boldsymbol{\psi})$ be an absolute continuous model with related pdf $f(y|\boldsymbol{\xi},\boldsymbol{\psi})$ The hazard function referring to $F(y|\boldsymbol{\xi},\boldsymbol{\psi})$ is defined by $\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})$

$$\hbar(y|.)=\frac{f(y|.)}{1-F(y|.)},\qquad Supp\in F.\tag{7}$$

Here, this is noticeable that the hrf of a twice differentiable distribution function follows given differential equation of first order

$$\frac{\hbar'(y|.)}{\hbar(y|.)}-\hbar(y|.)=k_1(y|.),\tag{8}$$

Where k_1 is a suitable integrable function. Though this differential equation has a specific form because

$$\frac{f'(y|.)}{f(y|.)}=\frac{\hbar'(y|.)}{\hbar(y|.)}-h(y|.).\tag{9}$$

For several continuous, univariate distributions (7) appears to be only differential equation in view of the hazard function . The purpose here is to create a differential equation that has the simplest possible form and not of the trivial form (8). However that may not be feasible for certain general distribution families. Here is the consequence of our characterization for the distribution of ATF distribution.

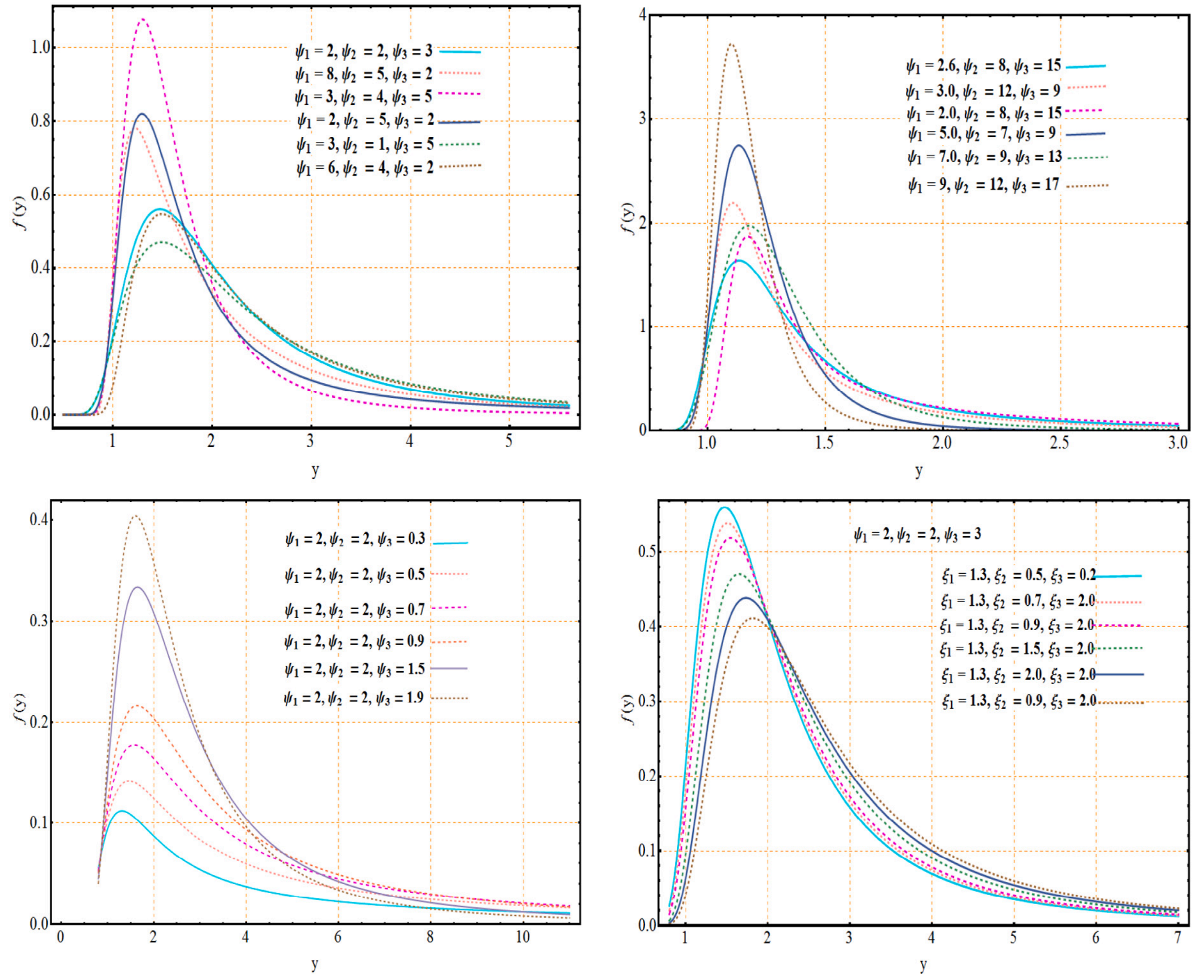

Fig. 1. Plots of density curves of ATF at different parameter values.

Proposition 1. *Let $Y : \chi \to (0,\infty)$ be a continuous r.v. The pdf of Y is (2) iff its hazard function $\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})$ satisfies the differential equation*

$$\frac{d\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})}{dy} - \left[\frac{\xi_1\psi_1 y^{-\psi_1-1} + \xi_2\psi_2 y^{-\psi_2-1} + \xi_3\psi_3 y^{-\psi_3-1}}{1-e^{-(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3})}}\right] h$$
$$= -\frac{\xi_1\psi_1(\psi_1+1)y^{-\psi_1-2} + \xi_2\psi_2(\psi_2+1)y^{-\psi_2-2} + \xi_3\psi_3(\psi_3+1)y^{-\psi_3-2}}{e^{(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3})}-1}. \quad (10)$$

Proof. : If Y follows (2), then obviously (8) holds. If $h(y|\boldsymbol{\xi},\boldsymbol{\psi})$ satisfies above mention equation, then, after some algebraic calculation, we can show that

$$\frac{d}{dy}\left[\hbar\left(1-e^{-(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3})}\right)\right]$$
$$= -\left(\xi_1\psi_1(\psi_1+1)y^{-\psi_1-2} + \xi_2\psi_2(\psi_2+1)y^{-\psi_2-2}\right.$$
$$\left.+\xi_3\psi_3(\psi_3+1)y^{-\psi_3-2}\right),$$

$$\frac{d}{dy}\left[\hbar\left(1-e^{-(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3})}\right)\right]$$
$$= \frac{d}{dy}\left(\xi_1\psi_1 y^{-\psi_1-1} + \xi_2\psi_2 y^{-\psi_2-1} + \xi_3\psi_3 y^{-\psi_3-1}\right).$$

Integrating both sides we get

$$\hbar\left(1-e^{-(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3})}\right) = \xi_1\psi_1 y^{-\psi_1-1} + \xi_2\psi_2 y^{-\psi_2-1} + \xi_3\psi_3 y^{-\psi_3-1},$$

$$\hbar = \frac{f(y|\boldsymbol{\xi},\boldsymbol{\psi})}{1-F(y|\boldsymbol{\xi},\boldsymbol{\psi})} = \frac{\xi_1\psi_1 y^{-\psi_1-1} + \xi_2\psi_2 y^{-\psi_2-1} + \xi_3\psi_3 y^{-\psi_3-1}}{\left(1-e^{-(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3})}\right)}. \quad (11)$$

Integrating both sides of (11) with respect to y from 0 to y we obtain

$$-\ln(1-F(y|\boldsymbol{\xi},\boldsymbol{\psi})) = -\ln\left[1-\exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right)\right].$$

Taking antilog on both sides

$$1-F(y|\boldsymbol{\xi},\boldsymbol{\psi}) = 1-\exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right),$$

and hence

$$F(y|\boldsymbol{\xi},\boldsymbol{\psi}) = \exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right),$$

from which we arrive at $\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi})$

$$\hbar(y|\boldsymbol{\xi},\boldsymbol{\psi}) = \frac{f(y|\boldsymbol{\xi},\boldsymbol{\psi})}{1-F(y|\boldsymbol{\xi},\boldsymbol{\psi})} = \frac{\sum_{i=1}^{3}\xi_i\psi_i y^{-(\psi_i+1)}}{\exp\left(\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right)-1}. \quad \square$$

Order statistics

Let $Y_{(1)} \le Y_{(1)} \ldots \le Y_{(n)}$ be the order statistics of size n from model (1). Then, for $i = 1,2,3$, and $m = 1,2,\ldots,n$,the pdf of the $Y_{(m)}$ (mth order statistic) is

$$f_{(m)}(y|\boldsymbol{\xi},\boldsymbol{\psi}) = \Psi\mathcal{F}(y|\boldsymbol{\xi},\boldsymbol{\psi})^{m-1}\{1-\mathcal{F}(y|\boldsymbol{\xi},\boldsymbol{\psi})\}^{n-m} f(y|\boldsymbol{\xi},\boldsymbol{\psi}), \quad (12)$$

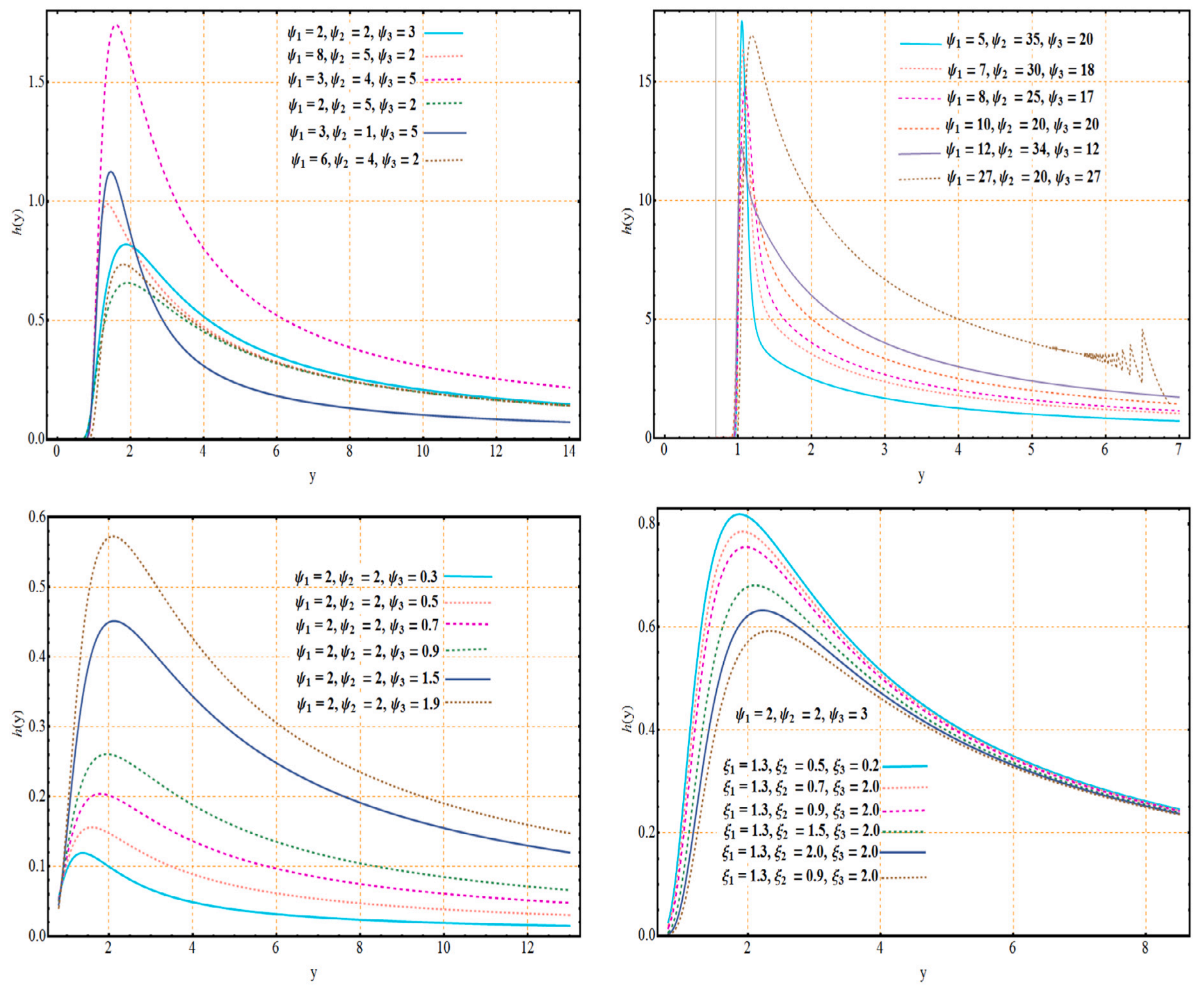

Fig. 2. Plots of hrf curves of ATF at different qualities of parameter.

where $\Psi = \frac{n!}{(m-1)!(n-m)!}$. From (2), (1) and (12) the pdf of $Y_{(m)}$.

$$f_{(m)}\left(y|\,\boldsymbol{\xi},\boldsymbol{\psi}\right) = \hat{\Psi} \sum_{i=1}^{3} \xi_i \psi_i y^{-(\psi_i+1)} \exp\left\{-(m+f-1)\sum_{i=1}^{3} \xi_i y^{-\psi_i}\right\}. \tag{13}$$

where $\hat{\Psi} = \Psi \sum_{f=0}^{n-m} \sum_{i=0}^{3} \binom{n-m}{f} (-1)^f$. The cdf of $Y_{(m)}$ results as

$$\mathcal{F}_{(m)}\left(y|\,\boldsymbol{\xi},\boldsymbol{\psi}\right) = \sum_{j=m}^{n} \binom{n}{j} \mathcal{F}\left(y|\,\xi_i,\psi_i\right)^j \left\{1-\mathcal{F}\left(y|\,\xi_i,\psi_i\right)\right\}^{n-j}, \tag{14}$$

then the cdf of $Y_{(m)}$ is

$$\mathcal{F}_{(m)}\left(y|\,\boldsymbol{\xi},\boldsymbol{\psi}\right) = \sum_{j=m}^{n}\sum_{g=1}^{n-j} \binom{n}{j}\binom{n-j}{g} (-1)^g \mathcal{F}\left(y|\,\xi_i,\psi_i\right)^{j+g} \times \exp\left\{-(j+g)\sum_{i=1}^{3} \xi_i y^{-\psi_i}\right\}. \tag{15}$$

Generally, cdfs of $Y_{(n)}$ and $Y_{(1)}$ are obtained as

$$\mathcal{F}_{(n)}(y) = F^n(y), \qquad \mathcal{F}_{(1)}(y) = 1-[1-F(y)]^n,$$

$$\mathcal{F}_{(n)}\left(y|\,\boldsymbol{\xi},\boldsymbol{\psi}\right) = \exp\left\{-(n)\sum_{i=1}^{3} \xi_i y^{-\psi_i}\right\}, \tag{16}$$

$$\mathcal{F}_{(1)}\left(y|\,\boldsymbol{\xi},\boldsymbol{\psi}\right) = 1-\sum_{l=1}^{n}\binom{n}{l}(-1)^l \exp\left\{-l\sum_{i=1}^{3} \xi_i y^{-\psi_i}\right\}. \tag{17}$$

Let (for $0<\zeta<1$) $\mathbb{Q}_{(m)}(\zeta)$ be qf of $Y_{(m)}$. Eventually from (16) and (17)

$$\mathbb{Q}_{(n)}(\zeta) = \mathbb{Q}\left(\zeta^{1/n}\right), \qquad \mathbb{Q}_{(1)}(y) = \mathbb{Q}\left\{1-[1-\zeta]^{1/n}\right\}, \tag{18}$$

where $\mathbb{Q}(.)$ is the qf of Y. Thus, from (24) and (18), the qfs of $Y_{(n)}$ and $Y_{(1)}$ are not in closed-form. In the case of i.i.d. random values, it is possible to attain an expression for the rth ordinary moment of the order statistics when $\acute{\mu}_r < \infty$. So, as [28], we can represent the rth moment of the mth order statistic as

$$\rho^r_{(m)} = \mathrm{E}\left\{Y^r_{(m)}\right\} = \sum_{j=n-m+1}^{n} \binom{j-1}{n-m}\binom{n}{j} (-1)^{j-n+m-1}\, \mathrm{I}_j(r), \tag{19}$$

where $\mathrm{I}_j(r) = r\int_0^\infty y^{r-1}\left[1-\mathcal{F}(y)\right]^j dy$. For the ATF model, we obtain

Proposition 2. *Let $Y_{(1)} \le Y_{(1)} \ldots \le Y_{(n)}$ be the order statistics of a sample of size n from the distribution ATF model.*

The next outcome shows the rth moment of the $Y_{(m)}$ can be described as

$$\rho^r_{(m)} = \hat{R}_{t,p,q} \frac{r t^{p+q} \xi_2^p \xi_3^q \Gamma\left\{\left(\psi_2 p+\psi_3 q-r\right)/\psi_1\right\}}{\psi_1 p! q! \left\{t\xi_1\right\}^{\left\{(\psi_2 p+\psi_3 q-r)/\psi_1\right\}}}. \tag{20}$$

where $\Bbbk_{j,m,n} = \sum_{j=n-m+1}^{n} (-1)^{j-n+m-1}\binom{j-1}{n-m}\binom{n}{j}$ and $\hat{R}_{t,p,q} = \Bbbk_{j,m,n} \sum_{t=1}^{j}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\binom{t}{j}(-1)^{t+p+q}$.

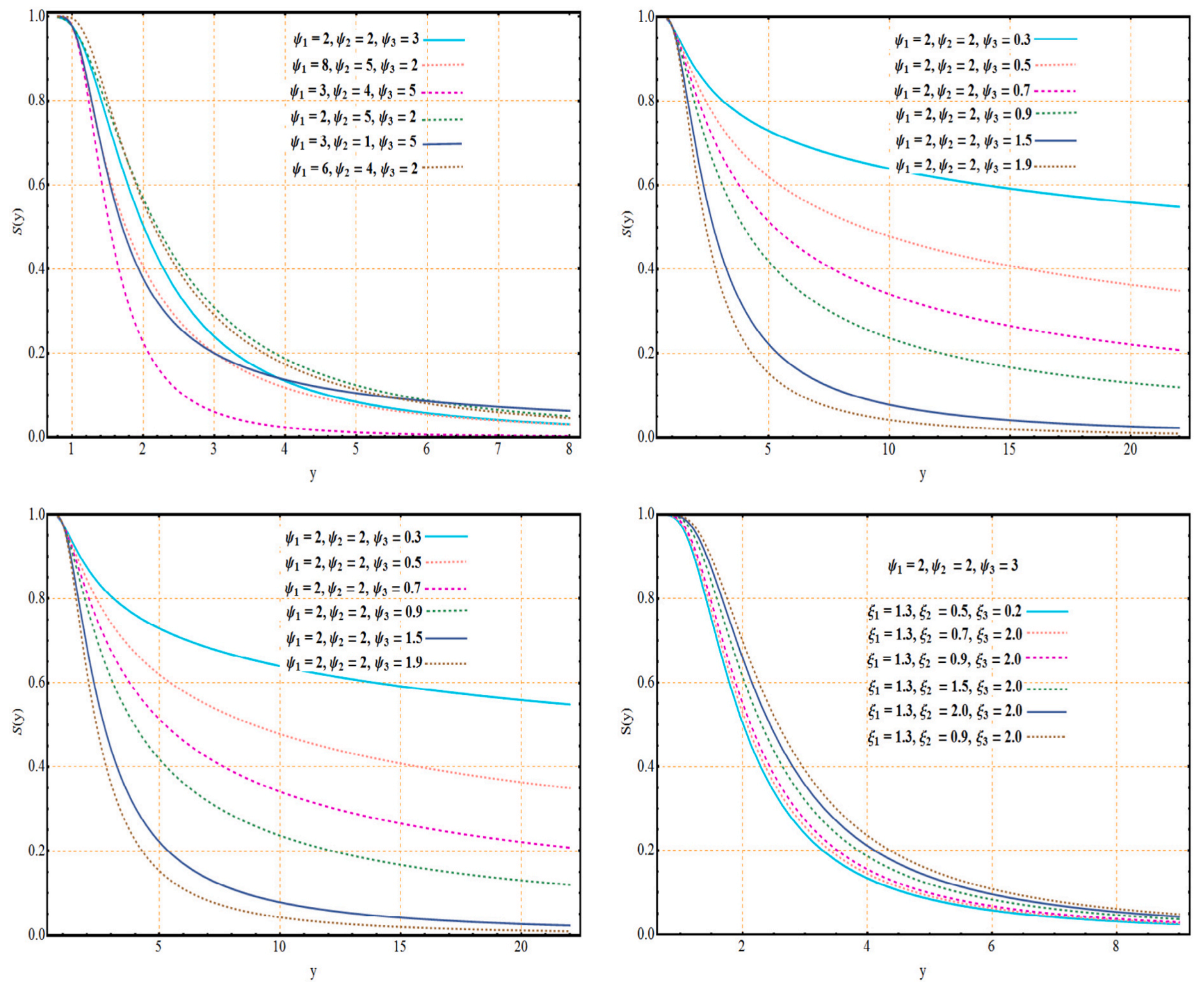

Fig. 3. Survival curves of ATF model at different parametric values.

Proof.

$$\rho^r_{(m)} = \mathrm{E}\left\{Y^r_{(m)}\right\} = \sum_{j=n-m+1}^{n} \binom{j-1}{n-m}\binom{n}{j}(-1)^{j-n+m-1}\,\mathrm{I}_j(r),$$

Now, consider $\mathrm{I}_j(r) = r\int_0^\infty y^{r-1}\left[1-\mathcal{F}(y)\right]^j dy$, where $\mathcal{F}(y)$ is in (1). By incorporating the $\mathcal{F}(y)$, we have

$$\mathrm{I}_j(r) = r\int_0^\infty \sum_{t=0}^{j}\binom{j}{t}(-1)^t y^{r-1}\exp\left\{-t\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)\right\}dy, \tag{21}$$

Letting $z = y^{-\psi_1}$ using the result $dy = \frac{-1}{\psi_1 z^{1+\frac{1}{\psi_1}}}dz$ in (21) and after some algebraic manipulation, we obtain

$$\mathrm{I}_j(r) = r\sum_{t=1}^{j}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\binom{t}{j}(-1)^{t+p+q}\frac{t^{p+q}\xi_2^p\xi_3^q}{\psi_1 p!q!}$$
$$\times\int_0^\infty z^{\left\{(\psi_2 p+\psi_3 q-r)/\psi_1-1\right\}}\exp\left(-t\xi_1 z\right)dz.$$

By incorporating the equalities listed above together, the proof of the proposition is completed. □

Stochastic ordering

For random variables X and Y, we state, $X \preccurlyeq_{lr} Y$, if the ratio of the two respective pdfs a reducing function in x. An key technique for analyzing relative behavior is stochastic ordering of continuous positive random variables. It is supposed that random variable X is smaller than random variable Y in a

(i) stochastic order $X \preccurlyeq_{st} Y$. if $F_X(y) \preccurlyeq F_Y(y)$ for all y; (ii) hazard rate order $X \preccurlyeq_{hr} Y$ if $h_X(y) \succcurlyeq h_Y(y)$ for all y; (iii) likelihood ratio (LR) order $X \preccurlyeq_{lr} Y$, if $\frac{f_X(y)}{f_Y(y)}$ reduces in y. The following implications are well-known, as described in [29] chapter 9:

$$X \preccurlyeq_{lr} Y \Rightarrow X \preccurlyeq_{hr} Y \Rightarrow X \preccurlyeq_{st} Y. \tag{22}$$

The ATF models are ordered with respect to the strongest "LR" ordering as demonstrated by the next theorem.

Theorem 3. *Let $X \sim ATF\left(\xi_1,\xi_2,\xi_3,\psi_1,\psi_2,\psi_3\right)$, and $Y \sim ATF\left(\breve{\xi}_1,\breve{\xi}_2,\breve{\xi}_3,\psi_1,\psi_2,\psi_3\right)$. Case (i) If $\xi_2 = \breve{\xi}_2$, $\xi_3 = \breve{\xi}_3$, and $\xi_1 \le \breve{\xi}_1$, Case (ii) If $\xi_1 = \breve{\xi}_1$, $\xi_3 = \breve{\xi}_3$, and $\xi_2 \le \breve{\xi}_2$, Case (iii) If $\xi_1 = \breve{\xi}_1$, $\xi_2 = \breve{\xi}_2$, and $\xi_3 \le \breve{\xi}_3$, then $X \preccurlyeq_{lr} Y$ $\left(X \preccurlyeq_{hr} Y,\ X \preccurlyeq_{st} Y\right)$ in all three cases exits.*

Proof. : The LR is

$$\frac{f_X(y)}{f_Y(y)} = \frac{e^{y^{-\psi_1}\left(\breve{\xi}_1-\xi_1\right)+y^{-\psi_2}\left(\breve{\xi}_2-\xi_2\right)+y^{-\psi_3}\left(\breve{\xi}_3-\xi_3\right)}}{y^{\gamma_2}\psi_1\breve{\xi}_1+y^{\gamma_3}\psi_2\breve{\xi}_2+y^{\gamma_1}\psi_3\breve{\xi}_3}\left(y^{\gamma_2}\gamma_4+y^{\gamma_3}\gamma_5+y^{\gamma_1}\gamma_6\right), \tag{23}$$

where $\gamma_1 = \psi_1+\psi_2$, $\gamma_2 = \psi_2+\psi_3$, $\gamma_3 = \psi_1+\psi_3$, $\gamma_4 = \psi_1\xi_1$, $\gamma_5 = \psi_2\xi_2$, $\gamma_6 = \psi_3\xi_3$,

Case (i) Thus if $\xi_2 = \breve{\xi}_2$, $\xi_3 = \breve{\xi}_3$, and $\xi_1 \le \breve{\xi}_1$, then

$$\frac{d}{dy}\left[\frac{f_X(y)}{f_Y(y)}\right] = -\frac{e^{y^{-\psi_1}\left(\breve{\xi}_1-\xi_1\right)}y^{-\psi_1-1}\psi_1\left(\breve{\xi}_1-\xi_1\right)}{\left(y^{\gamma_2}\psi_1\breve{\xi}_1+y^{\gamma_3}\gamma_5+y^{\gamma_1}\gamma_6\right)^2}$$
$$\times\left[y^{2\gamma_2}\breve{\xi}_1\gamma_4\psi_1+y^{\psi_1+\gamma_2}\left(\breve{\xi}_1+\xi_1\right)\right.$$

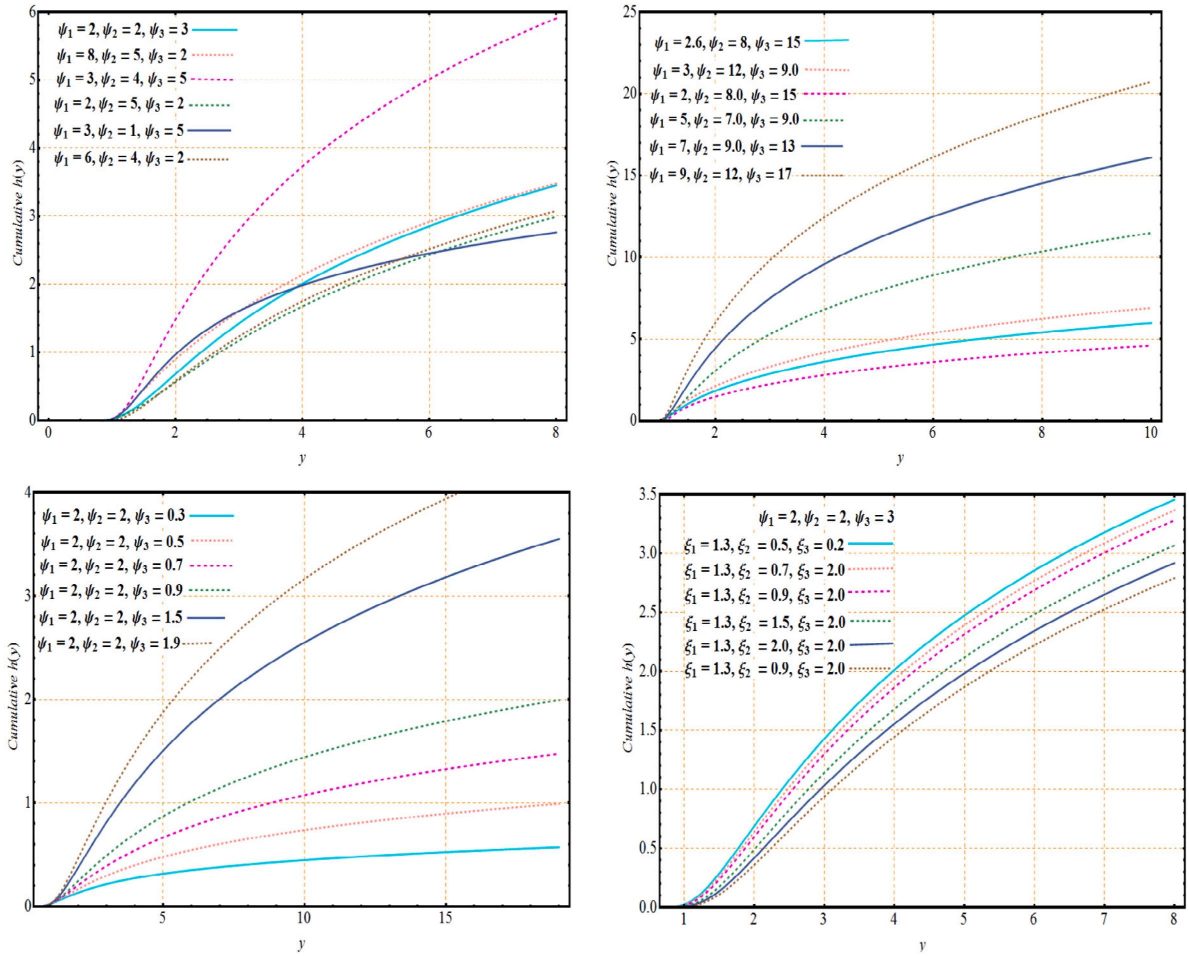

Fig. 4. Plots of cumulative hrf curves of ATF model at different parameter values.

$$\psi_1\left(y^{\psi_3}\gamma_5+y^{\psi_2}\gamma_6\right)+y^{2\psi_1}\left\{y^{2\psi_3}\gamma_5^2\right.$$
$$+y^{\gamma_2}\gamma_5\left(y^{\psi_3}\left(\psi_2-\psi_1\right)+2\gamma_6\right)$$
$$\left.\left.+y^{2\psi_2}\gamma_6\left(\gamma_6+y^{\psi_3}\left(\psi_3-\psi_1\right)\right)\right\}\right]\leq 0.$$

where $\gamma_1=\psi_1+\psi_3$, $\gamma_1=\psi_1+\psi_3$, As a result, it demonstrates that $X\preccurlyeq_{lr} Y$, and in accordance with (22) these both are $X\preccurlyeq_{hr} Y$, $X\preccurlyeq_{st} Y$ also hold.

Case (ii) Thus if $\xi_1=\check{\xi}_1$, $\xi_3=\check{\xi}_3$, and $\xi_2\leq\check{\xi}_2$, then

$$\frac{d}{dy}\left[\frac{f_X(y)}{f_Y(y)}\right]=-\frac{e^{y^{-\psi_2}\left(\check{\xi}_2-\xi_2\right)}y^{-\psi_2-1}\psi_2\left(\check{\xi}_2-\xi_2\right)}{\left(y^{\gamma_2}\gamma_4+y^{\gamma_3}\psi_2\check{\xi}_2+y^{\gamma_1}\gamma_6\right)^2}\left[y^{2\gamma_3}\check{\xi}_2\gamma_5\psi_2+y^{\psi_1+\gamma_2}\left(\check{\xi}_2+\xi_2\right)\right.$$
$$\psi_2\left(y^{\psi_3}\gamma_4+y^{\psi_1}\gamma_6\right)+y^{2\psi_2}\left\{y^{2\psi_3}\gamma_4^2+y^{\gamma_3}\gamma_4\left(y^{\psi_3}\left(\psi_1-\psi_2\right)+2\gamma_6\right)\right.$$
$$\left.\left.+y^{2\psi_1}\gamma_6\left(\gamma_6+y^{\psi_3}\left(\psi_3-\psi_2\right)\right)\right\}\right]\leq 0.$$

As a result, it represents that $X\preccurlyeq_{lr} Y$, and in accordance with (22) these $X\preccurlyeq_{hr} Y$, $X\preccurlyeq_{st} Y$ are also hold.

Case (iii) Thus if $\xi_1=\check{\xi}_1$, $\xi_2=\check{\xi}_2$, and $\xi_3\leq\check{\xi}_3$, then

$$\frac{d}{dy}\left[\frac{f_X(y)}{f_Y(y)}\right]=-\frac{e^{y^{-\psi_3}\left(\check{\xi}_3-\xi_3\right)}y^{-\psi_3-1}\psi_3\left(\check{\xi}_3-\xi_3\right)}{\left(y^{\psi_2+\psi_3}\gamma_4+y^{\psi_1+\psi_3}\gamma_5+y^{\psi_1+\psi_2}\psi_3\check{\xi}_3\right)^2}$$
$$\left[y^{2\gamma_1}\check{\xi}_3\gamma_6\psi_3+y^{\psi_1+\gamma_2}\left(\check{\xi}_3+\xi_3\right)\right.$$
$$\psi_3\left(y^{\psi_2}\gamma_4+y^{\psi_1}\gamma_5\right)+y^{2\psi_3}\left\{y^{2\psi_1}\gamma_5^2+y^{\psi_2+\psi_3}\right.$$
$$\times\gamma_5\left(y^{\psi_1}\left(\psi_2-\psi_3\right)+2\gamma_4\right)$$
$$\left.\left.+y^{2\psi_2}\gamma_4\left(\gamma_4+y^{\psi_1}\left(\psi_1-\psi_3\right)\right)\right\}\right]\leq 0. \tag{24}$$

Hence it shows that $X\preccurlyeq_{lr} Y$, and in accordance with (22) these $X\preccurlyeq_{hr} Y$, $X\preccurlyeq_{st} Y$ are also hold. □

Random number generator

Let ζ be a consideration such that $\zeta\sim U(0,1)$. An observation of Y can be provided as follows a solution of nonlinear equation

$$\left(\xi_1 y^{-\psi_1}+\xi_2 y^{-\psi_2}+\xi_3 y^{-\psi_3}\right)+\log(\zeta)=0. \tag{25}$$

For the determination of Y from (25), computational algorithms like Newton–Raphson techniques can be used. For exceptional cases, however, one may get the solution.

Special cases of quantile function

Case 1: When $\psi_1=0, \psi_2=1, \psi_3=2$

$$y=Q\left(y|\,\xi_i,\psi_i\right)=\frac{-\xi_2\pm\sqrt{\xi_2^2-4\xi_1\xi_3-4\xi_3\log(\zeta)}}{2(\xi_1+\log(\zeta))}. \tag{26}$$

Case 2: When $\psi_1=\psi_2=1, \psi_3=2$

$$y=Q\left(y|\,\xi_i,\psi_i\right)=\frac{-\left(\xi_1+\xi_2\right)\pm\sqrt{\left(\xi_1+\xi_2\right)^2-4\xi_3\log(\zeta)}}{2(\log(\zeta))}. \tag{27}$$

Case 3: When $\psi_1=\psi_2=\psi_3=1$

$$Y=Q\left(y|\,\xi_i,\psi_i\right)=\frac{-\left(\xi_1+\xi_2+\xi_3\right)}{\log(\zeta)}. \tag{28}$$

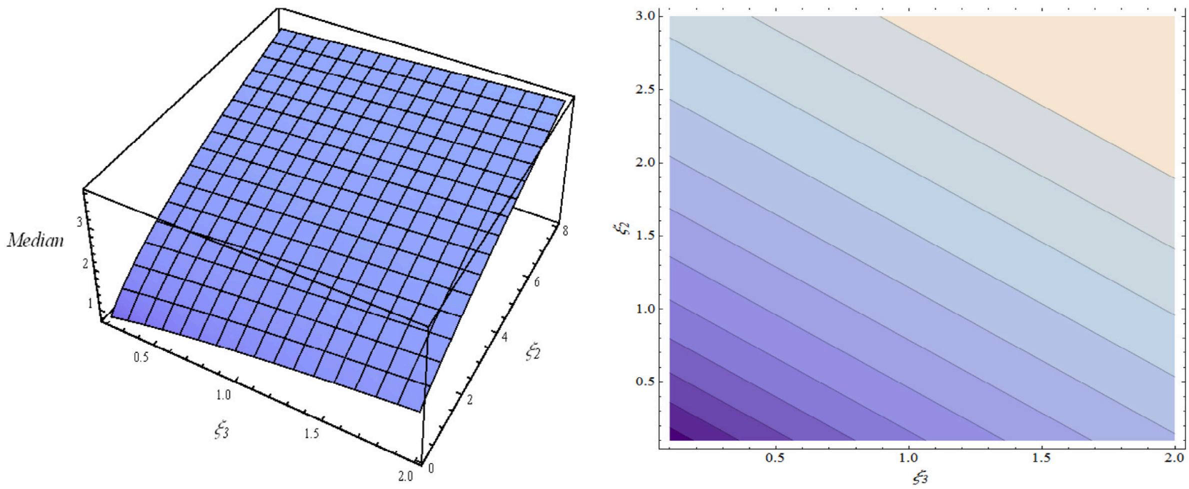

Fig. 5. Plots of median curves of ATF model at various parametric quantities.

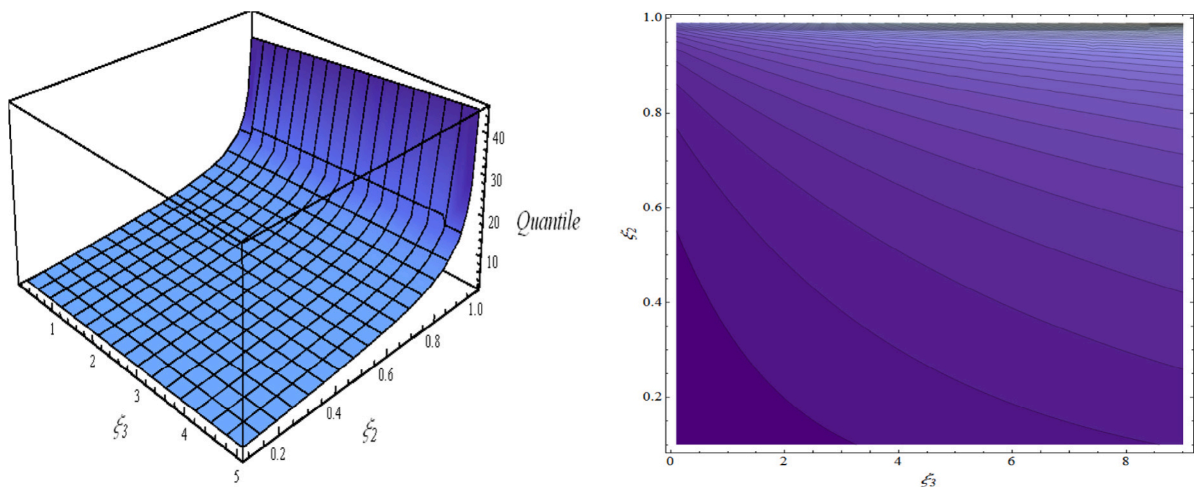

Fig. 6. Plots of Quantile function and contour plot of Quantile function of ATF distribution at different parameter values.

Case 4: When $\psi_1 = \psi_2 = \psi_3 = 2$

$$Y = Q\left(y|\,\xi_i, \psi_i\right) = \pm\sqrt{\frac{\left(-\xi_1 - \xi_2 - \xi_3\right)}{\log\left(\zeta\right)}}. \tag{29}$$

Case 5: When $\psi_1 = \psi_2 = 2$, $\psi_3 = 1$

$$y = Q\left(y|\,\xi_i, \psi_i\right) = -\frac{\xi_3 \pm \sqrt{\xi_3^2 - 4\left(\xi_1 + \xi_2\right)\log\left(\zeta\right)}}{2\log\left(\zeta\right)}. \tag{30}$$

Case 6: When $\psi_1 = \psi_2 = 2$, $\psi_3 = 0$

$$Y = Q\left(y|\,\xi_i, \psi_i\right) = \pm\sqrt{\frac{\left(-\xi_1 - \xi_2\right)}{\xi_3 + \log\left(\zeta\right)}}. \tag{31}$$

By putting $\zeta = (0.25, 0.50, 0.75)$ in (24), first three quartiles Q_1, Q_2, and Q_3 are obtained. Fig. 5 shows the median behavior of AFT model for certain parameters values.

The graphs of median and quantile function of the ATF distribution along with the corresponding contour plots are presented in Figs. 5 and 6, respectively. From Figs. 5 and 6, it is clear that for fixed values of ψ_i, the median is decreased when ξ_i is increased. (ii) For fixed values of ψ_i, the quantile function is increased when ξ_i increased. Meaningful measurements of $S_{skewness}$ and $K_{kurtosis}$ are given by $\phi_3 = \mu_3/\sigma^3$ and $\phi_4 = \mu_4/\sigma^4$, respectively, where μ_κ (κ^{th} moment) and σ is the standard deviation. The measures that are more robust and do occur for distributions without moments are Bowley's Skewness measure S_B and Moors 'Kurtosis measure K_M and are specified by

$$S_{B|\xi,\psi} = \frac{\mathbb{Q}\left(6/8\,|\xi,\boldsymbol{\psi}\right) + \mathbb{Q}\left(2/8|\,\xi,\boldsymbol{\psi}\right) - 2\mathbb{Q}\left(4/8|\,\xi,\boldsymbol{\psi}\right)}{\mathbb{Q}\left(6/8|\,\xi,\boldsymbol{\psi}\right) - \mathbb{Q}\left(2/8|\,\xi,\boldsymbol{\psi}\right)}, \tag{32}$$

$$K_{M|\xi,\psi} = \frac{\mathbb{Q}\left(7/8|\,\xi,\boldsymbol{\psi}\right) - \mathbb{Q}\left(5/8|\,\xi,\boldsymbol{\psi}\right) + \mathbb{Q}\left(3/8|\,\xi,\boldsymbol{\psi}\right) - \mathbb{Q}\left(1/8|\,\xi,\boldsymbol{\psi}\right)}{\mathbb{Q}\left(6/8|\,\xi,\boldsymbol{\psi}\right) - \mathbb{Q}\left(2/8|\,\xi,\boldsymbol{\psi}\right)}, \tag{33}$$

where $\mathbb{Q}(.|.)$ represents qf. If $S_{B|\xi,\psi} < 0$ distribution is left skewed, if $S_{B|\xi,\psi} > 0$, distribution is right skewed, and if $S_{B|\xi,\psi} = 0$, distribution is symmetrical. Instead, a high quality of $K_{M|\xi,\psi}$ signifies a heavy tail for distribution and a low quality of $K_{M|\xi,\psi}$ means a mild tail instead.

Figs. 5 and 6 display the median and quantile function behavior in connection of model of ATF and parametric values. Figs. 7 and 8 assess the pattern graphically of these two measures in the light of ATF model and as per qualities of the parameter.

Reliability in multicomponent stress–strength model

If Y, Y_1, $Y_2, \ldots, Y_k$ be a samples that makes $\mathcal{F}_X(x|.)$ is CDF of common stress, Y, and Y_1, $Y_2, \ldots, Y_k$ are independently and identically distributed (iid) with $\mathcal{F}_Y(y|.)$, subject to X, then, the reliability in the multicomponent stress–strength model is described as follows

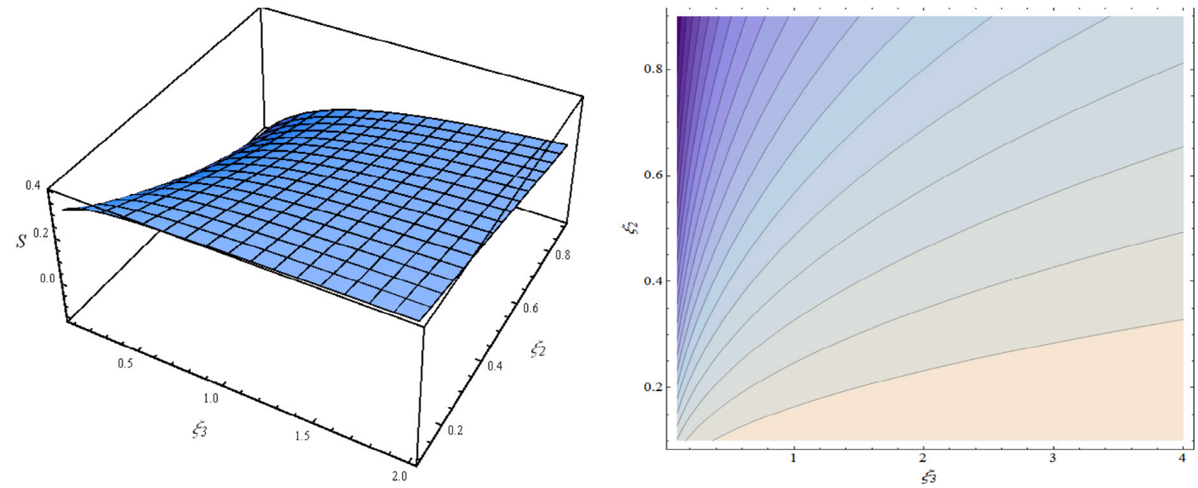

Fig. 7. Plots of Skewness function and contour plot of Skewness function of ATF distribution at different parameter values.

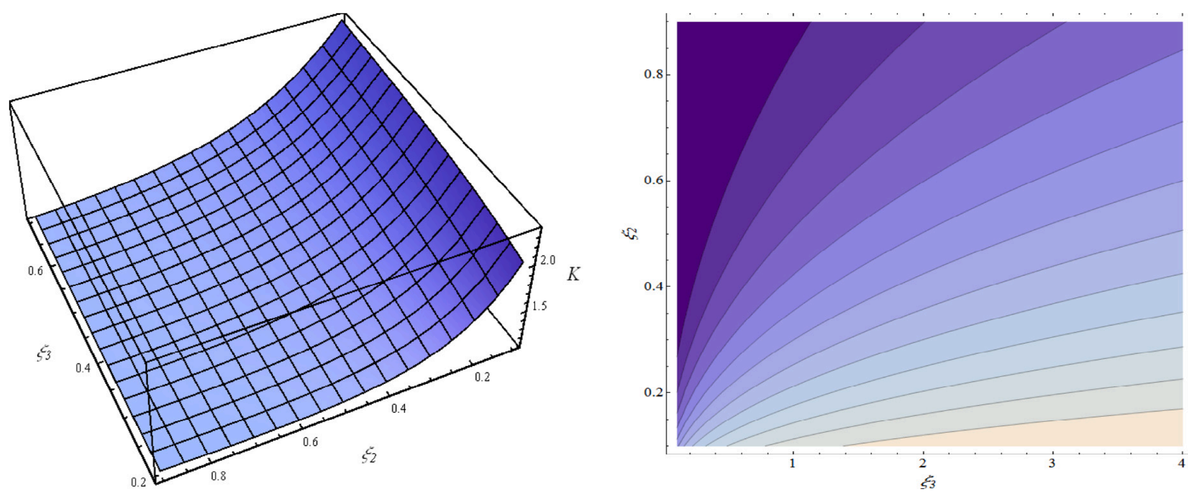

Fig. 8. Plots of Kurtosis function and contour plot of Quantile function of ATF distribution at different parameter values.

(see [30])

$$\mathbb{R}_{s,.k} = \Pr\left(\text{at least }\ \left(Y_1, Y_2, \ldots, Y_k\right) > X\right)$$
$$= \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}\left[1-\mathcal{F}_Y\left(x|.\right)\right]^i\left[\mathcal{F}_Y\left(x|.\right)\right]^{k-i}d\mathcal{F}_X\left(x|.\right).$$

The goal of this section is to figure out how to describe reliability in the multi-component stress-power model using two random variables: strength (Y) and stress (X), where $Y \sim$ ATF $(\xi_1, \xi_2, \xi_3, \psi_1, \psi_2, \psi_3)$ and $X \sim$ ATF $(\bar{\xi}_1, \xi_2, \xi_3, \psi_1, \psi_2, \psi_3)$. If at least s out of k components are active at the same time, the system will function; otherwise, it will fail. Eqs. (1), (2), and (34), respectively, can be used to establish the reliability of an AFT distribution in a multicomponent stress–strength model:

$$\mathbb{R}_{s,.k} = \sum_{i=s}^{k}\sum_{m=0}^{i}\binom{k}{i}\binom{i}{m}(-1)^m\int_0^{\infty}e^{-(m+k-i)\xi_1 x^{-\psi_1}}e^{(-(m+k-i+1))\left(\xi_2 x^{-\psi_2}+\xi_3 x^{-\psi_3}\right)}$$
$$e^{-\bar{\xi}_1 x^{-\psi_1}}\left\{\left(\psi_1\xi_1 x^{-\psi_1-1}+\psi_2\xi_2 x^{-\psi_2-1}+\psi_3\xi_3 x^{-\psi_3-1}\right)\right\}\Big]\,dx. \tag{34}$$

After making the transformation $z = x^{-\psi_1}$ using the result $dx = \frac{-1}{\psi_1 z^{1+\frac{1}{\psi_1}}}dz$ in (35) and after some algebraic manipulation, we obtain

$$\mathbb{R}_{s,\ k} = Y_{p,q,i,m}\int_0^{\infty} z^{\frac{p\psi_2+q\psi_3}{\psi_1}}e^{-z\left[(m+k-i)\xi_1+\bar{\xi}_1\right]} + \frac{\xi_2\psi_2}{\psi_1}z^{\frac{(p+1)\psi_2+q\psi_3}{\psi_1}-1}e^{-z\left[(m+k-i)\xi_1+\bar{\xi}_1\right]}$$
$$+\frac{\xi_3\psi_3}{\psi_1}\ z^{\frac{p\psi_2+(q+1)\psi_3}{\psi_1}-1}e^{-z\left[(m+k-i)\xi_1+\bar{\xi}_1\right]}\Bigg]\,dz,$$

where $Y_{p,q,i,m} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{i=s}^{k}\sum_{m=0}^{i}\binom{k}{i}\binom{i}{m}(-1)^{m+p+q}\frac{\xi_2^p\xi_3^q}{p!q!}(m+k-i+1)^{p+q}$.

Now

$$\mathbb{R}_{s,.k} = Y_{p,q,i,m}\left[\frac{\bar{\xi}_1\Gamma(\varsigma_{p,q}+1)}{\left(\vartheta_{\xi_1,\bar{\xi}_1}\right)^{(\varsigma_{p,q}+1)}} + \frac{\xi_2\psi_2}{\psi_1}\frac{\bar{\xi}_1\Gamma(\varsigma_{p,q}+\frac{\psi_2}{\psi_1})}{\left(\vartheta_{\xi_1,\bar{\xi}_1}\right)^{(\varsigma_{p,q}+\frac{\psi_2}{\psi_1})}} + \frac{\xi_3\psi_3}{\psi_1}\frac{\bar{\xi}_1\Gamma(\varsigma_{p,q}+\frac{\psi_3}{\psi_1})}{\left(\vartheta_{\xi_1,\bar{\xi}_1}\right)^{(\varsigma_{p,q}+\frac{\psi_3}{\psi_1})}}\right], \tag{35}$$

where $\varsigma_{p,q} = \frac{p\psi_2+q\psi_3}{\psi_1}, \vartheta_{\xi_1,\bar{\xi}_1} = \left[(m+k-i)\,\xi_1+\bar{\xi}_1\right]$.

If $s = k = 1$, and $\xi_2 = \xi_3 = 0$, then the model of stress strength is reduced to the equation:

$$\mathbb{R}_{1,1} = \sum_{m=0}^{1}\binom{i}{m}(-1)^m\frac{\bar{\xi}_1}{\vartheta_{\xi_1,\bar{\xi}_1}},$$

Finally,

$$\mathbb{R}_{1,1} = 1-\frac{\bar{\xi}_1}{\xi_1+\bar{\xi}_1} = \frac{\xi_1}{\xi_1+\bar{\xi}_1}, \tag{36}$$

Notice that we consider the well known value in the identically distributed case where $\xi_1 = \bar{\xi}_1$, is $\mathbb{R}_{1,1} = 0.5$, that stress and strength are same in magnitude.

Non-central moments and generating function

For the ATF model, we get rth non-central moment ρ_r. If Y has pdf (2), it is as follows.

$$\rho^r(y|\xi,\psi) = \int_0^\infty y^r dF(y|\xi,\psi); \; r = 1,2,\ldots, \tag{37}$$

$$\rho^r(y|\xi,\psi) = \int_0^\infty y^r \sum_{i=1}^{3} \psi_i \xi_i y^{-(\psi_i+1)} \exp\left(-\sum_{i=1}^{3} \xi_i y^{-\psi_i}\right) dy, \; i = 1,2,3. \tag{38}$$

The rth moment ρ_r of Y is expressed in terms of the gamma function in the following result.

Proposition 4. *For $\psi_i, \xi_i > 0$, the rth moment of Y is*

$$\rho^r(y|\xi,\psi) = \Theta_{j,k}^{\phi_{j,k}} \left\{ \Gamma\{\phi_{j,k}+1\} + \frac{\psi_2\xi_2}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi_2}{\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_2}{\psi_1}}} + \frac{\psi_3\xi_3}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi 3}{\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_3}{\psi_1}}} \right\}, \tag{39}$$

where $\phi_{j,k} = \frac{j\psi_2+k\psi_3-r}{\psi_1}$, *and* $\Theta_{j,k}^{\phi_{j,k}} = \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^{j+k} \frac{\xi_2^j \xi_3^k}{j!k!\{\xi_1\}^{\phi_{j,k}}}$.

Proof. Allowing $z = y^{-\psi_1}$ using the result $dy = \frac{-1}{\psi_1 z^{1+\frac{1}{\psi_1}}} dz$ and we have

$$\rho^r(y|\xi,\psi) = \int_0^\infty z^{\frac{-r}{\psi_1}} \left[\xi_1 + \frac{\xi_2\psi_2}{\psi_1} z^{\left(\frac{\psi_2}{\psi_1}-1\right)} + \frac{\xi_3\psi_3}{\psi_1} z^{\left(\frac{\psi_3}{\psi_1}-1\right)}\right] \exp\left\{-\left(\xi_1 z + \frac{\xi_2\psi_2}{\psi_1} z^{\left(\frac{\psi_2}{\psi_1}-1\right)} + \frac{\xi_3\psi_3}{\psi_1} z^{\left(\frac{\psi_3}{\psi_1}-1\right)}\right)\right\} dz,$$

and after some algebraic manipulation We get,

$$\rho^r(y|\xi,\psi) = \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^{j+k} \frac{\xi_2^j \xi_3^k}{j!k!} \left[\int_0^\infty \xi_1 z^{\left(\frac{j\psi_2+k\psi_3-r}{\psi_1}\right)} e^{-\xi_1 z} dz + \frac{\xi_2\psi_2}{\psi_1} \int_0^\infty z^{\left(\frac{(j+1)\psi_2+k\psi_3-r}{\psi_1}-1\right)} e^{-\xi_1 z} dz + \frac{\xi_3\psi_3}{\psi_1} \int_0^\infty z^{\left(\frac{(k+1)\psi_3+j\psi_2-r}{\psi_1}-1\right)} e^{-\xi_1 z} dz\right].$$

Finally, we obtain the above result. That fills out the proof. □

The MG function of ATF model may be indicated as

$$M(y|\xi,\psi) = \sum_{\varsigma=0}^\infty \sum_{n=r}^\infty \frac{t^\varsigma}{\varsigma!} \rho^r(y|\xi,\psi), \tag{40}$$

where ρ^r given in (39).

Characteristic function (CF)

The CFof Y is

$$\Phi(\tau y|\xi,\psi) = \int_0^\infty e^{i\tau y} dF(y|\xi,\psi). \tag{41}$$

After using exponential series, we have

$$\Phi(\tau y|\xi,\psi) = \sum_{\upsilon=0}^\infty \frac{(i\tau)^\upsilon}{\upsilon!} \int_0^\infty y^\upsilon dF(y|\xi,\psi).$$

Hence, we obtain

$$\Phi(\tau y|\xi,\psi) = \sum_{\upsilon=0}^\infty \frac{(i\tau)^\upsilon}{\upsilon!} \Theta_{j,k}^{\phi_{j,k}} \left\{ \Gamma\{\phi_{j,k}+1\} + \frac{\psi_2\xi_2}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi_2}{\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_2}{\psi_1}}} + \frac{\psi_3\xi_3}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi 3}{\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_3}{\psi_1}}} \right\}, \tag{42}$$

where $\phi_{j,k} = \frac{j\psi_2+k\psi_3-\upsilon}{\psi_1}$, and $\Theta_{j,k}^{\phi_{j,k}} = \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^{j+k} \frac{\xi_2^j \xi_3^k}{j!k!\{\xi_1\}^{\phi_{j,k}}}$. That concludes the objective evidence. □

Factorial generating function (FGF)

The FGF of ATF model is

$$F_y(\tau y|\xi,\psi) = \int_0^\infty e^{\log(1+\tau)^y} dF(y|\xi,\psi) = \sum_{\upsilon=0}^\infty \frac{(\log(1+\tau))^\upsilon}{\upsilon!} \int_0^\infty y^\upsilon dF(y|\xi,\psi). \tag{43}$$

So we can compose the integral in (43) as

$$F_y(\tau y|\xi,\psi) = \sum_{\upsilon=0}^\infty \frac{(\log(1+\tau))^\upsilon}{\upsilon!} \Theta_{j,k}^{\phi_{j,k}} \left\{ \Gamma\{\phi_{j,k}+1\} + \frac{\psi_2\xi_2}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi_2}{\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_2}{\psi_1}}} + \frac{\psi_3\xi_3}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi 3}{\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_3}{\psi_1}}} \right\}, \tag{44}$$

where $\phi_{j,k} = \frac{j\psi_2+k\psi_3-\upsilon}{\psi_1}$, and $\Theta_{j,k}^{\phi_{j,k}} = \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^{j+k} \frac{\xi_2^j \xi_3^k}{j!k!\{\xi_1\}^{\phi_{j,k}}}$.

Incomplete non-central moments

First, some notation are initiated. The upper incomplete gamma function symbolized by $\Gamma(\varrho,x)$ is defined as

$$\Gamma(\varrho,x) = \int_x^\infty u^{\varrho-1} e^{-u} du, \; x > 0, \; \varrho \in \mathbb{R}. \tag{45}$$

Furthermore the exponential integral function can be specified regarding the upper incomplete gamma function as follows (cf. Olver et al. [31])

$$E_\hbar(x) = \int_1^\infty t^{-\hbar} e^{-tx} dt, \; x > 0, \; \hbar \in \mathbb{R}, \tag{46}$$

$$E_\hbar(x) = x^{\hbar-1} \Gamma(1-\hbar, x), \; \hbar, \; x \in \mathbb{R}. \tag{47}$$

Proposition 5. *The rth incomplete moment $\rho_{Y,r}(z)$ of Y is*

$$\rho_{Y,r}(z) = \Theta_{j,k}^{\phi_{j,k}} \left\{ \Gamma\{1-\phi_{j,k}, \xi_1 z^{-\psi_1}\} + \frac{\psi_2\xi_2}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi_2}{\psi_1}, \xi_1 z^{-\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_2}{\psi_1}}} + \frac{\psi_3\xi_3}{\psi_1} \frac{\Gamma\left\{\phi_{j,k}+\frac{\psi_3}{\psi_1}, \xi_1 z^{-\psi_1}\right\}}{\{\xi_1\}^{\frac{\psi_3}{\psi_1}}} \right\}, \tag{48}$$

where $\phi_{j,k} = \frac{j\psi_2+k\psi_3-r}{\psi_1}$, *and* $\Theta_{j,k}^{\phi_{j,k}(\psi_1,\psi_2,\psi_3)} = \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^{j+k} \frac{\xi_2^j \xi_3^k}{j!k!\{\xi_1\}^{\phi_{j,k}}}$.

Further rth incomplete moment $\rho_{Y,r}(z)$ in terms of the exponential integral function

$$\rho_{Y,r}(z) = \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^{j+k} \frac{\xi_2^j \xi_3^k}{j!k!} \left[\frac{E_{-\phi_{j,k}}(\xi_1 z^{-\psi_1})}{\psi_1(\phi_{j,k}+1)} + \frac{E_{1-\frac{\psi_2}{\psi_1}-\phi_{j,k}}(\xi_1 z^{-\psi_1})}{\psi_1\phi_{j,k}+\psi_2} + \frac{E_{1-\frac{\psi_3}{\psi_1}-\phi_{j,k}}(\xi_1 z^{-\psi_1})}{\psi_1\phi_{j,k}+\psi_3}\right], \tag{49}$$

where $E_\hbar(x) = \int_1^\infty e^{-tx} t^{-\hbar} dt$.

Table 1
First four non central moments, variance and coefficient of variation for a set of values of (ξ, ψ).

$(\xi_1,\ \xi_2,\ \xi_3,\ \psi_1,\ \psi_2,\ \psi_3)$	$\mu_1'\|_{\xi,\psi}$	$\mu_2'\|_{\xi,\psi}$	$\mu_3'\|_{\xi,\psi}$	$\mu_4'\|_{\xi,\psi}$	$\sigma^2\|_{\xi,\psi}$	$CV\|_{\xi,\psi}$
(0.8, 0.5, 0.8, 2, 1.5, 2.0)	0.372822	0.422800	0.495134	0.826426	0.283804	1.42892
(1.2, 0.5, 0.8, 2, 1.5, 2.0)	0.301762	0.322743	0.42798	0.814547	0.231683	1.59508
(1.5, 0.5, 0.8, 2, 1.5, 2.0)	0.23521	0.274132	0.395014	0.812991	0.218808	1.98873
(1.2, 0.2, 0.8, 2, 1.5, 2.0)	0.129165	0.132892	0.168136	0.298518	0.116209	2.63922
(1.2, 0.5, 0.8, 2, 1.5, 2.0)	0.301762	0.322743	0.427980	0.814547	0.231683	1.59508
(1.2, 0.9, 0.8, 2, 1.5, 2.0)	0.504935	0.563847	0.788414	1.621110	0.308887	1.10069
(1.2, 0.9, 0.3, 2, 1.5, 2.0)	0.202191	0.213808	0.283219	0.555504	0.172926	2.05668
(1.2, 0.9, 0.6, 2, 1.5, 2.0)	0.387537	0.424369	0.581751	1.175200	0.274183	1.35116
(1.2, 0.9, 0.9, 2, 1.5, 2.0)	0.562228	0.633435	0.893865	1.853340	0.317334	1.00195
(1.2, 0.9, 0.9, 3, 1.5, 2.0)	0.538163	0.650538	0.929483	1.808390	0.360919	1.11633
(1.2, 0.9, 0.9, 4, 1.5, 2.0)	0.543832	0.670881	0.953937	1.802200	0.375128	1.12622
(1.2, 0.9, 0.9, 5, 1.5, 2.0)	0.554626	0.688070	0.971390	1.803710	0.380459	1.11213
(1.2, 0.9, 0.9, 3, 1.1, 2.0)	0.544835	0.701090	1.149250	5.023560	0.404245	1.16696
(1.2, 0.9, 0.9, 3, 1.5, 2.0)	0.538163	0.650538	0.929483	1.808390	0.360919	1.11633
(1.2, 0.9, 0.9, 3, 2.0, 2.0)	0.535175	0.608508	0.783882	1.216200	0.322096	1.06047
(1.2, 0.9, 0.9, 3, 2.1, 1.5)	0.536150	0.638943	0.891106	1.642310	0.351486	1.10578
(1.2, 0.9, 0.9, 3, 2.1, 2.0)	0.535454	0.602367	0.763325	1.150490	0.315656	1.04926
(1.2, 0.9, 0.9, 3, 2.1, 2.5)	0.541718	0.581433	0.687562	0.926152	0.287974	0.99061

Proof. By definition

$$\rho_{Y,\ r}(z) = \int_0^z y^r F(y|\,\boldsymbol{\xi},\boldsymbol{\psi}),\ \ r = 1,2,\ldots, \tag{50}$$

we have

$$\begin{aligned}\rho_{Y,\ r}(z) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\frac{\xi_2^j\xi_3^k}{j!k!}\Bigg[\int_{\xi_1 z^{-\psi_1}}^{\infty}\frac{1}{\xi_1^{\phi_{j,k}}}t^{\phi_{j,k}}e^{-t}dt \\ +\frac{\xi_2\psi_2}{\psi_1}\int_0^{\infty}\frac{1}{\xi_1^{\phi_{j,k}+\frac{\psi_2}{\psi_1}}}t^{\frac{\psi_2}{\psi_1}+\phi_{j,k}-+1}e^{-t}dt \\ +\frac{\xi_3\psi_3}{\psi_1}\int_0^{\infty}\frac{1}{\xi_1^{\phi_{j,k}+\frac{\psi_3}{\psi_1}}}t^{\frac{\psi_3}{\psi_1}+\phi_{j,k}-+1}e^{-t}dt\Bigg].\end{aligned} \tag{51}$$

After using (45) and (47), we have final results. Which acquire the required outcome. □

As a numerical illustration, Table 1 gives values $\mu_1'\big|_{\xi,\psi}$, $\mu_2'\big|_{\xi,\psi}$, $\mu_3'\big|_{\xi,\psi}$, $\mu_4'\big|_{\xi,\psi}$, $\sigma^2\big|_{\xi,\psi}$ and $CV|_{\xi,\psi}$ of ATF model. We observe that there are significant impact of ξ_i and ψ_i on first four non central moments, variance and coefficient of variation. For the fixed levels of ξ_i and ψ_i; it can be observed that, mixed behavior of $\mu_1'\big|_{\xi,\psi}$, $\mu_2'\big|_{\xi,\psi}$, $\mu_3'\big|_{\xi,\psi}$, $\mu_4'\big|_{\xi,\psi}$, $\sigma^2\big|_{\xi,\psi}$ and $CV|_{\xi,\psi}$ with the few exceptions are observed. The ATF distribution is simply demonstrated to be over-dispersed when $\sigma|_{\xi,\psi} > \mu$, equi-dispersed $\sigma|_{\xi,\psi} = \mu$, as well as under-dispersed $\sigma|_{\xi,\psi} < \mu$. We still have $\sigma|_{\xi,\psi} > \mu$, so the distribution is over-dispersed with small variations for $\mu_j'\big|_{\xi,\psi}$, $j = 1,2,3,4$.

Conditional moments and mean deviations

The rth conditional moment of Y is

$$\begin{aligned}E(Y^r|\,Y > t) &= \frac{1}{S(t)}\left[E(Y^r) - \int_0^t y^r f(y)\,dy\right], \\ &= \frac{\rho^r(y|\,\boldsymbol{\xi},\boldsymbol{\psi})|_{r=1} - \rho_{Y,\ r}(t)}{1-\exp\left\{-\xi_1 t^{-\psi_1} + \xi_2 t^{-\psi_2} + \xi_3 t^{-\psi_3}\right\}}.\end{aligned} \tag{52}$$

The mean deviations include valuable knowledge on a population's features and can be estimated from first incomplete moment. Moreover, the amount of dispersion in a data may be measured to certain degree by all deviations from the mean and median. The mean deviations of Y about the mean $\rho^1 = E(Y)$ and about the median M_{ed} can be stated as $\boldsymbol{\Phi} = 2F\left(\rho^1\right) - 2\lambda_1\rho^1$ and $\boldsymbol{\Psi} = \rho^1 - 2\lambda_1 M_{ed}$, where $\lambda_1(\tau) = \int_0^{\tau} y f(y)\,dy$ and $F\left(\rho^1\right)$ is specified in (1).

Uncertainty measures

Information generating function, Shannon entropy, Renyi entropy and other entropies for the distribution of ATF are being investigated in this section.

Information generating function

For the ATF model the information generating function for Y is estimated as:

$$\tilde{\mathrm{I}}(f) = E\left[f^{\eta-1}(y|\,\boldsymbol{\xi},\boldsymbol{\psi})\right] = \int_0^{\infty} f^{\eta}(y|\,\boldsymbol{\xi},\boldsymbol{\psi})\,dy, \tag{53}$$

$$\tilde{\mathrm{I}}(f) = \int_0^{\infty}\left(\sum_{i=1}^{3}\psi_i\xi_i y^{-(\psi_i+1)}\exp\left(-\sum_{i=1}^{3}\xi_i y^{-\psi_i}\right)\right)^{\eta}dy, \tag{54}$$

Now making the transformation $z = y^{-\psi_1}$ using the result $dy = \frac{-1}{\psi_1 z^{1+\frac{1}{\psi_1}}}dz$ in Eq. (54) and after a little simplification we got

$$\tilde{I}(f) = \frac{\Theta_{r,r_1}^{\phi_{r,r_1}}}{\psi_1}\frac{\Gamma\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}{\left\{\delta\xi_1\right\}^{\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}}, \tag{55}$$

where $\phi_{r,r_1} = \sum_{r=0}^{\infty}\sum_{r_1=0}^{\infty}\binom{\eta}{r}\binom{r}{r_1}\left(\xi_1\psi_1\right)^{\eta-r}\left(\psi_2\xi_2\right)^{r-r_1}\left(\psi_3\xi_3\right)^{r_1}$, $\Theta_{r,r_1}^{\phi_{r,r_1}} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\xi_2^p\xi_3^q}{p!q!}\phi_{r,r_1}$, and $\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3) = \left\{\frac{(\eta+r)\psi_1+(r-r_1+p)\psi_2+(r+q)\psi_3-1}{\psi_1}\right\}$.

Entropy measures

Entropy is an useful concept in various fields such as communications, statistical mechanics, information theory, thermodynamics, topological dynamics, measure-preserving dynamical systems, and so on. There are various definitions of entropy, and none of them are ideal for all purposes.

Renyi entropy

The Renyi entropy $\tilde{\mathrm{I}}_{\delta}(Y)$ for Y with ATF is

$$\tilde{I}_{\delta}(Y) = \frac{1}{1-\delta}\log\int_0^{\infty} f^{\delta}(y|\,\boldsymbol{\xi},\boldsymbol{\psi})\,dy, \delta \neq 1, \delta > 0. \tag{56}$$

Now

$$\begin{aligned}f^{\delta}(y|\,\boldsymbol{\xi},\boldsymbol{\psi}) = \left\{\xi_1\psi_1 y^{-(\psi_1+1)} + \xi_2\psi_2 y^{-(\psi_2+1)} + \xi_3\psi_3 y^{-(\psi_3+1)}\right\}^{\delta} \\ \exp\left\{-\delta\left(\xi_1 y^{-\psi_1} + \xi_2 y^{-\psi_2} + \xi_3 y^{-\psi_3}\right)\right\}.\end{aligned} \tag{57}$$

By putting the above transformation, we have

$$\int_0^\infty f^\delta(y|\boldsymbol{\xi},\boldsymbol{\psi})\,dy=\frac{\Theta_{r,r_1}^{\phi_{r,r_1}}}{\psi_1}\int_0^\infty t^{\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)-1}e^{-\delta\xi_1 t}dt,$$

$$\tilde{I}_\delta(Y)=\frac{1}{1-\delta}\log\left\{\frac{\Theta_{r,r_1}^{\phi_{r,r_1}}\,\Gamma\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}{\psi_1\{\delta\xi_1\}^{\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}}\right\}, \tag{58}$$

where $\phi_{r,r_1}=\sum_{r=0}^\infty\sum_{r_1=0}^\infty\binom{\delta}{r}\binom{r}{r_1}(\xi_1\psi_1)^{\delta-r}(\psi_2\xi_2)^{r-r_1}(\psi_3\xi_3)^{r_1}$, $\Theta_{r,r_1}^{\phi_{r,r_1}}=\sum_{p=0}^\infty\sum_{q=0}^\infty\frac{\xi_2^p\xi_3^q}{p!q!}\phi_{r,r_1}$, and $\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)=\frac{[(\delta+r)\psi_1+(r-r_1+p)\psi_2+(r+q)\psi_3+\delta-1]}{\psi_1}$. Finally we have

$$\tilde{I}_\delta(Y)=\frac{1}{1-\delta}\Big[\log\left\{\Theta_{r,r_1}^{\phi_{r,r_1}}\right\}+\log\Gamma\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)-\log\{\psi_1\}$$
$$-\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)\log\{\delta\xi_1\}\Big]. \tag{59}$$

The Shannon entropy is the special case of Renyi entropy and is characterized by $S_\delta(Y)=E\{-\ln[f(Y)]\}$. It can be obtained by the formula $S_\delta(Y)=\lim_{\delta\to1^+}\tilde{I}_\delta(Y)$.

Tsallis entropy

Tsallis entropy of Y is defined by

$$T_\delta(Y)=\frac{1}{\delta-1}\left(1-\int_0^\infty f^\delta(y|\boldsymbol{\xi},\boldsymbol{\psi})\,dy\right),\delta\neq1. \tag{60}$$

Using the above results we have

$$T_\delta(Y)=\frac{1}{\delta-1}\left(1-\left\{\frac{\Theta_{r,r_1}^{\phi_{r,r_1}}\,\Gamma\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}{\psi_1\{\delta\xi_1\}^{\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}}\right\}\right). \tag{61}$$

Mathai–Houbold entropy

Classical Shannon entropy has been expanded in various ways one of them is δ generalized entropy developed by [32] and is

$$\tilde{I}_{MH}(Y)=\frac{1}{\delta-1}\left(\int_0^\infty f^{2-\delta}(y|\boldsymbol{\xi},\boldsymbol{\psi})\,dy-1\right),\delta\neq1. \tag{62}$$

Similar arguments to (f^δ) gives

$$f^{2-\delta}(y|\boldsymbol{\xi},\boldsymbol{\psi})=\left\{\xi_1\psi_1y^{-(\psi_1+1)}+\xi_2\psi_2y^{-(\psi_2+1)}+\xi_3\psi_3y^{-(\psi_3+1)}\right\}^{2-\delta}$$
$$\exp\left\{-(2-\delta)\left(\xi_1y^{-\psi_1}+\xi_2y^{-\psi_2}+\xi_3y^{-\psi_3}\right)\right\}. \tag{63}$$

Therefore,

$$\int_0^\infty f^{2-\delta}(y|\boldsymbol{\xi},\boldsymbol{\psi})\,dy=\frac{\hat{\Theta}_{r,r_1}^{\phi_{r,r_1}}}{\psi_1}\int_0^\infty t^{\hat{\vartheta}_{r,r_1}(\psi_1,\psi_2,\psi_3)-1}e^{-(2-\delta)\xi_1 t}dt.$$

The final form is

$$\tilde{I}_{MH}(Y)=\frac{1}{1-\delta}\left\{\frac{\hat{\Theta}_{r,r_1}^{\phi_{r,r_1}}\,\Gamma\vartheta_{r,r_1}(\psi_1,\psi_2,\psi_3)}{\psi_1\{\delta\xi_1\}^{\hat{\vartheta}_{r,r_1}(\psi_1,\psi_2,\psi_3)}}-1\right\}, \tag{64}$$

where $\hat{\phi}_{r,r_1}=\sum_{r=0}^\infty\sum_{r_1=0}^\infty\binom{2-\delta}{r}\binom{r}{r_1}(\xi_1\psi_1)^{-(2-\delta)-r}(\psi_2\xi_2)^{r-r_1}(\psi_3\xi_3)^{r_1}$, $\hat{\Theta}_{r,r_1}^{\phi_{r,r_1}}=\sum_{p=0}^\infty\sum_{q=0}^\infty\frac{\xi_2^p\xi_3^q}{p!q!}\hat{\phi}_{r,r_1}$, and $\hat{\vartheta}_{r,r_1}(\psi_1,\psi_2,\psi_3)=\frac{[(2-\delta+r)\psi_1+(r-r_1+p)\psi_2+(r+q)\psi_3-\delta+1]}{\psi_1}$.

Residual life function with a certain measure of reliability

Random variables of residual life and inverted residual life are widely practiced in risk investigation. Hence, in connection with the ATF distribution, we explore some associated statistical features, like variance, survival function and mean. The residual life is explained by $R(t)=Y-t|Y>t,\ t\geq0$, and described as the period between the moment t and moment of failure. The reversed residual life (or time since failure) can also be described as $\ddot{R}(t)=t-Y|Y\leq t$, This refers to the moment elapsed due to the component's failure, given that its lifetime $\leq t$ (cf. Suchismita and Nanda [33], Tang et al. [34], and Siddiqui and Çağ lar [35]).

Characteristic of residual lifetime function

The survival function of $R(t)$ (for $t\geq0$ and $y>0$) for the ATF model is

$$S_{R(t)}(y)=\frac{S(y+t)}{S(t)}=\frac{1-\exp\left\{-\left(\xi_1(y+t)^{-\psi_1}+\xi_2(y+t)^{-\psi_2}+\xi_3(y+t)^{-\psi_3}\right)\right\}}{1-\exp\left\{-\left(\xi_1t^{-\psi_1}+\xi_2t^{-\psi_2}+\xi_3t^{-\psi_3}\right)\right\}}. \tag{65}$$

The density function of $R(t)$ then simplifies to

$$f_{R(t)}(y|\boldsymbol{\xi},\boldsymbol{\psi})=\frac{\left\{\xi_1\psi_1(y+t)^{-(\psi_1+1)}+\xi_2\psi_2(y+t)^{-(\psi_2+1)}+\xi_3\psi_3(y+t)^{-(\psi_3+1)}\right\}}{\exp\left\{\xi_1t^{-\psi_1}+\xi_2t^{-\psi_2}+\xi_3t^{-\psi_3}\right\}-1}. \tag{66}$$

The hrf of $R(t)$ is

$$h_{R(t)}(y|\boldsymbol{\xi},\boldsymbol{\psi})=\frac{\left\{\xi_1\psi_1(y+t)^{-(\psi_1+1)}+\xi_2\psi_2(y+t)^{-(\psi_2+1)}+\xi_3\psi_3(y+t)^{-(\psi_3+1)}\right\}}{\exp\left\{\xi_1t^{-\psi_1}+\xi_2t^{-\psi_2}+\xi_3t^{-\psi_3}\right\}}$$
$$\times\left\{\frac{1}{\left(1-\exp\left\{-\left(\xi_1(y+t)^{-\psi_1}+\xi_2(y+t)^{-\psi_2}+\xi_3(y+t)^{-\psi_3}\right)\right\}\right)}\right\}. \tag{67}$$

The average residual life function (M_{RL}) has many applications, like in insurance, maintenance and quality control of products, economics and social studies. For ATF distribution, we can represent its mean residual life as

$$\Lambda(t)=E\{R(t)\}=\frac{1}{1-F(t)}\int_t^\infty yf(y)\,dy-t,\ t\geq0,$$
$$=\frac{1}{1-F(t)}\left[E(Y)-\rho_{Y,\,1}(z)\right]-t,\ t\geq0, \tag{68}$$

where $F(y)$, $f(y)$, are specified in (1), (2) and

$E(Y)=\rho^r(y|\boldsymbol{\xi},\boldsymbol{\psi})|_{r=1}$.

The variance residual life V_{RL} is another measure of concern that has increased attention in latest years (Khorashadizadeh et al. [36] and Gupta [37]).

$$V_{RL}=V(R(t))=\frac{2}{S(t)}\int_t^\infty yS(y)\,dy-2t\Lambda(t)-\Lambda^2(t),$$
$$=\frac{1}{S(t)}\left[E\left(Y^2\right)-\rho_{t,\,2}(z)\right]-t^2-2t\Lambda(t)-\Lambda^2(t), \tag{69}$$

where $E\left(Y^2\right)=\rho^r(y|\boldsymbol{\xi},\boldsymbol{\psi})|_{r=2}$. and $\rho_{t,\,2}(z)$ specified in Eq. (48) by setting $r=2$.

Characteristic of reversed residual lifetime function

The survival function of reversed residual lifetime $\ddot{R}(t)$ (for $0\leq y<t$) for ATF model is

$$S_{\ddot{R}(t)}(y)=\frac{F(t-y)}{F(t)}=\frac{\exp\left\{-\left(\xi_1(t-y)^{-\psi_1}+\xi_2(t-y)^{-\psi_2}+\xi_3(t-y)^{-\psi_3}\right)\right\}}{\exp\left\{-\left(\xi_1t^{-\psi_1}+\xi_2t^{-\psi_2}+\xi_3t^{-\psi_3}\right)\right\}}. \tag{70}$$

Then the $\ddot{R}(t)$ pdf will become

$$f_{\ddot{R}(t)}(y|\boldsymbol{\xi},\boldsymbol{\psi})$$
$$=\frac{\left[\left\{\xi_1\psi_1(t-y)^{-(\psi_1+1)}+\xi_2\psi_2(t-y)^{-(\psi_2+1)}+\xi_3\psi_3(t-y)^{-(\psi_3+1)}\right\}\exp\left\{-\left(\xi_1(t-y)^{-\psi_1}+\xi_2(t-y)^{-\psi_2}+\xi_3(t-y)^{-\psi_3}\right)\right\}\right]}{\exp\left\{-\left(\xi_1t^{-\psi_1}+\xi_2t^{-\psi_2}+\xi_3t^{-\psi_3}\right)\right\}}. \tag{71}$$

The hrf of $\ddot{R}(t)$ is therefore reduced to

$$h_{\ddot{R}(t)}\left(y|\,\xi,\psi\right) = \frac{\left[\left\{\xi_1\psi_1\left(t-y\right)^{-(\psi_1+1)}+\xi_2\psi_2\left(t-y\right)^{-(\psi_2+1)}+\xi_3\psi_3\left(t-y\right)^{-(\psi_3+1)}\right\}\exp\left\{-\left(\xi_1\left(t-y\right)^{-\psi_1}+\xi_2\left(t-y\right)^{-\psi_2}+\xi_3\left(t-y\right)^{-\psi_3}\right)\right\}\right]}{\exp\left\{-\left(\xi_1\left(t-y\right)^{-\psi_1}+\xi_2\left(t-y\right)^{-\psi_2}+\xi_3\left(t-y\right)^{-\psi_3}\right)\right\}}. \quad (72)$$

The $\ddot{R}(t)$ mean and variance is provided by

$$\begin{aligned}\ddot{A}(t) &= E\left\{\ddot{R}(t)\right\} = t-\frac{1}{F(t)}\int_0^t yf(y)\,dy,\ 0< y<t,\\ &= t-\frac{1}{F(t)}\left[\rho_{t,\,1}(z)\right],\ 0< y<t. \quad (73)\end{aligned}$$

and

$$\begin{aligned}V_{\ddot{R}L} &= V\left(\ddot{R}(t)\right) = 2t\ddot{A}(t)-\ddot{A}^2(t)-\frac{2}{F(t)}\int_0^t yF(y)\,dy,\\ &= 2t\ddot{A}(t)-\ddot{A}^2(t)-t^2+\frac{1}{F(t)}\left[\rho_{t,\,2}(z)\right], \quad (74)\end{aligned}$$

where $F(t)$, $f(y)$ and $\rho_{t,\,2}(z)$ can be identified from (1), (2) and () by setting $r=2$, respectively.

Reliability measures

The curves of Bonferroni and Lorenz are income inequality measures that are commonly helpful and beneficial to certain other fields having reliability, demography, medicine and insurance and medicine. The Bonferroni curve $B_{F(y)}$ of Y is

$$B_{F(y)} = \frac{1}{E(Y)\,F(y)}\int_0^y yf(y)\,dy = \frac{\rho_{Y,\,1}(z)}{E(Y)\,F(y)}. \quad (75)$$

Groves-Kirkby et al. [38] highlights the significance of the Lorenz curve for applications in various scientific fields. he Lorenz curve $L_{F(y)}$ of Y is

$$L_{F(y)} = \frac{1}{E(Y)}\int_0^y yf(y)\,dy = \frac{\rho_{Y,\,1}(z)}{E(Y)}. \quad (76)$$

Estimation

The parameters of the ATF model can be assessed utilizing the log-likelihood based on the sample using Matlab (log lik), R (optimum and MaxLik features), the Ox programme (subroutine MaxBFGS), or SAS (PROC NLMIXED). Additionally, certain goodness-of-fit statistics are included for comparing density estimates and model selection.

Maximum likelihood estimation

The maximum likelihood estimates (MLEs) are provided by optimizing this equation according to ξ_i, and ψ_i, $i=1,2,3$. They are also characterized as the maximum of the log-likelihood function defined by $l_{\mathbf{y}|\xi,\psi} = \log L(\mathbf{y}|\,\xi,\psi)$.

The log-likelihood function for the ATF model is provided by the data set $y_1,\ldots,y_n$.

$$\begin{aligned}L\left(\mathbf{y}|\ \psi_1,\ \psi_2,\psi_3,\ \xi_1,\ \xi_2,\xi_3\right) = {}&\prod_{i=1}^n\left(\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}\right)\\ &\times\exp\left\{-\left(\xi_1y_i^{-\psi_1}+\xi_2y_i^{-\psi_2}+\xi_3y_i^{-\psi_3}\right)\right\}, \quad (77)\end{aligned}$$

$$\begin{aligned}l_{\mathbf{y}|\ \psi_1,\ \psi_2,\psi_3,\ \xi_1,\ \xi_2,\xi_3} = {}&\sum_{i=1}^n\log\left(\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}\right)\\ &-\sum_{i=1}^n\left(\xi_1y_i^{-\psi_1}+\xi_2y_i^{-\psi_2}+\xi_3y_i^{-\psi_3}\right). \quad (78)\end{aligned}$$

Table 2
Descriptive statistics.

Min.	1st Quartile	Median	Mean	3rd Quartile	Max.
1.516	2.789	3.178	3.282	3.637	6.869

Table 3
Maximum likelihood estimates.

Distribution	Estimates					
ATF (ξ,ψ)	7.52620	8.3147	7.5625	3.1695	3.1689	3.1693
AWD	0.02098	0.06647	0.00054	0.0007	2.0877	1.34626
AED	0.01569	0.03564	0.25345			

We obtain the components of parameter vector $\Lambda_{\xi,\psi} = \left(\Lambda_{\psi_1},\ \Lambda_{\psi_2},\ \Lambda_{\psi_3},\ \Lambda_{\xi_1},\ \Lambda_{\xi_2},\ \Lambda_{\xi_3}\right)^\tau$, and set them zero and given by

$$\Lambda_{\psi_1} = \frac{\partial l_{\mathbf{y}|\ \xi,\psi}}{\partial l_{\mathbf{y}|\psi_1}} = \sum_{i=1}^n\xi_1y_i^{-\psi_1}\log\left(y_i\right)+\sum_{i=1}^n\frac{\xi_1y_i^{-\psi_1-1}-\xi_1\psi_1y_i^{-\psi_1-1}\log\left(y_i\right)}{\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}}, \quad (79a)$$

$$\Lambda_{\psi_2} = \frac{\partial l_{\mathbf{y}|\ \xi,\psi}}{\partial l_{\mathbf{y}|\psi_2}} = \sum_{i=1}^n\xi_2y_i^{-\psi_2}\log\left(y_i\right)+\sum_{i=1}^n\frac{\xi_2y_i^{-\psi_2-1}-\xi_2\psi_2y_i^{-\psi_2-1}\log\left(y_i\right)}{\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}}, \quad (79b)$$

$$\Lambda_{\psi_3} = \frac{\partial l_{\mathbf{y}|\ \xi,\psi}}{\partial l_{\mathbf{y}|\psi_3}} = \sum_{i=1}^n\xi_3y_i^{-\psi_3}\log\left(y_i\right)+\sum_{i=1}^n\frac{\xi_3y_i^{-\psi_3-1}-\xi_3\psi_3y_i^{-\psi_3-1}\log\left(y_i\right)}{\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}}, \quad (79c)$$

$$\Lambda_{\xi_1} = \frac{\partial l_{\mathbf{y}|\ \xi,\psi}}{\partial l_{\mathbf{y}|\xi_1}} = -\sum_{i=1}^n y_i^{-\psi_1}+\sum_{i=1}^n\frac{\psi_1y_i^{-\psi_1-1}}{\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}}, \quad (79d)$$

$$\Lambda_{\xi_2} = \frac{\partial l_{\mathbf{y}|\ \xi,\psi}}{\partial l_{\mathbf{y}|\xi_2}} = -\sum_{i=1}^n y_i^{-\psi_2}+\sum_{i=1}^n\frac{\psi_2y_i^{-\psi_2-1}}{\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}}, \quad (79e)$$

$$\Lambda_{\xi_3} = \frac{\partial l_{\mathbf{y}|\ \xi,\psi}}{\partial l_{\mathbf{y}|\xi_3}} = -\sum_{i=1}^n y_i^{-\psi_3}+\sum_{i=1}^n\frac{\psi_3y_i^{-\psi_3-1}}{\xi_1\psi_1y_i^{-\psi_1-1}+\xi_2\psi_2y_i^{-\psi_2-1}+\xi_3\psi_3y_i^{-\psi_3-1}}. \quad (79f)$$

To obtain MLE $(\xi,\psi) = \left(\hat{\xi},\hat{\psi}\right)$ setting $\Lambda_{\psi_1} = \Lambda_{\psi_2} = \Lambda_{\psi_3} = \Lambda_{\xi_1} = \Lambda_{\xi_2} = \Lambda_{\xi_3} = 0$ and solving them simultaneously. Since they are not solvable in closed form, different numerical iterative methods available and applied like Newton–Raphson kind algorithms. In order to estimate the intervals of the parameters, we need the 6×6 information matrix $J(\Lambda) = J_{\xi,\psi}(\Lambda)$. The asymptotic distribution $\left(\hat{\Lambda}_{\xi,\psi}-\Lambda_{\xi,\psi}\right)$ is $N_6\left(0,\ \Delta(\Lambda)^{-1}\right)$, where $\Delta(\Lambda) = E\{J(\Lambda)\}$. The approximate multivariate normal $N_6\left(0,\ J(\Lambda)^{-1}\right)$ distribution, where $J(\Lambda)^{-1}$ is the inverse of information matrix at $\Lambda_{\xi,\psi} = \hat{\Lambda}_{\xi,\psi}$ can be implemented under standard regularity conditions to develop approximate confidence intervals for the model parameters.

Real data implementation

Let us just look at a real data set to evaluate if our new model provides better fit for the data than some other distributions. Using goodness of fit criteria including Akaike Information Parameters (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) and -Log-likelihood ($-LL$), the ATF's goodness of fit is compared to the Additive Weibull Distribution (AWD), and Additive exponential Distribution (AED). As expected, the lower these criteria's values are the better the fit. The AFT distribution does have the smallest statistics. The AFT distribution then provides the best fit among the distributions compared. This data represents drought mortality rate. The data contains 36 days of COVID-19 data (Canada), from 10 April to 15 May 2020 [39]. See Abu El Azm et al. [40], Shafiq et al. [41], Sindhu et al. [42,43] and Almongy et al. [44] for other examples of COVID-19 data applications.

The data summary, MLEs of ATF model and goodness-of-fit (GoF) measures are provided in Tables 2–4. The outcomes of these Tables clearly show that ATF is the best model as it has the smaller values of the -LL, AIC, BIC, and CAIC. In comparison to AWD and AED model provides a very good fit for this data, as seen in the Tables. Fig. 9 shows the profiles of the log-likelihood function (PLLF) based on data set.

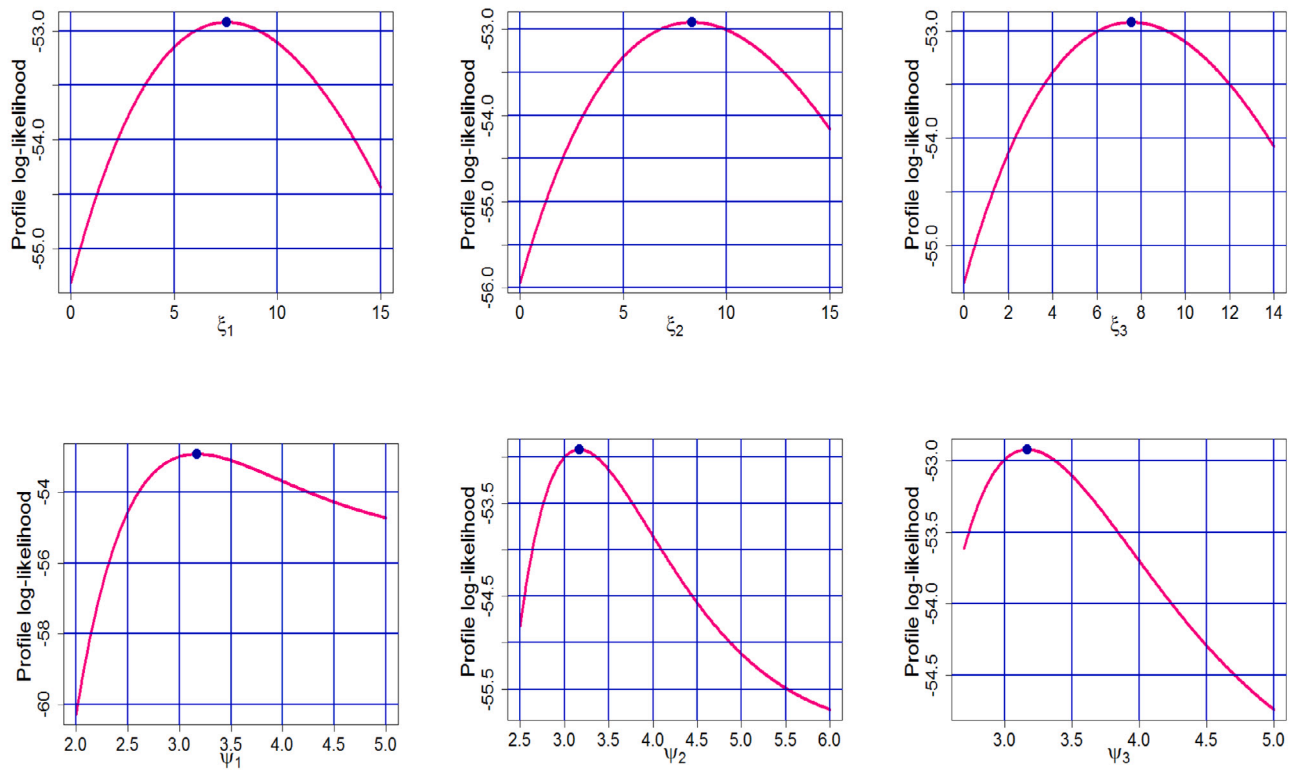

Fig. 9. Plots of profiles of the log-likelihood function.

Table 4
Goodness of fit criteria: AIC, CAIC, BIC, -Log-likelihood ($-LL$).

Distribution	$-LL$	AIC	BIC	CAIC
ATF (ξ, ψ)	-52.92007	117.8401	127.3413	120.7367
AWD	-58.55368	129.1074	138.6085	132.0039
AED	-78.77977	163.5595	168.3101	164.3095

Concluding remarks

We implement the six-parameter lifetime model recognized as the ATF distribution. Different mathematical properties were discussed with discussion involving quantile function, stochastic ordering and related measures. Under the certain restrictions, we can obtain random variables from the novel model. We include some figures for pdf, cdf, hazard function, quantile function, median, skewness and kurtosis. The general non-central complete, incomplete moments, characteristic function factorial generating function and residual life function with a certain measure of reliability are also discussed. For the generating function, generating function, non-moment (complete and incomplete), conditional moments and mean deviations, residual lifetime and reversed residual life functions, we also get explicit expression for the suggested model. Through using the classical goodness of fit indicators, we evaluate the efficiency of the new model with its significant counterparts. These findings are in line with the fact that the current distribution is quite suitable for real-life data applications.

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