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# An approximate analytical solution of the Navier–Stokes equations within Caputo operator and Elzaki transform decomposition method

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## Abstract

In this article, a hybrid technique of Elzaki transformation and decomposition method is used to solve the Navier–Stokes equations with a Caputo fractional derivative. The numerical simulations and examples are presented to show the validity of the suggested method. The solutions are determined for the problems of both fractional and integer orders by a simple and straightforward procedure. The obtained results are shown and explained through graphs and tables. It is observed that the derived results are very close to the actual solutions of the problems. The fractional solutions are of special interest and have a strong relation with the solution at the integer order of the problems. The numerical examples in this paper are nonlinear and thus handle its solutions in a sophisticated manner. It is believed that this work will make it easy to study the nonlinear dynamics, arising in different areas of research and innovation. Therefore, the current method can be extended for the solution of other higher-order nonlinear problems.

**Keywords:** Elzaki transformation; Adomian decomposition method; Navier–Stokes equations; Caputo operator

## 1 Introduction

Leibnitz conceived of a fraction in the derivative and it was discovered that fractional calculus (FC) is better suited to model various scientific processes than classical calculus. The researchers are motivated because the theory of fractional calculus interprets nature's truth in an excellent and systematic way [1–3]. In this connection, the researchers have also investigated that fractional calculus of non-integer-order derivatives are very useful in describing numerous problems of scientific value, such as diffusion processes, damping laws and rheology [4–8]. Various aspects of fractional calculus are given by Podlubny [2], Caputo [5], Kiryakova [6], Jafari and Seifi [7, 8], Momani and Shawagfeh [9], Oldham and Spanier [10], Diethelm et al. [11], Miller and Ross [1], Kemple and Beyer [12], Kilbas and Trujillo [13].

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Fractional differential equations (FDEs) as a part of FC are considered to be the most popular and important tool to describe and model various phenomena in nature such as earthquake nonlinear oscillations, and the involvement of fractional derivatives in fluid-dynamic traffic model eliminates the insufficiency arising in the process of continuum traffic flow. FDEs are also used in the simulations of mathematical biology, chemical and many other engineering and physical processes [14–24]. For engineers, physicists, and mathematicians, nonlinear problems are important, namely because in nature most of the physical systems are nonlinear. Nonlinear equations, however, are hard to solve and lead to interesting phenomena. The actual or exact solutions of the evolution processes have an important role in the study of high-order nonlinear problems.

Recently, mathematicians have had much attention for the approximate and analytical solutions of FDEs and had developed important mathematical techniques to solve FDEs. The well-known techniques regarding the solution of FDEs are the Adomian decomposition method (ADM) [25, 26], finite difference method (FDM) [27], the differential transform method (DTM) [28, 29], the homotopy perturbation transform method (HPTM) [30–32], the Haar wavelet method (HWM) [33, 34], the differential transform method (DTM) [35–37], the variational iteration transform method (VIM) [38] and many others.

In 1822, Claude Louis and Gabriel Stokes were the first to develop the Navier–Stokes (N–S) equation. The N–S model is considered to be an important model as it explained many physical processes, such as ocean currents, weather, air flow around a wing and water flow in pipes, which are arising in different areas of applied sciences [45]. The relation of viscous fluid versus rigid bodies is also investigated with the help of the N–S equation and considered to be the best tool in the field of meteorology and other related subjects [46].

Several mathematicians have used various techniques to solve the N–S equation. Kumar et al. have introduced a modified Laplace decomposition technique for finding an analytical solution of the Navier–Stokes fractional equation [47]. The combination of fractional complex transform (FCT) and He–Laplace transform (HLT) approach is implemented for solving the N–S equation [48]. The fractional reduced differential transformation method (FRDM) is also used for finding a time-fractional N–S equation numerical solution [49]; see also [50].

In the present work, we have investigated the solutions of the N–S equations of fractional order with the help of Elzaki transform decomposition method. The proposed method is a mixture of Elzaki transformation [39] and ADM [40, 41]. The Elzaki transformation [42–44] and ADM [40, 41] have been used separately for the solutions linear and nonlinear ordinary and partial differential equations (PDEs) and provide the actual solutions in the form of convergent series. In this research work, the analytical solutions of nonlinear N–S equations are calculated by using ETDM. The solutions are calculated for both fractional and integer orders of the problems. The results are explained and verified with the help of graphs and tables. It is analyzed that the present technique provides the solutions of fractional-order problems in a very simple and straightforward procedure. The present method allows one to calculate the solutions of other high nonlinear problems in various branches of applied sciences.

## 2 Definitions and preliminaries concepts

We have provided some clear and most important concepts in this unit concerning fractional calculus.

### 2.1 Definition

The operator  $D^\delta$  of order  $\delta$  defined by Abel–Riemann (A–R) as

$$D^\delta \mu(\psi) = \begin{cases} \frac{d^m}{d\psi^m} \mu(\psi), & \delta = m, \\ \frac{1}{\Gamma(m-\delta)} \frac{d}{d\psi^m} \int_0^\psi \frac{\mu(\mathcal{T})}{(\psi-\mathcal{T})^{\delta-m+1}} d\mathcal{T}, & m-1 < \delta < m, \end{cases}$$

where  $m \in \mathbb{Z}^+$ ,  $\delta \in \mathbb{R}^+$  and

$$D^{-\delta} \mu(\psi) = \frac{1}{\Gamma(\delta)} \int_0^\psi (\psi - \mathcal{T})^{\delta-1} \mu(\mathcal{T}) d\mathcal{T}, \quad 0 < \delta \leq 1.$$

### 2.2 Definition

The A–R integration operator  $J^\delta$  of fractional order is defined as

$$J^\delta \mu(\psi) = \frac{1}{\Gamma(\delta)} \int_0^\psi (\psi - \mathcal{T})^{\delta-1} \mu(\mathcal{T}) d\mathcal{T}, \quad \mathcal{T} > 0, \delta > 0.$$

Following Podlubny we may have

$$J^\delta \mathcal{T}^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} \mathcal{T}^{n+\delta},$$

$$D^\delta \mathcal{T}^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} \mathcal{T}^{n-\delta}.$$

### 2.3 Definition

The operator  $D^\delta$  in Caputo sense having order  $\delta$  is defined as

$$D^\delta \mu(\psi) = \begin{cases} \frac{1}{\Gamma(m-\delta)} \int_0^\psi \frac{\mu^m(\mathcal{T})}{(\psi-\mathcal{T})^{\delta-m+1}} d\mathcal{T}, & m-1 < \delta < m, \\ \frac{d^m}{d\mathcal{T}^m} \mu(\psi), & \delta = m, \end{cases}$$

having the following properties:

- (a)  $D_T^\delta J_T^\delta h(\mathcal{T}) = h(\mathcal{T})$ ,
- (b)  $J_T^\delta D_T^\delta h(\mathcal{T}) = h(\mathcal{T}) - \sum_{k=0}^m h^k(0^+) \frac{\mathcal{T}^k}{k!}$ , for  $\mathcal{T} > 0$ , and  $m-1 < \delta \leq m, m \in \mathbb{N}$ .

## 3 Elzaki transform (ET)

Modified Sumudu transform or ET definition for the function  $f(t)$  is given as

$$E[h(\mathcal{T})] = H(q) = q \int_0^\infty h(\mathcal{T}) e^{-\frac{\mathcal{T}}{q}} d\mathcal{T}, \quad \mathcal{T} > 0.$$

The Elzaki transform is a very efficient and strong technique to solve the integral equation that the Sumudu transform method cannot match.

Integration by parts can be used in order to find ET of partial derivatives as follows.

1.  $E\left[\frac{\partial h(\psi, \mathcal{T})}{\partial \mathcal{T}}\right] = \frac{1}{q} H(\psi, q) - q h(\psi, 0)$ .
2.  $E\left[\frac{\partial^2 h(\psi, \mathcal{T})}{\partial \mathcal{T}^2}\right] = \frac{1}{q^2} H(\psi, q) - h(\psi, 0) - q \frac{\partial h(\psi, 0)}{\partial \mathcal{T}}$ .

3.  $E\left[\frac{\partial h(\psi, T)}{\partial \psi}\right] = \frac{d}{d\psi} H(\psi, q).$
4.  $E\left[\frac{\partial^2 h(\psi, T)}{\partial \psi^2}\right] = \frac{d^2}{d\psi^2} H(\psi, q).$

### 3.1 ET of Caputo fractional derivative

**Theorem 1** Let  $G(s)$  be the Laplace transform of  $h(T)$ ; then  $ET H(q)$  of  $h(T)$  is defined as

$$H(q) = qG\left(\frac{1}{q}\right).$$

**Theorem 2** If  $H(q)$  is the ET of the function  $h(T)$ , then

$$E[D^\delta h(T)] = \frac{H(q)}{q^\delta} - \sum_{k=0}^{n-1} q^{k-\delta+2} h^{(k)}(0), \quad n-1 < \delta \leq n.$$

### 4 The procedure of ETDM

In this section we define the solution of ETDM for the system of fractional partial differential equations,

$$\begin{aligned} D_{\mathcal{T}}^\delta \mu(\psi, T) + \bar{\mathcal{G}}_1(\mu, v) + \mathcal{N}_1(\mu, v) - \mathcal{P}_1(\psi, T) &= 0, \\ D_{\mathcal{T}}^\delta v(\psi, T) + \bar{\mathcal{G}}_2(\mu, v) + \mathcal{N}_2(\mu, v) - \mathcal{P}_2(\psi, T) &= 0, \end{aligned} \tag{1}$$

$m-1 < \delta \leq m,$

having initial conditions

$$\mu(\psi, 0) = g_1(\psi), \quad v(\psi, 0) = g_2(\psi), \tag{2}$$

where  $D_{\mathcal{T}}^\delta = \frac{\partial^\delta}{\partial \mathcal{T}^\delta}$  is the Caputo fractional derivative of order  $\delta$ ,  $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2$  and  $\mathcal{N}_1, \mathcal{N}_2$  are linear and non-linear functions, respectively, and  $\mathcal{P}_1, \mathcal{P}_2$  are source operators.

Taking the Elzaki transform on both sides of Eq. (1), we get

$$\begin{aligned} E[D_{\mathcal{T}}^\delta \mu(\psi, T)] + E[\bar{\mathcal{G}}_1(\mu, v) + \mathcal{N}_1(\mu, v) - \mathcal{P}_1(\psi, T)] &= 0, \\ E[D_{\mathcal{T}}^\delta v(\psi, T)] + E[\bar{\mathcal{G}}_2(\mu, v) + \mathcal{N}_2(\mu, v) - \mathcal{P}_2(\psi, T)] &= 0. \end{aligned} \tag{3}$$

Using the Elzaki transform differentiation property, we obtain

$$\begin{aligned} E[\mu(\psi, T)] &= s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \mu(\psi, T)}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} \\ &\quad + s^\delta E[\mathcal{P}_1(\psi, T)] - s^\delta E\{\bar{\mathcal{G}}_1(\mu, v) + \mathcal{N}_1(\mu, v)\}, \\ E[v(\psi, T)] &= s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k v(\psi, T)}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} \\ &\quad + s^\delta E[\mathcal{P}_2(\psi, T)] - s^\delta E\{\bar{\mathcal{G}}_2(\mu, v) + \mathcal{N}_2(\mu, v)\}, \end{aligned} \tag{4}$$

ETDM defines the infinite series solution of  $\mu(\psi, T)$  and  $v(\psi, T)$ ,

$$\mu(\psi, T) = \sum_{m=0}^{\infty} \mu_m(\psi, T), \quad v(\psi, T) = \sum_{m=0}^{\infty} v_m(\psi, T). \tag{5}$$

The decomposition of the Adomian polynomials of nonlinear terms  $\mathcal{N}_1$  and  $\mathcal{N}_2$  is defined as

$$\mathcal{N}_1(\mu, \nu) = \sum_{m=0}^{\infty} \mathcal{A}_m, \quad \mathcal{N}_2(\mu, \nu) = \sum_{m=0}^{\infty} \mathcal{B}_m. \tag{6}$$

All types of nonlinearity can be represented by the Adomian polynomials as

$$\begin{aligned} \mathcal{A}_m &= \frac{1}{m!} \left[ \frac{\partial^m}{\partial \lambda^m} \left\{ \mathcal{N}_1 \left( \sum_{k=0}^{\infty} \lambda^k \mu_k, \sum_{k=0}^{\infty} \lambda^k \nu_k \right) \right\} \right]_{\lambda=0}, \\ \mathcal{B}_m &= \frac{1}{m!} \left[ \frac{\partial^m}{\partial \lambda^m} \left\{ \mathcal{N}_2 \left( \sum_{k=0}^{\infty} \lambda^k \mu_k, \sum_{k=0}^{\infty} \lambda^k \nu_k \right) \right\} \right]_{\lambda=0}. \end{aligned} \tag{7}$$

Substituting Eq. (5) and Eq. (7) into (4) gives

$$\begin{aligned} E \left[ \sum_{m=0}^{\infty} \mu_m(\psi, \mathcal{T}) \right] &= s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \mu(\psi, \mathcal{T})}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} + s^\delta E \\ &\quad \times \{ \mathcal{P}_1(\psi, \mathcal{T}) \} - s^\delta E \left\{ \bar{\mathcal{G}}_1 \left( \sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m \right) + \sum_{m=0}^{\infty} \mathcal{A}_m \right\}, \\ E \left[ \sum_{m=0}^{\infty} \nu_m(\psi, \mathcal{T}) \right] &= s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \nu(\psi, \mathcal{T})}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} + s^\delta E \\ &\quad \times \{ \mathcal{P}_2(\psi, \mathcal{T}) \} - s^\delta E \left\{ \bar{\mathcal{G}}_2 \left( \sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m \right) + \sum_{m=0}^{\infty} \mathcal{B}_m \right\}. \end{aligned} \tag{8}$$

Using the Elzaki inverse on both sides of Eq. (8), we get

$$\begin{aligned} \sum_{m=0}^{\infty} \mu_m(\psi, \mathcal{T}) &= E^- \left[ s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \mu(\psi, \mathcal{T})}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} + s^\delta E \right. \\ &\quad \left. \times \{ \mathcal{P}_1(\psi, \mathcal{T}) \} - s^\delta E \left\{ \bar{\mathcal{G}}_1 \left( \sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m \right) + \sum_{m=0}^{\infty} \mathcal{A}_m \right\} \right], \\ \sum_{m=0}^{\infty} \nu_m(\psi, \mathcal{T}) &= E^- \left[ s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \nu(\psi, \mathcal{T})}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} + s^\delta E \right. \\ &\quad \left. \times \{ \mathcal{P}_2(\psi, \mathcal{T}) \} - s^\delta E \left\{ \bar{\mathcal{G}}_2 \left( \sum_{m=0}^{\infty} \mu_m, \sum_{m=0}^{\infty} \nu_m \right) + \sum_{m=0}^{\infty} \mathcal{B}_m \right\} \right], \end{aligned} \tag{9}$$

we describe the following terms:

$$\begin{aligned} \mu_0(\psi, \mathcal{T}) &= E^- \left[ s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \mu(\psi, \mathcal{T})}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} + s^\delta E^+ \{ \mathcal{P}_1(\psi, \mathcal{T}) \} \right], \\ \nu_0(\psi, \mathcal{T}) &= E^- \left[ s^\delta \sum_{k=0}^{m-1} s^{2+k-\delta} \frac{\partial^k \nu(\psi, \mathcal{T})}{\partial^k \mathcal{T}} \Big|_{\mathcal{T}=0} + s^\delta E^+ \{ \mathcal{P}_2(\psi, \mathcal{T}) \} \right], \end{aligned} \tag{10}$$

$$\begin{aligned} \mu_1(\psi, \mathcal{T}) &= -E^- [s^\delta E^+ \{ \bar{\mathcal{G}}_1(\mu_0, \nu_0) + \mathcal{A}_0 \}], \\ \nu_1(\psi, \mathcal{T}) &= -E^- [s^\delta E^+ \{ \bar{\mathcal{G}}_2(\mu_0, \nu_0) + \mathcal{B}_0 \}], \end{aligned}$$

the general case, for  $m \geq 1$ , is given by

$$\begin{aligned} \mu_{m+1}(\psi, \mathcal{T}) &= -E^- [s^\delta E^+ \{ \bar{\mathcal{G}}_1(\mu_m, \nu_m) + \mathcal{A}_m \}], \\ \nu_{m+1}(\psi, \mathcal{T}) &= -E^- [s^\delta E^+ \{ \bar{\mathcal{G}}_2(\mu_m, \nu_m) + \mathcal{B}_m \}]. \end{aligned}$$

### 5 Numerical examples

#### 5.1 Problem 1

Consider the two-dimensional fractional order Navier–Stokes equation

$$\begin{aligned} D_{\mathcal{T}}^\delta(\mu) + \mu \frac{\partial \mu}{\partial \psi} + \nu \frac{\partial \mu}{\partial \zeta} &= \rho \left[ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right] + q, \\ D_{\mathcal{T}}^\delta(\nu) + \mu \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial \nu}{\partial \zeta} &= \rho \left[ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} \right] - q, \end{aligned} \tag{11}$$

with initial conditions

$$\begin{cases} \mu(\psi, \zeta, 0) = -\sin(\psi + \zeta), \\ \nu(\psi, \zeta, 0) = \sin(\psi + \zeta). \end{cases} \tag{12}$$

After the Elzaki transformation of Eq. (11), we get

$$\begin{aligned} E \left\{ \frac{\partial^\delta \mu}{\partial \mathcal{T}^\delta} \right\} &= E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + \nu \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} + q \right], \\ E \left\{ \frac{\partial^\delta \nu}{\partial \mathcal{T}^\delta} \right\} &= E \left[ - \left( \mu \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial \nu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} \right\} - q \right], \\ \frac{1}{s^\delta} E \{ \mu(\psi, \zeta, \mathcal{T}) \} - s^{2-\delta} \mu(\psi, \zeta, 0) &= E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + \nu \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} + q \right], \\ \frac{1}{s^\delta} E \{ \nu(\psi, \zeta, \mathcal{T}) \} - s^{2-\delta} \nu(\psi, \zeta, 0) &= E \left[ - \left( \mu \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial \nu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} \right\} - q \right]. \end{aligned}$$

The simplified form of the above algorithm is

$$\begin{aligned} E \{ \mu(\psi, \zeta, \mathcal{T}) \} &= s^2 \{ \mu(\psi, \zeta, 0) \} + s^\delta E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + \nu \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} + q \right], \\ E \{ \nu(\psi, \zeta, \mathcal{T}) \} &= s^2 \{ \nu(\psi, \zeta, 0) \} + s^\delta E \left[ - \left( \mu \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial \nu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} \right\} - q \right]. \end{aligned} \tag{13}$$

Using the inverse Elzaki transformation, we obtain

$$\begin{aligned} \mu(\psi, \zeta, T) &= \mu(\psi, \zeta, 0) + E^{-}[s^\delta E\{q\}] \\ &\quad + E^{-}\left[s^\delta E\left[-\left(\mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta}\right) + \rho \left\{\frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2}\right\}\right]\right], \\ v(\psi, \zeta, T) &= v(\psi, \zeta, 0) - E^{-}[s^\delta E\{q\}] \\ &\quad + E^{-}\left[s^\delta E\left[-\left(\mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta}\right) + \rho \left\{\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2}\right\}\right]\right]. \end{aligned} \tag{14}$$

Assume that the unknown functions  $\mu(\psi, \zeta, T)$  and  $v(\psi, \zeta, T)$  have the following solution in infinite series form:

$$\mu(\psi, \zeta, T) = \sum_{m=0}^{\infty} \mu_m(\psi, \zeta, T) \quad \text{and} \quad v(\psi, \zeta, T) = \sum_{m=0}^{\infty} v_m(\psi, \zeta, T).$$

Remember that  $\mu\mu_\psi = \sum_{m=0}^{\infty} \mathcal{A}_m$ ,  $v\mu_\zeta = \sum_{m=0}^{\infty} \mathcal{B}_m$ ,  $\mu v_\psi = \sum_{m=0}^{\infty} \mathcal{C}_m$  and  $v v_\zeta = \sum_{m=0}^{\infty} \mathcal{D}_m$  are the Adomian polynomials and the nonlinear terms were characterized. Equation (14) can be rewritten in the form using certain terms

$$\begin{aligned} \sum_{m=0}^{\infty} \mu_m(\psi, \zeta, T) &= \mu(\psi, \zeta, 0) + E^{-}[s^\delta E\{q\}] \\ &\quad + E^{-}\left[s^\delta E\left[-\left(\sum_{m=0}^{\infty} \mathcal{A}_m + \sum_{m=0}^{\infty} \mathcal{B}_m\right) + \rho \left\{\frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2}\right\}\right]\right], \\ \sum_{m=0}^{\infty} v_m(\psi, \zeta, T) &= v(\psi, \zeta, 0) - E^{-}[s^\delta E\{q\}] \\ &\quad + E^{-}\left[s^\delta E\left[-\left(\sum_{m=0}^{\infty} \mathcal{C}_m + \sum_{m=0}^{\infty} \mathcal{D}_m\right) + \rho \left\{\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2}\right\}\right]\right], \end{aligned} \tag{15}$$

$$\begin{aligned} \sum_{m=0}^{\infty} \mu_m(\psi, \zeta, T) &= -\sin(\psi + \zeta) + \frac{qT^\delta}{\Gamma(\delta + 1)} + E^{-}\left[s^\delta E\left[-\left(\sum_{m=0}^{\infty} \mathcal{A}_m + \sum_{m=0}^{\infty} \mathcal{B}_m\right)\right]\right] \\ &\quad + E^{-}\left[s^\delta E\left[\rho \left\{\sum_{m=0}^{\infty} \frac{\partial^2 \mu_m}{\partial \psi^2} + \sum_{m=0}^{\infty} \frac{\partial^2 \mu_m}{\partial \zeta^2}\right\}\right]\right], \\ \sum_{m=0}^{\infty} v_m(\psi, \zeta, T) &= \sin(\psi + \zeta) - \frac{qT^\delta}{\Gamma(\delta + 1)} - E^{-}\left[s^\delta E\left[-\left(\sum_{m=0}^{\infty} \mathcal{C}_m + \sum_{m=0}^{\infty} \mathcal{D}_m\right)\right]\right] \\ &\quad + E^{-}\left[s^\delta E\left[\rho \left\{\sum_{m=0}^{\infty} \frac{\partial^2 v_m}{\partial \psi^2} + \sum_{m=0}^{\infty} \frac{\partial^2 v_m}{\partial \zeta^2}\right\}\right]\right]. \end{aligned}$$

According to Eq. (7), all types of nonlinearity can be represented by the Adomian polynomials as

$$\begin{aligned} \mathcal{A}_0 &= \mu_0 \frac{\partial \mu_0}{\partial \psi}, & \mathcal{A}_1 &= \mu_0 \frac{\partial \mu_1}{\partial \psi} + \mu_1 \frac{\partial \mu_0}{\partial \psi}, \\ \mathcal{B}_0 &= v_0 \frac{\partial \mu_0}{\partial \zeta}, & \mathcal{B}_1 &= v_0 \frac{\partial \mu_1}{\partial \zeta} + v_1 \frac{\partial \mu_0}{\partial \zeta}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_0 &= \mu_0 \frac{\partial v_0}{\partial \psi}, & \mathcal{C}_1 &= \mu_0 \frac{\partial v_1}{\partial \psi} + \mu_1 \frac{\partial v_0}{\partial \psi}, \\ \mathcal{D}_0 &= v_0 \frac{\partial v_0}{\partial \psi}, & \mathcal{D}_1 &= v_0 \frac{\partial v_1}{\partial \psi} + v_1 \frac{\partial v_0}{\partial \psi}. \end{aligned}$$

Thus, by comparing both sides of Eq. (15) we can get easily the recursive relationship

$$\mu_0(\psi, \zeta, \mathcal{T}) = -\sin(\psi + \zeta) + \frac{q\mathcal{T}^\delta}{\Gamma(\delta + 1)}, \quad v_0(\psi, \zeta, \mathcal{T}) = \sin(\psi + \zeta) - \frac{q\mathcal{T}^\delta}{\Gamma(\delta + 1)}.$$

For  $m = 0$

$$\mu_1(\psi, \zeta, \mathcal{T}) = \sin(\psi + \zeta) \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta + 1)}, \quad v_1(\psi, \zeta, \mathcal{T}) = -\sin(\psi + \zeta) \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta + 1)}.$$

For  $m = 1$

$$\mu_2(\psi, \zeta, \mathcal{T}) = -\sin(\psi + \zeta) \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)}, \quad v_2(\psi, \zeta, \mathcal{T}) = \sin(\psi + \zeta) \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)}.$$

For  $m = 2$

$$\begin{aligned} \mu_3(\psi, \zeta, \mathcal{T}) &= \sin(\psi + \zeta) \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)}, \\ v_3(\psi, \zeta, \mathcal{T}) &= -\sin(\psi + \zeta) \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)}, \end{aligned}$$

⋮

In the same manner, the remaining  $\mu_m$  and  $v_m$  ( $m > 3$ ) elements of the ETDM solution are easy to obtain. So we describe the alternatives sequence as

$$\begin{aligned} \mu(\psi, \zeta, \mathcal{T}) &= \sum_{m=0}^{\infty} \mu_m(\psi, \zeta) = \mu_0(\psi, \zeta) + \mu_1(\psi, \zeta) + \mu_2(\psi, \zeta) + \mu_3(\psi, \zeta) + \dots, \\ v(\psi, \zeta, \mathcal{T}) &= \sum_{m=0}^{\infty} v_m(\psi, \zeta) = v_0(\psi, \zeta) + v_1(\psi, \zeta) + v_2(\psi, \zeta) + v_3(\psi, \zeta) + \dots, \\ \mu(\psi, \zeta, \mathcal{T}) &= -\sin(\psi + \zeta) + \frac{q\mathcal{T}^\delta}{\Gamma(\delta + 1)} + \sin(\psi + \zeta) \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta + 1)} \\ &\quad - \sin(\psi + \zeta) \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)} + \sin(\psi + \zeta) \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)} - \dots \\ &\quad - \sin(\psi + \zeta) \sum_{m=0}^{\infty} \frac{(-2\rho)^m\mathcal{T}^{m\delta}}{\Gamma(m\delta + 1)}, \\ v(\psi, \zeta, \mathcal{T}) &= \sin(\psi + \zeta) - \frac{q\mathcal{T}^\delta}{\Gamma(\delta + 1)} - \sin(\psi + \zeta) \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta + 1)} \\ &\quad \times \sin(\psi + \zeta) \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)} - \sin(\psi + \zeta) \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)} + \dots \\ &\quad + \sin(\psi + \zeta) \sum_{m=0}^{\infty} \frac{(-2\rho)^m\mathcal{T}^{m\delta}}{\Gamma(m\delta + 1)}. \end{aligned}$$



At  $\delta = 1$  and  $q = 0$ , the exact solution of Eq. (11) is

$$\begin{aligned} \mu(\psi, \zeta, T) &= -e^{-2\rho T} \sin(\psi + \zeta), \\ v(\psi, \zeta, T) &= e^{-2\rho T} \sin(\psi + \zeta). \end{aligned} \tag{16}$$

**5.2 Problem 2**

Consider the system of fractional order Navier–Stokes equation

$$\begin{aligned} D_{\mathcal{T}}^{\delta}(\mu) + \mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} &= \rho \left[ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right] + q, \\ D_{\mathcal{T}}^{\delta}(v) + \mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} &= \rho \left[ \frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} \right] - q, \end{aligned} \tag{17}$$

with initial conditions

$$\begin{cases} \mu(\psi, \zeta, 0) = -e^{\psi+\zeta}, \\ v(\psi, \zeta, 0) = e^{\psi+\zeta}. \end{cases} \tag{18}$$

After the Elzaki transformation of Eq. (17), we get

$$\begin{aligned} E \left\{ \frac{\partial^{\delta} \mu}{\partial \mathcal{T}^{\delta}} \right\} &= E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} + q \right], \\ E \left\{ \frac{\partial^{\delta} v}{\partial \mathcal{T}^{\delta}} \right\} &= E \left[ - \left( \mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} \right\} - q \right], \\ \frac{1}{s^{\delta}} E \{ \mu(\psi, \zeta, T) \} - s^{2-\delta} \mu(\psi, \zeta, 0) &= E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} + q \right], \\ \frac{1}{s^{\delta}} E \{ v(\psi, \zeta, T) \} - s^{2-\delta} v(\psi, \zeta, 0) &= E \left[ - \left( \mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} \right\} - q \right]. \end{aligned}$$

The simplified form of the above algorithm is

$$\begin{aligned} E \{ \mu(\psi, \zeta, T) \} &= s^2 \{ \mu(\psi, \zeta, 0) \} + s^{\delta} E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} + q \right], \\ E \{ v(\psi, \zeta, T) \} &= s^2 \{ v(\psi, \zeta, 0) \} + s^{\delta} E \left[ - \left( \mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} \right\} - q \right]. \end{aligned} \tag{19}$$

Using the inverse Elzaki transformation, we obtain

$$\begin{aligned} \mu(\psi, \zeta, T) &= \mu(\psi, \zeta, 0) + E^{-} [ s^{\delta} E^{+} \{ q \} ] \\ &\quad + E^{-} \left[ s^{\delta} E \left[ - \left( \mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} \right] \right], \\ v(\psi, \zeta, T) &= v(\psi, \zeta, 0) - E^{-} [ s^{\delta} E \{ q \} ] \\ &\quad + E^{-} \left[ s^{\delta} E \left[ - \left( \mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} \right) + \rho \left\{ \frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} \right\} \right] \right]. \end{aligned} \tag{20}$$

Assume that the unknown functions  $\mu(\psi, \zeta, T)$  and  $\nu(\psi, \zeta, T)$  have the following solution in infinite series form:

$$\mu(\psi, \zeta, T) = \sum_{m=0}^{\infty} \mu_m(\psi, \zeta, T) \quad \text{and} \quad \nu(\psi, \zeta, T) = \sum_{m=0}^{\infty} \nu_m(\psi, \zeta, T).$$

Remember that  $\mu\mu_\psi = \sum_{m=0}^{\infty} \mathcal{A}_m$ ,  $\nu\mu_\zeta = \sum_{m=0}^{\infty} \mathcal{B}_m$ ,  $\mu\nu_\psi = \sum_{m=0}^{\infty} \mathcal{C}_m$  and  $\nu\nu_\zeta = \sum_{m=0}^{\infty} \mathcal{D}_m$  are the Adomian polynomials and the nonlinear terms were characterized. Equation (20) can be rewritten in the form using certain terms

$$\begin{aligned} \sum_{m=0}^{\infty} \mu_m(\psi, \zeta, T) &= \mu(\psi, \zeta, 0) + E^- [s^\delta E\{q\}] \\ &\quad + E^- \left[ s^\delta E \left[ - \left( \sum_{m=0}^{\infty} \mathcal{A}_m + \sum_{m=0}^{\infty} \mathcal{B}_m \right) + \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} \right\} \right] \right], \\ \sum_{m=0}^{\infty} \nu_m(\psi, \zeta, T) &= \nu(\psi, \zeta, 0) - E^- [s^\delta E\{q\}] \\ &\quad + E^- \left[ s^\delta E \left[ - \left( \sum_{m=0}^{\infty} \mathcal{C}_m + \sum_{m=0}^{\infty} \mathcal{D}_m \right) + \rho \left\{ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} \right\} \right] \right], \tag{21} \\ \sum_{m=0}^{\infty} \mu_m(\psi, \zeta, T) &= -e^{(\psi+\zeta)} + \frac{qT^\delta}{\Gamma(\delta+1)} + E^- \left[ s^\delta E \left[ - \left( \sum_{m=0}^{\infty} \mathcal{A}_m + \sum_{m=0}^{\infty} \mathcal{B}_m \right) \right] \right] \\ &\quad + E^- \left[ s^\delta E \left[ \rho \left\{ \sum_{m=0}^{\infty} \frac{\partial^2 \mu_m}{\partial \psi^2} + \sum_{m=0}^{\infty} \frac{\partial^2 \mu_m}{\partial \zeta^2} \right\} \right] \right], \\ \sum_{m=0}^{\infty} \nu_m(\psi, \zeta, T) &= e^{(\psi+\zeta)} - \frac{qT^\delta}{\Gamma(\delta+1)} - E^- \left[ s^\delta E \left[ - \left( \sum_{m=0}^{\infty} \mathcal{C}_m + \sum_{m=0}^{\infty} \mathcal{D}_m \right) \right] \right] \\ &\quad + E^- \left[ s^\delta E \left[ \rho \left\{ \sum_{m=0}^{\infty} \frac{\partial^2 \nu_m}{\partial \psi^2} + \sum_{m=0}^{\infty} \frac{\partial^2 \nu_m}{\partial \zeta^2} \right\} \right] \right]. \end{aligned}$$

According to Eq. (7), all types of nonlinearity can be represented by the Adomian polynomials as

$$\begin{aligned} \mathcal{A}_0 &= \mu_0 \frac{\partial \mu_0}{\partial \psi}, & \mathcal{A}_1 &= \mu_0 \frac{\partial \mu_1}{\partial \psi} + \mu_1 \frac{\partial \mu_0}{\partial \psi}, \\ \mathcal{B}_0 &= \nu_0 \frac{\partial \mu_0}{\partial \zeta}, & \mathcal{B}_1 &= \nu_0 \frac{\partial \mu_1}{\partial \zeta} + \nu_1 \frac{\partial \mu_0}{\partial \zeta}, \\ \mathcal{C}_0 &= \mu_0 \frac{\partial \nu_0}{\partial \psi}, & \mathcal{C}_1 &= \mu_0 \frac{\partial \nu_1}{\partial \psi} + \mu_1 \frac{\partial \nu_0}{\partial \psi}, \\ \mathcal{D}_0 &= \nu_0 \frac{\partial \nu_0}{\partial \psi}, & \mathcal{D}_1 &= \nu_0 \frac{\partial \nu_1}{\partial \psi} + \nu_1 \frac{\partial \nu_0}{\partial \psi}. \end{aligned}$$

Thus, by comparing both sides of Eq. (21) we can get easily the recursive relationship

$$\mu_0(\psi, \zeta, T) = -e^{\psi+\zeta} + \frac{qT^\delta}{\Gamma(\delta+1)}, \quad \nu_0(\psi, \zeta, T) = e^{\psi+\zeta} - \frac{qT^\delta}{\Gamma(\delta+1)}.$$

For  $m = 0$

$$\mu_1(\psi, \zeta, \mathcal{T}) = -e^{\psi+\zeta} \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta+1)}, \quad \nu_1(\psi, \zeta, \mathcal{T}) = e^{\psi+\zeta} \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta+1)}.$$

For  $m = 1$

$$\mu_2(\psi, \zeta, \mathcal{T}) = -e^{\psi+\zeta} \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta+1)}, \quad \nu_2(\psi, \zeta, \mathcal{T}) = e^{\psi+\zeta} \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta+1)}.$$

For  $m = 2$

$$\mu_3(\psi, \zeta, \mathcal{T}) = -e^{\psi+\zeta} \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta+1)}, \quad \nu_3(\psi, \zeta, \mathcal{T}) = e^{\psi+\zeta} \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta+1)}.$$

⋮

In the same manner, the remaining  $\mu_m$  and  $\nu_m$  ( $m > 3$ ) elements of the ETDM solution are easy to obtain. So we describe the alternatives sequence as

$$\mu(\psi, \zeta, \mathcal{T}) = \sum_{m=0}^{\infty} \mu_m(\psi, \zeta) = \mu_0(\psi, \zeta) + \mu_1(\psi, \zeta) + \mu_2(\psi, \zeta) + \mu_3(\psi, \zeta) + \dots,$$

$$\nu(\psi, \zeta, \mathcal{T}) = \sum_{m=0}^{\infty} \nu_m(\psi, \zeta) = \nu_0(\psi, \zeta) + \nu_1(\psi, \zeta) + \nu_2(\psi, \zeta) + \nu_3(\psi, \zeta) + \dots,$$

$$\begin{aligned} \mu(\psi, \zeta, \mathcal{T}) &= -e^{\psi+\zeta} + \frac{q\mathcal{T}^\delta}{\Gamma(\delta+1)} - e^{\psi+\zeta} \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta+1)} \\ &\quad - e^{\psi+\zeta} \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta+1)} - e^{\psi+\zeta} \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta+1)} - \dots - e^{\psi+\zeta} \sum_{m=0}^{\infty} \frac{(-2\rho)^m\mathcal{T}^{m\delta}}{\Gamma(m\delta+1)}, \end{aligned}$$

$$\begin{aligned} \nu(\psi, \zeta, \mathcal{T}) &= e^{\psi+\zeta} - \frac{q\mathcal{T}^\delta}{\Gamma(\delta+1)} + e^{\psi+\zeta} \frac{2\rho\mathcal{T}^\delta}{\Gamma(\delta+1)} \\ &\quad + e^{\psi+\zeta} \frac{(2\rho)^2\mathcal{T}^{2\delta}}{\Gamma(2\delta+1)} + e^{\psi+\zeta} \frac{(2\rho)^3\mathcal{T}^{3\delta}}{\Gamma(3\delta+1)} + \dots - e^{\psi+\zeta} \sum_{m=0}^{\infty} \frac{(-2\rho)^m\mathcal{T}^{m\delta}}{\Gamma(m\delta+1)}. \end{aligned}$$

At  $\delta = 1$  and  $q = 0$ , the exact solution of Eq. (17) is

$$\begin{aligned} \mu(\psi, \zeta, \mathcal{T}) &= -e^{\psi+\zeta+2\rho\mathcal{T}}, \\ \nu(\psi, \zeta, \mathcal{T}) &= e^{\psi+\zeta+2\rho\mathcal{T}}. \end{aligned} \tag{22}$$

### 5.3 Problem 3

Consider the system of fractional order Navier–Stokes equations

$$\begin{aligned} D_{\mathcal{T}}^\delta(\mu) + \mu \frac{\partial \mu}{\partial \psi} + \nu \frac{\partial \mu}{\partial \zeta} + \omega \frac{\partial \mu}{\partial \gamma} &= \rho \left[ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \gamma^2} \right] + q_1, \\ D_{\mathcal{T}}^\delta(\nu) + \mu \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial \nu}{\partial \zeta} + \omega \frac{\partial \nu}{\partial \gamma} &= \rho \left[ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} + \frac{\partial^2 \nu}{\partial \gamma^2} \right] + q_2, \\ D_{\mathcal{T}}^\delta(\omega) + \mu \frac{\partial \omega}{\partial \psi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \frac{\partial \omega}{\partial \gamma} &= \rho \left[ \frac{\partial^2 \omega}{\partial \psi^2} + \frac{\partial^2 \omega}{\partial \zeta^2} + \frac{\partial^2 \omega}{\partial \gamma^2} \right] + q_3, \end{aligned} \tag{23}$$

with initial conditions

$$\begin{cases} \mu(\psi, \zeta, \gamma, 0) = -0.5\psi + \zeta + \gamma, \\ v(\psi, \zeta, \gamma, 0) = \psi - 0.5\zeta + \gamma, \\ \omega(\psi, \zeta, \gamma, 0) = \psi + \zeta - 0.5\gamma. \end{cases} \tag{24}$$

Furthermore, if  $\rho$  is known, then  $q_1 = -\frac{1}{\rho} \frac{\partial g}{\partial \psi}$ ,  $q_2 = -\frac{1}{\rho} \frac{\partial g}{\partial \zeta}$  and  $q_3 = -\frac{1}{\rho} \frac{\partial g}{\partial \gamma}$  can be determined.

After the Elzaki transformation of Eq. (23), we get

$$\begin{aligned} E\left\{\frac{\partial^\delta \mu}{\partial \mathcal{T}^\delta}\right\} &= E\left[-\left(\mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} + \omega \frac{\partial \mu}{\partial \gamma}\right)\right] + E\left\{\rho \left\{\frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \gamma^2}\right\} + q_1\right\}, \\ E\left\{\frac{\partial^\delta v}{\partial \mathcal{T}^\delta}\right\} &= E\left[-\left(\mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} + \omega \frac{\partial v}{\partial \gamma}\right)\right] + E\left\{\rho \left\{\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial^2 v}{\partial \gamma^2}\right\} + q_2\right\}, \\ E\left\{\frac{\partial^\delta \omega}{\partial \mathcal{T}^\delta}\right\} &= E\left[-\left(\mu \frac{\partial \omega}{\partial \psi} + v \frac{\partial \omega}{\partial \zeta} + \omega \frac{\partial \omega}{\partial \gamma}\right)\right] + E\left\{\rho \left\{\frac{\partial^2 \omega}{\partial \psi^2} + \frac{\partial^2 \omega}{\partial \zeta^2} + \frac{\partial^2 \omega}{\partial \gamma^2}\right\} + q_3\right\}, \\ \frac{1}{s^\delta} E\{\mu(\psi, \zeta, \gamma, \mathcal{T})\} - s^{2-\delta} \mu(\psi, \zeta, \gamma, 0) &= E\left[-\left(\mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} + \omega \frac{\partial \mu}{\partial \gamma}\right)\right] \\ &\quad + E\left\{\rho \left\{\frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \gamma^2}\right\} + q_1\right\}, \\ \frac{1}{s^\delta} E\{v(\psi, \zeta, \gamma, \mathcal{T})\} - s^{2-\delta} v(\psi, \zeta, \gamma, 0) &= E\left[-\left(\mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} + \omega \frac{\partial v}{\partial \gamma}\right)\right] \\ &\quad + E\left\{\rho \left\{\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial^2 v}{\partial \gamma^2}\right\} + q_2\right\}, \\ \frac{1}{s^\delta} E\{\omega(\psi, \zeta, \gamma, \mathcal{T})\} - s^{2-\delta} \omega(\psi, \zeta, \gamma, 0) &= E\left[-\left(\mu \frac{\partial \omega}{\partial \psi} + v \frac{\partial \omega}{\partial \zeta} + \omega \frac{\partial \omega}{\partial \gamma}\right)\right] \\ &\quad + E\left\{\rho \left\{\frac{\partial^2 \omega}{\partial \psi^2} + \frac{\partial^2 \omega}{\partial \zeta^2} + \frac{\partial^2 \omega}{\partial \gamma^2}\right\} + q_3\right\}. \end{aligned}$$

The simplified form of the above algorithm is

$$\begin{aligned} E\{\mu(\psi, \zeta, \gamma, \mathcal{T})\} &= s^2 \mu(\psi, \zeta, \gamma, 0) + s^\delta E\left[-\left(\mu \frac{\partial \mu}{\partial \psi} + v \frac{\partial \mu}{\partial \zeta} + \omega \frac{\partial \mu}{\partial \gamma}\right)\right] \\ &\quad + s^\delta E\left\{\rho \left\{\frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \gamma^2}\right\} + q_1\right\}, \\ E\{v(\psi, \zeta, \gamma, \mathcal{T})\} &= s^2 v(\psi, \zeta, \gamma, 0) + s^\delta E\left[-\left(\mu \frac{\partial v}{\partial \psi} + v \frac{\partial v}{\partial \zeta} + \omega \frac{\partial v}{\partial \gamma}\right)\right] \\ &\quad + s^\delta E\left\{\rho \left\{\frac{\partial^2 v}{\partial \psi^2} + \frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial^2 v}{\partial \gamma^2}\right\} + q_2\right\}, \end{aligned}$$

$$E\{\omega(\psi, \zeta, \gamma, T)\} = s^2\omega(\psi, \zeta, \gamma, 0) + s^\delta E\left[-\left(\mu \frac{\partial \omega}{\partial \psi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \frac{\partial \omega}{\partial \gamma}\right)\right] + s^\delta E\left[\rho \left\{\frac{\partial^2 \omega}{\partial \psi^2} + \frac{\partial^2 \omega}{\partial \zeta^2} + \frac{\partial^2 \omega}{\partial \gamma^2}\right\} + q_3\right].$$

Using the inverse Elzaki transformation, we obtain

$$\begin{aligned} \mu(\psi, \zeta, \gamma, T) &= \mu(\psi, \zeta, \gamma, 0) + E^- \left[ s^\delta E \left[ -\left( \mu \frac{\partial \mu}{\partial \psi} + \nu \frac{\partial \mu}{\partial \zeta} + \omega \frac{\partial \mu}{\partial \gamma} \right) \right] \right] \\ &\quad + E^- \left[ s^\delta E \left\{ \rho \left\{ \frac{\partial^2 \mu}{\partial \psi^2} + \frac{\partial^2 \mu}{\partial \zeta^2} + \frac{\partial^2 \mu}{\partial \gamma^2} \right\} + q_1 \right\} \right], \\ \nu(\psi, \zeta, \gamma, T) &= \nu(\psi, \zeta, \gamma, 0) + E^- \left[ s^\delta E \left[ -\left( \mu \frac{\partial \nu}{\partial \psi} + \nu \frac{\partial \nu}{\partial \zeta} + \omega \frac{\partial \nu}{\partial \gamma} \right) \right] \right] \\ &\quad + E^- \left[ s^\delta E \left\{ \rho \left\{ \frac{\partial^2 \nu}{\partial \psi^2} + \frac{\partial^2 \nu}{\partial \zeta^2} + \frac{\partial^2 \nu}{\partial \gamma^2} \right\} + q_2 \right\} \right], \\ \omega(\psi, \zeta, \gamma, T) &= \omega(\psi, \zeta, \gamma, 0) + E^- \left[ s^\delta E \left[ -\left( \mu \frac{\partial \omega}{\partial \psi} + \nu \frac{\partial \omega}{\partial \zeta} + \omega \frac{\partial \omega}{\partial \gamma} \right) \right] \right] \\ &\quad + E^- \left[ s^\delta E \left\{ \rho \left\{ \frac{\partial^2 \omega}{\partial \psi^2} + \frac{\partial^2 \omega}{\partial \zeta^2} + \frac{\partial^2 \omega}{\partial \gamma^2} \right\} + q_3 \right\} \right]. \end{aligned} \tag{25}$$

Assume that the unknown functions  $\mu(\psi, \zeta, \gamma, T)$ ,  $\nu(\psi, \zeta, \gamma, T)$  and  $\omega(\psi, \zeta, \gamma, T)$  have the following solution in infinite series form:

$$\begin{aligned} \mu(\psi, \zeta, \gamma, T) &= \sum_{m=0}^\infty \mu_m(\psi, \zeta, \gamma, T), \\ \nu(\psi, \zeta, \gamma, T) &= \sum_{m=0}^\infty \nu_m(\psi, \zeta, \gamma, T) \quad \text{and} \\ \omega(\psi, \zeta, \gamma, T) &= \sum_{m=0}^\infty \omega_m(\psi, \zeta, \gamma, T). \end{aligned}$$

The Adomian polynomials of non-linear terms as, using such terms, Eq. (26) can be rewritten in the form

$$\begin{aligned} \mu\mu_\psi &= \sum_{m=0}^\infty \mathcal{A}_m, & \nu\mu_\zeta &= \sum_{m=0}^\infty \mathcal{B}_m, & \omega\mu_\gamma &= \sum_{m=0}^\infty \mathcal{C}_m, & \mu\nu_\psi &= \sum_{m=0}^\infty \mathcal{D}_m, \\ \nu\nu_\zeta &= \sum_{m=0}^\infty \mathcal{E}_m, & \omega\nu_\zeta &= \sum_{m=0}^\infty \mathcal{F}_m, & \mu\omega_\psi &= \sum_{m=0}^\infty \mathcal{G}_m, \\ \nu\omega_\zeta &= \sum_{m=0}^\infty \mathcal{H}_m \quad \text{and} \quad \omega\omega_\zeta &= \sum_{m=0}^\infty \mathcal{I}_m, \\ & \sum_{m=0}^\infty \mu_m(\psi, \zeta, \gamma, T) \\ &= \mu(\psi, \zeta, \gamma, 0) + E^- [s^\delta E\{q_1\}] + E^- \left[ s^\delta E \left[ -\left( \sum_{m=0}^\infty \mathcal{A}_m + \sum_{m=0}^\infty \mathcal{B}_m + \sum_{m=0}^\infty \mathcal{C}_m \right) \right] \right] \end{aligned}$$

$$\begin{aligned}
 & + E^- \left[ s^\delta E \left\{ \rho \left\{ \frac{\partial^2 \mu_m}{\partial \psi^2} + \frac{\partial^2 \mu_m}{\partial \zeta^2} + \frac{\partial^2 \mu_m}{\partial \gamma^2} \right\} + q_1 \right\} \right], \\
 & \sum_{m=0}^{\infty} v_m(\psi, \zeta, \gamma, \mathcal{T}) \\
 & = v(\psi, \zeta, \gamma, 0) - E^- [s^\delta E\{q_2\}] + E^- \left[ s^\delta E \left[ - \left( \sum_{m=0}^{\infty} \mathcal{D}_m + \sum_{m=0}^{\infty} \mathcal{E}_m + \sum_{m=0}^{\infty} \mathcal{F}_m \right) \right] \right] \quad (26) \\
 & + E^- \left[ s^\delta E \left\{ \rho \left\{ \frac{\partial^2 v_m}{\partial \psi^2} + \frac{\partial^2 v_m}{\partial \zeta^2} + \frac{\partial^2 v_m}{\partial \gamma^2} \right\} + q_2 \right\} \right], \\
 & \sum_{m=0}^{\infty} \omega_m(\psi, \zeta, \gamma, \mathcal{T}) \\
 & = \omega(\psi, \zeta, \gamma, 0) - E^- [s^\delta E\{q_3\}] + E^- \left[ s^\delta E \left[ - \left( \sum_{m=0}^{\infty} \mathcal{G}_m + \sum_{m=0}^{\infty} \mathcal{H}_m + \sum_{m=0}^{\infty} \mathcal{I}_m \right) \right] \right] \\
 & + E^- \left[ s^\delta E \left\{ \frac{\partial^2 \omega}{\partial \psi^2} + \frac{\partial^2 \omega}{\partial \zeta^2} + \frac{\partial^2 \omega}{\partial \gamma^2} \right\} + q_3 \right].
 \end{aligned}$$

According to Eq. (7), all types of nonlinearity can be represented by the Adomian polynomials as

$$\begin{aligned}
 \mathcal{A}_0 &= \mu_0 \frac{\partial \mu_0}{\partial \psi}, & \mathcal{A}_1 &= \mu_0 \frac{\partial \mu_1}{\partial \psi} + \mu_1 \frac{\partial \mu_0}{\partial \psi}, & \mathcal{B}_0 &= v_0 \frac{\partial \mu_0}{\partial \zeta}, & \mathcal{B}_1 &= v_0 \frac{\partial \mu_1}{\partial \zeta} + v_1 \frac{\partial \mu_0}{\partial \zeta}, \\
 \mathcal{C}_0 &= \omega_0 \frac{\partial \mu_0}{\partial \zeta}, & \mathcal{C}_1 &= \omega_0 \frac{\partial \mu_1}{\partial \zeta} + \omega_1 \frac{\partial \mu_0}{\partial \zeta}, & \mathcal{D}_0 &= \mu_0 \frac{\partial v_0}{\partial \psi}, & \mathcal{D}_1 &= \mu_0 \frac{\partial v_1}{\partial \psi} + \mu_1 \frac{\partial v_0}{\partial \psi}, \\
 \mathcal{E}_0 &= v_0 \frac{\partial v_0}{\partial \psi}, & \mathcal{E}_1 &= v_0 \frac{\partial v_1}{\partial \psi} + v_1 \frac{\partial v_0}{\partial \psi}, & \mathcal{F}_0 &= \omega_0 \frac{\partial v_0}{\partial \psi}, & \mathcal{F}_1 &= \omega_0 \frac{\partial v_1}{\partial \psi} + \omega_1 \frac{\partial v_0}{\partial \psi}, \\
 \mathcal{G}_0 &= \mu_0 \frac{\partial \omega_0}{\partial \psi}, & \mathcal{G}_1 &= \mu_0 \frac{\partial \omega_1}{\partial \psi} + \mu_1 \frac{\partial \omega_0}{\partial \psi}, & \mathcal{H}_0 &= v_0 \frac{\partial \omega_0}{\partial \psi}, & \mathcal{H}_1 &= v_0 \frac{\partial \omega_1}{\partial \psi} + v_1 \frac{\partial \omega_0}{\partial \psi}, \\
 \mathcal{I}_0 &= \omega_0 \frac{\partial \omega_0}{\partial \psi}, & \mathcal{I}_1 &= \omega_0 \frac{\partial \omega_1}{\partial \psi} + \omega_1 \frac{\partial \omega_0}{\partial \psi}.
 \end{aligned}$$

Thus, by comparing the two sides of Eq. (27) we can get easily the recursive relationship

$$\begin{aligned}
 \mu_0(\psi, \zeta, \gamma, \mathcal{T}) &= -0.5\psi + \zeta + \gamma, \\
 v_0(\psi, \zeta, \gamma, \mathcal{T}) &= \psi - 0.5\zeta + \gamma, \\
 \omega_0(\psi, \zeta, \gamma, \mathcal{T}) &= \psi + \zeta - 0.5\gamma.
 \end{aligned}$$

For  $m = 0$

$$\begin{aligned}
 \mu_1(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{-2.25\psi \mathcal{T}^\delta}{\Gamma(\delta + 1)}, & v_1(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{-2.25\zeta \mathcal{T}^\delta}{\Gamma(\delta + 1)}, \\
 \omega_1(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{-2.25\gamma \mathcal{T}^\delta}{\Gamma(\delta + 1)}.
 \end{aligned}$$

For  $m = 1$

$$\begin{aligned} \mu_2(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{2(2.25)\psi \mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)}(-0.5\psi + \zeta + \gamma), \\ \nu_2(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{2(2.25)\zeta \mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)}(\psi - 0.5\zeta + \gamma), \\ \omega_2(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{2(2.25)\gamma \mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)}(\psi + \zeta - 0.5\gamma). \end{aligned}$$

For  $m = 2$

$$\begin{aligned} \mu_3(\psi, \zeta, \gamma, \mathcal{T}) &= -\frac{(2.25)^2\psi(4(\Gamma(\delta + 1))^2 + \Gamma(2\delta + 1))\mathcal{T}^{3\delta}}{\Gamma(2\delta + 1)(\Gamma(\delta + 1))^2}, \\ \nu_3(\psi, \zeta, \gamma, \mathcal{T}) &= -\frac{(2.25)^2\zeta(4(\Gamma(\delta + 1))^2 + \Gamma(2\delta + 1))\mathcal{T}^{3\delta}}{\Gamma(2\delta + 1)(\Gamma(\delta + 1))^2}, \\ \omega_3(\psi, \zeta, \gamma, \mathcal{T}) &= -\frac{(2.25)^2\gamma(4(\Gamma(\delta + 1))^2 + \Gamma(2\delta + 1))\mathcal{T}^{3\delta}}{\Gamma(2\delta + 1)(\Gamma(\delta + 1))^2}, \\ &\dots \\ &\dots \end{aligned}$$

In the same manner, the remaining  $\mu_m, \nu_m$  and  $\omega_m$  ( $m > 3$ ) elements of the ETDM solution are easy to obtain. So we describe the alternative sequence as

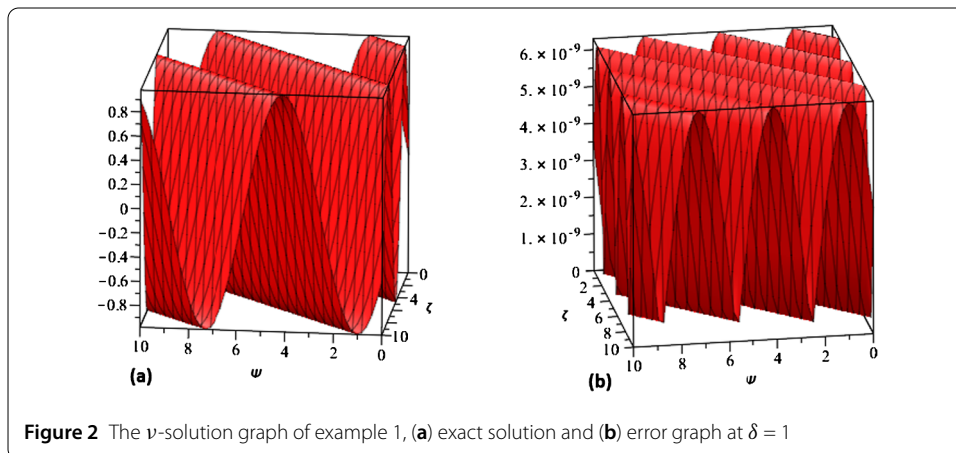
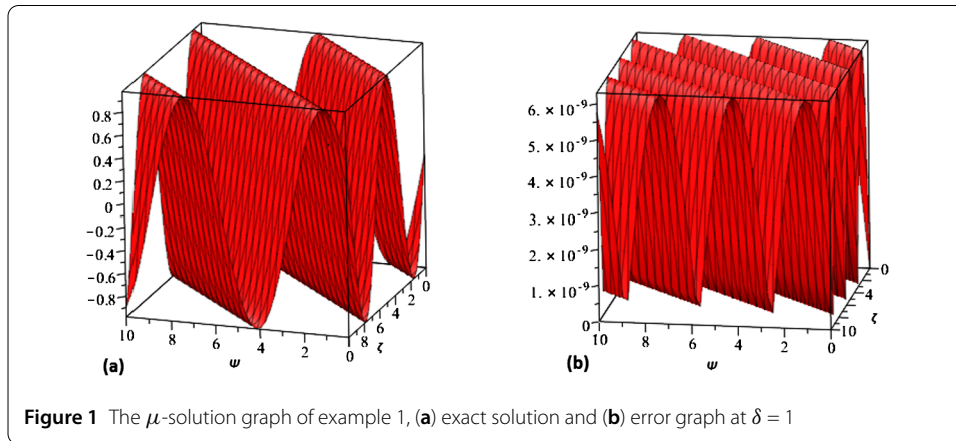
$$\begin{aligned} \mu(\psi, \zeta, \gamma, \mathcal{T}) &= \sum_{m=0}^{\infty} \mu_m(\psi, \zeta) = \mu_0(\psi, \zeta) + \mu_1(\psi, \zeta) + \mu_2(\psi, \zeta) + \mu_3(\psi, \zeta) + \dots, \\ \nu(\psi, \zeta, \gamma, \mathcal{T}) &= \sum_{m=0}^{\infty} \nu_m(\psi, \zeta) = \nu_0(\psi, \zeta) + \nu_1(\psi, \zeta) + \nu_2(\psi, \zeta) + \nu_3(\psi, \zeta) + \dots, \\ \omega(\psi, \zeta, \gamma, \mathcal{T}) &= \sum_{m=0}^{\infty} \omega_m(\psi, \zeta) = \omega_0(\psi, \zeta) + \omega_1(\psi, \zeta) + \omega_2(\psi, \zeta) + \omega_3(\psi, \zeta) + \dots, \\ \mu(\psi, \zeta, \gamma, \mathcal{T}) &= -0.5\psi + \zeta + \gamma - \frac{2.25\psi \mathcal{T}^\delta}{\Gamma(\delta + 1)} + \frac{2(2.25)\psi \mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)} \\ &\quad \times (-0.5\psi + \zeta + \gamma) - \frac{(2.25)^2\psi \mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)} \left( 4 + \frac{\Gamma(2\delta + 1)}{(\Gamma(\delta + 1))^2} \right) + \dots, \\ \nu(\psi, \zeta, \gamma, \mathcal{T}) &= \psi - 0.5\zeta + \gamma - \frac{2.25\zeta \mathcal{T}^\delta}{\Gamma(\delta + 1)} + \frac{2(2.25)\zeta \mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)} \\ &\quad \times (\psi - 0.5\zeta + \gamma) - \frac{(2.25)^2\zeta \mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)} \left( 4 + \frac{\Gamma(2\delta + 1)}{(\Gamma(\delta + 1))^2} \right) + \dots, \\ \omega(\psi, \zeta, \gamma, \mathcal{T}) &= \psi + \zeta - 0.5\gamma - \frac{2.25\gamma \mathcal{T}^\delta}{\Gamma(\delta + 1)} + \frac{2(2.25)\gamma \mathcal{T}^{2\delta}}{\Gamma(2\delta + 1)} \\ &\quad \times (\psi + \zeta - 0.5\gamma) - \frac{(2.25)^2\gamma \mathcal{T}^{3\delta}}{\Gamma(3\delta + 1)} \left( 4 + \frac{\Gamma(2\delta + 1)}{(\Gamma(\delta + 1))^2} \right) + \dots. \end{aligned}$$

At  $\delta = 1$  and  $q_1 = q_2 = q_3 = 0$ , the exact solution of Eq. (23) is

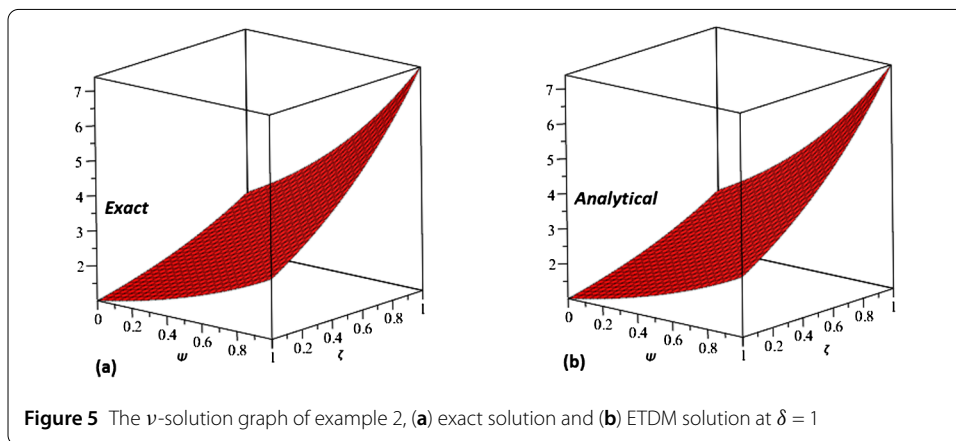
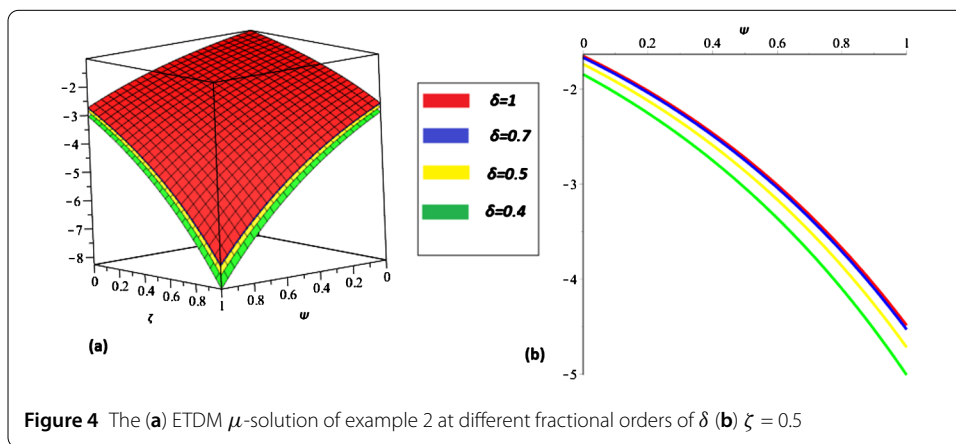
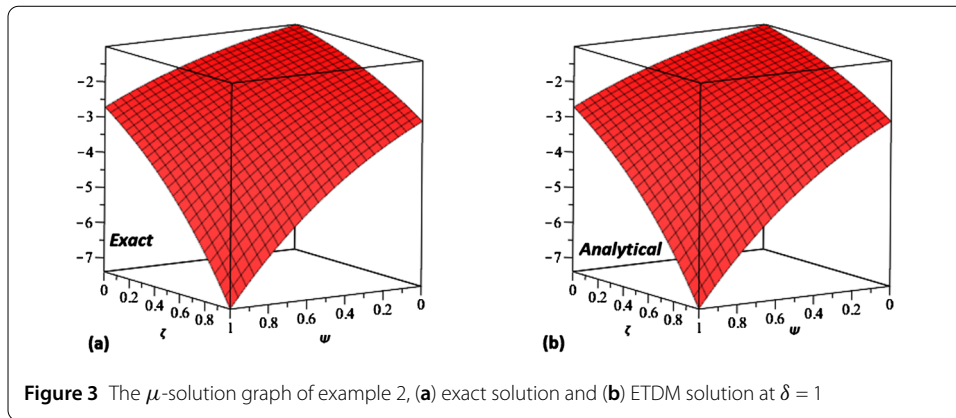
$$\begin{aligned} \mu(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{-0.5\psi + \zeta + \gamma - 2.25\psi\mathcal{T}}{1 - 2.25\mathcal{T}^2}, \\ \nu(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{\psi - 0.5\zeta + \gamma - 2.25\zeta\mathcal{T}}{1 - 2.25\mathcal{T}^2}, \\ \omega(\psi, \zeta, \gamma, \mathcal{T}) &= \frac{\psi + \zeta - 0.5\gamma - 2.25\gamma\mathcal{T}}{1 - 2.25\mathcal{T}^2}. \end{aligned} \tag{27}$$

### 6 Results and discussion

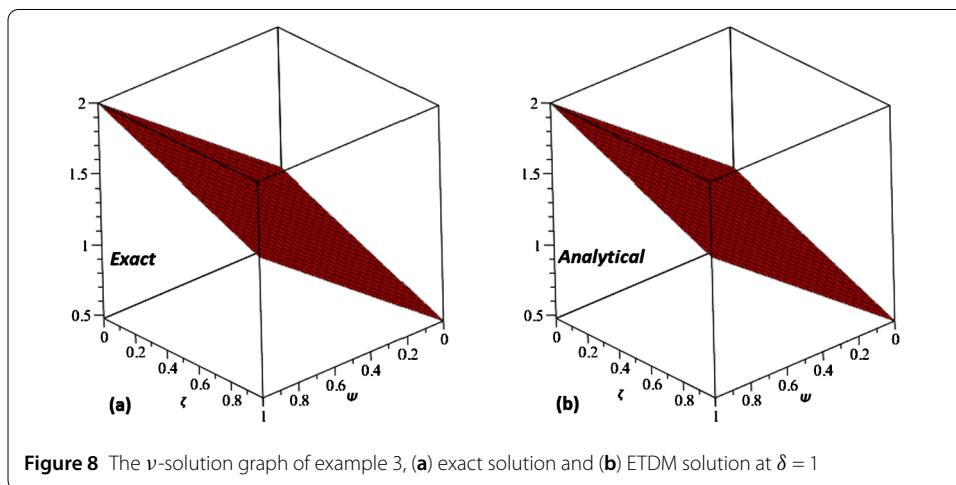
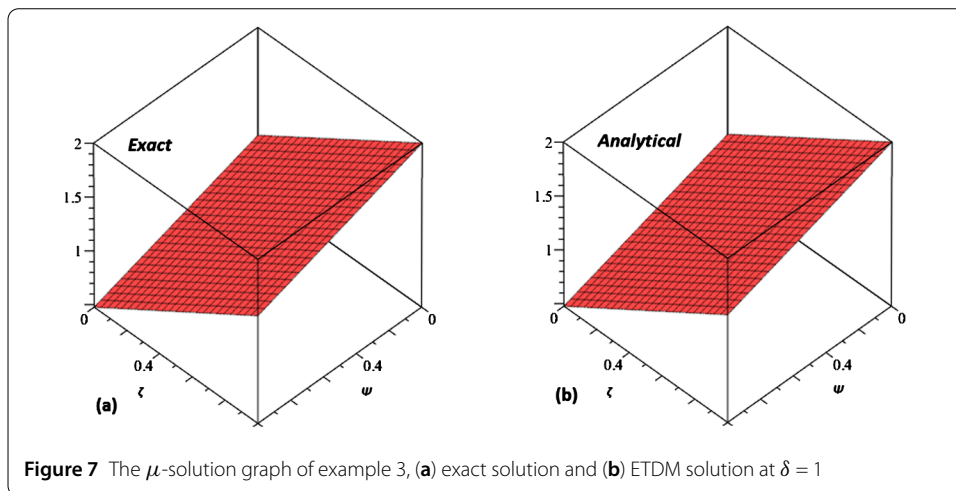
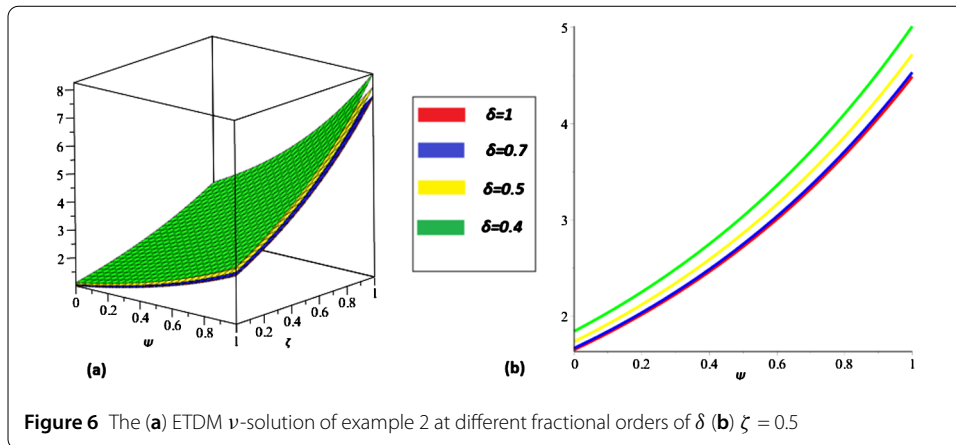
In Fig. 1, the subgraphs (a) and (b) represent the exact  $\mu$ -solution and associated ETDM error, respectively. In Fig. 2, the subgraphs (a) and (b) denotes the exact  $\nu$ -solution and associated error of ETDM, respectively, at  $\delta = 1$ . The error-graphs in Figs. 1 and 2 confirmed the higher accuracy of the proposed method. In Fig. 3, the comparison of the exact and ETDM  $\mu$ -solutions are displaced by using subgraphs (a) and (b) respectively for example 2. The exact and ETDM solutions are in closed contacts shown by their graphs. In Fig. 4, the  $\mu$ -solutions of example 2 at different fractional-orders are presented in both two and three dimensional sub-plots (a) and (b) respectively. In Fig. 5, the  $\nu$ -solutions comparison is done with sufficient degree of accuracy of example 2. Figure 6, represents the various



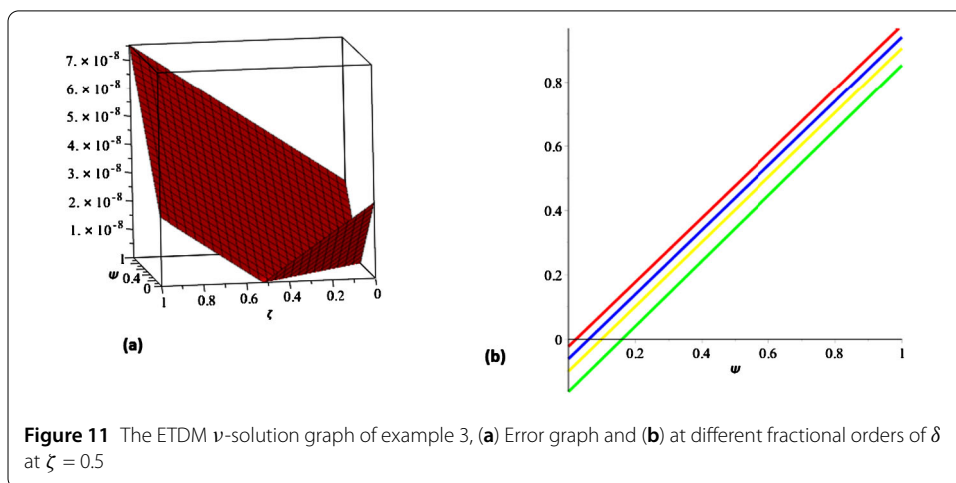
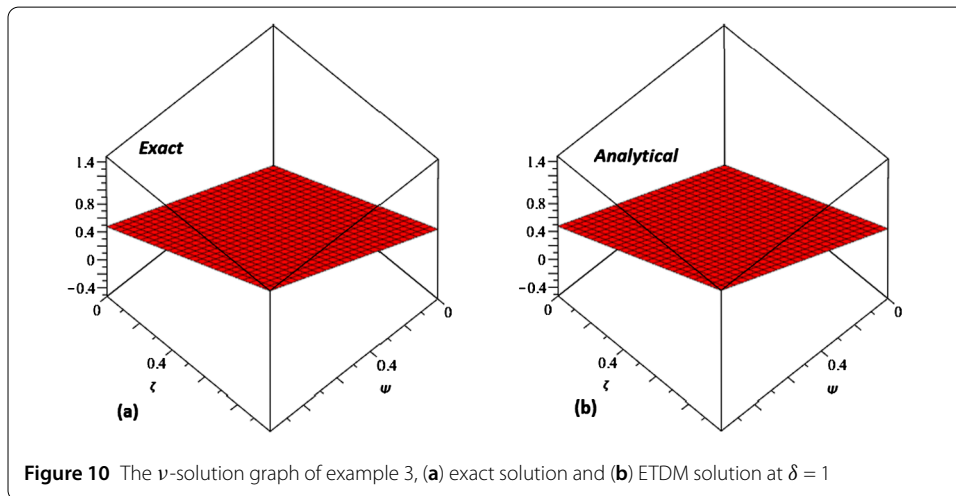
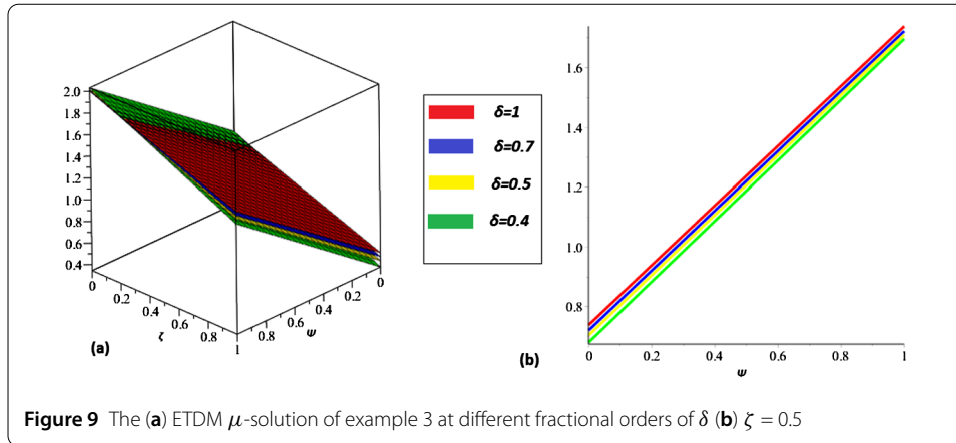




fractional solutions for variable  $\nu$  by using sub-graphs (a) and (b) of example 2 in both two and three dimensions. It is observed that the ETDM solutions are in good contact with the actual solutions of example 2. In Fig. 7, the exact and ETDM  $\mu$ -solutions are compared at  $\delta = 1$  of example 3. The comparison has shown a very close relation between the actual and ETDM solutions. In Fig. 8, the exact and ETDM solutions at  $\delta = 1$  of example 3 are represented by the sub-graphs (a) and (b) respectively. In Fig. 9, various fractional-order solutions of example 3 are presented in both two and three dimensional space. In Fig. 10, the  $\omega$ -exact and ETDM solutions are plotted which confirmed to close relation between



ETDM and exact solutions of example 3. Similarly in Fig. 11, the ETDM error is analyzed and solution at different fractional-orders are presented in two-dimensional graph for example 3. Tables 1 and 2, represent the exact, ETDM and the associated ETDM absolute error of example 1 at  $\delta = 1$  for  $\mu$  and  $\nu$  variables respectively. Both Tables 1 and 2 have



shown the sufficient degree of accuracy. Tables 3 and 4 provide the comparison of exact and ETDM solutions in term of absolute error at  $\delta = 1$ , at time  $\tau = 0.0005$  for variable  $\mu$  and  $\nu$  respectively.

**Table 1**  $\mu(\psi, \zeta, \mathcal{T})$  Comparison of Exact solution, ETDM and ETDM Error of example 1 at  $\rho = 1$

$\mathcal{T} = 0.01$ $\psi$	Exact solution $\delta = 1$	ETDM solution $\delta = 1$	AE of ETDM $\delta = 1$
0	-0.824808742900000	-0.824808737600000	5.3012672040E-09
1	0.891292131400000	-0.891292125700000	5.7285737890E-09
2	-0.138325644700000	-0.138325643800000	8.8905605100E-10
3	0.741816801800000	0.741816797100000	4.7678557200E-09
4	0.939936301900000	0.939936295800000	6.0412229310E-09
5	0.273882700600000	0.273882698900000	1.7603176390E-09
6	-0.643977392400000	-0.643977388300000	4.1390155720E-09
7	-0.969767640700000	-0.969767634500000	6.2329569540E-09
8	-0.403957992400000	-0.403957989800000	2.5963464570E-09
9	0.533248771200000	0.533248767700000	3.4273329990E-09
10	0.980189073800000	0.980189067500000	6.2999383020E-09

**Table 2**  $\nu(\psi, \zeta, \mathcal{T})$  Comparison of Exact solution, ETDM and ETDM Error of example 1 at  $\rho = 1$

$\mathcal{T} = 0.01$ $\psi$	Exact solution $\delta = 1$	ETDM solution $\delta = 1$	AE of ETDM $\delta = 1$
0	0.824808742900000	0.824808737600000	5.3012672040E-09
1	0.891292131400000	0.891292125700000	5.7285737890E-09
2	0.138325644700000	0.138325643800000	8.8905605100E-10
3	-0.741816801800000	-0.741816797100000	4.7678557200E-09
4	-0.939936301900000	-0.939936295800000	6.0412229310E-09
5	-0.273882700600000	-0.273882698900000	1.7603176390E-09
6	0.643977392400000	0.643977388300000	4.1390155720E-09
7	0.969767640700000	0.969767634500000	6.2329569540E-09
8	0.403957992400000	0.403957989800000	2.5963464570E-09
9	-0.533248771200000	-0.533248767700000	3.4273329990E-09
10	-0.980189073800000	-0.980189067500000	6.2999383020E-09

**Table 3**  $\mu(\psi, \zeta, \mathcal{T})$  Comparison of Exact solution, ETDM and ETDM Error of example 1 at  $\rho = 1$

$\mathcal{T} = 0.0005$ $\psi$	Exact solution $\delta = 1$	ETDM solution $\delta = 1$	AE of ETDM $\delta = 1$
0	-1.650370817000000	-1.650370817000000	0.0000000000E+00
1	-4.486173001000000	-4.486173000000000	1.0000000000E-09
2	-12.194682550000000	-12.194682550000000	0.0000000000E+00
3	-33.148583970000000	-33.148583970000000	0.0000000000E+00
4	-90.107193460000000	-90.107193440000000	2.0000000000E-08
5	-244.936746600000000	-244.936746600000000	0.0000000000E+00
6	-665.807107400000000	-665.807107200000000	2.0000000000E-07
7	-1809.851361000000000	-1809.851360000000000	1.0000000000E-06
8	-4919.686067000000000	-4919.686066000000000	1.0000000000E-06
9	-13373.093240000000000	-13373.093240000000000	0.0000000000E+00
10	-36351.836340000000000	-36351.836330000000000	1.0000000000E-05

**Table 4**  $\nu(\psi, \zeta, \mathcal{T})$  Comparison of Exact solution, ETDM and ETDM Error of example 1 at  $\rho = 1$

$\mathcal{T} = 0.0005$ $\psi$	Exact solution $\delta = 1$	ETDM solution $\delta = 1$	AE of ETDM $\delta = 1$
0	1.650370817000000	1.650370817000000	0.0000000000E+00
1	4.486173001000000	4.486173000000000	1.0000000000E-09
2	12.194682550000000	12.194682550000000	0.0000000000E+00
3	33.148583970000000	33.148583970000000	0.0000000000E+00
4	90.107193460000000	90.107193440000000	2.0000000000E-08
5	244.936746600000000	244.936746600000000	0.0000000000E+00
6	665.807107400000000	665.807107200000000	2.0000000000E-07
7	1809.851361000000000	1809.851360000000000	1.0000000000E-06
8	4919.686067000000000	4919.686066000000000	1.0000000000E-06
9	13373.093240000000000	13373.093240000000000	0.0000000000E+00
10	36351.836340000000000	36351.836330000000000	1.0000000000E-05

## 7 Conclusion

It is always difficult to investigate the solution of nonlinear fractional mathematical models which frequently occur in science and engineering. In this paper, we attempted with success to find the analytical solutions of some nonlinear fractional Navier–Stokes equations. The obtained results are found to be accurate and close to the exact solutions of the problems. The solution presentation has been done with the help of tables and graphs which confirmed the reliability of the proposed method. The solutions at different fractional orders are determined and found to be interesting as regards explaining the various dynamical behaviors of the suggested problems. To handle the nonlinearity of the problems and then solutions calculation are the novelty of the current research work. In conclusion, this work will contribute to investigating other nonlinear dynamics in science and engineering.

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### Availability of data and materials

Not applicable.

### Competing interests

The authors declare that there are no conflicts of interest regarding the publication of this article.

### Authors' contributions

The authors declare that this study was accomplished in collaboration with the same responsibility. All authors read and approved the final manuscript.

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