Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(6): 10387–10399.
DOI: 10.3934/math.2022579
Received: 13 February 2022
Revised: 22 March 2022
Accepted: 23 March 2022
Published: 28 March 2022

Research article

Analysing discrete fractional operators with exponential kernel for positivity in lower boundedness

Sarkhel Akbar Mahmood[1], **Pshtiwan Othman Mohammed**[1,*], **Dumitru Baleanu**[2,3,4,*], **Hassen Aydi**[5,6,7,*] **and Yasser S. Hamed**[8]

[1] Department of Mathematics, College of Education, University of Sulaimani, Sulaimani 46001, Kurdistan Region, Iraq

[2] Department of Mathematics, Cankaya University, Balgat 06530, Ankara, Turkey

[3] Institute of Space Sciences, R76900 Magurele-Bucharest, Romania

[4] Department of Mathematics, King Abdul Aziz University, Jeddah 21577, Saudi Arabia

[5] Dniversité de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

[6] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[7] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

[8] Department of Mathematics and Statistics, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: pshtiwansangawi@gmail.com, dumitru@cankaya.edu.tr, hassen.aydi@isima.rnu.tn.

Abstract: In this paper we study the positivity analysis problems for discrete fractional operators with exponential kernel, namely the discrete Caputo-Fabrizio operators. The results are applied to a discrete Caputo-Fabrizio-Caputo fractional operator of order ω of another discrete Caputo-Fabrizio-Riemann fractional operator of order β. Furthermore, the results are obtained for these operators with having the same orders. The conditions for the discrete fractional operators with respect to negative lower bound conditions are expressed in terms of a positive epsilon.

Keywords: discrete fractional operators; positivity analyses; exponential kernel; negative lower bound
Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A12, 39B62

1. Introduction

In recent years, the studies of discrete fractional calculus operators have received much attentions, as the investigation of these operators allow us to get better understanding the interactions between fractional differences and fractional sums (see e.g. [1–4]). Furthermore, in many applications discrete fractional difference operators present more accurate models of phenomena than in the continues operator cases. Therefore, they have obtained quality and importance due to their applications in recent science and engineering problems such as mechanics, fluid dynamic, physics, chemistry, etc. (see e.g. [5–8]). In the meantime, there have appeared many research articles dealing with the existence and uniqueness of solutions for different types of boundary value problems in the discrete fractional framework (see e.g. [9–13]).

Monotonicity and positivity analyses represent a special class of mathematical analysis which arise in the study of discrete fractional calculus. Moreover, these analyses on the discrete fractional operators over a time scale set $\mathcal{N}_0 := \{0, 1, \ldots\}$ have been well developed ever since Dahal and Goodrich [14], and Atici and Uyanik [15] proved the several theorems for analysing the monotonicity of the discrete fractional operators of Riemann-Liouville type in 2014 and 2015, respectively. A key point of their study is to first introduce the $\omega-$monotonicity concept for the functions defined on $\mathcal{N}_b := \{b, b + 1, \ldots\}$ induced by the positivity of the discrete Riemann-Liouville fractional operators, then prove $\omega-$monotonicity increasing for the corresponding discrete fractional differences of Riemann-Liouville type. For more monotonicity analysis on discrete Riemann-Liouville fractional operators we refer to (see e.g. [16–18] and references within). Recently, there have been extensive studies by many researchers for different discrete operators such as Riemann, Caputo, Caputo-Fabrizio and Attangana-Baleanu on $\mathcal{N}_b$ (see e.g. [19–22]). Besides, plenty of researchers have extended these results to generalized discrete fractional operators defined on $\mathcal{N}_b^h := \{b, b + h, b + 2h, \ldots\}$ (see e.g. [23–26]). For further results in this direction (see e.g. [27–32]).

Motivated by all these works together with the results in [33] which are established for the composition of two discrete Caputo-Fabrizio-Caputo fractional operators, in this paper we focus on the analysis of the discrete Caputo-Fabrizio-Caputo fractional operator of order ω of another discrete Caputo-Fabrizio-Riemann fractional operator of order β. Also, we the analysis of the operators for the same order. In both cases, we use a set of conditions in their lower bounds.

A brief outline of the study is designed as follows. In Section 2, we recall the main notations and basic definitions about discrete Caputo-Fabrizio fractional operators, as well as we present the basic lemmas and give their proofs. Section 3 includes a detailed description of the model under our study and the main results are also established here. Finally, some concluding remarks and future extensions related to the main result are stated in Section 4.

2. Preliminaries and basic lemmas

For the basics of discrete fractional calculus we refer e.g. to Refs. [1, 2]. Below we provide necessary definitions and lemmas which we use in the study: Let $(\nabla \mathtt{h})(\tau) = \mathtt{h}(\tau) - \mathtt{h}(\tau - 1)$ be the forward difference operator for $\tau \in \mathcal{N}_b$ with $b \in \mathsf{R}$. Then, for any function $\mathtt{h}$ defined on $\mathcal{N}_b$, the discrete delta Caputo-Fabrizio-Caputo and Caputo-Fabrizio-Riemann fractional differences are defined by

$$\left({}^{CFC}_{\ \ b}\nabla^{\omega}\mathrm{h}\right)(\tau) = \mathsf{M}(\omega)\left[\sum_{\mathrm{x}=b+1}^{\tau}\left(\nabla_{\mathrm{x}}\mathrm{h}\right)(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}}\right] \quad [\forall\ \tau \in \mathcal{N}_{b+1}], \tag{2.1}$$

and

$$\left({}^{CFR}_{\ \ b}\nabla^{\omega}\mathrm{h}\right)(\tau) = \mathsf{M}(\omega)\,\nabla_{\tau}\left[\sum_{\mathrm{x}=b+1}^{\tau}\mathrm{h}(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}}\right] \quad [\forall\ \tau \in \mathcal{N}_{b+1}], \tag{2.2}$$

respectively, where $\lambda = -\frac{\omega}{1-\omega}$ for $\omega \in [0,1)$, and $\mathsf{M}(\omega)$ is a normalizing positive constant. Moreover, for the higher order when $q-1 < \omega < q$, we have

$$\left({}^{CFR}_{\ \ b}\nabla^{\omega}\mathrm{h}\right)(\tau) = \left({}^{CFR}_{\ \ b}\nabla^{\omega-q}\,\nabla^{q}\mathrm{h}\right)(\tau) \quad [\forall\ \tau \in \mathcal{N}_{b+1}]. \tag{2.3}$$

Lemma 2.1. *Let* $\mathscr{M} := \{(\omega,\beta) \in \mathsf{R}\times\mathsf{R};\ 0 < \omega, \beta < 1 \text{ and } 1 \leq \omega+\beta < 2 \text{ for } \beta \neq \omega\}$. *Then, we have*

$$P(\jmath) := \frac{1}{\omega-\beta}\Big[\beta(1-\beta)^{\jmath} - \omega(1-\omega)^{\jmath}\Big] \geq 0, \tag{2.4}$$

for each $\jmath \in \mathcal{N}_1$ *and* $(\omega,\beta) \in \mathscr{M}$. *Moreover, we have*

$$Q(\jmath) := \frac{1}{\omega-\beta}\Big[(1-\beta)^{\jmath} - (1-\omega)^{\jmath}\Big] > 0, \tag{2.5}$$

for each $\jmath \in \mathcal{N}_1$ *and* $(\omega,\beta) \in \mathscr{M}$.

Proof. We use induction to proof this lemma. For the basic step $\jmath = 1$, we have

$$P(1) := \frac{1}{\omega-\beta}\Big[\beta(1-\beta) - \omega(1-\omega)\Big] = -1+\omega+\beta \geq 0,$$

since $\omega+\beta \geq 1$. Now, we suppose that

$$P(m) = \frac{1}{\omega-\beta}\Big[\beta(1-\beta)^{m} - \omega(1-\omega)^{m}\Big] \geq 0, \tag{2.6}$$

for some $m \in \mathcal{N}_1$. Then, we shall show that $P(m+1) \geq 0$. To prove this step, we have two cases: The first case if $\omega > \beta$, we have

$$P(m+1) = \underbrace{\frac{1}{\omega-\beta}}_{>0}\Big[\beta(1-\beta)^{m+1} - \omega(1-\omega)^{m+1}\Big],$$

and we only need to show that $\Big[\beta(1-\beta)^{m+1} - \omega(1-\omega)^{m+1}\Big] \geq 0$. But, we have

$$\beta(1-\beta)^{m+1} \underset{(2.6)}{\overset{by}{\geq}} \omega(1-\omega)^{m}(1-\beta) \geq \omega(1-\omega)^{m+1},$$

where we have used that $1-\beta > 1-\omega > 0$. Therefore, $P(m+1) \geq 0$ for $\omega > \beta$. Similarly, we can show that $P(m+1) \geq 0$ for $\omega < \beta$. Thus, $P(\jmath) \geq 0$ for each $(\omega,\beta) \in \mathscr{M}$ and $\jmath \in \mathcal{N}_1$. This completes the first part of the lemma. For the second part (2.5), we have for each $\jmath \geq 1$:

$$\frac{1}{\omega-\beta}\Big[(1-\beta)^\jmath - (1-\omega)^\jmath\Big] > \underbrace{\frac{1}{\omega-\beta}}_{>0}\underbrace{\Big[(1-\beta)^\jmath - (1-\beta)^\jmath\Big]}_{=0} = 0,$$

for $\omega > \beta$ ($\Longrightarrow 1-\beta > 1-\omega > 0$), and

$$\frac{1}{\omega-\beta}\Big[(1-\beta)^\jmath - (1-\omega)^\jmath\Big] = \frac{1}{\beta-\omega}\Big[(1-\omega)^\jmath - (1-\beta)^\jmath\Big] > \underbrace{\frac{1}{\beta-\omega}}_{>0}\underbrace{\Big[(1-\beta)^\jmath - (1-\beta)^\jmath\Big]}_{=0} = 0,$$

for $\beta > \omega$ (and hence $1-\omega > 1-\beta > 0$). Thus, the proof is completed. □

Lemma 2.2. *Let $\omega \in \left[\frac{1}{2}, 1\right)$. Then, we have*

$$J(\jmath) := (1-\omega)^{\jmath-1}[\omega\,\jmath - (1-\omega)] \geq 0, \tag{2.7}$$

for each $\jmath \in \mathcal{N}_1$.

Proof. Again, we proceed by induction. If $\jmath = 1$, then we have $J(1) = 2\omega - 1$ and this is nonnegative since $\omega \in \left[\frac{1}{2}, 1\right)$. We assume that $J(m) \geq 0$, that is,

$$(1-\omega)^{m-1}[\omega\, m - (1-\omega)] \geq 0, \tag{2.8}$$

for some $m \in \mathcal{N}_1$. Then, we have to show that $J(m+1) \geq 0$. But, we see that

$$(1-\omega)^m[\omega\,(m+1) - (1-\omega)] = \underbrace{(1-\omega)}_{>0}\,\underbrace{(1-\omega)^{m-1}[\omega\, m - (1-\omega)]}_{\geq 0 \text{ by claim (2.8)}} + \underbrace{\omega\,(1-\omega)^m}_{>0} \geq 0,$$

which implies that $J(m+1) \geq 0$. Consequently, we find that (2.7) is true for each $\jmath \in \mathcal{N}_1$. Hence, the proof is completed. □

Lemma 2.3. *For any function $\mathtt{h} : \mathcal{N}_b \to \mathsf{R}$, we have*

$$\nabla\left({}^{CFR}_{\;\;b}\nabla^{\omega}\mathtt{h}\right)(\tau) = \mathsf{M}(\omega)\left[(\nabla\mathtt{h})(\tau) - \omega\,\mathtt{h}(b)(1-\omega)^{\tau-2-b} - \omega\sum_{\mathtt{x}=b+1}^{\tau-1}(\nabla\mathtt{h})(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}-1}\right],$$

for $\omega \in (0,1)$ and τ in $\mathcal{N}_{b+2}$.

Proof. The use of (2.2) give us

$$\begin{aligned}\left({}^{CFR}_{\;\;b}\nabla^{\omega}\mathtt{h}\right)(\tau) &= \mathsf{M}(\omega)\left[\sum_{\mathtt{x}=b+1}^{\tau}\mathtt{h}(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}} - \sum_{\mathtt{x}=b+1}^{\tau-1}\mathtt{h}(\mathtt{x}-1)(1-\omega)^{\tau-\mathtt{x}-1}\right]\\ &= \mathsf{M}(\omega)\left[\mathtt{h}(\tau) - \omega\sum_{\mathtt{x}=b+1}^{\tau-1}\mathtt{h}(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}-1}\right].\end{aligned}$$

It follows from this,

$$\begin{aligned}
&\nabla\left({}^{CFR}_{\ \ b}\nabla^{\omega}\mathrm{h}\right)(\tau)\\
&=\mathsf{M}(\omega)\left[(\nabla\mathrm{h})(\tau)-\omega\sum_{\mathrm{x}=b+1}^{\tau-1}\mathrm{h}(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}-1}+\omega\sum_{\mathrm{x}=b+1}^{\tau-2}\mathrm{h}(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}-2}\right]\\
&=\mathsf{M}(\omega)\Bigg[(\nabla\mathrm{h})(\tau)-\omega\,\mathrm{h}(b)(1-\omega)^{\tau-2-b}-\omega\sum_{\mathrm{x}=b+1}^{\tau-1}\mathrm{h}(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}-1}\\
&\qquad+\omega\sum_{\mathrm{x}=b+1}^{\tau-1}\mathrm{h}(\mathrm{x}-1)(1-\omega)^{\tau-\mathrm{x}-1}\Bigg]\\
&=\mathsf{M}(\omega)\left[(\nabla\mathrm{h})(\tau)-\omega\,\mathrm{h}(b)(1-\omega)^{\tau-2-b}-\omega\sum_{\mathrm{x}=b+1}^{\tau-1}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}-1}\right], \qquad (2.9)
\end{aligned}$$

for each $\tau \in \mathcal{N}_{b+2}$, and this completes the proof. □

3. Positivity results

The aim of this section is to prove the positivity analysis results for the CFR operator (2.2) with negative lower bound conditions. Our first result is defined on the set $\mathscr{M}$ as we have defined in the previous section. Also, we have shown graphically this set in Figure 1. Note that the dashed dot diagonal line represents the set $\mathscr{D}$ that we will define later, which is excluded from the set $\mathscr{M}$.

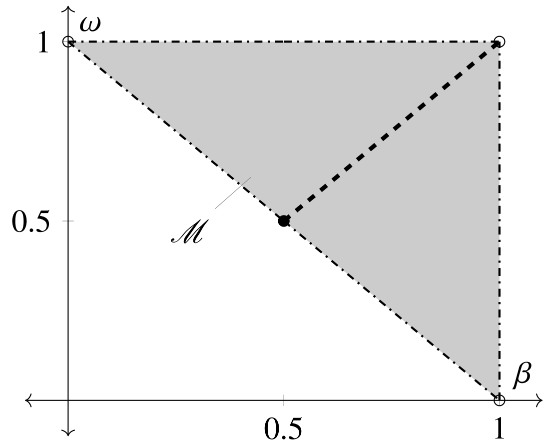

Figure 1. Plot illustration for the set $\mathscr{M}$.

Theorem 3.1. *Let* $\epsilon \geq 0$ *and* $\mathcal{N}_b^T := \{b, b+1, \ldots, T\}$. *If a function* $\mathrm{h} : \mathcal{N}_b \to \mathsf{R}$ *satisfies*

(i) $\mathrm{h}(b+1) \geq \mathrm{h}(b) \geq 0$;

(ii) $\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{\ \ b}\nabla^{\beta}\mathrm{h}\right)(\tau) \geq -\epsilon\,\mathsf{M}(\omega)\mathsf{M}(\beta)\mathrm{h}(b) \quad \left[\forall\ \tau \in \mathcal{N}_{b+2}^T\right]$;

(iii) $\beta\left(\dfrac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right) \geq \epsilon \quad \left[\forall\ \tau \in \mathcal{N}_{b+2}^T\right]$, *for each* $(\omega,\beta) \in \mathscr{M}$ *and for some* $T \in \mathcal{N}_{b+2}$*, then,* $(\nabla \mathrm{h})(\tau) \geq 0$ *for every* $\tau \in \mathcal{N}_{b+1}^{T}$.

Proof. We know that $(\nabla \mathrm{h})(b+1) \geq 0$ from the assumption (i). Now, we need to show that $(\nabla \mathrm{h})(\tau) \geq 0$ for $\tau \in \mathcal{N}_{b+2}^{T}$. In view of (2.9) and definition (2.2), we have

$$\begin{aligned}
&\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{b}\nabla^{\beta}\mathrm{h}\right)(\tau) \\
&= \mathsf{M}(\omega)\left[\sum_{\mathrm{x}=b+2}^{\tau}\left(\nabla\ {}^{CFR}_{b}\nabla^{\beta}\mathrm{h}\right)(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}}\right] \\
&= \mathsf{M}(\omega)\mathsf{M}(\beta)\Bigg[\sum_{\mathrm{x}=b+2}^{\tau}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}} - \mathrm{h}(b)\beta\sum_{\mathrm{x}=b+2}^{\tau}(1-\beta)^{\mathrm{x}-b-2}(1-\omega)^{\tau-\mathrm{x}} \\
&-\beta\sum_{\mathrm{x}=b+2}^{\tau}\sum_{\kappa=b+1}^{\mathrm{x}-1}(\nabla\mathrm{h})(\kappa)(1-\beta)^{\mathrm{x}-\kappa-1}(1-\omega)^{\tau-\mathrm{x}}\Bigg] \\
&:= \mathsf{M}(\omega)\mathsf{M}(\beta)[A_1 - A_2].
\end{aligned} \tag{3.1}$$

Calculating A_1 and A_2 successively, we have

$$\begin{aligned}
A_1 &:= \sum_{\mathrm{x}=b+2}^{\tau}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}} \\
&\quad - \mathrm{h}(b)\beta\sum_{\mathrm{x}=b+2}^{\tau}(1-\beta)^{\mathrm{x}-b-2}(1-\omega)^{\tau-\mathrm{x}} \\
&= (\nabla\mathrm{h})(\tau) + \sum_{\mathrm{x}=b+2}^{\tau-1}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}} \\
&\quad - \mathrm{h}(b)\beta(1-\beta)^{-b-2}(1-\omega)^{\tau}\sum_{\mathrm{x}=b+2}^{\tau}\left(\frac{1-\beta}{1-\omega}\right)^{\mathrm{x}} \\
&= (\nabla\mathrm{h})(\tau) + \sum_{\mathrm{x}=b+2}^{\tau-1}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}} \\
&\quad - \mathrm{h}(b)\beta(1-\omega)^{\tau-2-b}\cdot\frac{1-\left(\frac{1-\beta}{1-\omega}\right)^{\tau-1-b}}{1-\frac{1-\beta}{1-\omega}} \\
&= (\nabla\mathrm{h})(\tau) + \sum_{\mathrm{x}=b+2}^{\tau-1}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}} \\
&\quad - \mathrm{h}(b)\beta\left(\frac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right),
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
A_2 &:= \beta\sum_{\mathrm{x}=b+2}^{\tau}\sum_{\kappa=b+1}^{\mathrm{x}-1}(\nabla\mathrm{h})(\kappa)(1-\beta)^{\mathrm{x}-\kappa-1}(1-\omega)^{\tau-\mathrm{x}} \\
&= \beta\sum_{\kappa=b+1}^{\tau-1}\left[(\nabla\mathrm{h})(\kappa)\frac{(1-\omega)^{\tau}}{(1-\beta)^{\kappa+1}}\sum_{\mathrm{x}=\kappa+1}^{\tau}\left(\frac{1-\beta}{1-\omega}\right)^{\mathrm{x}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \beta \sum_{\kappa=b+1}^{\tau-1} (\nabla \mathrm{h})(\kappa)(1-\omega)^{\tau-1-\kappa} \cdot \frac{1-\left(\frac{1-\beta}{1-\omega}\right)^{\tau-\kappa}}{1-\frac{1-\beta}{1-\omega}} \\
&= \beta \sum_{\kappa=b+1}^{\tau-1} (\nabla \mathrm{h})(\kappa) \left(\frac{(1-\beta)^{\tau-\kappa}-(1-\omega)^{\tau-\kappa}}{\omega-\beta}\right) \\
&= \beta \left(\frac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right)(\nabla \mathrm{h})(b+1) \\
&+ \beta \sum_{\kappa=b+2}^{\tau-1} (\nabla \mathrm{h})(\kappa) \left(\frac{(1-\beta)^{\tau-\kappa}-(1-\omega)^{\tau-\kappa}}{\omega-\beta}\right). \qquad (3.3)
\end{aligned}
$$

Here, it is worth mentioning that (3.3) is well defined since $(\omega,\beta) \notin \mathscr{D}$. By using (3.2) and (3.3) and (ii) in (3.1), we get

$$
\begin{aligned}
(\nabla \mathrm{h})(\tau) &\geq -\epsilon\, \mathrm{h}(b) + \underbrace{\mathrm{h}(b)\beta \left(\frac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right)}_{\geq \epsilon\, \mathrm{h}(b) \text{ by (iii)}} \\
&+ \beta \left(\frac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right)(\nabla \mathrm{h})(b+1) - \sum_{\mathtt{x}=b+2}^{\tau-1} (\nabla \mathrm{h})(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}} \\
&+ \beta \sum_{\kappa=b+2}^{\tau-1} (\nabla \mathrm{h})(\kappa) \left(\frac{(1-\beta)^{\tau-\kappa}-(1-\omega)^{\tau-\kappa}}{\omega-\beta}\right) \\
&\geq \beta \left(\frac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right)(\nabla \mathrm{h})(b+1) \\
&+ \sum_{\mathtt{x}=b+2}^{\tau-1} (\nabla \mathrm{h})(\mathtt{x}) \left(\frac{\beta(1-\beta)^{\tau-\mathtt{x}}-\beta(1-\omega)^{\tau-\mathtt{x}}}{\omega-\beta} - (1-\omega)^{\tau-\mathtt{x}}\right) \\
&= \underbrace{\beta \left(\frac{(1-\beta)^{\tau-1-b}-(1-\omega)^{\tau-1-b}}{\omega-\beta}\right)}_{>0 \text{ by (2.5)}}(\nabla \mathrm{h})(b+1) \\
&+ \sum_{\mathtt{x}=b+2}^{\tau-1} (\nabla \mathrm{h})(\mathtt{x}) \underbrace{\frac{1}{\omega-\beta}\Big[\beta(1-\beta)^{\tau-\mathtt{x}}-\omega(1-\omega)^{\tau-\mathtt{x}}\Big]}_{\geq 0 \text{ by (2.4)}}. \qquad (3.4)
\end{aligned}
$$

By substituting $\tau = b+2$ into (3.4), we get

$$
(\nabla \mathrm{h})(b+2) \geq \beta \left(\frac{(1-\beta)-(1-\omega)}{\omega-\beta}\right)(\nabla \mathrm{h})(b+1) + \underbrace{\sum_{\mathtt{x}=b+2}^{b+1} (\cdot)}_{=0} \geq \beta\, (\nabla \mathrm{h})(b+1) \geq 0.
$$

Thus, by using $(\nabla \mathrm{h})(b+1) \geq 0$, $(\nabla \mathrm{h})(b+2) \geq 0$, (2.5) and (2.4), we get $(\nabla \mathrm{h})(\tau) \geq 0$ for each $\tau \in \mathcal{N}_{b+2}^{T}$ as desired. □

Having defining the set $\mathscr{M}$ in the previous section and Figure 1, we here define another set $\mathscr{D} := \{(\omega,\beta) \in \mathsf{R}\times\mathsf{R};\ 0 < \omega, \beta < 1 \text{ and } 1 \leq \omega + \beta < 2 \text{ for } \beta = \omega\}$, which our next theorem is defined on this set. As we have shown in Figure 1, the set $\mathscr{D}$ (the dashed dot diagonal) is excluded from the set $\mathscr{M}$. Furthermore, the following Figure 2 illustrates the set $\mathscr{D}$ more clearly.

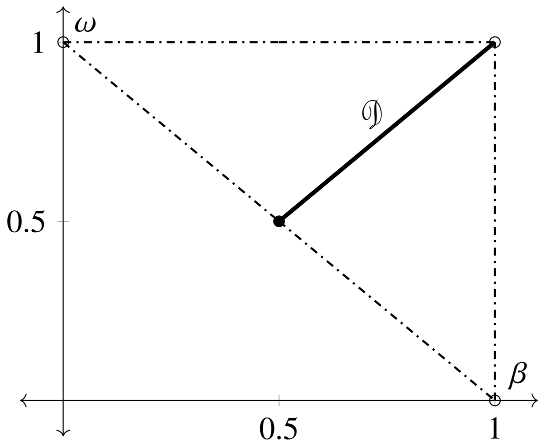

Figure 2. Plot illustration for the set $\mathscr{D}$.

Theorem 3.2. *Let* $\epsilon \geq 0$ *and* $\frac{1}{2} \leq \omega < 1$. *If a function* $\mathrm{h} : \mathcal{N}_b \to \mathsf{R}$ *satisfies*

(i) $\mathrm{h}(b+1) \geq \mathrm{h}(b) \geq 0$;

(ii) $\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{b}\nabla^{\omega}\mathrm{h}\right)(\tau) \geq -\epsilon\,\mathsf{M}^2(\omega)\,\mathrm{h}(b) \quad \left[\forall\ \tau \in \mathcal{N}_{b+2}^{T}\right]$;

(iii) $\omega\,(\tau - 1 - b)\,(1-\omega)^{\tau-2-b} \geq \epsilon \quad \left[\forall\ \tau \in \mathcal{N}_{b+2}^{T}\right]$,

for some $T \in \mathcal{N}_{b+2}$, *then,* $(\nabla\mathrm{h})(\tau) \geq 0$ *for every* $\tau \in \mathcal{N}_{b+1}^{T}$.

Proof. From the assumption (i), we know that $(\nabla\mathrm{h})(b+1) \geq 0$. Now, we nwill try to prove that $(\nabla\mathrm{h})(\tau) \geq 0$ for $\tau \in \mathcal{N}_{b+2}^{T}$. Due to (2.9) and definition (2.2), we have

$$\begin{aligned}
\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{b}\nabla^{\omega}\mathrm{h}\right)(\tau) &= \mathsf{M}(\omega)\left[\sum_{\mathrm{x}=b+2}^{\tau}\left(\nabla\ {}^{CFR}_{b}\nabla^{\omega}\mathrm{h}\right)(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}}\right] \\
&= \mathsf{M}(\omega)\mathsf{M}(\omega)\Bigg[\sum_{\mathrm{x}=b+2}^{\tau}(\nabla\mathrm{h})(\mathrm{x})(1-\omega)^{\tau-\mathrm{x}} \\
&- \omega\,\mathrm{h}(b)\sum_{\mathrm{x}=b+2}^{\tau}(1-\omega)^{\mathrm{x}-b-2}(1-\omega)^{\tau-\mathrm{x}} \\
&- \omega\sum_{\mathrm{x}=b+2}^{\tau}\sum_{\kappa=b+1}^{\mathrm{x}-1}(\nabla\mathrm{h})(\kappa)(1-\omega)^{\mathrm{x}-\kappa-1}(1-\omega)^{\tau-\mathrm{x}}\Bigg] \\
&:= \mathsf{M}^2(\omega)[C_1 - C_2].
\end{aligned} \tag{3.5}$$

Calculating C_1 and C_2 successively, we have

$$\begin{aligned} C_1 &:= \sum_{\mathtt{x}=b+2}^{\tau} (\nabla \mathtt{h})(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}} - \omega(1-\omega)^{\tau-2-b}\, \mathtt{h}(b) \sum_{\mathtt{x}=b+2}^{\tau} (1) \\ &= (\nabla \mathtt{h})(\tau) + \sum_{\mathtt{x}=b+2}^{\tau-1} (\nabla \mathtt{h})(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}} - \omega\,(1-\omega)^{\tau-2-b}\, \mathtt{h}(b)(\tau-1-b), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} C_2 &:= \omega \sum_{\mathtt{x}=b+2}^{\tau} \sum_{\kappa=b+1}^{\mathtt{x}-1} (\nabla \mathtt{h})(\kappa)(1-\omega)^{\mathtt{x}-\kappa-1}(1-\omega)^{\tau-\mathtt{x}} \\ &= \omega \sum_{\kappa=b+1}^{\tau-1} \left[(\nabla \mathtt{h})(\kappa)(1-\omega)^{\tau-1-\kappa} \sum_{\mathtt{x}=\kappa+1}^{\tau} (1) \right] \\ &= \omega \sum_{\kappa=b+1}^{\tau-1} (\nabla \mathtt{h})(\kappa)(1-\omega)^{\tau-1-\kappa}\,(\tau-\kappa) \\ &= \omega\,(\tau-1-b)(1-\omega)^{\tau-2-b}(\nabla \mathtt{h})(b+1) \\ &+ \omega \sum_{\kappa=b+2}^{\tau-1} (\nabla \mathtt{h})(\kappa)(1-\omega)^{\tau-1-\kappa}\,(\tau-\kappa). \end{aligned} \tag{3.7}$$

Due to (3.6) and (3.7) and (ii) in (3.5), we have

$$\begin{aligned} (\nabla \mathtt{h})(\tau) \geq -\epsilon\, \mathtt{h}(b) + &\underbrace{\omega\,(1-\omega)^{\tau-2-b}\,(\tau-1-b)\mathtt{h}(b)}_{\geq \epsilon\, \mathtt{h}(b) \text{ by (iii)}} \\ &+ \omega\,(\tau-1-b)(1-\omega)^{\tau-2-b}(\nabla \mathtt{h})(b+1) - \sum_{\mathtt{x}=b+2}^{\tau-1} (\nabla \mathtt{h})(\mathtt{x})(1-\omega)^{\tau-\mathtt{x}} \\ &+ \omega \sum_{\kappa=b+2}^{\tau-1} (\nabla \mathtt{h})(\kappa)(1-\omega)^{\tau-1-\kappa}\,(\tau-\kappa) \\ &\geq \underbrace{\omega\,(\tau-1-b)(1-\omega)^{\tau-2-b}(\nabla \mathtt{h})(b+1)}_{\geq 0 \text{ by (iii)}} \\ &+ \sum_{\mathtt{x}=b+2}^{\tau-1} (\nabla \mathtt{h})(\mathtt{x})\, \underbrace{(1-\omega)^{\tau-\mathtt{x}-1}\left(\omega\,(\tau-\mathtt{x})-(1-\omega)^{\tau-\mathtt{x}}\right)}_{\geq 0 \text{ by (2.7)}}. \end{aligned} \tag{3.8}$$

If we take $\tau = b+2$ into (3.8), we get

$$(\nabla \mathtt{h})(b+2) \geq \omega\,(\nabla \mathtt{h})(b+1) + \underbrace{\sum_{\mathtt{x}=b+2}^{b+1} (\cdot)}_{=0} \geq \omega\,(\nabla \mathtt{h})(b+1) \geq 0.$$

We can continue by the same process by using $(\nabla \mathtt{h})(b+1) \geq 0$, $(\nabla \mathtt{h})(b+2) \geq 0$ and (2.7), to reach $(\nabla \mathtt{h})(\tau) \geq 0$ for each $\tau \in \mathcal{N}_{b+2}^{T}$ as desired. □

The following example can verify the applicability of the above theorem.

Example 3.3. *Considering the identity* (3.5) *with* $\tau = b + 2$*, we get*

$$\begin{aligned}\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{b}\nabla^{\omega}\mathrm{h}\right)(b+2) &= \mathsf{M}^2(\omega)\Bigg[\sum_{\mathrm{x}=b+2}^{b+2}(\nabla \mathrm{h})(\mathrm{x})(1-\omega)^{b+2-\mathrm{x}} - \omega\,\mathrm{h}(b)\sum_{\mathrm{x}=b+2}^{b+2}(1-\omega)^{\mathrm{x}-b-2}(1-\omega)^{b+2-\mathrm{x}} \\ &\quad - \omega\sum_{\mathrm{x}=b+2}^{b+2}\sum_{\kappa=b+1}^{\mathrm{x}-1}(\nabla \mathrm{h})(\kappa)(1-\omega)^{\mathrm{x}-\kappa-1}(1-\omega)^{b+2-\mathrm{x}}\Bigg] \\ &= \mathsf{M}^2(\omega)\Big[\mathrm{h}(b+2) - (\omega+1)\mathrm{h}(a+1)\Big].\end{aligned}$$

If we choose $b = 0$, $\mathrm{h}(0) = 1, \mathrm{h}(1) = 1.5, \mathrm{h}(2) = 2$*,* $\epsilon = 0.5$ *and* $\omega = 0.6$ *in the above identity, we get*

$$\left({}^{CFC}_{1}\nabla^{\omega}\ {}^{CFR}_{0}\nabla^{\omega}\mathrm{h}\right)(2) = \mathsf{M}^2(0.6)\Big[2-(1.6)(1.5)\Big] = -0.4\mathsf{M}^2(0.6) \geq -0.5\mathsf{M}^2(0.6).$$

Moreover, $1.5 = \mathrm{h}(1) \geq 1 = \mathrm{h}(0)$ *and*

$$\omega\,(\tau - 1 - b)\,(1-\omega)^{\tau-2-b} = \omega \geq \epsilon = 0.5.$$

Thus, Theorem 3.2 confirms that $(\nabla \mathtt{f})(1)$ *and* $(\nabla \mathtt{f})(2)$ *are positive.*

4. Conclusions and future extensions

In this study, we have considered the positivity analysis of $\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{b}\nabla^{\beta}\mathrm{h}\right)(\tau)$ on the set

$$\mathscr{M} = \{(\omega, \beta) \in \mathsf{R}\times\mathsf{R};\ 0 < \omega, \beta < 1 \text{ and } 1 \leq \omega + \beta < 2 \text{ for } \beta \neq \omega\},$$

and $\left({}^{CFC}_{b+1}\nabla^{\omega}\ {}^{CFR}_{b}\nabla^{\omega}\mathrm{h}\right)(\tau)$ for $\frac{1}{2} \leq \omega < 1$. In which, we have obtained that $(\nabla \mathrm{h})(\tau)$ is positive on a finite set $\mathcal{N}^{T}_{b+2}$ for some $T \in \mathcal{N}_{b+1}$ by using some negative lower bound conditions and an initial condition $\mathrm{h}(b+1) \geq \mathrm{h}(b) \geq 0$. For this reason, the induction process as stated in Lemmas 2.1 and 2.2 has been used in obtaining our main results as examined in Theorems 3.1 and 3.2.

The results obtained in this study can be extended to the discrete fractional operators and the discrete generalized fractional operators defined using Mitta-Leffler kernels (see for more on these operators [2, 4]). Moreover, our results can be applied for the delta operators as established in [34].

Acknowledgments

This work was supported by the Taif University Researchers Supporting Project (No. TURSP-2020/155), Taif University, Taif, Saudi Arabia.

Conflict of interest

The authors declare no potential conflicts of interest in this work.

References

1. C. Goodrich, A. C. Peterson, *Discrete fractional calculus*, New York: Springer, 2015. https://doi.org/10.1007/978-3-319-25562-0
2. P. O. Mohammed, T. Abdeljawad, Discrete generalized fractional operators defined using h-discrete Mittag-Leffler kernels and applications to AB fractional difference systems, *Math. Meth. Appl. Sci.*, 2020, 1–26. https://doi.org/10.1002/mma.7083
3. D. Mozyrska, D. F. M. Torres, M. Wyrwas, Solutions of systems with the Caputo-Fabrizio fractional delta derivative on time scales, *Nonlinear Anal. Hybrid Syst.*, **32** (2019), 168–176. https://doi.org/10.1016/j.nahs.2018.12.001
4. T. Abdeljawad, Different type kernel h-fractional differences and their fractional h-sums, *Chaos Solit. Fract.*, **116** (2018), 146–156. https://doi.org/10.1016/j.chaos.2018.09.022
5. F. M. Atici, M. Atici, M. Belcher, D. Marshall, A new approach for modeling with discrete fractional equations, *Fund. Inform.*, **151** (2017), 313–324. https://doi.org/10.3233/FI-2017-1494
6. F. M. Atici, N. Nguyen, K. Dadashova, S. E. Pedersen, G. Koch, Pharmacokinetics and pharmacodynamics models of tumor growth and anticancer effects in discrete time, *Comput. Math. Biophys.*, **8** (2020), 114–125. https://doi.org/10.1515/cmb-2020-0105
7. C. S. Goodrich, On discrete sequential fractional boundary value problems, *J. Math. Anal. Appl.*, **385** (2012), 111–124. https://doi.org/10.1016/j.jmaa.2011.06.022
8. Z. Wang, B. Shiri, D. Baleanu, Discrete fractional watermark technique, *Front. Inf. Technol. Electron. Eng.*, **21** (2020), 880–883. https://doi.org/10.1631/FITEE.2000133
9. K. Ahrendt, L. Castle, M. Holm, K. Yochman, Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula, *Commun. Appl. Anal.*, **16** (2012), 317–347.
10. C. S. Goodrich, On discrete sequential fractional boundary value problems, *J. Math. Anal. Appl.*, **385** (2012), 111–124. https://doi.org/10.1016/j.jmaa.2011.06.022
11. H. M. Srivastava, P. O. Mohammed, J. L. G. Guirao, Y. S. Hamed, Link theorem and distributions of solutions to uncertain Liouville-Caputo difference equations, *DCDS-S*, **15** (2021), 427–440. https://doi.org/10.3934/dcdss.2021083
12. H. M. Srivastava, P. O. Mohammed, A correlation between solutions of uncertain fractional forward difference equations and their paths, *Front. Phys.*, **8** (2020), 280. https://doi.org/10.3389/fphy.2020.00280
13. H. M. Srivastava, P. O. Mohammed, C. S. Ryoo, Y. S. Hamed, Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations, *J. King Saud Univ. Sci.*, **33** (2021), 101497. https://doi.org/10.1016/j.jksus.2021.101497
14. R. Dahal, C. S. Goodrich, A monotonicity result for discrete fractional difference operators, *Arch. Math.*, **102** (2014), 293–299. https://doi.org/10.1007/s00013-014-0620-x
15. F. M. Atici, M. Uyanik, Analysis of discrete fractional operators, *Appl. Anal. Discr. Math.*, **9** (2015), 139–149. https://doi.org/10.2298/AADM150218007A
16. I. Suwan, S. Owies, T. Abdeljawad, Monotonicity results for h-discrete fractional operators and application, *Adv. Differ. Equ.*, **2018** (2018), 207. https://doi.org/10.1186/s13662-018-1660-5

17. R. Dahal, C. S. Goodrich, An almost sharp monotonicity result for discrete sequential fractional delta differences, *J. Differ. Equ. Appl.*, **23** (2017), 1190–1203. https://doi.org/10.1080/10236198.2017.1307351

18. B. Jia, L. Erbe, A. Peterson, Convexity for nabla and delta fractional differences, *J. Differ. Equ. Appl.*, **21** (2015), 360–373. https://doi.org/10.1080/10236198.2015.1011630

19. T. Abdeljawad, B. Abdallaa, Monotonicity results for delta and nabla Caputo and Riemann fractional differences via dual identities, *Filomat*, **31** (2017), 3671–3683. https://doi.org/10.2298/FIL1712671A

20. P. O. Mohammed, T. Abdeljawad, F. K. Hamasalh, On Riemann-Liouville and Caputo fractional forward difference monotonicity analysis, *Mathematics*, **9** (2021), 1303. https://doi.org/10.3390/math9111303

21. T. Abdeljawad, D. Baleanu, Monotonicity results for fractional difference operators with discrete exponential kernels, *Adv. Differ. Equ.*, **2017** (2017), 78. https://doi.org/10.1186/s13662-017-1126-1

22. T. Abdeljawad, D. Baleanu, Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel, *Chaos Solition Fract.*, **102** (2017), 106–110. https://doi.org/10.1016/j.chaos.2017.04.006

23. P. O. Mohammed, T. Abdeljawad, F. K. Hamasalh, On discrete delta Caputo-Fabrizio fractional operators and monotonicity analysis, *Fractal Fract.*, **5** (2021), 116. https://doi.org/10.3390/fractalfract5030116

24. P. O. Mohammed, C. S. Goodrich, A. B. Brzo, D. Baleanu Y. S. Hamed, New classifications of monotonicity investigation for discrete operators with Mittag-Leffler kernel, *Math. Biosci. Eng.*, **19** (2022), 4062–4074. https://doi.org/10.3934/mbe.2022186

25. I. Suwan, T. Abdeljawad, F. Jarad, Monotonicity analysis for nabla h-discrete fractional Atangana-Baleanu differences, *Chaos Solition Fract.*, **117** (2018), 50–59. https://doi.org/10.1016/j.chaos.2018.10.010

26. P. O. Mohammed, F. K. Hamasalh, T. Abdeljawad, Difference monotonicity analysis on discrete fractional operators with discrete generalized Mittag-Leffler kernels, *Adv. Differ. Equ.*, **2021** (2021), 213. https://doi.org/10.1186/s13662-021-03372-2

27. F. Du, B. Jia, L. Erbe, A. Peterson, Monotonicity and convexity for nabla fractional (q, h)-differences, *J. Differ. Equ. Appl.*, **22** (2016), 1224–1243. https://doi.org/10.1080/10236198.2016.1188089

28. R. Dahal, C. S. Goodrich, Theoretical and numerical analysis of monotonicity results for fractional difference operators, *Appl. Math. Lett.*, **117** (2021), 107104. https://doi.org/10.1016/j.aml.2021.107104

29. C. S. Goodrich, C. Lizama, Positivity, monotonicity, and convexity for convolution operators, *Discrete Contin. Dyn. Syst.*, **40** (2020), 4961–4983.

30. C. S. Goodrich, C. Lizama, A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity, *Israel J. Math.*, **236** (2020), 533–589. https://doi.org/10.1007/s11856-020-1991-2

31. C. S. Goodrich, J. M. Jonnalagadda, B. Lyons, Convexity, monotonicity and positivity results for sequential fractional nabla difference operators with discrete exponential kernels, *Math. Meth. Appl. Sci.*, **44** (2021), 7099–7120. https://doi.org/10.1002/mma.7247

32. P. O. Mohammed, O. Almutairi, R. P. Agarwal, Y. S. Hamed, On convexity, monotonicity and positivity analysis for discrete fractional operators defined using exponential kernels, *Fractal Fract.*, **6** (2022), 55. https://doi.org/10.3390/fractalfract6020055

33. C. S. Goodrich, J. M. Jonnalagadda, Monotonicity results for CFC nabla fractional differences with negative lower bound, *Analysis*, **41** (2021), 221–229. https://doi.org/10.1515/anly-2021-0011

34. C. S. Goodrich, B. Lyons, M. T. Velcsov, Analytical and numerical monotonicity results for discrete fractional sequential differences with negative lower bound, *Commun. Pure Appl. Anal.*, **20** (2021), 339–358. https://doi.org/10.3934/cpaa.2020269