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Research article

Analysis of positive measure reducibility for quasi-periodic linear systems under Brjuno-Rüssmann condition

Muhammad Afzal[1], Tariq Ismaeel[2], Riaz Ahmad[3,*], Ilyas Khan[4,*] and Dumitru Baleanu[5,6,7]

[1] Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54770, Pakistan
[2] Department of Mathematics, Government College University, Lahore 54000, Pakistan
[3] Faculty of Science, Yibin University, Yibin 644000, China
[4] Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah, P.O. Box 66, Majmaah 11952, Saudi Arabia
[5] Department of Mathematics, Cankaya University, Ankara 06790, Turkey
[6] Institute of Space Sciences, 077125 Magurele, Romania
[7] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40447, Taiwan

* **Correspondence:** Email: riazgill2007@gmail.com, i.said@mu.edu.sa.

Abstract: In this article, we discuss the positive measure reducibility for quasi-periodic linear systems close to a constant which is defined as:

$$\frac{dx}{dt} = (A(\lambda) + Q(\varphi, \lambda))x, \dot{\varphi} = \omega,$$

where ω is a Brjuno vector and parameter $\lambda \in (a, b)$. The result is proved by using the KAM method, Brjuno-Rüssmann condition, and non-degeneracy condition.

Keywords: quasi-periodic; Brjuno-Rüssmann condition; reducibility; KAM method
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1. Introduction

Suppose the quasi-periodic linear system

$$\frac{dx}{dt} = A(\omega_1 t, \omega_2 t, \ldots, \omega_r t)x \tag{1.1}$$

in which $t \in \mathbb{R}$, $x \in \mathbb{C}^r$, $A(\omega_1 t, \omega_2 t, \ldots, \omega_r t)$ is quasi-periodic(q-p) time dependent $r \times r$ matrix and the basic frequencies $\omega_1, \ldots, \omega_r$ are rational independent.

The system (1.1) is said to be reducible, if there exists a so called quasi-periodic Lyapunov-Perron (L-P) transformation $x = P(\omega_1 t, \ldots, \omega_r t)y$, so that the transformed system is a linear system with constant coefficients. We call the transformation $x = P(\omega_1 t, \ldots, \omega_r t)y$ is quasi-periodic L-P transformation, if $P(t)$ is non singular and P, P^{-1} and $\dot{P}$ are quasi-periodic and are bounded in $t \in \mathbb{R}$.

Many researchers have discussed the reducibility problems for quasi-periodic linear systems. For $r = 1$, i.e., the periodic case, the well known Floquet theorem states that there always exists a periodic change of variables $x = P(\omega_1 t)y$ so that the system $\dot{x} = A(\omega_1 t)x$ is reducible to a constant coefficient system $\dot{y} = By, \dot{\varphi} = \omega$, where B is a constant matrix. For $r > 1$, i.e., quasi-periodic case, there is an example in [1] which shows that the system (1.1) is not always reducible. Earlier for q-p case, Coppel [2] proved that a linear differential equation with bounded coefficient matrix is pseudo-autonomous if and only if it is almost reducible and Johnson and Sell [3] showed that if (1.1) satisfies the full spectrum assumption, then there is a quasi-periodic linear change of variables $x = P(\omega_1 t, \ldots, \omega_r t)y$ that transforms (1.1) to a constant coefficient system $\dot{y} = By$, where B is a constant matrix. Their results failed for the pure imaginary spectrum [4].

The first reducibility result by KAM method was given by Dinaburg and Sinai [5] who proved that the linear Schrödinger equation $\frac{d^2x}{dt^2} + q(\omega_1 t, \omega_2 t, \ldots, \omega_r t)x = \lambda x$ is reducible for 'most' large enough λ in measure sense , where ω is fixed satisfying the Diophantine condition: $|\langle k, \omega \rangle| > \frac{\alpha^{-1}}{|k|^\tau}$, $0 \neq k \in \mathbb{Z}^r$, where α, τ are positive constants. See also Rüssmann [6] for a refined result.

In 1992 Jorba and Simó [7] considered the following linear differential system

$$\frac{dx}{dt} = \left(A + \lambda \bar{Q} + \lambda^2 Q(\omega_1 t, \ldots, \omega_r t)\right) x, \quad x \in \mathbb{R}^d, \tag{1.2}$$

in which $A, \bar{Q}$ are constant diagonal matrices, and Q is an analytic q-p matrix having r basic frequencies, and with a small parameter λ. Using the KAM method, They proved that there exists a positive measure Cantor subset $E \subset (0, \lambda_0)$, $\lambda_0 \ll 1$ such that for any $\lambda \in E$, the system (1.2) is reducible, provided that the following non-degeneracy conditions

$$|\alpha_i(\lambda) - \alpha_j(\lambda)| > \delta > 0, \quad |\frac{d}{d\lambda}(\alpha_i(\lambda) - \alpha_j(\lambda))| > \chi > 0, \quad \forall 1 \leq i < j \leq d \tag{1.3}$$

where $\alpha_i(\lambda)$, $1 \leq i \leq m$, are the eigenvalues of $\bar{A} = A + \lambda \bar{Q}$. In 1999, Xu [8] improved the result for the weaker non-degeneracy conditions.

Eliasson [9] considered the following linear Shrödinger equation

$$\frac{d^2x}{dt^2} + (\lambda + Q(\omega t)x = 0.$$

For almost all $\lambda \in (a, b)$, the full measure reducibility result is proved in a Lebesgue measure sense provided that Q is small. On the other hand, Krikorian [10] generalized the work for linear systems with coefficients in $so(3)$. Then, Eliasson [11] discussed the full measure reducibility result for the following parameter dependent systems

$$\frac{dx}{dt} = (A(\lambda) + Q(\omega_1 t, \ldots, \omega_r t, \lambda))x, \tag{1.4}$$

in which $t \in \mathbb{R}$, $x \in \mathbb{C}^d$, a constant matrix A of dimension $d \times d$, the parameter $\lambda \in (a, b)$, and an analytic mapping $Q : T^r \times (a, b) \to gl(m, \mathbb{C})$, a Diophantine vector $(\omega_1, \ldots, \omega_r)$ and for sufficiently small $|Q|$.

He and You [12] proved the positive measure reducibility result for the following quasi-periodic skew-product systems: $\frac{dx}{dt} = (A(\lambda) + Q(\varphi, \lambda))x, \dot{\varphi} = \omega$, close to constant. The result is proved by using KAM method, under weaker non-resonant conditions and non-degeneracy conditions.

All the above mentioned results only discuss the reducibility of linear systems with the Diophantine condition

$$|\langle k, \omega \rangle| \geq \frac{\alpha^{-1}}{|k|^{\tau}}, \quad 0 \neq k \in \mathbb{Z}^d, \tag{1.5}$$

where $\alpha > 1$ and $\tau > d - 1$.

In our problem, we are going to focussed on the Brjuno-Rüssmann condition (see [13, 14]) which is slightly weaker than the Diophantine condition (1.5), if the frequencies $\omega = (\omega_1, \ldots, \omega_d)$ satisfy

$$|\langle k, \omega \rangle| \geq \frac{\alpha^{-1}}{\Delta(|k|)}, \quad 0 \neq k \in \mathbb{Z}^d, \tag{1.6}$$

where $\alpha > 1$ and some Rüssmann approximation function Δ. These are continuous, increasing and unbounded functions $\Delta : [0, +\infty) \to [1, +\infty)$ such that $\Delta(0) = 1$ and

$$\int_1^{+\infty} \frac{ln\Delta(t)}{t^2} dt < \infty.$$

Remark: If we have $\Delta(t) = t^{\tau}$, then the Brjuno-Rüssmann conditions (1.6) becomes the Diophantine conditions (1.5).

Furthermore, in this article we will generalize the result of He and You [12] for quasi-periodic linear systems using Brjuno-Rüssmann non-resonant condition which is slightly weaker than the Diophantine condition.

This article is organized as: at the end of Section 1, the statement of the main result is given and in Section 2 proof of the main result is given.

To state our main result, we now give some definitions and results.

Definition 1.1. *([15, 16])*
A vector $\omega \in \mathbb{R}^d$ is Brjuno if the following condition is satisfied

$$\sum_{n=1}^{\infty} 2^{-n} ln(\frac{1}{\Omega_n}) < \infty, \quad \Omega_n = \min_{\nu \in \mathbb{Z}^d, 0 < |\nu| \leq 2^n} |\langle \omega, \nu \rangle|.$$

The set of Brjuno vectors is of full Lebesgue measure. In particular, it contains all Diophantine vectors. Conversely, there are vectors that are Brjuno and are not Diophantine.

This article aims to discuss the positive measure reducibility for q-p linear systems like (1.4) proposed by He and You [12]. The existed positive measure reducibility is discussed by using the Diophantine conditions, but we will discuss the positive measure reducibility using the Brjuno-Rüssmann condition.

Equivalently, for the system (1.4), we suppose the following skew-product system

$$\frac{dx}{dt} = (A(\lambda) + Q(\varphi, \lambda))x, \quad \dot{\varphi} = \omega, \tag{1.7}$$

where $x \in \mathbb{C}^r$, the parameter $\lambda \in \Lambda = (a, b)$, A is a $r \times r$ constant matrix, and $Q(\varphi, \lambda)$ is an analytic mapping from $\mathbb{T}^r \times (a, b)$ to $gl(m, \mathbb{C})$, $(\omega_1, \omega_2, \cdots, \omega_n)$ is a Brjuno vector and $|Q|$ is sufficiently small.

In our discussion, we will use the following equivalent formulation of reducibility:

Consider

$$\frac{dZ}{dt} = b(t)Z \tag{1.8}$$

be an analytic q-p linear system. For the skew-product system, it can be rewritten as:

$$\frac{dZ}{dt} = B(\varphi)Z, \ \ \dot{\varphi} = \omega, \tag{1.9}$$

where b, B are in the Lie algebra $g = g(m, \mathbb{C})$ and their solutions have values in the Lie group $G = GL(m, \mathbb{C})$. For a complex neighbourhood $W_h(\mathbb{T}^r)$ if B is an analytic on $W_h(\mathbb{T}^r)$, then we represent $B \in C_h^\omega(\mathbb{T}^r, g)$. It is said that the analytic g-valued functions $B_1, B_2 \in C_h^\omega(\mathbb{T}^r, g)$ are conjugated, if $\exists$ a L-P transformation G-valued function $P \in C_h^\omega(\mathbb{T}^r, G)$, s.t. for the solutions Z_1, Z_2 corresponding to B_1, B_2, we have the following relation

$$Z_2 = P(\varphi)Z_1$$

and the conjugate relation can be denoted by:

$$B_1 \equiv B_2(modP).$$

It is easy to prove that $B_1 \equiv B_2(modP)$ can equivalently be written in the form of following equality

$$B_2 = D_\omega P \cdot P^{-1} + PB_1P^{-1}, \tag{1.10}$$

where $D_\omega = \frac{\partial}{\partial \varphi} \cdot \dot{\varphi}$ denotes the derivative in the direction of frequency vector ω. B_1 is known to be reducible if it conjugates to a constant B_2.

In our article, we shall prove that, for any $\lambda \in \Lambda = (a, b)$, where λ is the parameter and Λ is a positive measure set,then $\exists$ a L-P transformation $P(\varphi)$, such that the system $A + Q(\varphi)$ is transformed into a constant system A^*.

For the positive measure reducibility, we will use the non-degeneracy conditions (or the transverse conditions as in Eliasson and Krikorian terminology) . Without loss of generality, let's suppose a block-diagonal matrix $A(\lambda) = \mathrm{diag}(A_1(\lambda), \cdots, A_s(\lambda))$ with

$$\mathrm{dist}(\sigma(A_i), \sigma(A_j)) > \varrho > 0, \ \ \text{for} \ \ i \neq j,$$

where $\sigma(A_i)$ represents the eigenvalues set for A_i. Let (see in [12] for definition)

$$J_{ij}(k, \lambda) = i\langle k, \omega\rangle I_{l_i l_j} + (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda) \otimes I_{l_j}),$$

$$J(k, \lambda) = i\langle k, \omega\rangle I_{n^2} + (I_n \otimes A(\lambda) - A^T(\lambda) \otimes I_n),$$

$$d_{ij}(k, \lambda) = \det[i\langle k, \omega\rangle I_{l_i l_j} + (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda) \otimes I_{l_j})].$$

For the skew-product system (1.7), by using Lemmas 1.1 and 1.2 in [12], we set for $\forall\ \langle k,\omega\rangle \in \mathbb{R}$

$$g_{ij}(k,\lambda) = \begin{cases} \prod_{\alpha_u\in\sigma(A_i),\beta_v\in\sigma(A_j)}(i\langle k,\omega\rangle - (\alpha_u(\lambda)-\beta_v(\lambda)), & i \neq j; \\ \prod_{\alpha_u,\alpha_v\in\sigma(A_i),u\neq v}(i\langle k,\omega\rangle - (\alpha_u(\lambda)-\alpha_v(\lambda)), & \text{i=j}. \end{cases}$$

Remark: It is easily seen that if $A \in C^{\omega}(\Lambda, g)$ and the division of $\sigma(A)$ is sufficiently separated , then all g_{ij} are analytic functions of λ, $\forall\ 1 \le i, j \le s$.

For the proof of this remark (see in [17]).

Thus, we assume the following:

Non-degeneracy Conditions: There exist an integer $d \ge 0$ and $\varsigma \ge 0$ such that

$$\max_{0\le l\le d} |\frac{\partial^l}{\partial\lambda^l} g_{ij}(k,\lambda)| > \varsigma, \quad \text{for all} \ \ 1 \le i, j \le s \tag{1.11}$$

uniformly hold $\forall\ \lambda \in \Lambda$ and $\langle k,\omega\rangle \in \mathbb{R}$.

Remark: The condition (1.11) will assure that the small denominator condition always holds for the "most" parameter λ. Here, we take only those k in which $|\langle k,\omega\rangle| \le 2\delta_0$, because for the large enough $|\langle k,\omega\rangle|$, we always see that the matrix $i\langle k,\omega\rangle I_{l_i l_j} - (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda) \otimes I_{l_j})$ is automatically non-singular and the small denominator condition is satisfied. It can easily be seen that the condition (1.11) is weaker than the non-degeneracy condition (1.3) used by Jorba and Simó.

Moreover, the property that $g_{ij}(k,\lambda)$ depends analytically on λ can be preserved under small perturbations, and at each iterative step, we will preserve the non-degeneracy conditions.

1.1. Statement of the main result

To state the main result, consider Q as an analytic g-valued function that can be defined on a complex neighbourhood of $\mathbb{T}^r \times \Lambda$:

$$W_h(\mathbb{T}^r \times \Lambda) = \{(\vartheta,\lambda) \in \mathbb{C}^r \times \Lambda | dist(\vartheta, \mathbb{T}^r) < h\},$$

where $\lambda \in \Lambda = (a, b)$. Defined the norm of Q as:

$$\|Q\|_h = \max_{0\le l\le d} \sup_{(\vartheta,\lambda)\in W_h(\mathbb{T}^r\times\Lambda)} |\frac{\partial^l Q}{\partial\lambda^l}|$$

similarly

$$\|A\| = \max_{0\le l\le d} \sup_{\lambda\in\Lambda} |\frac{\partial^l A(\lambda)}{\partial\lambda^l}|$$

where $\|\cdot\|$ denotes the matrix norm.

Theorem 1.1. *Consider the skew-product system (1.7) in which ω is a fixed Brjuno vector and it satisfies the Brjuno-Rüssmann condition (1.6) and $A(\lambda)$ satisfies the non-degeneracy condition (1.11), and there exists $K > 0$ such that $\|A\| \le K$. Then there exist $\varepsilon > 0, h > 0$, such that if $\|Q(\cdot,\cdot)\|_h = \varepsilon_1 < \varepsilon$, the measure of the set of parameter λ's for which the system (1.7) is non-reducible is no larger than $CL(10\varepsilon_1)^c$, with some positive constants C, c, and L denotes the length of the parameter interval Λ.*

2. Proof of the Theorem 1.1

Theorem 1.1 will be proven by KAM iteration. At each iterative step, we have a L-P transformation close to identity as

$$P(\varphi) = I + Z(\varphi), \tag{2.1}$$

where $Z(\varphi) \in C_h^\omega(\mathbb{T}^r, g)$, $P(\varphi) \in C_h^\omega(\mathbb{T}^r, G)$ and by using the L-P transformation (2.1), the quasi-periodic system $\frac{dx}{dt} = (A + Q)x$ is changed into

$$\frac{dx}{dt} = (D_\omega P \cdot P^{-1} + P(A + Q)P^{-1})x.$$

Since Z is very small and in the expansion form P^{-1} can be written as:

$$P^{-1} = I - Z + Z^2 + O(\|Z\|^3).$$

So, we have

$$\begin{aligned} &D_\omega P \cdot P^{-1} + P(A + Q)P^{-1} \\ &= D_\omega Z(I - Z + Z^2 + O(\|Z\|^3)) + (I + Z)(A + Q)(I - Z + Z^2 + O(\|Z\|^3)) \\ &= A + D_\omega Z + [Z, A] + Q - D_\omega Z \cdot Z + [Z, Q] + AZ^2 - ZAZ + O(\|Z\|^3). \end{aligned} \tag{2.2}$$

In general, we have to find a small Z in which the transformed system is still of the form $\frac{dx}{dt} = (A^+ + Q^+)x$, where A^+ is block-diagonal as A and Q^+ is much smaller than Q.
To do this, we have to calculate Z solving the following linearized equation

$$D_\omega Z - [A, Z] = -Q \tag{2.3}$$

where $[A, Z] = AZ - ZA$ and to prove

$$Q^+ = -D_\omega Z \cdot Z + [Z, Q] + AZ^2 - ZAZ + O(\|Z\|^3)$$

is more smaller.

2.1. Solution of the linearized equation

In this subsection, we will solve the linearized equation, for this we need the following:
Definition: Let $u = (u_1, \cdots, u_m) \in \mathbb{T}^m$. Its norm is denoted by $\|u\|$ and is defined as:

$$\|u\| = \max_{1 \le i \le m} |u_i|.$$

Definition: For a $m \times m$ matrix $S = (s_{ij})$, its operator norm is denoted by $\|S\|$ and is equivalent to $m \times \max |s_{ij}|$.
Notation: Let $F \in C_h^\omega(\mathbb{T}^r \times \Lambda, g)$ and its Fourier series is $F = \sum_{k \in \mathbb{Z}^r} F_k e^{i\langle k, \varphi \rangle}$,then the k^{th} Fourier coefficients of F denoted by F_k, given by $F_k = \int_{\mathbb{T}^r} e^{-i\langle k, \varphi \rangle} F(\varphi) d\varphi$.
Remark 2.1. For $F \in C_h^\omega(\mathbb{T}^r \times \Lambda, g)$, we have

$$|F_k| \le |F|_h e^{-|k|h}.$$

Note. For $k \in \mathbb{Z}^d$, we denote $|k| = \sum_{n=1}^{d} |k_n|$. Similarly, for a function f , its modulus is denoted by $|f|$.

Throughout the discussion, to simplify notations, the letters c, C denote different positive constants.

By substituting the Fourier series expansions of Z, Q into the Eq (2.3), and then by equating the corresponding Fourier coefficients on both sides, we obtain

$$i\langle k,\omega\rangle Z_k - (AZ_k - Z_k A) = -Q_k. \tag{2.4}$$

suppose that the eigenvalues of the linear operator $i\langle k,\omega\rangle I_d + [A,\cdot]$ in the left part are

$$i\langle k,\omega\rangle - (\alpha_i - \alpha_j),\ \ 1 \le i, j \le n,\ \ \alpha_i, \alpha_j \in \sigma(A).$$

The eigenvalues will be $\alpha_i - \alpha_j$ for $k = 0$. As the considered matrix $A = diag(A_1, \cdots, A_s)$ is a block-diagonal with different blocks A_i, A_j and each block have different eigenvalues, .i.e. $\alpha_u \neq \beta_v$ if $\alpha_u \in A_i, \beta_v \in A_j$ for $i \neq j$, from conclusions as seen from other researchers [12, 17–20] , we see that the matrix $I_{l_i} \otimes A_j - A_i^T \otimes I_{l_j}$ is non-singular if $i \neq j$.

In block-diagonal form, let Q_k can be written as (Q_{kij}), where Q_{kij} is a matrix of order $l_i \times l_j$,$1 \le i, j \le s$ and l_i, l_j are the orders of matrices A_i, A_j respectively.

Now, for $k = 0$, we solve the equation (2.4). Suppose

$$Q_0^d = (Q_{011}, \cdots, Q_{0ss})$$

and

$$Q_0^* = Q_0 - Q_0^d.$$

For $k = 0$, the equation (2.4) can be written as

$$AZ_0 - Z_0 A = Q_0 \tag{2.5}$$

Equation (2.5) can not be solved completely because the eigenvalues involved the multiplicity. However, the following equation

$$AZ_0 - Z_0 A = Q_0^*$$

has a solution $Z_0 = (Z_{0ij})$ with $Z_{0ii} = 0$ and

$$A_i Z_{0ij} - Z_{0ij} A_j = Q_{0ij},\ \ \text{for}\ \ i \neq j$$

has the unique solution Z_{0ij}.

Moreover, we have the estimate [12]

$$\begin{aligned} \|J_{ij}^{-1}(0,\lambda)\| &\le \max_{i\neq j} \|[I_{l_j} \otimes A_i(\lambda) - A_j^T(\lambda) \otimes I_{l_i}]^{-1}\| \\ &\le c\frac{nK^{l_i l_j}}{\varrho^{l_i l_j}} \le C(\varrho, n)K^{l_i l_j}, \end{aligned} \tag{2.6}$$

and

$$\max_{0\le l\le r} \|\frac{\partial^l}{\partial\lambda^l} J_{ij}^{-1}(0,\lambda)\| = \max_{1\le l\le r} \|\frac{\partial^l}{\partial\lambda^l}(\frac{adJ_{ij}}{detJ_{ij}})\|$$

$$\leq C(\varrho, n, r)K^{(l_i l_j)^2}$$

as $\mathrm{dist}(\sigma(A_i(\lambda)), \sigma(A_j(\lambda))) > \varrho > 0$, for $i \neq j$. Moreover, we get

$$\max_{0\leq l\leq r} \|\frac{\partial^l}{\partial\lambda^l} Z_0(\lambda)\| \leq C \max_{0\leq l\leq r} \|\frac{\partial^l}{\partial\lambda^l}(J_{ij}^{-1}(0,\lambda)\| \cdot \|\frac{\partial^l}{\partial\lambda^l} Q_0(\lambda)\|$$
$$\leq C(\varrho, n, r)K^{n^4} \max_{0\leq l\leq r} \|\frac{\partial^l}{\partial\lambda^l} Q_0(\lambda)\|. \tag{2.7}$$

Now, we solve the Eq (2.4) for $k \neq 0$. From Lemma 3.2 as seen in [12], the solution of (2.4) is equivalent to the solution of the following vector equation

$$J(k,\lambda)Z_k^{'}(\lambda) = -Q_k^{'}(\lambda) \tag{2.8}$$

By using corollaries [12], Eq (2.8) is solvable $\iff$ the matrix $J(k,\lambda)$ is invertible. Suppose $P = I + \sum Z_k$ is a L-P transformation. Then by using the L-P transformation, the new system becomes

$$\frac{dx}{dt} = (A^+ + Q^+)x$$

where

$$A^+ = A + Q_0^d$$
$$Q^+ = -D_\omega Z \cdot Z^{-1} + [Z, Q] + AZ^2 - ZAZ + O(\|Z\|^3) \tag{2.9}$$

Since A and Q_0^d are block-diagonal matrices, therefore A^+ is also a block-diagonal. Next, we will show that in a smaller domain Q^+ is much smaller and the non-degeneracy condition is satisfied by A^+.
Estimation of Q^+.

First of all, we estimate Z_k. Actually, to control the solution of Z_k, we need the following small denominator condition, .i.e. if there exist $N > 0$ such that $\forall i, j$

$$|g_{ij}(k,\lambda)| \geq \frac{N^{-1}}{\Delta(|k|)}, \quad 1 \leq i, j \leq s. \tag{2.10}$$

where Δ is an approximation function as defined above.

In order to estimate Z_k, we need to estimate the operator $J_{ij}^{-1}(k,\lambda)$ for $k \neq 0$.

Lemma 2.1. *For $k \neq 0$ and the small denominator conditions (2.10) are satisfied by all parameters λ, then we have*

$$\|J_{ij}^{-1}(k,\lambda)\| \leq cK^{l_i l_j} N(\Delta(|k|))^{l_i l_j}, \quad i \neq j, \tag{2.11}$$

$$\|J^{-1}(k,\lambda)\| \leq cK^{n^2}\alpha^n N^{n^2}(\Delta(|k|))^{n^2}, \tag{2.12}$$

$$\max_{0\leq l\leq r} \|\frac{\partial^l}{\partial\lambda^l} J^{-1}(k,\lambda)\| \leq cK^{n^4}\alpha^{2^r n} N^{2^r n^2}(\Delta(|k|))^{2^r n^2}. \tag{2.13}$$

where c denotes constant.

Proof. Since J_{ij} is a non-singular matrix, so its inverse is defined as $J_{ij}^{-1} = adJ_{ij}/detJ_{ij}$. By the small denominator conditions (2.10), we have

$$|J(k,\lambda)| = |[i\langle k,\omega\rangle I_{n^2} + (I_n \otimes A(\lambda) - A^T(\lambda)\otimes I_n)]| \geq (N^{-1})^{n^2}(\frac{\alpha^{-1}}{\Delta(|k|)})^n$$

and

$$|J_{ij}(k,\lambda)| = |[i\langle k,\omega\rangle I_{l_il_j} + (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda)\otimes I_{l_j})]| \geq (N^{-1})^{l_il_j}(\frac{\alpha^{-1}}{\Delta(|k|)})^{l_i}$$

using the definition of the norm $\|J_{ij}\|$ and the small denominator condition (2.10), the estimate (2.11)can be found easily. Also as $detJ = \prod_{1\leq i,j\leq s} detJ_{ij}$, similarly we can calculate the estimations (2.12) and (2.13).
For $k \neq 0$, from Eq (2.8), we have

$$Z_k^{'}(\lambda) = -J^{-1}(k,\lambda)Q_k^{'}(\lambda) \tag{2.14}$$

as $Z_k^{'}, Q_k^{'}$ are the transpose of Z_k and Q_k respectively, therefore it is easy to prove $\|Z_k\| = \|Z_k^{'}\|, \|Q_k\| = \|Q_k^{'}\|$ (see in [12] for the proof).
In our article, we represent $F(\lambda)$ a λ-dependent matrix as:

$$|F(\lambda)| = \max_{0\leq l\leq r} \|\frac{\partial^l F(\lambda)}{\partial\lambda^l}\|.$$

Since $Q \in C_h^{\omega}(\mathbb{T}^r \times \Lambda, g)$, then by the Remark 2.1, we have

$$|Q_k| \leq |Q|_h e^{-|k|h}.$$

As a result, for $k \neq 0$ and for any $0 < \bar{h} < h$, we have

$$\begin{aligned}|Z_k(\lambda)| &\leq |J^{-1}(k,\lambda)||Q_k(\lambda)|\\ &\leq CK^{n^4}\alpha^{2^r n}N^{2^r n^2}(\Delta(|k|))^{2^r n^2}|Q|_h e^{-|k|h}\end{aligned}$$

or

$$|Z_k(\lambda)| \leq CK^{n^4}\alpha^{2^r n}N^{2^r n^2}(\Delta(|k|))^{2^r n^2}|Q|_h e^{-|k|(h-\bar{h})}e^{-|k|\bar{h}}. \tag{2.15}$$

In particular, take an approximation function $\Delta(t) = e^{t^\delta}, \delta < 1$, which satisfy the Brjuno-Rüssmann condition (1.6), since the function $e^{t^\delta 2^r n^2} \cdot e^{-t(h-\bar{h})}$ has the maximal value at $t = (\frac{2^r n^2\delta}{h-\bar{h}})^{\frac{-1}{\delta-1}}$, one has

$$\begin{aligned}|Z_k(\lambda)| &\leq CK^{n^4}\alpha^{2^r n}N^{2^r n^2}|Q|_h e^{[2^r n^2(\frac{2^r n^2\delta}{h-\bar{h}})^{\frac{-\delta}{\delta-1}} - (\frac{2^r n^2\delta}{h-\bar{h}})^{\frac{-1}{\delta-1}}(h-\bar{h})]}e^{-|k|\bar{h}}\\ &\leq C(n,r,\delta,\alpha)K^{n^4}N^{2^r n^2}[\frac{|Q|_h}{(h-\bar{h})^{\frac{\delta^2-\delta-1}{\delta-1}}} - \frac{|Q|_h}{(h-\bar{h})^{\frac{\delta}{\delta-1}}}]e^{-|k|\bar{h}}.\end{aligned} \tag{2.16}$$

Consider

$$Z(t,\lambda) = \sum_{k\in\mathbb{Z}^r} Z_k(\lambda)e^{i\langle k,t\rangle}$$

choose $h^{'} : 0 < h^{'} < \bar{h}$ s.t. if $\bar{h} - h^{'} = h - h^{'} < 1$. So, using the Lemma 4 in [7], we obtain

$$\begin{aligned}
|Z|_{h'} &\leq \sum_{k\in\mathbb{Z}^r} |Z_k| e^{|k|h^{'}} \\
&\leq CK^{n^4}|Q_0| + CK^{n^4}N^{2^r n^2}\Big[\frac{|Q|_h}{(h-\bar{h})^{\frac{\delta^2-\delta-1}{\delta-1}}} - \frac{|Q|_h}{(h-\bar{h})^{\frac{\delta}{\delta-1}}}\Big] \sum_{k\in\mathbb{Z}^r\setminus\{0\}} e^{-(\bar{h}-h^{'})|k|} \\
&\leq CK^{n^4}N^{2^r n^2}\Big[\frac{|Q|_h}{(h-\bar{h})^{\frac{\delta^2-\delta-1}{\delta-1}}} - \frac{|Q|_h}{(h-\bar{h})^{\frac{\delta}{\delta-1}}}\Big]\Big(\frac{2}{\bar{h}-h^{'}}\Big)^m e^{\frac{(\bar{h}-h^{'})m}{2}} \\
&\leq C(n,r,\delta,\alpha,m)K^{n^4}N^{2^r n^2}\Big[\frac{1}{(h-h')^{\frac{\delta^2-\delta-1}{\delta-1}+m}} - \frac{1}{(h-h')^{\frac{\delta}{\delta-1}+m}}\Big]|Q|_h. \qquad (2.17)
\end{aligned}$$

Let $s = \frac{\delta^2-\delta-1}{\delta-1} + m$, and $s^{'} = \frac{\delta}{\delta-1} + m$, we get

$$|Z|_{h'} \leq CK^{n^4}N^{2^r n^2}\Big[\frac{1}{(h-h')^s} - \frac{1}{(h-h')^{s^{'}}}\Big]|Q|_h. \qquad (2.18)$$

similarly, we can find

$$|D_\omega Z|_{h'} \leq CK^{n^4}N^{2^r n^2}\Big[\frac{1}{(h-h')^{s+1}} - \frac{1}{(h-h')^{s^{'}+1}}\Big]|Q|_h$$

$$|D_\omega Z\cdot Z|_{h'} \leq CK^{n^4}N^{2^r n^2}\Big[\frac{1}{(h-h')^{2s+1}} - \frac{1}{(h-h')^{2s^{'}+1}}\Big]|Q|_h^2$$

$$|AZ^2|_{h'} = |ZAZ|_{h'} \leq CK^{n^4}N^{2^r n^2}\Big[\frac{1}{(h-h')^{2s}} - \frac{1}{(h-h')^{2s^{'}}}\Big]|Q|_h^2$$

$$|[Z,Q]|_{h'} \leq 2|Z|_{h'}\cdot|Q|_h \leq CK^{n^4}N^{2^r n^2}\Big[\frac{1}{(h-h')^{s}} - \frac{1}{(h-h')^{s^{'}}}\Big]|Q|_h^2$$

Hence, from Eq (2.9), we get

$$|Q^+|_{h'} \leq CK^{2n^4+1}N^{2^{r+1}n^2}\Big[\frac{1}{(h-h')^{2s+1}} - \frac{1}{(h-h')^{2s^{'}+1}}\Big]|Q|_h^2. \qquad (2.19)$$

Verification of the non-degeneracy conditions for A^+.

Since

$$A^+ = A + Q_0^d = \mathrm{diag}(A_1 + Q_{011}, \cdots, A_s + Q_{0ss}).$$

Let

$$D_{ij}^+(k,\lambda) = \det[i\langle k,\omega\rangle I_{l_i l_j} + (I_{l_i}\otimes(A_j(\lambda) + Q_{0jj}(\lambda))) - (A_i^T(\lambda) + Q_{0ii}^T(\lambda))\otimes I_j)].$$

The new determinant D_{ij}^+ is analytic with respect to λ as well.

The above determinant can be rewritten as

$$D^{+}_{ij}(k,\lambda) = D_{ij}(k,\lambda) + Y_{ij}(k,\lambda).$$

where $D_{ij}(k,\lambda) = \det[i\langle k,\omega\rangle I_{l_i l_j} + (I_{l_i} \otimes A_j(\lambda) - A_i^T(\lambda)) \otimes I_j)]$ and $Y_{ij}(k,\lambda)$ is a summary of $2^{l_i l_j} - 1$ determinants denoted by $y_t(k,\lambda)(1 \le t \le 2^{l_i l_j} - 1)$. Furthermore, there exist at least one column in each determinant y_t such that the entries in this column are either 0 or of the form $c - d$, where c and d are entries of Q_{0jj} and Q_{0ii} respectively.

As $|Q_0^d|_h \le |Q|_h < \varepsilon$, we get

$$|\frac{\partial^l}{\partial\lambda^l} D^{+}_{ij}(k,\lambda)| \le C|A|\varepsilon, \text{for } 1 \le l \le r.$$

similarly,

$$|\frac{\partial^l}{\partial\lambda^l}(g^{+}_{ij}(k,\lambda) - g_{ij}(k,\lambda))| \le C|A|\varepsilon, \text{for } 1 \le l \le r. \tag{2.20}$$

So, we have

$$|\frac{\partial^l}{\partial\lambda^l} g^{+}_{ij}(k,\lambda)| \ge \varsigma - C|A|\varepsilon \ge \varsigma - CK\varepsilon = \varsigma', \text{for } 1 \le l \le r. \tag{2.21}$$

The proof is obvious. Note that, here we only need to choose such $k's$ so that $|(k,\lambda)|$ is not large enough, .i.e., $|(k,\lambda)| \le CK$,where $|A| \le K$, because for large enough $|(k,\lambda)|$, the matrix $J(k,\lambda)$ becomes automatically non-singular. So, when $|(k,\lambda)|$ has large values , then $J^{+}(k,\lambda)$ becomes naturally non-singular and no need to preserve non-degenerate property.

Alternatively, we know from the perturbation theory of matrices that the continuous change of eigenvalues depends on the entries, and by Ostrowski theorem (see [21]), the distance between eigenvalues of any two blocks can be estimated as

$$\min_{i \ne j} \text{dist}(\sigma(A_i^{+}), \sigma(A_j^{+})) = \varrho^{+} > \varrho - c\varepsilon^{\frac{1}{n}}.$$

Now, we summarize the above discussions in the following conclusion.

Conclusion 1.

Consider Λ subset of (a,b) be some parameter segment, a one parameter family of constant elements $A \in C^{\omega}(\Lambda, g)$, and $Q \in C_h^{\omega}(\mathbb{T}^r \times \Lambda, g)$ be the perturbation. Suppose that there exist $K, \varepsilon, N > 0$ s.t.

- $|A| \le K, \quad |Q|_h < \varepsilon$,
- for all $\lambda \in \Lambda$, the non-degeneracy conditions (1.11) and the small denominator conditions (2.10) hold.

Then, $\exists\ h' > 0$ and a map $Z \in C_{h'}^{\omega}(\mathbb{T}^r \times \Lambda, g)$, and

$$A^{+} \in C^{\omega}(\Lambda, g)$$

$$Q^{+} \in C_{h'}^{\omega}(\mathbb{T}^r \times \Lambda, g),$$

such that

1) $A^{+} = A + Q_0^d, \quad A^{+} + Q^{+} \equiv A + Q$
2) We have the estimation (2.19), .i.e. $|Q^{+}|_{h'} \le CK^{2n^4+1} N^{2^{r+1}n^2}[\frac{1}{(h-h')^{2s+1}} - \frac{1}{(h-h')^{2s'+1}}]|Q|_h^2$.
3) We have preserved the non-degeneracy conditions .i.e., $\max_{0 \le l \le r} |\frac{\partial^l}{\partial\lambda^l} g^{+}_{ij}(k,\lambda)| \ge \varsigma'$.
4) $\varrho^{+} > \varrho - c\varepsilon^{\frac{1}{n}}, K^{+} < K + \varepsilon$.

2.2. Iteration

In this subsection, we will prove that the perturbation Q goes to zero very quickly provided that the small divisor conditions hold.

First of all, consider the following two iterative sequences:

$$h_m = (\frac{1}{2} + \frac{1}{2^m})h_1, \tag{2.22}$$

$$N_m = (\frac{(\frac{6}{5})^m + \frac{1}{\eta}}{h_{m-1} - h_m})^\gamma = (h_1)^{-\gamma} 2^{m\gamma}((\frac{6}{5})^m + \frac{1}{\eta})^\gamma \tag{2.23}$$

where $\gamma \geq r$ is a constant, and η will be considered as in the following lemma

Lemma 2.2. *There exist positive constants $\eta < 1, b$, s.t., if ε_1 is sufficiently small, then $\forall\ m \geq 1$*

$$\varepsilon_m \leq \eta^b e^{-(\frac{6}{5})^m},$$

$$K_m \leq 2^{m-1} K_1.$$

Proof. Suppose that if we do this up to m^{th} step, we have

$$|Q_m|_{h_m} \leq \varepsilon_m \leq \eta^b e^{-(\frac{6}{5})^m}$$

and

$$K_m \leq K_{m-1} + \varepsilon_{m-1} \leq 2^{m-1} K_1.$$

By induction, we need to prove that

$$|Q_{m+1}|_{h_{m+1}} \leq \eta^b e^{-(\frac{6}{5})^{m+1}} \tag{2.24}$$

and

$$K_{m+1} \leq 2^m K_1. \tag{2.25}$$

Indeed Eq (2.25) is satisfied as

$$K_{m+1} \leq K_m + \varepsilon_m \leq K_m + \eta^b e^{-(\frac{6}{5})^m} \leq K_m + 1 \leq 2K_m \leq 2 \cdot 2^{m-1} K_1 = 2^m K_1.$$

And from Eq (2.19), we have

$$\varepsilon_{m+1} \leq C K_m^{2n^4+1} N_m^{2^{r+1}n^2} [\frac{1}{(h_m - h_{m+1})^{2s+1}} - \frac{1}{(h_m - h_{m+1})^{2s'+1}}] \varepsilon_m^2.$$

To prove Eq (2.24), we need

$$C K_m^{2n^4+1} N_m^{2^{r+1}n^2} [\frac{1}{(h_m - h_{m+1})^{2s+1}} - \frac{1}{(h_m - h_{m+1})^{2s'+1}}] \eta^{2b} e^{-(\frac{6}{5})^{2m}} \leq \eta^b e^{-(\frac{6}{5})^{m+1}}.$$

Then by using Eqs (2.22) and (2.25), we have

$$CK_1^{2n^4+1}h_1^{-(2s+1)}2^{m(2n^4+1)+(m+1)(2s+1)}N_m^{2^{r+1}n^2}\eta^{2b}e^{-(4/5)(\frac{6}{5})^m} \leq 1. \tag{2.26}$$

Let $R_m(\eta) = N_m^{2^{r+1}n^2}\eta^{b-1}$, if we choose

$$b > 2^{r+1}n^2\gamma + 1, \tag{2.27}$$

then by Eq (2.23) we see that for smaller value of η, the value of R_m also goes smaller. Now, firstly we set $\eta = \eta_0 < 1$. As the sequence

$$2^{m(2n^4+1)+(m+1)(2s+1)+mr}R_m(\eta_0)e^{-(4/5)(\frac{6}{5})^m},$$

is bounded from above,let's denote its maximum by $\bar{\beta}$. In order to satisfy Eq (2.26), it is enough to choose η s.t.

$$CK_1^{2n^4+1}h_1^{-(2s+1)}\bar{\beta}\eta \leq 1.$$

Thus, define

$$\eta \leq \min\{CK_1^{-(2n^4+1)}h_1^{2s+1}\bar{\beta}^{-1}, \eta_0\},$$

and so we obtained the Eq (2.26). If we choose $\eta = (10\varepsilon_1)^{1/b}$, then it is enough to take

$$\varepsilon_1 \leq \min\{\frac{CK_1^{-b(2n^4+1)}h_1^{b(2s+1)}}{10\beta^b}, \eta^b e^{-\frac{6}{5}}\}. \tag{2.28}$$

Hence, the proof of lemma is finished.

From Eq (2.18), it can be seen that the sequence $|Z_m|_{h_m}$ converges to 0 with super-exponential velocity, then by the transformation $P_m = I + Z_m$, we have $P_m \to I$, and so the composition of transformations $P_m \circ P_{m-1} \circ \cdots \circ P_1$ will also be convergent. On the other hand, from conclusion 1, we have

$$\varsigma_m \geq \varsigma_{m-1} - CK_m\varepsilon_m,$$

so

$$\varsigma_m \geq \varsigma - C\sum_{1\leq i\leq m-1} K_i\varepsilon_i \geq \frac{\varsigma}{2}, \tag{2.29}$$

for small enough ε_1. Thus, the preservation of the non-degeneracy conditions is proved. By the way, for small enough ε_1, we also have the estimate

$$\varrho_m \geq \varrho - C\sum_{1\leq i\leq m-1} \varepsilon_i^{\frac{1}{n}} \geq \frac{\varrho}{2}. \tag{2.30}$$

2.3. Measure of the removed set

In this subsection, we will show that the set of parameters satisfying the small denominator conditions is of the large Lebesgue measure. In the end, we estimate the measure of the removed parameter set. At the m^{th} step, for $\forall i, j, 1 \le i, j \le s$, we denote the removed set as:

$$R_{kij}^{m} = \{\lambda : |g_{ij}^{m}(k,\lambda)| \le \frac{N_m^{-1}}{\Delta(|k|)}\}$$

and consider

$$R_k^m = \bigcup_{1\le i,j\le s} R_{kij}^m,$$

$$R^m = \bigcup_{0\ne k\in\mathbb{Z}^r} R_k^m.$$

To calculate the estimate for the measure of R_{kij}^m, the following lemma is needed:

Lemma 2.3. *Consider $g(x)$ is a C^M function on the closure $\bar{I}$, where $I \in R^1$ is an interval of length L. Let $I_h = \{x : |g(x)| \le h, h > 0\}$. If for some constant $r > 0, |g^{(M)}(x)| \ge r$ for $\forall x \in I$, then $|I_h| \le cLh^{1/M}$, where $|I_h|$ denotes the Lebesgue measure of I_h and constant $c = 2(2 + 3 + \cdots + M + r^{-1})$.*

For the proof, see [22].

Then, let L denotes the length of the parameter interval Λ, and using above Lemma 2.3, we obtain

$$mes(R_{kij}^m) \le cL(\frac{N_m^{-1}}{\Delta(|k|)})^{1/r}$$

where $c = 2(2 + 3 + \cdots + r + 2/\varsigma)$, as $g_{ij}^m(k,\lambda) \in C^m(\Lambda)$ and using the non-degeneracy conditions and Eq (2.30). Thus,

$$mes(R^m) \le Cn^2LN_m^{-\frac{1}{r}} \sum_{0\ne k\in\mathbb{Z}^r} (\frac{1}{\Delta(|k|)})^{1/r}.$$

For $\Delta(|k|) = e^{|k|^\delta}, \delta < 1$, we have

$$\begin{aligned} mes(R^m) &\le Cn^2LN_m^{-\frac{1}{r}} \sum_{0\ne k\in\mathbb{Z}^r} e^{-|k|^\delta/r} \\ &\le C(n,r,\delta,\varsigma)LN_m^{-\frac{1}{r}}. \end{aligned}$$

By Eq (2.23), $N_m > \frac{2^m\gamma}{\eta^\gamma}$, we have

$$N_m^{-\frac{1}{r}} \le \eta^{\frac{\gamma}{r}} \cdot \frac{1}{2^{\frac{m\gamma}{r}}}.$$

Therefore, for $\eta = (10\varepsilon_1)^{\frac{1}{b}}$ and $\gamma \ge r$, one has

$$\begin{aligned} mes(\bigcup_{m=1}^{\infty} R^m) &\le CL\eta^{\frac{\gamma}{r}} \sum_{m=1}^{\infty} 2^{\frac{-m\gamma}{r}} \le CL\eta^{\frac{\gamma}{r}} \\ &\le C(n,r,\delta,\varsigma,\gamma,\varrho)L(10\varepsilon_1)^c, \text{where,} \quad c = \frac{\gamma}{br}. \end{aligned}$$

Hence, the proof of the main result is completed.

3. Conclusions

In this article, we discussed the positive measure reducibility for quasi-periodic linear systems and proved that the system (1.7) is reduced to a constant coefficient system. The result was proved for a Brjuno vector ω and small parameter λ by using the KAM method, Brjuno-Rüssmann condition and non-degeneracy condition.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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