Appl. Set-Valued Anal. Optim. 4 (2022), No. 3, pp. 271-275
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.4.2022.3.02

ĆIRIĆ AND MEIR-KEELER FIXED POINT RESULTS IN SUPER METRIC SPACES

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Abstract. In this paper, we consider Meir Keeler and Ćirić contractions in the setting of super metric spaces which is an interesting generalization of standard metric space. We investigate the existence and uniqueness of fixed points for these operators in this new structure.
Keywords. Ćirić contraction; Fixed point; Meir-Keeler contraction; Super metric space.

1. Introduction

The metric fixed point theory has a plain statement. We underline here that plain does not mean it is simple. On the contrary, it contains practical techniques for solving a wide range of problems in the real world. Banach initiated the metric fixed point theory by reporting the first result in the setting of normed spaces. Caccioppoli gave the metrical version in 1931. Today, the known form of Banach's fixed theorem is indeed the result of Caccioppoli. The statement and proof of Banach's fixed theorem are plain and extraordinary. The proof of the theorem is essential as its statement. Banach's fixed theorem statement guarantees both the existence and uniqueness of a fixed point of contraction mappings in complete metric spaces. On the other hand, how to obtain this desired fixed point was not explained in the statement of the theorem. The proof of Banach guarantees the existence of the fixed point. Further, an iterative sequence constructs this fixed point. That is why the proof of Banach's fixed theorem is so crucial. Since the finding of the fixed point corresponds to the solution of the induced equation from real-world problems, it has a wide range of applications.

Therefore, fixed point theory is worthy of study in all quantitative sciences. Naturally, due to the requirement of (real-world) problems, the theory of fixed points need to be advanced. The theory has two main directions to be developed. First, the notion of the contraction mapping needs to be improved, generalized, and extended. The second direction is to refine the metric spaces to more abstract spaces according to different problems.

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Received April 22, 2022; Accepted May 8, 2022.

This paper focuses on the second direction and investigate a better abstract space to guarantee the existence and uniqueness of a fixed point of certain operators. Recently, researchers proposed several new distance functions in the literature to obtain a finer structure. Among those, we mention quasi-metric, symmetric, b-metric, fuzzy metric, partial metric, dislocated metric, D-metric, G-metric, F-metric, S-metric, cone metric, complex-valued metric, quaternion-valued metric, and C*-algebra valued metric; see [4] for more details.

In this study, we mention the super metric, which was introduced recently. In [4], Karapınar and Khojasteh defined the notion of super metric spaces. The super metric is an interesting extension of b-metric spaces; see, e.g., [1, 2, 3, 5]. It is also a generalization of the standard metric spaces. The super metric space has a proper topology and useful topological properties that make this space suitable for investigating fixed points.

In what follows, we state the definition of the super metric.

Definition 1.1. [4] Let $\mathfrak{X}$ be a nonempty set. We say that $\mathfrak{m}:\mathfrak{X}\times\mathfrak{X}\to[0,+\infty)$ is super-metric or s-metric if

(1) $\mathfrak{m}(x,y)=0$ if and only if $x=y$ for all $x,y\in\mathfrak{X}$,
(2) $\mathfrak{m}(x,y)=\mathfrak{m}(y,x)$ for all $x,y\in\mathfrak{X}$,
(3) there exists $s\geq 1$ such that $\mathfrak{m}(x,y)\leq s\mathfrak{m}(x,z)$ for all distinct $x,y,z\in\mathfrak{X}$.

Then, we call $(\mathfrak{X},\mathfrak{m})$ a s-metric space.

Suppose that $r>0$ and $x\in\mathfrak{X}$. Denote

$$\mathfrak{B}(x,r)=\{y\in\mathfrak{X}:\mathfrak{m}(x,y)<r\}$$

as an open ball in super metric space $(\mathfrak{X},\mathfrak{m})$.

Let $(\mathfrak{X},\mathfrak{m})$ be a super metric space. Define ([4])

$$\tau_{\mathfrak{X}}=\{A\subset\mathfrak{X}:x\in A\iff\exists\, r>0,\ \textit{such that}\ \mathfrak{B}(x,r)\subset A\}$$

Lemma 1.1. [4] *Let $(\mathfrak{X},\mathfrak{m})$ be a super space. Then each open ball is an open set.*

Definition 1.2. [4] Let $(\mathfrak{X},\mathfrak{m})$ be a super metric space and let $\{x_n\}$ be a sequence in $\mathfrak{X}$. We say that $\{x_n\}$ converges to x in $\mathfrak{X}$ if and only if $\mathfrak{m}(x_n,x)\to 0$ as $n\to\infty$.

Definition 1.3. [4] Let $(\mathfrak{X},\mathfrak{m})$ be a super metric space and let $\{x_n\}$ be a sequence in $\mathfrak{X}$. We say that $\{x_n\}$ is a Cauchy sequence in $\mathfrak{X}$ if and only if $\lim_{n\to\infty}\sup\{\mathfrak{m}(x_n,x_m):m>n\}=0$.

Definition 1.4. [4] Let $(\mathfrak{X},\mathfrak{m})$ be a super metric space. We say that $(\mathfrak{X},\mathfrak{m}$ is a complete super metric space if and only if every Cauchy sequence is convergent in $\mathfrak{X}$.

In this paper, we consider two celebrated contractions, Ćirić contractions and Meir-Keeler contractions in the context of super metric spaces.

2. Main Results

This section consists of two subsections. In the first subsection, we consider Ćirić contractions, and we examine Meir-Keeler contractions in the second subsection.

2.1. **Ćirić fixed point theorem in super metric spaces.** In this subsection, we state and prove the renowned Ćirić fixed point theorem in super metric spaces.

Theorem 2.1. *Let $(\mathfrak{X}, \mathfrak{m})$ be a super metric space by metric constant $s \geq 1$, and let $T : \mathfrak{X} \to \mathfrak{X}$ be a mapping. Suppose that $0 < \alpha < \frac{1}{s}$ such that*

$$\mathfrak{m}(Tx, Ty) \leq \alpha M_{\mathfrak{m}}(x, y) \tag{2.1}$$

for all $x, y \in \mathfrak{X}$, where

$$M_{\mathfrak{m}}(x, y) = \{\mathfrak{m}(x, y), \mathfrak{m}(x, Tx), \mathfrak{m}(y, Ty), \mathfrak{m}(x, Ty), \mathfrak{m}(y, Tx)\}.$$

Then T has a unique fixed point in $\mathfrak{X}$.

Proof. Let $x_0 \in \mathfrak{X}$ and let $x_1 = Tx_0$. If $x_0 = x_1$, then x_1 is the fixed point and the proof is completed. So, we suppose $x_0 \neq x_1$. Thus $\mathfrak{m}(x_0, x_1) > 0$. Without loss of generality, we can define $x_{n+1} = Tx_n$ such that $x_n \neq x_{n+1}$. So, $\mathfrak{m}(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Note that, for all $n \in \mathbb{N}$, $\mathfrak{m}(x_{n+1}, x_{n-1}) \leq s\mathfrak{m}(x_{n-1}, x_n)$. Thus

$$\begin{aligned} \mathfrak{m}(x_n, x_{n+1}) &\leq \alpha M_{\mathfrak{m}}(x_n, x_{n-1}) \\ &= \alpha \max\{\mathfrak{m}(x_{n-1}, x_n), \mathfrak{m}(x_{n-1}, x_{n+1})\} \\ &\leq \alpha \max\{\mathfrak{m}(x_{n-1}, x_n), s\mathfrak{m}(x_{n-1}, x_n)\} \\ &= \alpha s\mathfrak{m}(x_{n-1}, x_n) \\ &\leq (\alpha s)^2 \mathfrak{m}(x_{n-1}, x_{n-2}) \\ & \\ &\leq (\alpha s)^n \mathfrak{m}(x_1, x_0). \end{aligned} \tag{2.2}$$

Taking the limit from both side of (2.2) implies that $\lim_{n\to\infty} \mathfrak{m}(x_n, x_{n+1}) = 0$. Now we suppose that $m > n$ for all $m, n \in \mathbb{N}$. If $x_n = x_m$, we have $T^m(x_0) = T^n(x_0)$. Thus $T^{m-n}(T^n(x_0)) = T^n(x_0)$. It follows that $T^n(x_0)$ is the fixed point of T^{m-n}, and

$$T(T^{m-n}(T^n(x_0))) = T^{m-n}(T(T^n(x_0))) = T(T^n(x_0)).$$

This means that $T(T^n(x_0))$ is the fixed point of T^{m-n}. Thus $T(T^n(x_0)) = T^n(x_0)$. Hence, $T^n(x_0)$ is the fixed point of T. Without loss of generality, we can suppose that $x_n \neq x_m$. Therefore,

$$\mathfrak{m}(x_n, x_m) \leq s\mathfrak{m}(x_n, x_{n+1}). \tag{2.3}$$

Taking limit from both side of (2.3), one can conclude that $\lim_{n\to\infty} \sup\{\mathfrak{m}(x_n, x_m) : m > n\} = 0$. It means that $\{x_n\}$ is a Cauchy sequence. Since $(\mathfrak{X}, \mathfrak{m})$ is a complete super metric space, one has that sequence $\{x_n\}$ converges to $z \in \mathfrak{X}$. We claim that z is the fixed point of T. On the contrary, assume $\mathfrak{m}(z, Tz) > 0$. Note that

$$\begin{aligned} \mathfrak{m}(x_{n+1}, Tz) &= \mathfrak{m}(Tx_n, Tz) \\ &\leq \alpha \max\{\mathfrak{m}(x_n, z), \mathfrak{m}(x_n, x_{n+1}), \mathfrak{m}(Tz, z), \mathfrak{m}(x_n, Tz), \mathfrak{m}(z, x_{n+1})\}. \end{aligned} \tag{2.4}$$

Taking limit from both side of (2.4), we have $\alpha > 1$, which is a contradiction. Thus we have that $Tz = z$ is the fixed point of T in $\mathfrak{X}$. Also, the uniqueness of the fixed point is straightforward from (2.1). This completes the proof. □

Example 2.1. Let $\mathfrak{X} = [0, 2] \cup [4, 5]$ and let

$$\mathfrak{m}(x, y) = \begin{cases} \frac{1}{x+y+4}, & x \neq y, \\ 0, & x = y. \end{cases}$$

It is easy to check that $(\mathfrak{X}, d)$ is a supper metric space with respect to $s = \frac{13}{4}$. Now let $T : \mathfrak{X} \to \mathfrak{X}$ be defined as follow:

$$Tx = \begin{cases} 1, & x \in [0,2], \\ 4, & x \in [4,5]. \end{cases}$$

Since $\mathfrak{m}(Tx,Ty) = \frac{1}{9}$, $\mathfrak{m}(x,Ty) = \frac{1}{x+8}$, $\mathfrak{m}(Tx,y) = \frac{1}{y+5}$, $\mathfrak{m}(x,Tx) = \frac{1}{x+5}$, and $\mathfrak{m}(y,Ty) = \frac{1}{y+8}$, we have

$$\max\{\mathfrak{m}(x,y), \mathfrak{m}(x,Ty), \mathfrak{m}(Tx,y), \mathfrak{m}(x,Tx), \mathfrak{m}(y,Ty)\} = \frac{1}{x+5}$$

and $\frac{x+5}{9} \le \frac{7}{9}$. Taking $\frac{7}{9} \le \alpha < 1$, we have

$$\mathfrak{m}(Tx,Ty) = \frac{1}{9} \le \alpha \max\{\mathfrak{m}(x,y), \mathfrak{m}(x,Ty), \mathfrak{m}(Tx,y), \mathfrak{m}(x,Tx), \mathfrak{m}(y,Ty)\}.$$

Note that $\frac{1}{s} = \frac{4}{13} < \alpha$. Thus T satisfies all the conditions of Theorem 2.1, and T has a unique fixed point ($x = 4$ in this example).

2.2. **Meir-Keeler in super metric spaces.** In the following, we investigate the Meir-Keeler contraction in supper metric spaces.

Theorem 2.2. *Let $(\mathfrak{X}, \mathfrak{m})$ be a super metric space by metric constant $s \ge 1$, and let $T : \mathfrak{X} \to \mathfrak{X}$ be a mapping. Suppose that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for $x, y \in \mathfrak{X}$,*

$$\varepsilon \le \mathfrak{m}(x,y) < \varepsilon + \delta \text{ implies } \mathfrak{m}(Tx,Ty) < \varepsilon.$$

Then T has a unique fixed point in $\mathfrak{X}$.

Proof. First we claim that $\mathfrak{m}(Tx,Ty) \le \mathfrak{m}(x,y)$ for all $x, y \in \mathfrak{X}$, with $x \ne y$. Suppose that there exists $x_0, y_0 \in \mathfrak{X}$ such that $\mathfrak{m}(x_0,y_0) < \mathfrak{m}(Tx_0,Ty_0)$. Consider $\varepsilon_0 = \frac{1}{2}\mathfrak{m}(Tx_0,Ty_0)$. There are two cases:

- $\mathfrak{m}(x_0,y_0) < \varepsilon_0$,
- $\varepsilon_0 < \mathfrak{m}(x_0,y_0)$.

In both cases, there exists $\delta > 0$ such that $\mathfrak{m}(x_0,y_0) < \varepsilon + \delta$ and it implies that $\mathfrak{m}(Tx_0,Ty_0) < \varepsilon_0$, which is a contradiction.

Now let $x_0 \in \mathfrak{X}$ and let $x_{n+1} = Tx_n$. We have $\mathfrak{m}(x_{n+1},x_n) \le \mathfrak{m}(x_n,x_{n-1})$. So, the sequence $\{\mathfrak{m}(x_n,x_{n-1})\}$ converges to some $r \ge 0$. We claim that $r = 0$. Suppose that $r > 0$. Since $r \le \mathfrak{m}(x_n,x_{n-1})$ for all $n > N$ (N can be chosen sufficiently large), there exists $\delta > 0$ such that

$$r \le \mathfrak{m}(x_N,x_{N-1}) < r + \delta,$$

which implies $\mathfrak{m}(x_N,x_{N+1}) < r$. This is again a contradiction. So $r = 0$. Now, we claim that x_n is a Cauchy sequence. On the contrary, assume $\{x_n\}$ is not a Cauchy sequence. Therefore, there exists an $\varepsilon_0 > 0$ such that, for all $n \in \mathbb{N}$, there exists $i_n, j_n \ge n$ such that $\mathfrak{m}(x_{i_n},x_{j_n}) \ge \varepsilon_0$. Given $d \in (0, \frac{2\varepsilon_0}{s})$, there exists $M > 0$ such that, for all $n \ge M$, $\mathfrak{m}(x_{i_M},x_{i_M-1}) < \frac{d}{s}$.

Now we have

$$\begin{aligned} \mathfrak{m}(x_{i_M-1},x_{i_M+1}) &\le \tfrac{s}{2}(\mathfrak{m}(x_{i_M-1},x_{i_M}) + \mathfrak{m}(x_{i_M},x_{i_M+1})) \\ &\le \tfrac{s}{2}(\tfrac{d}{s} + \tfrac{d}{s}) = \tfrac{d}{2} + \tfrac{d}{2} < d + \tfrac{\varepsilon_0}{s^{j_M-i_M-2}}. \end{aligned}$$

Thus

$$\mathfrak{m}(x_{i_M},x_{i_M+2}) < \frac{\varepsilon_0}{s^{j_M-i_M-2}}.$$

We also have

$$\begin{array}{rcl}\mathfrak{m}(x_{i_M-1},x_{i_M+2}) & \leq & \frac{s}{2}(\mathfrak{m}(x_{i_M-1},x_{i_M})+\mathfrak{m}(x_{i_M},x_{i_M+2})) \\ & \leq & \frac{s}{2}(\frac{d}{s}+\frac{\varepsilon_0}{s^{j_M-i_M-2}})=\frac{d}{2}+\frac{\varepsilon_0}{s^{j_M-i_M-3}}.\end{array}$$

Thus

$$\mathfrak{m}(x_{i_M},x_{i_M+3})<\frac{\varepsilon_0}{s^{j_M-i_M-3}}.$$

Repeating the procedure, we have

$$\mathfrak{m}(x_{i_M},x_{i_M+k})<\frac{\varepsilon_0}{s^{j_M-i_M-k}}$$

for each $k\in\{2,3,...,j_M-i_M\}$. So, taking $k=j_M-i_M$, we have $\mathfrak{m}(x_{i_M},x_{j_M})<\varepsilon_0$ which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. So the sequence is converges to $z\in\mathfrak{X}$. We claim that z is the fixed point of T. Since $\mathfrak{m}(Tx,Ty)\leq\mathfrak{m}(x,y)$, we have $\mathfrak{m}(x_{n+1},Tz)\leq\mathfrak{m}(x_n,z)$. Taking limit from both side, we obtain the desired result, and z is the fixed point of T. □

REFERENCES

[1] I.A. Bakhtin The contraction mapping principle in quasimetric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30 (1989) 26-37.

[2] V. Berinde, Generalized contractions in quasi-metric spaces, Seminar on Fixed Point Theory, Babeş-Bolyai University, Research Sem. pp. 3-9, 1993.

[3] S. Czerwik, Contraction mappings in *b*-metric spaces. Acta. Math. Inform. Univ. Ostraviensis. **1** (1993) 5–11.

[4] E. Karapınar, F. Khojasteh, Super metric spaces, Filomat, in press.

[5] E. Karapınar, Recent advances on metric fixed point theory: A review, Appl. Comput. Math. in press.