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ANALYTICAL AND NUMERICAL SOLUTIONS FOR TIME-FRACTIONAL NEW COUPLED MKDV EQUATION ARISING IN INTERACTION OF TWO LONG WAVES

ORKUN TASBOZAN[1], MEHMET ŞENOL[2], ALI KURT[3,*], DUMITRU BALEANU[4,5]

[1]Department of Mathematics, Faculty of Science and Art, Mustafa Kemal University, Hatay, Turkey

[2]Department of Mathematics, Faculty of Science and Art, Nevşehir University, Nevşehir, Turkey

[3]Department of Mathematics, Faculty of Science and Art, Pamukkale University, Denizli, Turkey

[4]Department of Mathematics and Computer Sciences, Faculty of Art and Sciences, Cankaya University, Ankara, Turkey

[5]Institute of Space Sciences, Magurele-Bucharest, Romania

*Corresponding Author: pau.dr.alikurt@gmail.com

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Abstract. The aim of this paper is to present new exact solution sets of nonlinear conformable time-fractional new coupled mKdV equations which arise in interaction of two long waves with different dispersion relations by means of sub-equation method. In addition, we also propose an analytical-approximate method namely residual power series method (RPSM) for the system. The fractional derivatives have been explained in newly defined conformable type, during the solution procedure. The exact solutions of the system obtained by the sub-equation method have been compared to approximate solutions derived by RPSM. The results showed that both methods are robust, dependable, easy to apply and a good alternative for seeking solutions of fractional partial differential equations.

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1. Introduction

The fractional differential equations have been widely used by more researchers to model real-world problems in recent years. It has been broadly studied and applied for various models in many branches of engineering and science such as dynamical systems [10], mathematical physics [34], fluid mechanics [28], biology [19], viscoelasticity [9] and control [35]. In addition, the investigations for analytical and approximate solutions of fractional partial differential equations (FPDEs) gives scientists the opportunity to define phenomena in applied sciences. Therefore, obtaining analytical and approximate solutions of FPDEs has significant and a special place in above cited fields. Some o the classical analytical and approximate methods for solving fractional differential equations (FDEs) are finite difference method (FDM) [27,32], Adomian decomposition method (ADM) [17,20],

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variational iteration method (VIM) [29], homotopy analysis method (HAM) [24,25] and perturbation-iteration algorithm [30,31].

In this article, the sub-equation [11,36] and residual power series methods [4–7] have been implemented to receive new exact and approximate solutions of time-fractional new coupled mKdV equations of the form [12]:

$$\begin{gathered}\partial_t^\alpha u-\frac{1}{2}u_{xxx}+3u^2u_x-3(vv_x)_x-3(uv^2)_x=0,\\ \partial_t^\alpha v+v_{xxx}-6uvu_x+3(vu_x)_x-(u^2-v^2)v_x=0\end{gathered}\tag{1.1}$$

The sub-equation method is a powerful tool for obtaining exact solutions of nonlinear FPDEs. It transforms the given system to an ordinary differential equation to solve it easily. Residual power series is also set up on the power series expansion. It may be applied to the equation directly in the absence of discretization, linearization or any transformation by choosing proper initial conditions.

The remainder of the study is organized as follows. Brief explanations of the methods has firstly been given and the implementation of the considered methods are presented on an example to show efficiency and reliability of the proposed methods. Also figures and tables have been presented in order to compare their numerical results. Finally, we discussed about the obtained results as a conclusion.

2. Preliminaries

There are a few definition of fractional derivative of order $\alpha > 0$. The most widely used are the Riemann-Liouville and Caputo fractional derivatives.

Definition 2.1. *The Riemann-Liouville fractional derivative operator $D^\alpha f(x)$ defined as [2,15,16]:*

$$D^\alpha f(x)=\frac{d^q}{dx^q}\left[\frac{1}{\Gamma(q-\alpha)}\int_\alpha^x\frac{f(t)}{(x-t)^{\alpha+1-q}}dt\right]\tag{2.1}$$

where $\alpha > 0$ and $q-1<\alpha<q$.

Definition 2.2. *The Caputo fractional derivative of order α defined as [13]:*

$$D_*^\alpha f(x)=J^{n-\alpha}D^nf(x)=\frac{1}{\Gamma(n-\alpha)}\int_\alpha^x(x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^nf(t)dt\tag{2.2}$$

where $\alpha>0$ for $n\in\mathbb{N}$, $n-1<\alpha<n$.

Recently, a new definition has been proposed by Khalil et al. [22] which is called "conformable fractional derivative".

Definition 2.3. *An $\alpha-th$ order "conformable fractional derivative" of a function $f:[0,\infty)\to\mathbb{R}$ defined by*

$$T_\alpha(f)(t)=\lim_{\varepsilon\to0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}\tag{2.3}$$

for all $\alpha\in(0,1)$ and $t>0$.

The following theorem gives the properties of this new definition [22].

Theorem 2.1. *Let f and g are α-differantiable functions for $\alpha \in (0,1]$ and $t > 0$. In that case*

1. $T_\alpha(mf + ng) = mT_\alpha(f) + nT_\alpha(g)$ for all $m, n \in \mathbb{R}$,
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all p,
3. $T_\alpha(f.g) = fT_\alpha(g) + gT_\alpha(f)$,
4. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$,
5. $T_\alpha(c) = 0$ for all $f(t) = c$ constant functions,
6. $T_\alpha(f)(t) = t^{1-\alpha}\frac{df(t)}{dt}$, if f is differentiable.

Definition 2.4. *The conformable partial derivatives of an f function of order $\alpha \in (0,1]$ with $x_1, ..., x_n$ variables, is defined as* [8]

$$\frac{d^\alpha}{dx_i^\alpha} f(x_1, ..., x_n) = \lim_{\varepsilon \to 0} \frac{f(x_1, ..., x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, ..., x_n) - f(x_1, ..., x_n)}{\varepsilon}. \tag{2.4}$$

Definition 2.5. *The conformable integral of an f function for $a \geqslant 0$ is defined as* [33]

$$I_\alpha^a(f)(s) = \int_a^s \frac{f(t)}{t^{1-\alpha}} dt. \tag{2.5}$$

In this section, some important definitions and theorems about residual power series will be given.

Theorem 2.2. *Suppose that f is an infinitely $\alpha-$differentiable function at a neigborhood of a point t_0 for some $0 < \alpha \leq 1$, then f has the fractional power series expansion of the form:*

$$f(t) = \sum_{k=0}^{\infty} \frac{\left(T_\alpha^{t_0} f\right)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!},\ t_0 < t < t_0 + R^{\frac{1}{\alpha}},\ R > 0. \tag{2.6}$$

Here $\left(T_\alpha^{t_0} f\right)^{(k)}(t_0)$ represents the application of the fractional derivative $k-$times [1].

Definition 2.6. *A multiple fractional power series about $t_0 = 0$ is defined by $\sum\limits_{n=0}^{\infty} f_n(x)t^{n\alpha}$ for $0 \leq m-1 < \alpha < m$, where t is a variable and $f_n(x)$ are functions called the coefficients of the series* [3, 18].

Theorem 2.3. *Assume that $u(x,t)$ has a multiple fractional power series representation at $t_0 = 0$ of the form* [3]

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)t^{n\alpha},\ 0 \leq m-1 < \alpha < m,\ x \in I,\ 0 \leq t \leq R^{\frac{1}{\alpha}}. \tag{2.7}$$

If $u_t^{(n\alpha)}(x,t)$, $n = 0, 1, 2, \ldots$ are continuous on $I \times (0, R^{\frac{1}{\alpha}})$, then $f_n(x) = \frac{u_t^{(n\alpha)}(x,0)}{\alpha^n n!}$.

3. Basics of the Sub-Equation Method

In this section, we present a short explanation of sub equation method [36]. For a given conformable fractional differential equation with two independent variables x and t, consider

$$P\left(u, D_t^\alpha u, D_x u, D_t^{2\alpha} u, D_x^2 u, ...\right) = 0 \tag{3.1}$$

where $D_t^{\alpha}u$ is conformable fractional derivatives of unknown function $u(x,t)$ and $D_t^{2\alpha}$ stand for two times conformable fractional derivative of it. By adopting the wave transformation [14]

$$u(x,t) = U(\xi),\ \xi = x - k\frac{t^{\alpha}}{\alpha} \tag{3.2}$$

and chain rule [1], Eq. (3.1) turns into nonlinear ordinary differential equation

$$G(U; U'; U''; ...) = 0 \tag{3.3}$$

where primes show integer order derivatives with respect to new wave variables ξ and k that will be examined later. We assume that equation (3.3) has a solution in the form

$$U(\xi) = \sum_{i=0}^{N} a_i \varphi^i(\xi),\ a_N \neq 0 \tag{3.4}$$

where a_i $(0 \leq i \leq N)$ are constant coefficients to be determined later. N is a positive integer which is going to be calculated by balancing the highest order derivatives of linear and nonlinear terms [26] in equation (3.3) and $\varphi(\xi)$ is a solution of Riccati equation

$$\varphi'(\xi) = \sigma + (\varphi(\xi))^2 \tag{3.5}$$

where σ is a constant. Some special solutions for the Riccati equation (3.5) are given by the following formulas.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), & \sigma < 0 \\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), & \sigma < 0 \\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right), & \sigma > 0 \\ \sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right), & \sigma > 0 \\ -\frac{1}{\xi+\varpi},\ \varpi\ is\ a\ cons., & \sigma = 0 \end{cases} \tag{3.6}$$

Subrogating equations (3.4) and (3.5) into equation (3.3) we obtain a polynomial with respect $\varphi(\xi)$. Setting all the coefficients of $\varphi^i(\xi)$ to zero, one gets nonlinear algebraic system in k, a_i $(i = 0, 1, ..., N)$. By solving these nonlinear algebraic system we determine the constants k, a_i $(i = 0, 1, ..., N)$. Then substituting obtained constants from the nonlinear algebraic system and the solutions of equation (3.5) into equation (3.4) by the help of the formulas (3.6) we acquire the exact solutions for equation (3.1).

4. Basics of the residual power series method

To illustrate the basic idea of RPSM, let's take the following nonlinear fractional differential equation [23]:

$$T_{\alpha}u(x,t) + N[x]u(x,t) + L[x]u(x,t) = c(x,t), \tag{4.1}$$

where $n - 1 < n\alpha \leq n$, $x \in \mathbb{R}$, $t > 0$ and given with the initial condition

$$f_0(x) = u(x,0) = f(x). \tag{4.2}$$

Here, $L[x]$ is a linear, $N[x]$ is a non-linear operator and $c(x,t)$ are continuous functions.

The RPSM method made up of stating the solution of the equation (4.1) subject to (4.2) as a fractional power series expansion around $t = 0$.

$$f_{(n-1)}(x) = T_t^{(n-1)\alpha} u(x,0) = h(x) \tag{4.3}$$

The expansion form of the solution is denoted by

$$u(x,t) = f(x) + \sum_{n=1}^{\infty} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \tag{4.4}$$

In the next step, the k.truncted series of $u(x,t)$, namely $u_k(x,t)$ can be written as:

$$u_k(x,t) = f(x) + \sum_{n=1}^{k} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}. \tag{4.5}$$

Since the $1st$ approximate solution $u_1(x,t)$ is

$$u_1(x,t) = f(x) + f_1(x) \frac{t^{\alpha}}{\alpha^n} \tag{4.6}$$

then $u_k(x,t)$ might be reformulated as

$$u_k(x,t) = f(x) + f_1(x) \frac{t^{\alpha}}{\alpha^n} + \sum_{n=2}^{k} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!},\ k = 2,3,4,... \tag{4.7}$$

for $0 < \alpha \leq 1$, $0 \leq t < \mathbb{R}^{\frac{1}{v}}$, $x \in I$.

Initially we express the residual function and the $k-th$ residual function

$$Resu(x,t) = T_{\alpha} u(x,t) + N[x]u(x,t) + L[x]u(x,t) - c(x,t), \tag{4.8}$$

$$Resu_k(x,t) = T_{\alpha} u_k(x,t) + N[x]u_k(x,t) + L[x]u_k(x,t) - g(x,t),\ k = 1,2,3,... \tag{4.9}$$

respectively. Obviously, $Res(x,t) = 0$ and $\lim_{k\to\infty} Resu_k(x,t) = Resu(x,t)$ for each $x \in I$ and $t \geq 0$. Indeed this bring about $\frac{\partial^{(n-1)\alpha}}{\partial t^{(n-1)\alpha}} Resu_k(x,t) = 0$ for $n = 1,2,3,...,k$. Since the fractional derivative of a constant is zero in the conformable sense [4,21]. Solving the equation $\frac{\partial^{(n-1)\alpha}}{\partial t^{(n-1)\alpha}} Resu_k(x,0) = 0$ gives us the required $f_n(x)$ coefficients. So the $u_n(x,t)$ approximate solutions can be obtained respectively in this fashion.

5. Implementation of the Considered Methods

5.1. **Wave Solutions for Fractional New Coupled mKdV Equation.** Let consider the fractional new coupled mKdv equation in Eq. (1.1) where $t \geq 0$, $0 < \alpha \leq 1$ and the derivatives are in conformable sense. With the aid of the chain rule [1] with the aid of wave transform $\xi = x - k\frac{t^{\alpha}}{\alpha}$ [14] led to

$$\begin{aligned} -kU - \frac{1}{2}U'' + U^3 - 3VV' - 3UV^2 &= 0, \\ -kV + V'' + V^3 - 3VU' - 3VU^2 &= 0. \end{aligned} \tag{5.1}$$

Using balancing procedure in equation (5.1), we obtain $N = 1$. So we set up

$$U(\xi) = a_0 + a_1\varphi(\xi), \quad V(\xi) = b_0 + b_1\varphi(\xi) \tag{5.2}$$

where $\varphi(\xi)$ satisfies the Riccati equation (3.5). Subrogating equations (5.2) with equation (3.5) into eq. (5.1), collecting and equating all the coefficients of $\varphi^i(\xi)$ to zero yield a set of algebraic equations with respect to a_0, a_1, b_0, b_1, k. These equations can be mentioned

$\varphi^0 : a_0^3 - 3a_0b_0^2 - a_0k - 3b_0b_1\sigma = 0, -3a_0^2b_0 + b_0^3 - b_0k + 3a_1b_0\sigma = 0,$

$\varphi^1 : 3a_0^2a_1 - 3a_1b_0^2 - 6a_0b_0b_1 - a_1k - a_1\sigma - 3b_1^2\sigma = 0, -6a_0a_1b_0 - 3a_0^2b_1 + 3b_0^2b_1 - b_1k + 2b_1\sigma + 3a_1b_1 = 0,$

$\varphi^2 : 3a_0a_1^2 - 3b_0b_1 - 6a_1b_0b_1 - 3a_0b_1^2 = 0, 3a_1b_0 - 3a_1^2b_0 - 6a_0a_1b_1 + 3b_0b_1^2 = 0,$

$\varphi^3 : -a_1 + a_1^3 - 3b_1^2 - 3a_1b_1^2 = 0, 2b_1 + 3a_1b_1 - 3a_1^2b_1 + b_1^3 = 0.$

Solving the above system equations with the help of computer software Mathematica, we get

$$a_0 = 0,\ a_1 = -\frac{1}{2},\ b_0 = 0,\ b_1 = \frac{1}{2},\ k = \frac{\sigma}{2}. \tag{5.3}$$

Using the values of above coefficients and the equalities (3.6) along with the wave transform $\xi = x - k\frac{t^\alpha}{\alpha}$ and (5.2) yields the following travelling wave solutions

$$\begin{aligned}
u_1(x,t) &= \frac{1}{2}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
v_1(x,t) &= -\frac{1}{2}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
u_2(x,t) &= \frac{1}{2}\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
v_2(x,t) &= -\frac{1}{2}\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
u_3(x,t) &= -\frac{1}{2}\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
v_3(x,t) &= \frac{1}{2}\sqrt{\sigma}\tan\left(\sqrt{\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
u_4(x,t) &= \frac{1}{2}\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right),\\
v_4(x,t) &= -\frac{1}{2}\sqrt{\sigma}\cot\left(\sqrt{\sigma}\left(x - \frac{\sigma t^\alpha}{2\alpha}\right)\right).
\end{aligned} \tag{5.4}$$

5.2. **Approximate Solutions for Fractional New Coupled mKdV Equation.** Consider the nonlinear time-fractional new coupled mKdV equation in Eq. (1.1) with the following initial equations

$$\begin{aligned}
u(x,0) &= f(x) = \frac{1}{2}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}x\right),\\
v(x,0) &= g(x) = -\frac{1}{2}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}x\right).
\end{aligned} \tag{5.5}$$

For residual power series

$$u(x,t) = f(x) + \sum_{n=1}^{\infty} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!},$$
$$v(x,t) = g(x) + \sum_{n=1}^{\infty} g_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \tag{5.6}$$

and k.truncated series of $u(x,t)$ and $v(x,t)$

$$u_k(x,t) = f(x) + \sum_{n=1}^{k} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}, \; k = 1, 2, 3, ...,$$
$$v_k(x,t) = g(x) + \sum_{n=1}^{k} g_n(x) \frac{t^{n\alpha}}{\alpha^n n!}, \; k = 1, 2, 3, ... \tag{5.7}$$

Therefore, the k-th residual functions of time-fractional new coupled mKdV equation are:

$$Resu_k(x,t) = \partial_t^\alpha u_k - \frac{1}{2} u_{k,xxx} + 3u_k^2 u_{k,x} - 3(v_k v_{k,x})_x - 3(u_k v_k^2)_x,$$
$$Resv_k(x,t) = \partial_t^\alpha v_k + v_{k,xxx} - 6u_k v_k u_{k,x} + 3(v_k u_{k,x})_x - (u_k^2 - v_k^2) v_{k,x}. \tag{5.8}$$

To determine the coefficients $f_1(x)$ and $g_1(x)$, in $u_1(x,t)$ and $v_1(x,t)$, we should replace the $1st$ truncated series $u_1(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\alpha}$ and $v_1(x,t) = g(x) + g_1(x)\frac{t^\alpha}{\alpha}$ into the $1st$ truncated residual functions as

$$\begin{aligned} Resu_1(x,t) &= f_1(x) + 3\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha}\right)^2 \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha}\right) - 3\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha}\right)^2 \\ &\quad \times \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha}\right) + 6\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha}\right)\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha}\right)\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha}\right) \\ &\quad -3\left(\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha}\right)^2 + \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha}\right)\left(g''(x) + \frac{t^\alpha g_1''(x)}{\alpha}\right)\right) \\ &\quad +\frac{1}{2}\left(-f^{(3)}(x) - \frac{t^\alpha f_1^{(3)}(x)}{\alpha}\right), \end{aligned} \tag{5.9}$$

$$\begin{aligned} Resv_1(x,t) &= g_1(x) - 6\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha}\right)\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha}\right)\left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha}\right) \\ &\quad -3\left(\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha}\right)^2 - \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha}\right)^2\right)\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha}\right) \\ &\quad +3\left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha}\right)\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha}\right) + \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha}\right)\left(f''(x) + \frac{t^\alpha f_1''(x)}{\alpha}\right) \\ &\quad +g^{(3)}(x) + \frac{t^\alpha g_1^{(3)}(x)}{\alpha}. \end{aligned} \tag{5.10}$$

Now for the substitution of $t = 0$ through equation $Resu_1(x,t)$ and $Resu_1(x,t)$ to obtain

$$Resu_1(x,0) = f_1(x) + 3f^2(x)f'(x) - 3\left(g^2(x)f'(x) + 2f(x)g(x)g'(x)\right) - 3\left(g'^2(x) + g(x)g''(x)\right) - \frac{1}{2}f^{(3)}(x), \tag{5.11}$$

$$Resv_1(x,0) = g_1(x) - 6f(x)g(x)f'(x) - 3\left(f^2(x) - g^2(x)\right)g'(x) + 3\left(f'(x)g'(x) + g(x)f''(x)\right) + g^{(3)}(x). \tag{5.12}$$

Thus for $Resu_1(x,0)=0$ and $Resv_1(x,0)=0$,

$$f_1(x) = -3f^2f' + 3g^2f' + 6fgg' + 3g'^2 + 3gg'' + \frac{1}{2}f^{(3)}, \tag{5.13}$$

$$g_1(x) = 6fgf' + 3f^2g' - 3g^2g' - 3f'g' - 3gf'' - g^{(3)}. \tag{5.14}$$

Therefore, we obtain the $1st$ RPS approximate solutions of time-fractional equation as

$$u_1(x,t) = f + \frac{t^\alpha\left(-3f^2f' + 3g^2f' + 6fgg' + 3g'^2 + 3gg'' + \frac{1}{2}f^{(3)}\right)}{\alpha}, \tag{5.15}$$

$$v_1(x,t) = g + \frac{t^\alpha\left(6fgf' + 3f^2g' - 3g^2g' - 3f'g' - 3gf'' - g^{(3)}\right)}{\alpha}. \tag{5.16}$$

Again, to determine the second unknown coefficients $f_2(x)$ and $g_2(x)$, we replace the $2nd$ truncated series solutions $u_2(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\alpha} + f_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ and $v_2(x,t) = g(x) + g_1(x)\frac{t^\alpha}{\alpha} + g_2(x)\frac{t^{2\alpha}}{2\alpha^2}$ into the $2nd$ truncated residual functions and obtain

$$\begin{aligned}
Resu_2(x,t) &= f_1(x) + \frac{t^\alpha f_2(x)}{\alpha} + 3\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2}\right)^2\left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} + \frac{t^{2\alpha} f_2'(x)}{2\alpha^2}\right) \\
&\quad -3\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2}\right)^2\left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} + \frac{t^{2\alpha} f_2'(x)}{2\alpha^2}\right) \\
&\quad +6\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2}\right)\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2}\right) \\
&\quad \times\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} + \frac{t^{2\alpha} g_2'(x)}{2\alpha^2}\right) - 3\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} + \frac{t^{2\alpha} g_2'(x)}{2\alpha^2}\right)^2 \\
&\quad +\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2}\right)\times\left(g''(x) + \frac{t^\alpha g_1''(x)}{\alpha} + \frac{t^{2\alpha} g_2''(x)}{2\alpha^2}\right) \\
&\quad +\frac{1}{2}\left(-f^{(3)}(x) - \frac{t^\alpha f_1^{(3)}(x)}{\alpha} - \frac{t^{2\alpha} f_2^{(3)}(x)}{2\alpha^2}\right),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
Resv_2(x,t) &= g_1(x) + \frac{t^\alpha g_2(x)}{\alpha} - 6\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2}\right)\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2}\right) \\
&\quad \times\left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} + \frac{t^{2\alpha} f_2'(x)}{2\alpha^2}\right) - 3\left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2}\right)^2 \\
&\quad -3\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2}\right)^2\times\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} + \frac{t^{2\alpha} g_2'(x)}{2\alpha^2}\right) \\
&\quad +3\left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} + \frac{t^{2\alpha} f_2'(x)}{2\alpha^2}\right)\left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} + \frac{t^{2\alpha} g_2'(x)}{2\alpha^2}\right) \\
&\quad +\left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2}\right)\left(f''(x) + \frac{t^\alpha f_1''(x)}{\alpha} + \frac{t^\alpha f_2''(x)}{2\alpha^2}\right) \\
&\quad +g^{(3)}(x) + \frac{t^\alpha g_1^{(3)}(x)}{\alpha} + \frac{t^{2\alpha} g_2^{(3)}(x)}{2\alpha^2}.
\end{aligned} \tag{5.18}$$

Now, applying T_α on both sides of $Resu_2(x,t)$ and $Resv_2(x,t)$ and equating to 0 for $t=0$ gives:

$$\begin{aligned} f_2(x) &= -\frac{\alpha^5}{2\alpha^5-9t^{5\alpha}g_2g_2'}\left(12ff_1f'-12gg_1f'-12gf_1g'-12fg_1g'+6f^2f_1'-6g^2f_1'-12fgg_1'-12g'g_1'\right) \\ &-\frac{\alpha^5}{2\alpha^5-9t^{5\alpha}g_2g_2'}\left(-6g_1g''-6gg_1''-f_1^{(3)}-12fgg_1'-12g'g_1'-6g_1g''-6gg_1''-f_1^{(3)}\right), \end{aligned} \tag{5.19}$$

$$\begin{aligned} g_2(x) &= 6gf_1f'+6fg_1f'+6ff_1g'-6gg_1g'+6fgf_1'-3g'f_1' \\ &+3f^2g_1'-3g^2g_1'-3f'g_1'-3g_1f''-3gf_1''-g_1^{(3)}. \end{aligned} \tag{5.20}$$

Therefore the $2nd$ RPS approximate solutions of time-fractional new coupled mKdV equation is obtained as:

$$\begin{aligned} u_2(x,t) &= f+\frac{t^\alpha f_1}{\alpha}-\frac{t^{2\alpha}\alpha^3}{2\left(2\alpha^5-9t^{5\alpha}g_2g_2'\right)}\left(12ff_1f'-12gg_1f'-12gf_1g'-12fg_1g'+6f^2f_1'\right) \\ &-\frac{t^{2\alpha}\alpha^3}{2\left(2\alpha^5-9t^{5\alpha}g_2g_2'\right)}\left(-6g^2f_1'-12fgg_1'-12g'g_1'-6g_1g''-6gg_1''-f_1^{(3)}\right), \end{aligned} \tag{5.21}$$

$$\begin{aligned} v_2(x,t) &= g(x)+\frac{t^\alpha g_1(x)}{\alpha}+\frac{t^{2\alpha}}{2\alpha^2}\left(6gf_1f'+6fg_1f'+6ff_1g'-6gg_1g'+6fgf_1'\right) \\ &+\frac{t^{2\alpha}}{2\alpha^2}\left(-3g'f_1'+3f^2g_1'-3g^2g_1'-3f'g_1'-3g_1f''-3gf_1''-g_1^{(3)}\right). \end{aligned} \tag{5.22}$$

In the same manner, we apply the same procedure for $n=3$ to obtain the following results.

$$\begin{aligned} f_3(x) &= -6f_1^2f'-6ff_2f'+6g_1^2f'+6gg_2f'+6gf_2g'+12f_1g_1g'+6fg_2g' \\ &-12ff_1f_1'+12gg_1f_1'-3f^2f_2'+3g^2f_2'+12gf_1g_1'+12fg_1g_1' \\ &+6g_1'^2+6fgg_2'+6g'g_2'+3g_2g''+6g_1g_1''+3gg_2''+\frac{1}{2}f_2^{(3)}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} g_3(x) &= 6gf_2f'+12f_1g_1f'+6fg_2f'+6f_1^2g'+6ff_2g'-6g_1^2g'-6gg_2g' \\ &+12gf_1f_1'+12fg_1f_1'+6fgf_2'-3g'f_2'+12ff_1g_1'-12gg_1g_1'-6f_1'g_1' \\ &+3f^2g_2'-3g^2g_2'-3f'g_2'-3g_2f''-6g_1f_1''-3gf_2''-g_2^{(3)}, \end{aligned} \tag{5.24}$$

$$\begin{aligned} u_3(x,t) &= f(x)+\frac{t^\alpha f_1}{\alpha}+\frac{t^{2\alpha}f_2}{2\alpha^2}+\frac{t^{3\alpha}}{6\alpha^3}\left(-6f_1^2f'-6ff_2f'+6g_1^2f'+6gg_2f'+6gf_2g'+12f_1g_1g'\right) \\ &+\frac{t^{3\alpha}}{6\alpha^3}\left(+6fg_2g'-12ff_1f_1'+12gg_1f_1'-3f^2f_2'+3g^2f_2'+12gf_1g_1'+12fg_1g_1'\right) \\ &+\frac{t^{3\alpha}}{6\alpha^3}\left(+6g_1'^2+6fgg_2'+6g'g_2'+3g_2g''+6g_1g_1''+3gg_2''+\frac{1}{2}f_2^{(3)}\right), \end{aligned} \tag{5.25}$$

$$\begin{aligned} v_3(x,t) &= g+\frac{t^\alpha g_1}{\alpha}+\frac{t^{2\alpha}g_2}{2\alpha^2}+\frac{t^{3\alpha}}{6\alpha^3}(6gf_2f'+12f_1g_1f'+6fg_2f'+6f_1^2g'+6ff_2g'-6g_1^2g'-6gg_2g') \\ &+\frac{t^{3\alpha}}{6\alpha^3}(12gf_1f_1'+12fg_1f_1'+6fgf_2'-3g'f_2'+12ff_1g_1'-12gg_1g_1'-6f_1'g_1') \\ &+\frac{t^{3\alpha}}{6\alpha^3}(3f^2g_2'-3g^2g_2'-3f'g_2'-3g_2f''-6g_1f_1''-3gf_2''-g_2^{(3)}). \end{aligned} \tag{5.26}$$

In Tables 1-2, the fourth-order approximate RPSM solutions of time-fractional new coupled mKdV equation are compared numerically with the exact solutions

$$u(x,t) = \frac{1}{2}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x - \frac{\sigma t^{\alpha}}{2\alpha}\right)\right), \tag{5.27}$$

$$v(x,t) = -\frac{1}{2}\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\left(x - \frac{\sigma t^{\alpha}}{2\alpha}\right)\right). \tag{5.28}$$

Table 1. Numerical results of third order RPSM $(u_3(x,t))$ solutions with absolute errors for $\sigma = -1/2$ and $t = 0.1$.

	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
x	$RPSM$	$Exact$	$Abs.\ Error$	$RPSM$	$Exact$	$Abs.\ Error$	$RPSM$	$Exact$	$Abs.\ Error$
0.0	0.132820	0.133616	7.96289E-4	0.039363	0.039364	8.19389E-7	0.014810	0.014810	6.09402E-9
0.1	0.160405	0.154454	5.95048E-3	0.063818	0.063821	3.06690E-6	0.039651	0.039651	5.64029E-8
0.2	0.183978	0.174045	9.93250E-3	0.087657	0.087662	5.16170E-6	0.064102	0.064102	1.02078E-7
0.3	0.201688	0.192320	9.36789E-3	0.110677	0.110684	6.98854E-6	0.087935	0.087935	1.39622E-7
0.4	0.214072	0.209244	4.82823E-3	0.132701	0.132709	8.40691E-6	0.110946	0.110946	1.66603E-7
0.5	0.223533	0.224810	1.27678E-3	0.153588	0.153598	9.28657E-6	0.132959	0.132959	1.81898E-7
0.6	0.232774	0.239038	6.26428E-3	0.173234	0.173243	9.55621E-6	0.153833	0.153833	1.85693E-7
0.7	0.243403	0.251971	8.56755E-3	0.191566	0.191575	9.23384E-6	0.173464	0.173464	1.79282E-7
0.8	0.255536	0.263664	8.12801E-3	0.208547	0.208556	8.42239E-6	0.191780	0.191780	1.64722E-7
0.9	0.268286	0.274188	5.90160E-3	0.224172	0.224179	7.27695E-6	0.208745	0.208745	1.44442E-7
1.0	0.280523	0.283619	3.09595E-3	0.238457	0.238463	5.96287E-6	0.224352	0.224352	1.20880E-7

Absolute errors are presented for $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$. The results indicate that as the x values increase the absolute errors also increase. Besides, as the α values increase, the absolute errors decrease. Also the Tables 1 and 2 show competitive solutions of the RPSM with highly approximate results. Moreover, in figures, the surface plots of the approximate solutions are illustrated for $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$.

Table 2. Numerical results of third order RPSM $(v_3(x,t))$ solutions with absolute errors for $\sigma = -1/2$ and $t = 0.1$.

	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
x	$RPSM$	$Exact$	$Abs.\ Error$	$RPSM$	$Exact$	$Abs.\ Error$	$RPSM$	$Exact$	$Abs.\ Error$
0.0	-0.133176	-0.133616	4.40433E-4	-0.039363	-0.039364	8.19364E-7	-0.014810	-0.014810	6.09402E-9
0.1	-0.154147	-0.154454	3.07373E-4	-0.063818	-0.063821	3.31577E-6	-0.039651	-0.039651	5.64978E-8
0.2	-0.173537	-0.174045	5.07560E-4	-0.087657	-0.087662	5.55303E-6	-0.064102	-0.064102	1.02223E-7
0.3	-0.191150	-0.192320	1.16995E-3	-0.110676	-0.110684	7.36827E-6	-0.087935	-0.087935	1.39755E-7
0.4	-0.207176	-0.209244	2.06771E-3	-0.132701	-0.132709	8.64380E-6	-0.110946	-0.110946	1.66675E-7
0.5	-0.222035	-0.224810	2.77502E-3	-0.153588	-0.153598	9.32228E-6	-0.132959	-0.132959	1.81889E-7
0.6	-0.23609	-0.239038	2.94836E-3	-0.173234	-0.173243	9.41358E-6	-0.153833	-0.153833	1.85615E-7
0.7	-0.249455	-0.251971	2.51534E-3	-0.191566	-0.191575	8.98895E-6	-0.173464	-0.173464	1.79169E-7
0.8	-0.261997	-0.263664	1.66669E-3	-0.208547	-0.208556	8.16298E-6	-0.191780	-0.191780	1.64611E-7
0.9	-0.273478	-0.274188	7.09964E-4	-0.224172	-0.224179	7.06952E-6	-0.208745	-0.208745	1.44359E-7
1.0	-0.283708	-0.283619	8.86102E-5	-0.238458	-0.238463	5.83883E-6	-0.224352	-0.224352	1.20836E-7

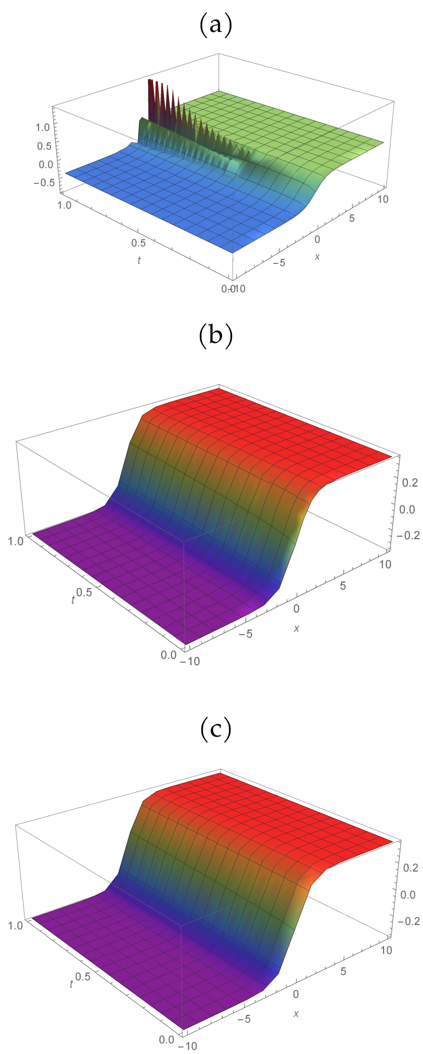

FIGURE 1. The surface plots of $u_3(x,t)$ for $\sigma = -1/2$ and $t = 0.1$ and for a.) $\alpha = 0.25$, b.) $\alpha = 0.50$, c.) $\alpha = 0.75$.

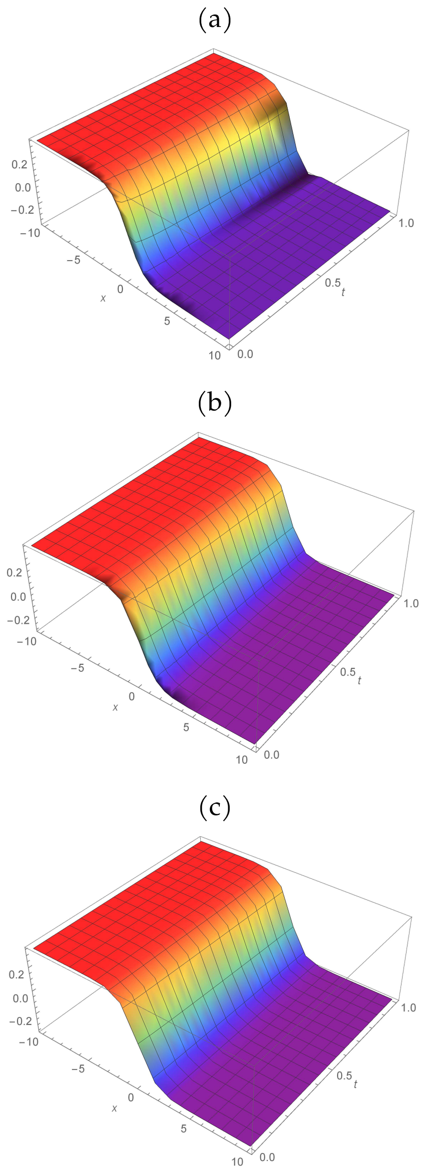

FIGURE 2. The surface plots of $v_3(x,t)$ for $\sigma = -1/2$ and $t = 0.1$ and for a.) $\alpha = 0.25$, b.) $\alpha = 0.50$, c.) $\alpha = 0.75$.

6. Conclusion

In this paper, firstly the exact solutions of the nonlinear time-fractional new coupled mKdV equations are obtained by the sub-equation method. Then the approximate solution of the model is demonstrated by the residual power series method. The fractional derivatives in the solution procedure is taken in the conformable sense. By the proposed methods and conformable fractional derivative definition, it is shown dependable ways of obtaining exact and approximate solutions for nonlinear fractional partial differential equations. Approximate solutions are compared with the exact solutions to show the reliability of the methods. Absolute errors are given with approximate and exact solutions with the help of figures and tables. Therefore, we can conclude that the methods are very effective tools for FDEs arising in different branches of applied sciences.

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