Hindawi
Journal of Function Spaces
Volume 2022, Article ID 3785584, 10 pages
https://doi.org/10.1155/2022/3785584

Research Article

Convergence of Generalized Quasi-Nonexpansive Mappings in Hyperbolic Space

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Received 13 November 2021; Revised 5 May 2022; Accepted 25 May 2022; Published 14 June 2022

Academic Editor: John R. Akeroyd

In this article, we consider a wider class of nonexpansive mappings (locally related quasi-nonexpansive) than monotone nonexpansive mappings. We obtained the convergence of fixed point for quasi ϱ-preserving locally related quasi-nonexpansive mappings in hyperbolic space. An iterative process is also used to obtain the convergence results for this mapping. Fixed point is approximated numerically in a nontrivial example by using Matlab.

1. Introduction

A first attempt to enrich metric spaces with convexity was essentially due to Takahashi [1] and is known as convex metric spaces. The class of convex metric spaces is general in nature and has constant curvature. Fixed point theory has many developments regarding convex metric spaces; for example, see [2–5].

The concept of hyperbolic space was introduced by Kohlenbach [6] in 2005, which is more general than the concept of hyperbolic space in [7] and more restrictive than the hyperbolic space defined in [8]. This definition is different from Takahashi's notion of convex metric space in the sense that every convex subset of a hyperbolic space is itself a hyperbolic space. These spaces are nonlinear in nature and more general than normed spaces. Fixed point theorems for nonexpansive mappings in hyperbolic space have been studied in [7, 9–16]. The existence of fixed point for nonexpansive mapping was initiated by Browder [17], Kirk [18], and Göhde [19] independently in 1965. In 1967, Diaz and Metcalf [20] gave an idea about quasi-nonexpansive mapping. Example of quasi-nonexpansive mappings was given by Doston [21] in 1972 which was not nonexpansive. Fixed point results for monotone nonexpansive mappings were presented by Bachar and Khamsi [22] in 2015. In 2019, the concept of ϱ-preserving was introduced by Al-Rawashdeh and Mehmood [23] which is generalized than the concept of monotone.

In this article, we will generalize the results of [23] in hyperbolic space. A nontrivial example is also given in which Picard [24], Mann [23, 25], Ishikawa [26, 27], Agarwal [28], Abbas and Nazir [29], and Noor [30] iteration schemes are used to approximate the fixed point.

2. Preliminaries

A hyperbolic space [6] is a metric space $(X, d)\,; X \neq \varnothing$, together with a mapping

$$H : X \times X \times [0, 1] \longrightarrow X, \tag{1}$$

which satisfies the following for all $u, x, y, z \in X$ and $\alpha, \beta \in [0, 1]$

(1) $d(u, H(x, y, \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y)$,

(2) $d(H(x, y, \alpha), H(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

(3) $H(x, y, \alpha) = H(y, x, (1-\alpha))$,

(4) $d(H(x, z, \alpha), H(y, u, \alpha)) \le (1-\alpha)d(x, y) + \alpha d(z, u)$.

If a space satisfies only Condition 1, it coincides with the convex metric space introduced by Takahashi [1].

Throughout this article, we consider

$$H(x, y, \alpha) = \alpha y \oplus (1-\alpha)x, \tag{2}$$

for all $x, y, z, H \in X$ and $\alpha, \beta \in [0, 1]$.

Let X be a hyperbolic space and K be a nonempty subset of X and $\Gamma : K \longrightarrow K$ be a mapping. According to [23], the underlying concepts are defined in hyperbolic space as follows: A mapping Γ is said to be *nonexpansive* if

$$d(\Gamma x, \Gamma y) \le d(x, y), \text{for all } x, y \in K, \tag{3}$$

and quasi-nonexpansive provided $F(\Gamma)$ is nonempty and for each $y \in F(\Gamma)$

$$d(\Gamma x, y) \le d(x, y), \text{for all } x \in K. \tag{4}$$

Let ϱ be a relation on X; a self-map Γ of X is said to be ϱ-*preserving* if

$$x\varrho y \Rightarrow \Gamma x \varrho \Gamma y, \text{for all } x, y \in X. \tag{5}$$

Let $(X, d, \preccurlyeq)$ be a partially order hyperbolic space, and a self map Γ of X is said to be *monotone nonexpansive* if Γ is monotone and

$$d(\Gamma x, \Gamma y) \le d(x, y) \text{whenever } x \preccurlyeq y, \text{for all } x, y \in X. \tag{6}$$

A mapping $\Gamma : K \longrightarrow K$ is said to be

(i) Locally related quasi-nonexpansive (abbreviated as L. R.Q.N) provided $F(\Gamma)$ is nonempty, and for each $y \in F(\Gamma)$

$$d(\Gamma x, y) \le d(x, y) \text{ and } x\varrho y, \text{for all } x \in K, \tag{7}$$

(ii) Quasi ϱ-preserving provided $F(\Gamma)$ is nonempty, and for each $y \in F(\Gamma)$

$$\Gamma x \varrho y \text{ whenever } x\varrho y, \text{for all } x \in K. \tag{8}$$

A quasi ϱ-preserving mapping which is also L.R.Q.N is called *quasi* ϱ-*preserving* L.R.Q.N.

Condition (S) A hyperbolic space X having a relation ϱ on it satisfying condition (S) if every convergent sequence $\{x_n\}, x_n \longrightarrow x$ where $x \in X$ has a subsequence $\{x_{n_k}\}$ such that $x\varrho x_{n_k}$ for all $k \in \mathbb{N}$.

Let X be a hyperbolic space and ϱ be the relation on X; ϱ is said to be *compatible* if for all $x, y \in X$,

(a) $x\varrho y$ implies $(x \oplus z)\varrho(y \oplus z)$,

(b) $x\varrho y$ implies $\alpha x \varrho \alpha y$ for $\alpha \in (0, 1)$.

Remark 1. In the above definitions, we consider only a relation which needs not to be a partial order relation necessarily. The condition S is utilized in Theorems 2, 5, 9, 13, 17, and 19 which is moderate than the conditions already been considered in the literature [31]. The condition

$$\liminf_{n\longrightarrow\infty} d(x_n, F(\Gamma)) = 0, \tag{9}$$

mentioned in each Theorems 2, 5, 9, 13, 17, and 19 is based on the condition S (see the last paragraph on page 4 of [23]).

3. Main Results

Let X be a hyperbolic space and K be a nonempty subset of X, I be the identity map, and $\Gamma : K \longrightarrow K$ be a mapping. For $x_0 \in K$, and $\alpha, \beta \in (0, 1)$, let $\{x_n\}$ be a sequence with iterations given as

$$x_n = \Gamma(x_{n-1}) = \Gamma^n(x_0), \text{(Picard iteration)}, \tag{10}$$

$$x_n = \Gamma_\alpha(x_{n-1}) = \Gamma_\alpha^n(x_0), \text{where } \Gamma_\alpha = H(\Gamma, I, \alpha), \text{(Mann iteration)}, \tag{11}$$

$$x_n = \Gamma_{\alpha,\beta}^n(x_{n-1}) = \Gamma_{\alpha,\beta}^n(x_0), \text{where } \Gamma_{\alpha,\beta} = H(\Gamma(\Gamma_\beta), I, \alpha), \text{(Ishikawa iteration)}. \tag{12}$$

Theorem 2. *Let X be hyperbolic space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose Γ be a quasi ϱ-preserving L.R.Q.N self map of K. If there exists some $c_0 \in K$ such that $c_0\varrho y$ for all $y \in F(\Gamma)$, then, (10) converges to a fixed point of Γ in K if and only if*

$$\liminf_{n\longrightarrow\infty} d(x_n, F(\Gamma)) = 0. \tag{13}$$

Proof. If $\{x_n\}$ converges to some fixed point of Γ, then $\liminf_{n\longrightarrow\infty} d(x_n, F(\Gamma)) = 0$ holds. Conversely, suppose $\liminf_{n\longrightarrow\infty} d(x_n, F(\Gamma)) = 0$ holds. As Γ is quasi ϱ-preserving and $c_0\varrho y$, we have $\Gamma(c_0)\varrho y$. Similarly,

$$\Gamma^{n-1}(c_0)\varrho y = x_{n-1}\varrho y \text{ for all } n \in \mathbb{N}. \tag{14}$$

□

Since Γ is L.R.Q.N, so for all $n \in \mathbb{N}$ and $y \in F(\Gamma)$,

$$d(x_n, y) = d(\Gamma(x_{n-1}), y) \le d(x_{n-1}, y), \tag{15}$$

taking inf over y

$$d(x_n, F(\Gamma)) \le d(x_{n-1}, F(\Gamma)), \tag{16}$$

implies $\{d(x_n, F(\Gamma)) \ge 0\}$ is a nonincreasing sequence and bounded as well. So

$$\lim_{n\longrightarrow\infty} d(x_n, F(\Gamma)) = 0. \tag{17}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. For a given $\varepsilon > 0$, there exists $k \in \mathbb{N}$, such that for all $n \geq k$,

$$d(x_n, F(\Gamma)) < \frac{\varepsilon}{2}. \tag{18}$$

For $x \in F(\Gamma)$ and all $l, m \geq k$, we have

$$\begin{aligned} d(x_l, x) &= d\left(\Gamma^l(c_0), x\right) \leq d\left(\Gamma^k(c_0), x\right),\\ d(x_m, x) &= d(\Gamma^m(c_0), x) \leq d\left(\Gamma^k(c_0), x\right), \end{aligned} \tag{19}$$

by adding

$$d(x_l, x) + d(x_m, x) \leq 2d\left(\Gamma^k(c_0), x\right). \tag{20}$$

Taking inf over x

$$d(x_l, x_m) \leq 2d(x_n, F(\Gamma)) < \varepsilon, \tag{21}$$

implies $\{x_n\}$ is a Cauchy sequence. Since K is complete, so there exists $x \in K$, such that

$$\lim_{n\longrightarrow\infty} x_n = x. \tag{22}$$

Next, we have to show that $F(\Gamma)$ is closed. Let $x \in K$ be a limit point of $F(\Gamma)$; then, there exists a sequence $\{x_n\} \subseteq F(\Gamma)$, and using condition (S), a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x and

$$x\varrho x_{n_k} \text{ for all } k \in \mathbb{N}. \tag{23}$$

Consider

$$\begin{aligned} d(\Gamma(x), x) &\leq d\left(\Gamma(x), x_{n_k}\right) + d\left(x, x_{n_k}\right)\\ &\leq 2d\left(x, x_{n_k}\right) \longrightarrow 0 \text{ as } k \longrightarrow \infty, \end{aligned} \tag{24}$$

so $x \in F(\Gamma)$.

Example 1. Let $X = \{(a_1, b_1) \in R^2\,; a_1, b_1 > 0\}$ be a hyperbolic space and ϱ be the relation defined as

$$(a_1, b_1)\varrho(a_2, b_2) \Leftrightarrow a_1 \leq a_2 \text{ and } b_1 \geq b_2. \tag{25}$$

Let $d : X \times X \longrightarrow \mathbb{R}$ be defined as

$$d(a, b) = |a_1 - a_2| + |a_1 b_1 - a_2 b_2| \text{ where } a = (a_1, b_1), b = (a_2, b_2). \tag{26}$$

Let $K = [1, 4] \times [1, 4] \subset X$ and $\Gamma : K \longrightarrow K$ be a mapping defined by

$$\Gamma(a, b) = \begin{pmatrix} (1, 2) & (a, b) = (4, 4)\\ (1, 1) & (a, b) \neq (4, 4) \end{pmatrix}. \tag{27}$$

As $(1, 1)$ is the fixed point of Γ, and $(a, b)\varrho(1, 1)$ implies $a \leq 1$ and $b \geq 1$ that is $(1, b)$, where $1 \leq b \leq 4$, we have

$$d(\Gamma(1, b), (1, 1)) = 0 \leq b - 1 = d((1, b), (1, 1)), \tag{28}$$

which shows that Γ is L.R.Q.N.

Next since $(1, 4)\varrho(4, 4)$ but $\Gamma(1, 4)\varrho\Gamma(4, 4)$ does not hold, which shows that Γ is not ϱ-preserving.

Also $(1, b)\varrho(1, 1)$ implies

$$\Gamma(1, b) = (1, 1)\varrho(1, 1) = \Gamma(1, 1), \tag{29}$$

so Γ is quasi ρ-preserving. As $(3.9{,}4)\varrho(4, 4)$

$$d(\Gamma(3.9{,}4), \Gamma(4, 4)) = 1 > 0.5 = d((3.9{,}4), (4, 4)), \tag{30}$$

hence, Γ is not nonexpansive. This example shows that L.R. Q.N mapping is not necessarily ϱ-preserving or ϱ-preserving nonexpansive.

Example 2. Let $X = \mathbb{R}^2$ be a hyperbolic space and ϱ be the relation defined as

$$(a_1, b_1)\varrho(a_2, b_2) \Leftrightarrow a_1 \leq a_2 \text{ and } b_1 = b_2. \tag{31}$$

Let $d : X \times X \longrightarrow \mathbb{R}$ be defined as

$$d(a, b) = |a_1 - a_2| + |a_1 b_1 - a_2 b_2| \text{ where } a = (a_1, b_1), b = (a_2, b_2). \tag{32}$$

Let $K = [0.1{,}0.5] \times [0.1{,}0.5] \subset X$ and $\Gamma : K \longrightarrow K$ be a mapping defined by

$$\Gamma(a, b) = \left\{\left(\frac{\cos a}{2}, \frac{\cos b}{2}\right); (a, b) \in K\right\}. \tag{33}$$

As $(0.450183611294874{,}0.450183611294874)$ is the fixed point of Γ, and $(a_1, b_1)\varrho(0.450183611294874{,}0.450183611294874)$ implies $a_1 \leq 0.4501836112948740$ and $b_1 = 0.450183611294874$, this shows that Γ is L.R.Q.N.

There are many examples in literature which shows that hyperbolic spaces are more general than Banach spaces for detail [1], so we have the following corollary which is Theorem 2.6 of [23].

Corollary 3. *Let X be Banach space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose $\Gamma : K \longrightarrow K$ be a quasi ϱ-preserving L.R.Q.N mapping. If there exists some $c_0 \in K$ such that $c_0\varrho y$ for all $y \in F(\Gamma)$, then, the sequence (10) converges to* *a fixed point of Γ in K if and only if*

$$\liminf_{n\longrightarrow\infty} d(x_n, F(\Gamma)) = 0. \tag{34}$$

All the results of [23] are the consequences of Theorem 2.

Following proposition from [1] will be helpful to proof the next results.

Proposition 4. *Let H be a Takahashi convex structure on metric space (X, d). If $x, y \in X$ and $t \in [0,1]$, then*

(1) $H(x, y, 1) = x$ *and* $H(x, y, 0) = y$,

(2) $H(x, x, t) = x$,

(3) $d(x, H(x, y, t)) = (1-t)d(x, y)$ *and* $d(y, H(x, y, t)) = td(x, y)$.

Theorem 5. *Let X be hyperbolic space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose $\Gamma : K \longrightarrow K$ be a quasi ϱ -preserving L.R.Q.N mapping. If there exists some $c_0 \in K$ such that $c_0 \varrho y$ for all $y \in F(\Gamma)$, then sequence (11) converges to a fixed point of Γ in K if and only if*

$$\lim_{n\longrightarrow\infty} \inf\, d(x_n, F(\Gamma)) = 0. \tag{35}$$

Proof. Let $q \in F(\Gamma)$, then

$$\Gamma_\alpha(q) = H(\Gamma(q), q, \alpha) = H(q, q, \alpha) = q, \tag{36}$$

so $F(\Gamma) \subset F(\Gamma_\alpha)$. Suppose $y \in F(\Gamma_\alpha)$ that is $\Gamma_\alpha(y) = y$ then from (11)

$$\begin{aligned} d(y, \Gamma(y)) &= d(H(\Gamma(y), y, \alpha), \Gamma(y)) \\ &\le (1-\alpha)d(\Gamma(y), \Gamma(y)) + \alpha d(y, \Gamma(y)), \end{aligned} \tag{37}$$

implies $d(y, \Gamma(y)) = 0$ and $y \in F(\Gamma)$. Hence, $F(\Gamma) = F(\Gamma_\alpha)$. □

Now, we show that Γ_α is quasi ϱ-preserving. Suppose $x \varrho y$ then $\Gamma(x) \varrho y$ and by the compatibility of ϱ,

$$\begin{aligned} &\alpha x \varrho \alpha y,\ (1-\alpha)\Gamma(x)\varrho(1-\alpha)y \text{ and} \\ &\quad \alpha x \oplus (1-\alpha)\Gamma(x) \varrho \alpha y \oplus (1-\alpha)\Gamma(x). \end{aligned} \tag{38}$$

Also

$$\begin{aligned} H(\Gamma(x), x, \alpha) &= \alpha x \oplus (1-\alpha)\Gamma(x) \varrho \alpha y \oplus (1-\alpha)\Gamma(y) \\ &= H(\Gamma(y), y, \alpha) = y, \end{aligned} \tag{39}$$

implies

$$\Gamma_\alpha(x) \varrho y. \tag{40}$$

Finally, we prove that Γ_α is L.R.Q.N. For this let $x \varrho y$, then

$$\begin{aligned} d(\Gamma_\alpha(x), y) &= d(H(\Gamma(x), x, \alpha), y) \\ &\le \alpha d(\Gamma(x), y) + (1-\alpha)d(x, y) \\ &\le \alpha d(x, y) + (1-\alpha)d(x, y) = d(x, y). \end{aligned} \tag{41}$$

Using Theorem 2, we get the result.

Remark 6. If Γ is ϱ-preserving, then Γ_α is also ϱ-preserving.

Proof. As Γ is ρ-preserving, $x \varrho y$ implies $\Gamma(x) \varrho \Gamma(y)$. Using the compatibility of ϱ, we get

$$\alpha x \varrho \alpha y \text{ and } (1-\alpha)\Gamma(x)\varrho(1-\alpha)\Gamma(y), \tag{42}$$

which implies

$$\begin{gathered} H(\Gamma(x), x, \alpha) = \alpha x \oplus (1-\alpha)\Gamma(x) \varrho \alpha y \oplus (1-\alpha)\Gamma(y) = H(\Gamma(y), y, \alpha) \\ \Gamma_\alpha(x) \varrho \Gamma_\alpha(y). \end{gathered} \tag{43}$$

□

Now, we discuss the convergence of the iterative scheme (12) given as

$$x_n = \Gamma^n_{\alpha,\beta}(x_{n-1}) = \Gamma^n_{\alpha,\beta}(x_0), \text{where } \Gamma_{\alpha,\beta} = H\left(\Gamma\left(\Gamma_\beta\right), I, \alpha\right). \tag{44}$$

Proposition 7. *For $\alpha, \beta \in (0,1)$, $F(\Gamma_{\alpha,\beta}) = F(\Gamma)$, whenever Γ is ϱ-preserving nonexpansive and $y \varrho \Gamma_\beta(y)$, for $y \in F(\Gamma_{\alpha,\beta})$.*

Proof. Let $x \in F(\Gamma)$ that is $\Gamma(x) = x$ then

$$\begin{aligned} \Gamma_{\alpha,\beta}(x) &= H\left(\Gamma\left(\Gamma_\beta(x)\right), x, \alpha\right) = H(\Gamma(H(\Gamma(x), x, \beta)), x, \alpha) \\ &= H(\Gamma(x), x, \alpha) = H(x, x, \alpha) = x, \end{aligned} \tag{45}$$

so

$$F(\Gamma) \subseteq F\left(\Gamma_{\alpha,\beta}\right). \tag{46}$$

□

For the other inclusion, suppose $y \in F(\Gamma_{\alpha,\beta})$ that is $\Gamma_{\alpha,\beta}(y) = y$, and consider

$$\begin{aligned} d(\Gamma(y), y) &= d\left(\Gamma(y), \Gamma_{\alpha,\beta}(y)\right) = d\left(\Gamma(y), H\left(\Gamma\left(\Gamma_\beta(y)\right), y, \alpha\right)\right) \\ &\le (1-\alpha)d(\Gamma(y), y) + \alpha d\left(\Gamma(y), \Gamma\left(\Gamma_\beta(y)\right)\right) \\ &\le (1-\alpha)d(\Gamma(y), y) + \alpha d\left(y, \Gamma_\beta(y)\right) \\ &\quad \cdot \Gamma \text{ is nonexpansive} = (1-\alpha)d(\Gamma(y), y) \\ &\quad + \alpha d(y, H(\Gamma(y), y, \beta)) = (1-\alpha)d(\Gamma(y), y) \\ &\quad + \alpha\beta d(\Gamma(y), y) \text{ using } d(y, H(x, y, t)) = td(x, y) \\ &= (1 - \alpha + \alpha\beta)d(\Gamma(y), y), \end{aligned} \tag{47}$$

which gives $\alpha(1-\beta)d(\Gamma(y),y)=0$ and therefore $y\in F(\Gamma)$ implies $F(\Gamma_{\alpha,\beta})\subseteq F(\Gamma)$. Hence,

$$F\left(\Gamma_{\alpha,\beta}\right)=F(\Gamma). \tag{48}$$

Proposition 8. *If Γ is quasi ϱ-preserving L.R.Q.N mapping, then $\Gamma_{\alpha,\beta}$ is also quasi ϱ-preserving L.R.Q.N mapping.*

Proof. Firstly we show that if Γ is quasi ϱ-preserving, then $\Gamma_{\alpha,\beta}$ is quasi ϱ-preserving. For this, let $x\in K$ and $y\in F(\Gamma)\subseteq F(\Gamma_{\alpha,\beta})$, such that $x\varrho y$ implies $\Gamma(x)\varrho y$. Also by Theorem 5, Γ_β is quasi ϱ-preserving, $y\in F(\Gamma_\beta)$; therefore, $\Gamma_\beta(x)\varrho y$. Further $\Gamma\Gamma_\beta(x)\varrho y$ and by using the compatibility of ϱ, we get

$$\begin{aligned}\Gamma_{\alpha,\beta}(x)&=H\left(\Gamma\Gamma_\beta(x),x,\alpha\right)=\alpha x\oplus(1-\alpha)\Gamma\Gamma_\beta(x)\varrho\alpha y\oplus\\&\quad\cdot(1-\alpha)y=H(y,y,\alpha)=y.\end{aligned} \tag{49}$$

□

Hence, $\Gamma_{\alpha,\beta}$ is quasi ϱ-preserving.

Let $x\in K$ and $y\in F(\Gamma)$, such that $x\varrho y$ then $\Gamma_\beta(x)\varrho\Gamma_\beta(y)=y$ as Γ_β is ϱ-preserving. Consider

$$\begin{aligned}d\left(\Gamma_{\alpha,\beta}(x),y\right)&=d\left(H\left(\Gamma\Gamma_\beta(x),x,\alpha\right),y\right)\le\alpha d\left(\Gamma\Gamma_\beta(x),y\right)\\&\quad+(1-\alpha)d(x,y)\le\alpha d\left(\Gamma_\beta(x),y\right)\\&\quad+(1-\alpha)d(x,y)\text{ as }\Gamma\text{ is quasi nonexpansive}\\&\le\alpha d(H(\Gamma(x),x,\beta),y)+(1-\alpha)d(x,y)\\&\le\alpha\beta d(\Gamma(x),y)+\alpha(1-\beta)d(x,y)\\&\quad+(1-\alpha)d(x,y)\le\alpha\beta d(x,y)\\&\quad+\alpha(1-\beta)d(x,y)+(1-\alpha)d(x,y)\\&\le(\alpha\beta+\alpha-\alpha\beta+1-\alpha)d(x,y)\le d(x,y).\end{aligned} \tag{50}$$

Theorem 9. *Let X be hyperbolic space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose $\Gamma:K\longrightarrow K$ be a quasi ϱ-preserving L.R.Q.N mapping. If there exists some $c_0\in K$ such that $c_0\varrho y$ for all $y\in F(\Gamma)$, then the sequence (12) converges to a fixed point of Γ in K if and only if*

$$\lim_{n\longrightarrow\infty}\inf d(x_n,F(\Gamma))=0. \tag{51}$$

Proof. As Γ is quasiϱ-preserving L.R.Q.N mapping, by the Proposition 8, $\Gamma_{\alpha,\beta}$ is also quasi ϱ-preserving L.R.Q.N mapping. For $y\in F(\Gamma)$,

$$\Gamma_{\alpha,\beta}(y)=y, \tag{52}$$

and $\Gamma^n_{\alpha,\beta}(c_0)\varrho y$ for all $n\in\mathbb{N}$. By Theorem 2, we get conclusion. □

Next, we will discuss the convergence of the Agarwal iteration process defined in [28] as

$$x_n=\left(\Gamma^\beta_\gamma\right)^n(x_0), \tag{53}$$

where

$$\begin{aligned}\Gamma^\beta_\gamma(x)&=(1-\beta)\Gamma(x)\oplus\beta\Gamma\Gamma_\gamma(x)=H\left(\Gamma(x),\Gamma\Gamma_\gamma(x),\beta\right),\\\Gamma_\gamma(x)&=(1-\gamma)\Gamma(x)\oplus\gamma x=H(\Gamma(x),x,\gamma).\end{aligned} \tag{54}$$

Proposition 10. *If Γ is quasi ϱ-preserving, then, Γ^β_γ is also quasi ϱ-preserving.*

Proof. Suppose Γ is quasi ρ-preservin,g then for $x\in K$ and $y\in F(\Gamma)$, such that $x\varrho y$ implies $\Gamma(x)\varrho y$,$\Gamma_\gamma(x)\varrho y$, and $\Gamma\Gamma_\gamma(x)\varrho y$. Then, by using the compatibility of ϱ, we get

$$\Gamma^\beta_\gamma(x)=(1-\beta)\Gamma(x)\oplus\beta\Gamma\Gamma_\gamma(x)\varrho(1-\beta)y\oplus\beta y=H(y,y,\beta)=y. \tag{55}$$

Hence, Γ^β_γ is quasi ρ-preserving. □

Proposition 11. *If Γ is quasi ϱ-preserving L.R.Q.N mapping, then Γ^β_γ is also quasi ϱ-preserving L.R.Q.N mapping.*

Proof. Let $x\in K$ and $q\in F(\Gamma)$, such that $x\varrho q$. Consider

$$\begin{aligned}d\left(\Gamma^\beta_\gamma(x),q\right)&=d\left(H\left(\Gamma(x),\Gamma\Gamma_\gamma(x),\beta\right),q\right)\le\beta d(\Gamma(x),q)\\&\quad+(1-\beta)d\left(\Gamma\Gamma_\gamma(x),q\right)\le\beta d(x,q)\\&\quad+(1-\beta)d\left(\Gamma_\gamma(x),q\right)\text{ as }\Gamma\text{ is quasi}\\&\quad\text{nonexpansive}\le\beta d(x,q)+(1-\beta)d(x,q)\\&\quad\text{as }\Gamma_\gamma\text{ is quasi nonexpansive}\le d(x,q).\end{aligned} \tag{56}$$

Hence, Γ^β_γ is also quasi ϱ-preserving L.R.Q.N mapping. □

Proposition 12. *For $\beta,\gamma\in(0,1)$, $F(\Gamma)\subseteq F(\Gamma^\beta_\gamma)$.*

Proof. Let $q\in F(\Gamma)$ that is $\Gamma(q)=q$ then

$$\begin{aligned}\Gamma^\beta_\gamma(q)&=H\left(\Gamma(q),\Gamma\Gamma_\gamma(q),\beta\right)=H(\Gamma(q),\Gamma(H(\Gamma(q),q,\gamma)),\beta)\\&=H(q,\Gamma(H(q,q,\gamma)),\beta)=H(q,\Gamma(q),\beta)\\&=H(q,q,\beta)=q,\end{aligned} \tag{57}$$

so

$$F(\Gamma) \subseteq F\left(\Gamma_\gamma^\beta\right). \tag{58}$$

□

The following theorem describes the necessary and sufficient conditions for convergence of iterative sequence (53) of quasi ϱ-preserving L.R.Q.N mappings.

Theorem 13. *Let X be hyperbolic space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose $\Gamma : K \longrightarrow K$ be a quasi ϱ-preserving L.R.Q.N mapping. If there exists some $c_0 \in K$ such that $c_0 \varrho y$ for all $y \in F(\Gamma)$, then, the sequence (53) converges to a fixed point of Γ in K if and only if*

$$\lim_{n\longrightarrow\infty} \inf d(x_n, F(\Gamma)) = 0. \tag{59}$$

Proof. As Γ is quasi ϱ-preserving L.R.Q.N mapping, by the Proposition 15, Γ_γ^β is also quasi ϱ-preserving L.R.Q.N mapping. For $y \in F(\Gamma)$,

$$\Gamma_\gamma^\beta(y) = y, \tag{60}$$

and $\left(\Gamma_\gamma^\beta\right)^n(c_0)\varrho y$ for all $n \in \mathbb{N}$. By Theorem 2, we get conclusion. □

Next, we will discuss the convergence of the iteration process (Abbas and Nazir) defined in [29] as

$$x_n = \left(\Gamma_\gamma^{\alpha,\beta}\right)^n(x_0), \tag{61}$$

where

$$\begin{aligned}\Gamma_\gamma^{\alpha,\beta}(x) &= (1-\alpha)\Gamma\Gamma_\gamma^\beta(x) \oplus \alpha\Gamma\Gamma_\gamma(x) = H\left(\Gamma\Gamma_\gamma^\beta(x), \Gamma\Gamma_\gamma(x), \alpha\right),\\ \Gamma_\gamma^\beta(x) &= (1-\beta)\Gamma(x) \oplus \beta\Gamma\Gamma_\gamma(x) = H(\Gamma(x), \Gamma\Gamma_\gamma(x), \beta),\\ \Gamma_\gamma(x) &= (1-\gamma)\Gamma(x) \oplus \gamma x = H(\Gamma(x), x, \gamma).\end{aligned} \tag{62}$$

Proposition 14. *For $\alpha, \beta, \gamma \in (0, 1)$, $F(\Gamma) \subseteq F(\Gamma_\gamma^{\alpha,\beta})$.*

Proof. Let $q \in F(\Gamma)$ that is $\Gamma(q) = q$ then

$$\begin{aligned}\Gamma_\gamma^{\alpha,\beta}(q) &= H\left(\Gamma\Gamma_\gamma^\beta(q), \Gamma\Gamma_\gamma(q), \alpha\right)\\ &= H(\Gamma(H(\Gamma(q), \Gamma\Gamma_\gamma(q), \beta)), \Gamma(H(\Gamma(q), \Gamma\Gamma_\gamma(q), \beta)), \alpha)\\ &= H(\Gamma(H(q, q, \beta)), \Gamma(H(q, q, \beta)), \alpha)\\ &= H(\Gamma(q), \Gamma(q), \alpha) = H(q, q, \alpha) = q,\end{aligned} \tag{63}$$

Table 1: Rate of convergence of Mann, Ishikawa, Agarwal, Noor, and Abbas iterations for the mapping given in Example 2.

Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas
Number of iterations	19	30	13	30	09

Table 2: Influence of parameters on convergence.

For $\alpha = \beta = \gamma = 0.1$					
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas
Number of iterations	15	17	7	14	11
For $\alpha = 0.2 = \beta, \gamma = 0.3$					
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas
Number of iterations	11	22	12	20	9
For $\alpha = 0.3, \beta = 0.4, \gamma = 0.5$					
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas
Number of iterations	19	27	15	26	5
For $\alpha = 0.4, \beta = 0.6, \gamma = 0.8$					
Iteration process	Mann	Ishikawa	Agarwal	Noor	Abbas
Number of iterations	28	33	19	33	11

so

$$F(\Gamma) \subseteq F\left(\Gamma_\gamma^{\alpha,\beta}\right). \tag{64}$$

□

Proposition 15. *If Γ is quasi ϱ-preserving, then $\Gamma_\gamma^{\alpha,\beta}$ is also quasi ϱ-preserving.*

Proof. Suppose Γ is quasi ϱ-preserving, then, by Proposition 10 for $\beta, \gamma \in (0, 1)$,Γ_γ and Γ_γ^β are also quasi ϱ-preserving. Suppose for $x \in K$ and $y \in F(\Gamma)$, such that $x\varrho y$ implies $\Gamma(x)\varrho y$,$\Gamma_\gamma(x)\varrho y$,$\Gamma\Gamma_\gamma(x)\varrho y$,$\Gamma_\gamma^\beta(x)\varrho y$, and $\Gamma\Gamma_\gamma^\beta(x)\varrho y$. Then, by using the compatibility of ϱ, we get

$$\begin{aligned}\Gamma_\gamma^{\alpha,\beta}(x) &= (1-\alpha)\Gamma\Gamma_\gamma^\beta(x) \oplus \alpha\Gamma\Gamma_\gamma(x)\varrho(1-\alpha)y \oplus \alpha y\\ &= H(y, y, \alpha) = y.\end{aligned} \tag{65}$$

□

Hence, $\Gamma_\gamma^{\alpha,\beta}$ is quasi ϱ-preserving.

Proposition 16. *If Γ is quasi ϱ-preserving L.R.Q.N mapping, then $\Gamma_\gamma^{\alpha,\beta}$ is also quasi ϱ-preserving L.R.Q.N mapping.*

Table 3: Table shows the behavior of different iterations of Example 2 toward fixed point along the y-component (as the values of both components are same).

Steps	Mann	Ishikawa	Agarwal	Noor	Abbas
0	0.1	0.1	0.1	0.1	0.1
1	0.378251457847309	0.349865685358071	0.469116310149775	0.353488188856970	0.448059476721477
2	0.438734562339734	0.420761766649737	0.448858705072300	0.423665139911046	0.450145841140597
3	0.448471847181185	0.441497041994946	0.450275252488784	0.442926124783628	0.450182937304719
4	0.449930314735026	0.447614030619777	0.450177267430227	0.448198567436677	0.450183599267090
5	0.450146188219705	0.449423071779080	0.450184050424117	0.449640755996101	0.450183611080230
6	0.450178083528851	0.449958470681449	0.450183580897748	0.449640755996101	0.450183611291043
7	0.450182794815804	0.450116960188063	0.450183613399004	0.450035161737148	0.450183611294805
8	0.450183490697381	0.450163879470642	0.450183611149223	0.450143016673408	0.450183611294872
9	0.450183593482115	0.450177769732867	0.450183611304956	0.450172510434213	0.450183611294874
10	0.450183608663854	0.450181881911416	0.450183611294176	0.450180575695778	0.450183611294874
11	0.450183610906261	0.450183099313956	0.450183611294922	0.450182781191706	0.450183611294874
12	0.450183611237474	0.450183459723810	0.450183611294870	0.450183384298093	0.450183611294874
13	0.450183611286395	0.450183566422521	0.450183611294874	0.450183549221219	0.450183611294874
14	0.450183611293621	0.450183598010491	0.450183611294874	0.450183594320454	0.450183611294874
15	0.450183611294689	0.450183607362055	0.450183611294874	0.450183606653115	0.450183611294874
16	0.450183611294846	0.450183610130569	0.450183611294874	0.450183610025556	0.450183611294874
17	0.450183611294870	0.450183610950183	0.450183611294874	0.450183610947771	0.450183611294874
18	0.450183611294873	0.450183611192829	0.450183611294874	0.450183611199956	0.450183611294874
19	0.450183611294874	0.450183611264663	0.450183611294874	0.450183611199956	0.450183611294874
20	0.450183611294874	0.450183611285930	0.450183611294874	0.450183611268918	0.450183611294874
21	0.450183611294874	0.450183611292226	0.450183611294874	0.450183611287776	0.450183611294874
22	0.450183611294874	0.450183611294090	0.450183611294874	0.450183611292933	0.450183611294874
23	0.450183611294874	0.450183611294642	0.450183611294874	0.450183611294343	0.450183611294874
24	0.450183611294874	0.450183611294805	0.450183611294874	0.450183611294728	0.450183611294874
25	0.450183611294874	0.450183611294853	0.450183611294874	0.450183611294834	0.450183611294874
26	0.450183611294874	0.450183611294868	0.450183611294874	0.450183611294863	0.450183611294874
27	0.450183611294874	0.450183611294872	0.450183611294874	0.450183611294871	0.450183611294874
28	0.450183611294874	0.450183611294873	0.450183611294874	0.450183611294873	0.450183611294874
29	0.450183611294874	0.450183611294873	0.450183611294874	0.450183611294873	0.450183611294874
30	0.450183611294874	0.450183611294874	0.450183611294874	0.450183611294874	0.450183611294874

Proof. Let $x \in K$ and $y \in F(\Gamma)$, such that $x\varrho y$, then, $\Gamma_\gamma(x)\varrho y$ as Γ_γ is ϱ-preserving. Consider

$$\begin{aligned} d\left(\Gamma_\gamma^{\alpha,\beta}(x), y\right) &= d\left(H\left(\Gamma\Gamma_\gamma^{\beta}(x), \Gamma\Gamma_\gamma(x), \alpha\right),\right) \\ &\leq \alpha d\left(\Gamma\Gamma_\gamma^{\beta}(x), y\right) + (1-\alpha) d\left(\Gamma\Gamma_\gamma(x), y\right) \\ &\leq \alpha d\left(\Gamma_\gamma^{\beta}(x), y\right) + (1-\alpha) d\left(\Gamma_\gamma(x), y\right) \\ &\quad \text{as } \Gamma \text{ is quasi non expansive} \\ &\leq \alpha d(x, y) + (1-\alpha) d(x, y) \leq d(x, y). \end{aligned} \tag{66}$$

□

Hence, $\Gamma_\gamma^{\alpha,\beta}$ is also quasi ρ-preserving L.R.Q.N mapping.

The following theorem describes the necessary and sufficient conditions for convergence of iterative sequence (61) of quasi ϱ-preserving L.R.Q.N mappings.

Theorem 17. *Let X be hyperbolic space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose $\Gamma : K \longrightarrow K$ be a quasi ϱ-preserving L.R.Q.N mapping. If there exists some $c_0 \in K$ such that $c_0 \varrho y$ for all $y \in F(\Gamma)$, then, the sequence (61) converges to a fixed point of Γ in K if and only if*

$$\lim_{n \longrightarrow \infty} \inf d(x_n, F(\Gamma)) = 0. \tag{67}$$

Proof. As Γ is quasi ϱ-preserving L.R.Q.N mapping, by the Proposition 15, $\Gamma_\gamma^{\alpha,\beta}$ is also quasi ϱ-preserving L.R.Q.N

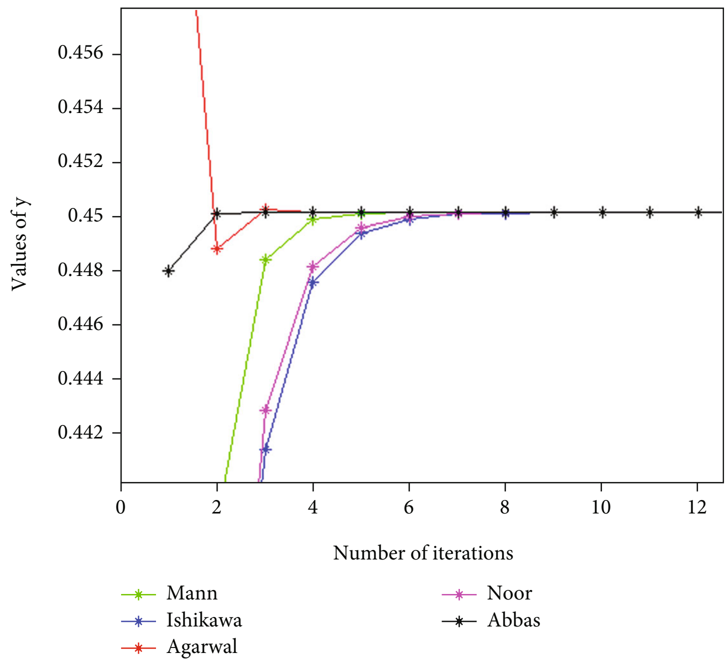

FIGURE 1: Convergence behavior of Mann, Ishikawa, Agarwal, Noor iterations with Abbas for $\alpha = \gamma = 0.3, \beta = 0.2$, initial values are $x_0 = 0.1$, $y_0 = 0.1$, and $tolerence = 10^{-18}$.

mapping. For $y \in F(\Gamma)$,

$$\Gamma_{\gamma}^{\alpha,\beta}(y) = y, \tag{68}$$

and $(\Gamma_{\gamma}^{\alpha,\beta})^n(c_0)\varrho y$ for all $n \in \mathbb{N}$. By Theorem 2, we get conclusion. □

Next, we will discuss the convergence of the Noor iteration process defined in [30] as

$$x_n = (\Gamma_{\alpha,\beta,\gamma})^n(x_0), \tag{69}$$

where

$$\begin{aligned} \Gamma_{\alpha,\beta,\gamma}(x) &= (1-\alpha)\Gamma\Gamma_{\beta,\gamma}(x) \oplus \alpha x = H(\Gamma\Gamma_{\beta,\gamma}(x), x, \alpha), \\ \Gamma_{\beta,\gamma}(x) &= (1-\beta)\Gamma\Gamma_{\gamma}(x) \oplus \beta x = H(\Gamma\Gamma_{\gamma}(x), x, \beta), \\ \Gamma_{\gamma}(x) &= (1-\gamma)\Gamma(x) \oplus \gamma x = H(\Gamma(x), x, \gamma). \end{aligned} \tag{70}$$

Proposition 18. *For $\alpha, \beta, \gamma \in (0,1)$, $F(\Gamma) \subseteq F(\Gamma_{\beta,\gamma})$ and $F(\Gamma) \subseteq F(\Gamma_{\alpha,\beta,\gamma})$.*

Proof. Let $q \in F(\Gamma)$ that is $\Gamma(q) = q$ then

$$\begin{aligned} \Gamma_{\beta,\gamma}(q) &= H(\Gamma\Gamma_{\gamma}(q), q, \beta) = H(\Gamma(H(\Gamma(q), q, \gamma)), q, \beta) \\ &= H(\Gamma(H(q, q, \gamma)), q, \beta) = H(\Gamma(q), q, \beta) \\ &= H(q, q, \beta) = q, \end{aligned} \tag{71}$$

so

$$F(\Gamma) \subseteq F(\Gamma_{\beta,\gamma}). \tag{72}$$

□

Now consider

$$\Gamma_{\alpha,\beta,\gamma}(q) = H(\Gamma\Gamma_{\beta,\gamma}(q), q, \alpha) = H(\Gamma(q), q, \alpha) = H(q, q, \alpha) = q, \tag{73}$$

which implies

$$F(\Gamma) \subseteq F(\Gamma_{\alpha,\beta,\gamma}). \tag{74}$$

The following theorem describes the necessary and sufficient conditions for convergence of iterative sequence (53) of quasi ϱ-preserving L.R.Q.N mappings.

Theorem 19. *Let X be hyperbolic space, having a compatible relation ϱ on it satisfying condition (S). Let K be a closed and convex subset of X. Suppose $\Gamma : K \longrightarrow K$ be a quasi ϱ-preserving L.R.Q.N mapping. If there exists some $c_0 \in K$ such that $c_0 \varrho y$ for all $y \in F(\Gamma)$, then, the sequence (12) converges to a fixed point of Γ in K if and only if*

$$\lim_{n \longrightarrow \infty} \inf d(x_n, F(\Gamma)) = 0. \tag{75}$$

Proof. As Γ is quasi ρ-preserving L.R.Q.N mapping, by the Proposition 18, $\Gamma_{\alpha,\beta,\gamma}$ is also quasi ρ-preserving L.R.Q.N mapping. For $y \in F(\Gamma)$,

$$\Gamma_{\alpha,\beta,\gamma}(y) = y, \tag{76}$$

and $(\Gamma_{\alpha,\beta,\gamma})^n(c_0)\varrho y$ for all $n \in \mathbb{N}$. By Theorem 2, we get conclusion. □

All iterations converges to $(x, y) = (0.450183611294874, 0.450183611294874)$.

Table 1 provides the rate of convergence of Mann, Ishikawa, Agarwal, Noor and Abbas iterations for the mapping given in Example 2. For $\alpha = \gamma = 0.3, \beta = 0.2$, initial values are $x_0 = 0.1, y_0 = 0.1$, and $tolerence = 10^{-18}$.

Using different values for α, β and γ, we can see that in Table 2 Abbas iteration not only converges faster but also stable than other iterations.

Remark 20. The numerical Example 2 validates the existence of L.R.Q.N mappings for hyperbolic spaces in the perspective of different iterations with the help of Table 3 and Figure 1.

4. Conclusion

In the present article, the concept of monotone has been generalized to ϱ-preserving in the framework of hyperbolic space. We also constructed a nontrivial example to show that locally related quasi-nonexpansive mapping is not necessarily ϱ-preserving or ϱ-preserving nonexpansive and approximate the fixed point numerically and compare the convergence result of different iterations with Abbas iteration by using Matlab.

Data Availability

No data were used to submit this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally.

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