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Convoluted fractional differentials of various forms utilizing the generalized Raina's function description with applications

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ABSTRACT

A generalized differential operator utilizing Raina's function is constructed in light of a certain type of fractional calculus. We next use the generalized operators to build a formula for analytic functions of type normalized. Our method is based on the concepts of subordination and superordination. As an application, a class of differential equations involving the suggested operator is studied. As seen, the solution is provided by a certain hypergeometric function. We also create a fractional coefficient differential operator. Its geometric and analytic features are discussed. Finally, we use the Jackson's calculus to expand the Raina's differential operator and investigate its properties in relation to geometric function theory.

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1. Introduction

Fractional calculus has grown in popularity in recent years, thanks to its applications in science and engineering. Fifty-first-order differential equations are used to model almost all nonlinear physical processes. In terms of the Mittag–Leffler function and its extensions, all classes of fractional differential equations have solutions in terms of this function (the Queen Function of Fractional Calculus) (see [1–3]).

Basic power sums and polynomials, particularly the Mittag–Leffler function and its generalizations (Raina's function), as well as polynomials and their implications, are recognized to have extensive applications in various areas of number theory, such as the theory of partitions. Vector calculus, statistical studies, particle physics, optics, fluid studies, mechanical studies, quantum theory and applications, thermal study, and measurements all benefit from these functions (see [4–12]). This function has been investigated in different types of inequalities and convex inequalities. Shu-Bo Chen et al. [13] presented an integral formula inequality containing the Raina's function. Chu et al. [14] generalized harmonically ψ-convex with respect to Raina's function on fractal set. Rashid et al. [15] extended the Mittag–Leffler kernel. Mohammed et al. [16–18] introduced various studies on the generalized Mittag–Leffler kernel.

In this study, we look at how Raina's function

$$\mathcal{A}^{\mu}_{\alpha,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{\mu(n)}{\Gamma(\alpha n+\beta)} \zeta^n$$

may be used to extend a differential operator in the open unit disk. The fractional differential operator is employed to explain a variety of innovative normalized analytic functions. Therefore, we utilize the convolution product between the normalized Raina's function and analytic function satisfying the normalization equality $\chi(0) = \chi'(0) - 1 = 0$. To investigate a collection of differential inequalities, the concept of differential subordination and superordination is employed. Furthermore, we investigate the geometric behaviour of the diffusive wave differential equation, a family of analytic functions. The novel convolution linear operator is used for a variety of applications.

2. Approaches

In this section, we'll go through the approaches we employed.

2.1. Geometric approaches

We'll go over some geometric function theory fundamentals covered in this book [19]

Definition 2.1: Introduce the set $\mathbb{K} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, which indicates the open unit disk. The analytic functions χ_1, χ_2 in $\mathbb{K}$ are subordinated $\chi_1 \prec \chi_2$ or

$$\chi_1(\zeta) \prec \chi_2(\zeta), \quad \zeta \in \mathbb{K}$$

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if for an analytic function $\psi, |\psi| \le |\zeta| < 1$ fulfilling

$$\chi_1(\zeta) = \chi_2(\psi(\zeta)), \quad \zeta \in \mathbb{K}.$$

Definition 2.2: Define the subclass of analytic functions

$$\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \quad \eta \in \mathbb{K}$$

denoting by Λ and satisfying $\chi(0) = \chi'(0) - 1 = 0$.

Moreover, two functions $\phi, \varphi \in \Lambda$ are convoluted $(\phi * \varphi)$ if they achieve the product [20]

$$(\phi * \varphi)(\zeta) = \left(\zeta + \sum_{n=2}^{\infty} \phi_n \zeta^n\right) * \left(\zeta + \sum_{n=2}^{\infty} \varphi_n \zeta^n\right)$$
$$= \zeta + \sum_{n=2}^{\infty} \phi_n \varphi_n \zeta^n.$$

Definition 2.3: Related to this class, the class $\mathcal{S}^*$ of starlike functions and the class $\mathcal{C}$ of convex functions. Moreover, the class $\mathcal{P} := \{\rho : \rho(\zeta) = 1 + \rho_1\zeta + \rho_2\zeta^2 + \cdots, \zeta \in \mathbb{K}\}$ is a special class of analytic functions in $\mathbb{K}$ with positive real part in $\mathbb{K}$ and $\rho(0) = 1$.

2.2. Raina's function

Integrals and outcomes of many kinds of differential equations fall within the category of special functions. As a result, most integral sets contain descriptions of special functions, and these special functions entail the most fundamental integrals; at the very least, the integral representation of special functions. Because symmetries of differential equations are significant in both physics and mathematics, the theory of special functions is closely connected to several mathematical physics issues [21]. To begin, we'll look at the Mittag–Leffler function, which is a well-known special function.

Definition 2.4: The power of the generalized Mittag–Leffler function is as follows: [4]

$$\mathcal{L}^{\mu}_{\alpha,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{\Gamma(\alpha n + \beta)} \frac{\zeta^n}{n!},$$

where

$$\Gamma(\zeta) = \int_0^{\infty} \chi^{\zeta - 1} \mathrm{e}^{-\chi} \, \mathrm{d}\chi, \quad \Re(\zeta) > 0$$

is the gamma function and

$$(\mu)_n = \frac{\Gamma(\mu + 1)}{\Gamma(\mu - n + 1)}$$

is the Pochhammer operator. Obviously, we have [10]

$$\mathcal{L}^{1}_{\alpha,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{\Gamma(\alpha n + \beta)}.$$

Continue by defining Raina's function.

Definition 2.5: The power series determines Raina's function as follows [22]:

$$\mathcal{A}^{\mu}_{\alpha,\beta}(\zeta) = \sum_{n=0}^{\infty} \frac{\mu(n)}{\Gamma(\alpha n + \beta)} \zeta^n, \quad \zeta \in \mathbb{K},$$

where $\alpha > 0, \beta \ge 1$ and $\mu := \{\mu(0), \mu(1), \ldots, \mu(n)\}$ is a bounded sequence of positive real numbers.

Remark 2.6:

- If $\mu(n) = 1$, then we have $\mathcal{L}^{1}_{\alpha,\beta}(\zeta)$;
- If $\mu(n) = \frac{(\mu)_n}{n!}$, then we obtain $\mathcal{L}^{\mu}_{\alpha,\beta}(\zeta)$;
- If $\alpha = 1, \beta = 1, \mu(n) = \frac{(a)_n (b)_n}{(c)_n}$, then we receive the hypergeometric function

$$_2G_1(a, b; c; \zeta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{\zeta^n}{\Gamma(n+1)}.$$

Utilizing the function $\mathcal{A}^{\mu}_{\alpha,\beta}(\zeta)$, we the convolution operator, for $\chi \in \Lambda$

$$\mathbb{A}^{\mu}_{\alpha,\beta}\chi(\zeta) = \left(\frac{\Gamma(\alpha+\beta)}{\mu(1)}\right) (\mathcal{A}^{\mu}_{\alpha,\beta} * \chi)(\zeta)$$
$$= \left(\left(\frac{\Gamma(\alpha+\beta)}{\mu(1)}\right) \mu(0) + \zeta + \sum_{n=2}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\mu(1)}\right) \left(\frac{\mu(n)}{\Gamma(\alpha n + \beta)}\right) \zeta^n\right)$$
$$* \left(\zeta + \sum_{n=2}^{\infty} a_n \zeta^n\right)$$
$$= \zeta + \sum_{n=2}^{\infty} \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha n + \beta)}\right) \left(\frac{\mu(n)}{\mu(1)}\right) a_n \zeta^n$$
$$:= \zeta + \sum_{n=2}^{\infty} \curlywedge_n a_n \zeta^n,$$

where

$$\curlywedge_n := \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha n + \beta)}\right) \left(\frac{\mu(n)}{\mu(1)}\right).$$

$$(\chi \in \Lambda, \ \zeta \in \mathbb{K}, \ \alpha > 0, \ \beta \ge 1, \ \mu = \{\mu(0), \ldots, \mu(n)\})$$

Now by using the Sàlàgean derivative [23], we have

$$\mathbb{A}^{\mu}_{\alpha,\beta}\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \curlywedge_n a_n \zeta^n$$
$$\mathbb{A}^{\mu,2}_{\alpha,\beta}\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} n \curlywedge_n a_n \zeta^n$$
$$\vdots$$
$$\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} n^k \curlywedge_n a_n \zeta^n.$$

Clearly, $\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta) \in \Lambda$. As a result, the Raina fractional differential operator can be studied geometrically.

Remark 2.7:

- The linear operator $\mathbb{A}^{\mu}_{\alpha,\beta}\chi(\zeta)$ is a natural transform of the analytic function $\chi(\zeta)$ $(\chi(\zeta) \to \mathbb{A}^{\mu}_{\alpha,\beta}\chi(\zeta))$. The Raina's summation, which is a generalization of the Mittag–Leffler summation, is the name for this function

$$\mathcal{L}_{\alpha}\chi(\zeta) \equiv \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(1+\alpha n)}\zeta^n.$$

And the Borel's summation

$$\mathcal{B}\chi(\zeta) \equiv \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n.$$

- In the geometric function theory, the operator $\mathbb{A}^{\mu}_{\alpha,\beta}\chi(\zeta)$ is a generalization of the well-known linear Carlson–Shaffer operator [24], when $\alpha = \beta = 1$, and

$$\mu(n) = \frac{(1)_n(\aleph)_n}{(\complement)_n}$$

such that

$$\complement(\aleph, c)\chi(\zeta) = \sum_{n=0}^{\infty} \left(\frac{(1)_n(\aleph)_n}{\Gamma(n+1)(c)_n} \right) a_n\zeta^n.$$

- When $\mu(n) = \Gamma(\alpha n + \beta)$ for all $n \geq 1$, we obtain the well known the Sàlàgean differential operator [23]

$$\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} n^k a_n\zeta^n.$$

2.3. Preparatory

The conclusions of this investigation into the differential subordination theory are established using the following preliminaries:

Lemma 2.8 ([19]): *Suppose that $f(\zeta)$ and $g(\zeta)$ are convex univalent defining in $\mathbb{K}$ with $f(0) = g(0)$. In addition, for a constant $\xi \neq 0$, $\Re(\xi) \geq 0$, the subordination*

$$f(\zeta) + (1/\xi)\zeta f'(\zeta) \prec g(\zeta)$$

yields

$$f(\zeta) \prec g(\zeta).$$

Lemma 2.9 ([19]): *Define the general class of analytic functions*

$$\Pi[\natural, n] = \{\upsilon : \upsilon(\zeta) = \natural + \natural_n\zeta^n + \natural_{n+1}\zeta^{n+1} + \cdots\},$$

where $\natural \in \mathbb{C}$ and n is a positive integer. If $" \in \mathbb{R}$, then

$$\Re\{\upsilon(\zeta) + "\zeta\upsilon'(\zeta)\} > 0 \Rightarrow \Re(\upsilon(\zeta)) > 0.$$

Furthermore, if $" > 0$ and $\upsilon \in \Pi[1, n]$, then there occurs two positive numbers $\xi_1 > 0$ and $\xi_2 > 0$ satisfying the relation

$$\upsilon(\zeta) + "\zeta\upsilon'(\zeta) \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\xi_1}$$

implies

$$\upsilon(\zeta) \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\xi_2}.$$

Lemma 2.10 (see [25]): *Let $\hbar, \upsilon \in \Pi[\natural, n]$, where υ is convex univalent in $\mathbb{K}$ and for $v_1, v_2 \in \mathbb{C}$, $v_2 \neq 0$, then*

$$v_1\hbar(\zeta) + v_2\zeta\hbar'(\zeta) \prec v_1\upsilon(\zeta) + v_2\zeta\upsilon'(\zeta)$$
$$\to \hbar(\zeta) \prec \upsilon(\zeta).$$

Lemma 2.11 (see [26]): *Let $\upsilon, \wp \in \Pi[\natural, n]$, where $\wp$ is convex univalent in $\mathbb{K}$ and the functional $\upsilon(\zeta) + \omega\zeta\upsilon'(\zeta)$ is univalent for $\omega > 0$. Then*

$$\wp(\zeta) + \omega\zeta\wp'(\zeta) \prec \upsilon(\zeta) + \omega\zeta\upsilon'(\zeta) \to \wp(\zeta) \prec \upsilon(\zeta).$$

Lemma 2.12 ([27]): *Assume that $\hbar$ analytic in $\mathbb{K}$ fulfilling $\hbar(0) = 0$. Then the upper value of $\hbar$ on the circle $|\zeta| = 1$ at the point $\zeta_0 = r\,e^{i\theta}$, $\theta \in [-\pi, \pi]$, $0 < q < 1$ is*

$$\zeta_0(\eth_q\hbar(\zeta_0)) = "\hbar(\zeta_0), \quad " \geq 1,$$

where $\eth_q$ represents the Jackson fractional derivative (or quantum fractional derivative).

3. Outcomes

In this study, we formulate the next class of normalized analytic functions and study its properties in view of the geometric function theory.

Definition 3.1: A function $\chi \in \Lambda$ is called to be in the class $\Omega^{\mu,k}_{\alpha,\beta}(\varsigma, \rho)$ if it fulfils the inequality

$$\left(\frac{1-\varsigma}{\zeta}\right)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' \prec \rho(\zeta). \quad (1)$$
$$(\zeta \in \mathbb{K},\ \varsigma \in [0,1],\ \rho(0) = 1,\ \alpha > 0,\ \beta \geq 1),$$

where ρ is convex univalent in $\mathbb{K}$.

Obviously, the convex univalent function

$$\rho(\zeta) = \frac{A\zeta + 1}{B\zeta + 1},$$

is a member in the class

$$\mathcal{P} := \left\{ \rho \in \mathbb{K} : \rho(\eta) = 1 + \sum_{n=1}^{\infty} \rho_i\zeta^n \right\}.$$

Consider the functional $\Sigma_\chi : \mathbb{K} \to \mathbb{K}$, as in the following structure:

$$\Sigma_\chi(\zeta) := \left(\frac{1-\varsigma}{\zeta}\right)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' \quad (2)$$

Based on Definition 3.1, we have the following inequality:

$$\Sigma_\chi(\zeta) \prec \frac{A\zeta+1}{B\zeta+1}, \quad \zeta \in \mathbb{K}.$$

Our study is as follows:

3.1. Inequalities outcomes

We start with the next property of Raina' s operator.

Theorem 3.2: *Let $\chi \in \Omega^{\mu,k}_{\alpha,\beta}(\varsigma,\rho)$ such that*

$$\Re\{\Sigma_\chi(\zeta)\} = \Re\left\{\left(\frac{1-\varsigma}{\zeta}\right)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'\right\}$$
$$:= \Re\left\{1+\sum_{n=1}^{\infty}\sigma_n\right\} > 0$$

Then the inequality is fully filled by the coefficient boundaries of Σ_χ with the probability measure $d\omega(\theta)$:

$$\frac{|\sigma_n|}{2} \le \int_0^{2\pi} |e^{-in\theta}|\, d\omega(\theta),$$

Moreover, if

$$\Re(e^{i\upsilon}\Sigma_\chi(\zeta)) > 0, \quad \zeta \in \mathbb{K},\ \upsilon \in \mathbb{R}$$

then $\chi \in \Omega^{\mu,k}_{\alpha,\beta}(\frac{A\zeta+1}{B\zeta+1})$ and

$$\Sigma_\chi(\zeta) = \frac{A\zeta+1}{B\zeta+1}, \quad \xi \in \mathbb{K},\ |A| = |B| = 1.$$

Proof: Since

$$\Re(\Sigma_\chi(\zeta)) = \Re\left(1+\sum_{n=1}^{\infty}\sigma_n\zeta^n\right) > 0,$$

then $\Sigma_\chi(\zeta)$ is a Carathéodory function in the open unit disk. Continuously, the Carathéodory positivist methodology brings that

$$|\sigma_n| \le 2\int_0^{2\pi} |e^{-in\theta}|\, d\omega(\theta),$$

where $d\omega$ is a probability measure. Additionally, if

$$\Re(e^{i\upsilon}\Sigma_\chi(\zeta)) > 0, \quad \zeta \in \mathbb{K},\ \upsilon \in \mathbb{R}$$

then in virtue of [20, Theorem 1.6] and for fixed number $\upsilon \in \mathbb{R}$, we get

$$\rho(\zeta) = \frac{A\zeta+1}{B\zeta+1}, \quad \zeta \in \mathbb{K},\ |A| = |B| = 1.$$

Moreover, we have from the proof of [20, Theorem 1.6]

$$\Sigma_\chi(\zeta) * \rho(\zeta) \neq 0,$$

and that such that the range $(\Sigma_\chi * \rho)(\mathbb{K})$ i s contained in the interior of $\rho(\mathbb{K})$. This yields $0 \notin (\Sigma_\chi * \rho)(\mathbb{K})$. Hence,

$$\chi \in \Omega^{\mu,k}_{\alpha,\beta}\left(\varsigma, \frac{A\zeta+1}{B\zeta+1}\right).$$

■

The next outcomes indicate the necessary and sufficient method for the functional sandwich theory.

Theorem 3.3: *Let the following conditions hold:*

$$\varsigma\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'' + [\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' \prec p_2(\zeta) + \zeta p_2'(\zeta), \quad (3)$$

where $p_2(0) = 1$ and convex in $\mathbb{K}$. Additionally, assume that $\Sigma_\chi(\zeta)$ is univalent in $\mathbb{K}$ such that $\Sigma_\chi \in \Pi[p_1(0),1] \cap \mathbb{O}$, where $\mathbb{O}$ presents the set of all univalent analytic functions g with $\lim_{\zeta\in\partial\mathbb{O}} g \neq \infty$ and

$$p_1(\zeta) + \zeta p_1'(\zeta) \prec \varsigma\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'' + [\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'. \quad (4)$$

Then

$$p_1(\zeta) \prec \Sigma_\chi(\zeta) \prec p_2(\zeta)$$

and $p_1(\zeta)$ is the best sub-dominant and $p_2(\zeta)$ is the best dominant.

Proof: Let

$$\Sigma_\chi(\zeta) = \left(\frac{1-\varsigma}{\zeta}\right)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'.$$

A computation implies

$$\begin{aligned}\Sigma_\chi(\zeta) + \zeta\Sigma_\chi'(\zeta) &= \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' \\ &+ \frac{\zeta(\varsigma\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'' - (\varsigma-1)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]') + (\varsigma-1)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta} \\ &+ \frac{(1-\varsigma)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta} \\ &= \varsigma\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'' + [\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'.\end{aligned}$$

As a result, the following double inequality is obtained

$$p_1(\zeta) + \zeta p_1'(\zeta) \prec \Sigma_\chi(\zeta) + \zeta\Sigma_\chi'(\zeta) \prec p_2(\zeta) + \zeta p_2'(\zeta).$$

As a conclusion, the desired result is yielded by Lemmas 2.10 and 2.11. ■

Theorem 3.4: *Let*

$$\Sigma_\chi(\zeta) = \frac{(1-\varsigma)}{\zeta}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'$$

then

$$\left(\frac{\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'}{\zeta}\right)\varepsilon_1 + [\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)][\varepsilon_1 + 3\varepsilon_2]$$

$$+\varepsilon_2\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'' \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\xi_1}$$

$$\Rightarrow \Sigma_\chi(\zeta) \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\xi_2}.$$

$$(\xi_1 > 0,\ \xi_2 > 0,\ \varepsilon_1 = 1-\varsigma,\ \varepsilon_2 = \varsigma > 0)$$

Proof: A computation yields

$$\begin{aligned}
&\Sigma_\chi(\zeta) + \zeta\Sigma'_\chi(\zeta)\\
&= \frac{(1-\varsigma)}{\zeta}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'\\
&\quad + \zeta\left(\frac{(1-\varsigma)}{\zeta}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'\right)'\\
&= \left(\frac{[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]'}{\zeta}\right)\varepsilon_1 + [\varepsilon_1 + 3\varepsilon_2][\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]\\
&\quad + \varepsilon_2\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]''\\
&\prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\xi_1}
\end{aligned}$$

According to Lemma 2.9 with ”= 1, we get

$$\Sigma_\chi(\zeta) \prec \left(\frac{1+\zeta}{1-\zeta}\right)^{\xi_2}.$$

■

3.2. *Fractional differential equation*

In this part, we continue our study using the convolution linear operator. We formulate the operator to present a generalized formula of the diffusive wave differential equation. When inertial acceleration is substantially lower than all other sources of acceleration, or when there is mostly sub-critical flow with low Froude values, the diffusive wave is viable.

In light of the suggested operator, we utilize the class $\Omega^{\mu,k}_{\alpha,\beta}(\varsigma, \frac{1+\zeta}{1-\zeta})$ to develop a class of fractional diffusive wave differential equations. We look at the upper bound of the diffusive wave equation. The formula is as follows:

$$\left(\frac{1-\varsigma}{\zeta}\right)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' = \frac{A\zeta+1}{B\zeta+1}, \quad ([\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(0)] = 0,\ \varsigma\in[0,1],\ \zeta\in\mathbb{K}). \tag{5}$$

The solution to (5) is given by the following result.

Theorem 3.5: *Let $\chi \in \Omega^{\mu,k}_{\alpha,\beta}(\varsigma, \frac{1+\zeta}{1-\zeta})$. Then (5) has a solution expressed by*

$$[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \zeta\left(\frac{2\zeta\ {}_2G_1(1, 1+\frac{1}{\varsigma}, 2+\frac{1}{\varsigma}, \zeta)}{\varsigma+1} + 1\right), \tag{6}$$

where ${}_2G_1(a,b,c;\zeta)$ indicates the hypergeometric function.

Proof: Assume that $\chi \in \Omega^{\mu,k}_{\alpha,\beta}(\varsigma, \frac{1+\zeta}{1-\zeta})$. Then it satisfies the differential equation

$$\left(\frac{1-\varsigma}{\zeta}\right)[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' = \frac{\varphi(\zeta)+1}{1-\varphi(\zeta)},$$

where φ is a Schwarz function with the property: $|\varphi| \le |\zeta| < 1$ and $\varphi(0) = 0$. Now, by using Schwarz lemma, equality $\varphi(\zeta) = \sigma\zeta, |\sigma| = 1$ (see [28, Theorem 5.34]) implies the differential equation

$$\frac{(1-\varsigma)}{\zeta}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \varsigma[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' = \frac{\zeta+1}{1-\zeta}.$$

Rearrange the above equation, we have

$$[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' + \frac{1-\varsigma}{\varsigma\zeta}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \left(\frac{1}{\varsigma}\right)\left(\frac{1+\zeta}{1-\zeta}\right).$$

Multiply the above equation by the functional

$$\mathbb{T}(\zeta) = \exp\left(\int\frac{1-\varsigma}{\varsigma\zeta}\,d\zeta\right) = \zeta^{\frac{1-\varsigma}{\varsigma}},$$

we have

$$\zeta^{\frac{1-\varsigma}{\varsigma}}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' + \left(\frac{1-\varsigma}{\varsigma}\right)\zeta^{\frac{1-2\varsigma}{\varsigma}}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \left(\frac{1}{\varsigma}\right)\left(\frac{1+\zeta}{1-\zeta}\right).$$

$$\zeta^{1/\varsigma-1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]' - \frac{[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]\left((1-\varsigma)\zeta^{1/\varsigma-2}\right)}{\varsigma} = \left(\frac{\zeta^{1/\varsigma-1}}{\varsigma}\right)\left(\frac{1+\zeta}{1-\zeta}\right).$$

The solution of the above first-order differential equation is

$$[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \zeta\left(\frac{2\zeta\ {}_2G_1(1, 1+\frac{1}{\varsigma}, 2+\frac{1}{\varsigma}, \zeta)}{\varsigma+1} + 1\right),$$

where ${}_2G_1(a,b,c;\zeta)$ indicates the hypergeometric function. This completes the proof. ■

Example 3.6: Let $\chi \in \Omega^{\mu,k}_{\alpha,\beta}(\varsigma, \frac{1+\zeta}{1-\zeta})$, where $\varsigma = 1/2$. Then in view of Theorem 3.5, we get the solution

$$\begin{aligned}
[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] &= \zeta\left(\frac{4\zeta\ {}_2G_1(1,3,4,\zeta)}{3} + 1\right)\\
&= \zeta + 1.3\zeta^2 + \zeta^3 + 0.8\zeta^4 + 0.67\zeta^5\\
&\quad + 0.57\zeta^6 + O(\zeta^7), \quad |\zeta| < 1.
\end{aligned}$$

3.3. *First order differential operator*

In the next study, we employ the Raina's operator to define a new generalized differential operator.

Definition 3.7: For non-negative real numbers λ let $[[\lambda]]$ be the integer part of λ. For $\chi \in \Lambda$, and by employing the Raina's operator $[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]$, we have the following extended linear differential operator:

$$\mathcal{A}^\lambda[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \mathcal{A}^{\lambda-[[\lambda]]}(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])$$

$$= \frac{\mathbb{k}_1(\lambda - [[\lambda]], \zeta)}{\mathbb{k}_1(\lambda - [[\lambda]], \zeta) + \mathbb{k}_0(\lambda - [[\lambda]], \zeta)} \times (\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) + \frac{\mathbb{k}_0(\lambda - [[\lambda]], \zeta)}{\mathbb{k}_1(\lambda - [[\lambda]], \zeta) + \mathbb{k}_0(\lambda - [[\lambda]], \zeta)} \times (\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])', \quad (7)$$

where for $\nu = \lambda - [[\lambda]] \in [0, 1)$,

$$\mathcal{A}^0[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = [\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]$$

$$\begin{aligned}\mathcal{A}^{\nu}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] &= \frac{\mathbb{k}_1(\nu,\zeta)}{\mathbb{k}_1(\nu,\zeta) + \mathbb{k}_0(\nu,\zeta)}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] + \frac{\mathbb{k}_0(\nu,\zeta)}{\mathbb{k}_1(\nu,\eta) + \mathbb{k}_0(\nu,\zeta)}(\zeta[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]') \\ &= \frac{\mathbb{k}_1(\nu,\zeta)}{\mathbb{k}_1(\nu,\eta) + \mathbb{k}_0(\nu,\zeta)}\left[\zeta + \sum_{n=2}^{\infty} \lambda_n n^k a_n \zeta^n\right] + \frac{\mathbb{k}_0(\nu,\zeta)}{\mathbb{k}_1(\nu,\zeta) + \mathbb{k}_0(\nu,\zeta)}\left([\zeta + \sum_{n=2}^{\infty} \lambda_n n^{k+1} a_n \zeta^n]\right) \\ &= \zeta + \sum_{n=2}^{\infty}\left(\frac{\mathbb{k}_1(\nu,\zeta) + n\mathbb{k}_0(\nu,\zeta)}{\mathbb{k}_1(\nu,\zeta) + \mathbb{k}_0(\nu,\zeta)}\right) n^k \lambda_n a_n \zeta^n \\ &:= \zeta + \sum_{n=2}^{\infty} \kappa_n n^k \lambda_n a_n \zeta^n\end{aligned}$$

$$\mathcal{A}^1[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \zeta([\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])', \ldots,$$

$$\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \mathcal{A}^1(\mathcal{A}^{[[\lambda]]-1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]),$$

where

$$\kappa_n := \left(\frac{\mathbb{k}_1(\nu,\zeta) + n\mathbb{k}_0(\nu,\zeta)}{\mathbb{k}_1(\nu,\zeta) + \mathbb{k}_0(\nu,\zeta)}\right);$$

and the functions $\mathbb{k}_1, \mathbb{k}_0 : [0,1] \times \mathbb{K} \to \mathbb{K}$ are analytic in $\mathbb{K}$ with

$$\mathbb{k}_1(\nu,\zeta) \neq -\mathbb{k}_0(\nu,\zeta),$$

$$\lim_{\nu\to 0} \mathbb{k}_1(\nu,\zeta) = 1, \quad \lim_{\nu\to 1} \mathbb{k}_1(\nu,\zeta) = 0, \quad \mathbb{k}_1(\nu,\zeta) \neq 0, \quad \forall \zeta \in \mathbb{K},\ \nu \in (0,1),$$

and

$$\lim_{\nu\to 0} \mathbb{k}_0(\nu,\zeta) = 0, \quad \lim_{\nu\to 1} \mathbb{k}_0(\nu,\zeta) = 1, \quad \mathbb{k}_0(\nu,\zeta) \neq 0, \quad \forall \zeta \in \mathbb{K},\ \nu \in (0,1).$$

It is clear that, for constant coefficients, $\mathcal{A}^{\nu}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] \in \Lambda$. For example $\mathbb{k}_0(\nu,\zeta) = \nu$ and $\mathbb{k}_1(\nu,\zeta) = 1 - \nu$.

Clearly, if λ assumes only non-negative integer values, that is if $\lambda - [[\lambda]] = 0$, $\alpha = \beta = 1$, $\mu(n) = n!$, $\forall n \geq 1$, then we have the Sàlàgean differential operator [23]. We also have the differential operator in [29], which is based on the same assumptions. In this section, we examine the geometric properties of the complex conformable derivative (7) when applied to functions with a positive real portion.

Theorem 3.8: *For a fixed number $\varepsilon \in (0,1)$ and $\lambda \in [0,\infty)$ let*

$$\mathbb{k}_0(\lambda - [[\lambda]], \zeta) = \left(\frac{\varepsilon}{1-\varepsilon}\right)\mathbb{k}_1(\lambda - [[\lambda]], \zeta).$$

Then

$$\frac{\mathcal{A}^{\lambda+2}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]} \in \mathcal{P} \implies \frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]} \in \mathcal{P}.$$

Proof: For $\mathbb{k}_0(\lambda - [[\lambda]], \zeta) = \frac{\varepsilon}{1-\varepsilon}\ \mathbb{k}_1(\lambda - [[\lambda]], \zeta)$ and by Definition 3.7, we get

$$\begin{aligned}\mathcal{A}^{\gamma}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] &= (1-\varepsilon)(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) + \varepsilon\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])' \\ \mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] &= \zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])' + \varepsilon\zeta^2(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])''\end{aligned}$$

and

$$\mathcal{A}^{\lambda+2}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])' + (1+2\varepsilon) \times \lambda^2(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'' + \varepsilon\zeta^3(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'''.$$

Obviously, we obtain

$$\Re\left(\frac{\mathcal{A}^{\lambda+2}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\gamma+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}\right) > 0$$

if and only if

$$\Re\left\{1 + \frac{(1+\varepsilon)\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'' + \varepsilon\zeta^2(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'''}{(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])' + \varepsilon\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])''}\right\} > 0.$$

Accordingly, if and only if

$$\Re\left\{1 + \frac{\zeta[(1-\varepsilon)(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) + \varepsilon\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])']''}{[(1-\varepsilon)(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) + \varepsilon\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])']'}\right\} > 0. \quad (8)$$

The convexity of a function is obtained by combining the inequality 8 with the idea of convex functions:

$$(1-\varepsilon)(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) + \varepsilon\zeta[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]'.$$

But all convex functions are starlike, then we obtain that

$$\Re\left\{\frac{\zeta[(1-\varepsilon)(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) +\varepsilon\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])']'}{(1-\varepsilon)(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]) +\varepsilon\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'}\right\} > 0. \qquad (9)$$

The inequality 9 occurs if and only if

$$\Re\left(\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}\right) > 0$$

and this ends the proof. ■

The main condition to put on the operator $\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]$ is computed by our second theorem, for the functional

$$\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'}$$

to be of positive real part.

Theorem 3.9: *For a positive number $\varepsilon \in (0,1)$ and $\lambda \in [0,\infty)$ let*

$$\Bbbk_1(\lambda-[[\lambda]],\zeta) = \left(\frac{\varepsilon}{1-\varepsilon}\right)\Bbbk_0(\lambda-[[\lambda]],\zeta).$$

If $\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] \in \mathcal{C}$, then

$$\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'} \in \mathcal{P}(\epsilon).$$

Proof: Applying the differential operator rule to

$$\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] = \mathcal{A}(\mathcal{A}^{\lambda}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])$$

implies

$$\begin{aligned}
&\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]\\
&= \mathcal{A}^{\lambda-[[\lambda]]}(\mathcal{A}^{[[\lambda]]+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])\\
&= \mathcal{A}^{\lambda-[[\lambda]]}\{\mathcal{A}[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]\}\\
&= \mathcal{A}^{\lambda-[[\lambda]]}\{\zeta[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]'\}\\
&= \frac{\Bbbk_1(\lambda-[[\lambda]],\zeta)}{\Bbbk_1(\lambda-[[\lambda]],\zeta)+\Bbbk_0(\lambda-[[\lambda]],\zeta)}\\
&\quad\times\{\zeta[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]'\}\\
&\quad+\frac{\Bbbk_0(\lambda-[[\lambda]],\zeta)}{\Bbbk_1(\lambda-[[\lambda]],\zeta)+\Bbbk_0(\lambda-[[\lambda]],\zeta)}\\
&\quad\times\{\zeta[(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'+\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'']\}\\
&= \frac{\Bbbk_1(\lambda-[[\lambda]],\zeta)}{\Bbbk_1(\lambda-[[\lambda]],\zeta)+\Bbbk_0(\lambda-[[\lambda]],\zeta)}\\
&\quad\times\{\zeta[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]'\}\\
&\quad+\frac{\Bbbk_0(\lambda-[[\lambda]],\zeta)}{\Bbbk_1(\lambda-[[\lambda]],\zeta)+\Bbbk_0(\lambda-[[\lambda]],\zeta)}\\
&\quad\times\{\zeta[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]'\}\\
&\quad+\frac{\Bbbk_0(\lambda-[[\lambda]],\zeta)}{\Bbbk_1(\lambda-[[\lambda]],\zeta)+\Bbbk_0(\lambda-[[\lambda]],\zeta)}\\
&\quad\times\{\zeta^2[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]''\}\\
&= \zeta[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]'\\
&\quad+\frac{\Bbbk_0(\lambda-[[\lambda]],\zeta)}{\Bbbk_1(\lambda-[[\lambda]],\zeta)+\Bbbk_0(\lambda-[[\lambda]],\zeta)}\\
&\quad\times\{\zeta^2[\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]]''\}. \qquad (10)
\end{aligned}$$

Dividing Equation 10 by the term $\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'$ and utilizing the relation

$$\Bbbk_1(\lambda-[[\lambda]],\zeta) = \left(\frac{\varepsilon}{1-\varepsilon}\right)\Bbbk_0(\lambda-[[\lambda]],\zeta),$$

we get

$$\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'} = 1+(1-\varepsilon)\frac{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])''}{(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'}.$$

The convexity of $\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]$, it becomes

$$\Re\left\{1+\frac{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])''}{(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'}\right\} > 0.$$

Hence, it yields that

$$\Re\left\{\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'}\right\} > \varepsilon.$$

This ends the proof. ■

3.4. *Quantum starlike methodology*

Quantum calculus (QC) is a novel field of mathematical analysis and its applications, with applications in physics and mathematics. Jackson [30, 31] originally defined and enhanced the functions of q-differentiation and *q*-integration. The geometric function theory idea of *q*-calculus was later incorporated by Ismail et al. [32]. QC is now being used by researchers to propose and build new Ma and Minda classes. Seoudy and Aouf [33] suggested a quantum starlike function subclass based on q-derivatives. Recently, Zainab et al. [34] employed a novel curve to create appropriate *q*-stralikeness criteria. Different types of *q*-stralik functions dominated by a 2D-Julia set were explored by Samir et al. [35]. Furthermore, QC is used to generalize a variety of differential and integral operators [36–42].

Definition 3.10: The Jackson derivative may be shown using the difference operator below.

$$(\eth_q)p(\zeta) = \frac{p(\zeta) - p(q\zeta)}{\zeta(1-q)}, \quad q \in (0,1) \tag{11}$$

such that

$$\eth_q(\zeta^c) = \left(\frac{1-q^c}{1-q}\right)\zeta^{c-1}.$$

The total of the numbers is also included in the Maclaurin's series representation.

$$(\eth_q p)(\zeta) = \sum_{k=0}^{\infty} p_k [k]_q \zeta^{k-1}, \tag{12}$$

where

$$[k]_q := \frac{1-q^k}{1-q}.$$

Note that

$$\eth_q \complement = 0, \quad \lim_{q\to 1^-} (\eth_q p)(\zeta) = p'(\zeta),$$

where $\complement$ is a constant. Then there's the multiplication rule, which is formulated by multiplying two numbers together

$$\begin{aligned}\eth_q(p_1(\zeta)p_2(\zeta)) &= p_2(\zeta)\eth_q p_1(\zeta) + p_1(q\zeta)\eth_q p_2(\zeta)\\ &= p_2(q\zeta)\eth_q p_1(\zeta) + p_1(\zeta)\eth_q p_2(\zeta).\end{aligned}$$

We then use the q-parametric Mandelbrot function to formulate our q-starlike class, linking it to the normalized function subclass in the process $\mathbb{K}$

$$\begin{aligned}\mathfrak{G}^{(\ell)}(\zeta) &= \ell + \zeta^2\\ (\ell \in \mathbb{C}&, \zeta \in \mathbb{K}).\end{aligned} \tag{13}$$

We aim to investigate the sufficient conditions on the two parameters ℓ and q to obtain the q-starlike function.

Theorem 3.11: *Assume that* $\flat \in \mathbb{K}$ *with* $\flat(0) = 1$ *and*

$$1 + \zeta(\eth_q \flat(\zeta)) \prec \sqrt{1+\zeta}, \quad \zeta \in \mathbb{K}. \tag{14}$$

If for some positive constant j achieves the inequalities

$$\jmath > 1 + \sqrt{\frac{3}{2}}, \quad 0 < q \le \frac{2\jmath^2 - 4\jmath - 1}{2\jmath^2}, \tag{15}$$

then for some $\ell \in \mathbb{C}$, *we have*

$$\flat(\zeta) \prec \mathfrak{G}^{(\ell)}(\zeta) = \ell + \zeta^2. \tag{16}$$

Proof: Formulate a function $\hbar$ by

$$\hbar(\zeta) := 1 + \zeta(\eth_q \flat(\zeta)).$$

The condition (14) implies that

$$\begin{aligned}1 + \zeta(\eth_q \flat(\zeta)) &= \sqrt{1 + \upsilon(\zeta)},\\ (\upsilon(0) = 0, \ |\upsilon(\zeta)| &\le |\zeta| < 1)\end{aligned}$$

A computation gives

$$\upsilon(\zeta) = \hbar^2(\zeta) - 1.$$

Our aim is to show that

$$|\upsilon(\zeta)| = |\hbar^2(\zeta) - 1| < 1,$$

where $\zeta_0 \in \mathbb{K}$ satisfying

$$\flat(\zeta) = [\ell + \upsilon^2(\zeta)].$$

Consume not; if so, the preceding conclusion applies

$$\hbar(\eta) = 1 + \eta(\eth_q[\ell + \upsilon^2(\eta)]).$$

Employing Jackson's derivative principles as well as the formula

$$\upsilon(q\zeta) = \upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta),$$

and

$$\eth_q\upsilon^2(\zeta) = \eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)],$$

we obtain

$$\hbar(\zeta) = 1 + \zeta\eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)].$$

Consider the existence of a point $\zeta_0 \in \mathbb{K}$ such that

$$\max_{|\zeta|\le|\zeta_0|} |\upsilon(\zeta)| = |\upsilon(\zeta_0)| = 1$$

and

$$\zeta_0(\eth_q\upsilon(\zeta_0)) = \jmath\upsilon(\zeta_0), \quad \jmath \ge 1.$$

We proceed to prove

$$|\upsilon(\zeta)| = |\hbar^2(\zeta) - 1| < 1,$$

utilizing Jack Lemma 2.12.

Letting $\upsilon(\zeta_0) = e^{i\theta}$, we get

$$\begin{aligned}&|\hbar^2(\zeta) - 1|\\ &\quad = |1 + 2t[1 + \zeta\eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)]]\\ &\qquad + [1 + \zeta\eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)]]^2\\ &\qquad - 1|_{\zeta=\zeta_0}\\ &\quad = |2 + 2\zeta\eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)]\\ &\qquad + 1 + 2\zeta\eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)]\\ &\qquad + [\zeta\eth_q\upsilon(\zeta)[2\upsilon(\zeta) - (1-q)\zeta\eth_q\upsilon(\zeta)]]^2|_{\zeta=\zeta_0}\\ &\quad = |3 + 2\zeta_0\eth_q\upsilon(\zeta_0)[2\upsilon(\zeta_0) - (1-q)\zeta_0\eth_q\upsilon(\zeta_0)]\\ &\qquad + 2\zeta_0\eth_q\upsilon(\zeta_0)[2\upsilon(\zeta_0) - (1-q)\zeta_0\eth_q\upsilon(\zeta_0)]\\ &\qquad + [\zeta_0\eth_q\upsilon(\zeta_0)[2\upsilon(\zeta_0) - (1-q)\zeta_0\eth_q\upsilon(\zeta_0)]]^2|\\ &\quad = |3 + 2\jmath\upsilon(\zeta_0)[2\upsilon(\zeta_0) - (1-q)\jmath\upsilon(\zeta_0)]\\ &\qquad + 2\jmath\upsilon(\zeta_0)[2\upsilon(\zeta_0) - (1-q)\jmath\upsilon(\zeta_0)]\\ &\qquad + [\jmath\upsilon(\zeta_0)[2\upsilon(\zeta_0) - (1-q)\jmath\upsilon(\zeta_0)]]^2|\end{aligned}$$

$$\geq |3 + 4\jmath\upsilon^2(\zeta_0)[2 - (1-q)\jmath]|$$
$$= |3 + 4\jmath\, e^{2i\theta}[2 - (1-q)\jmath]|$$
$$\geq 1,$$

where

$$\jmath > 1 + \sqrt{\frac{3}{2}}, \quad \frac{2\jmath^2 - 4\jmath - 1}{2\jmath^2} \leq q < 1,$$

which contradicts (15). Hence, we obtain (16). ∎

The following examples involve the Raina' s operator.

Example 3.12: For a positive number $\varepsilon \in (0,1)$ and $\lambda \in [0,\infty)$ let

$$\Bbbk_1(\lambda - [[\lambda]], \zeta) = \frac{\varepsilon}{1-\varepsilon}\Bbbk_0(\lambda - [[\lambda]], \zeta).$$

- If $\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)] \in \mathcal{C}$, then in view of Theorem 3.9, we have

$$\flat(\zeta) := \frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'} \in \mathcal{P}(\epsilon).$$

That is $\flat(0) = 1$. Moreover, if

$$1 + \zeta\eth_q\left(\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'}\right) \prec \sqrt{1+\zeta},$$

where q satisfies (15) then according to Theorem 3.11

$$\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta(\mathcal{A}^{[[\lambda]]}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)])'} \prec \mathfrak{G}^{(\ell)}(\zeta).$$

- If

$$\frac{\mathcal{A}^{\lambda+2}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]} \in \mathcal{P}$$

then in view of Theorem 3.8, we get

$$\flat(\zeta) := \frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]} \in \mathcal{P}.$$

That is $\flat(0) = 1$. In addition, if

$$1 + \zeta\eth_q\left(\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}\right) \prec \sqrt{1+\zeta},$$

where q satisfies (15) then by Theorem 3.11, we have

$$\frac{\mathcal{A}^{\lambda+1}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\mathcal{A}^{\lambda}[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]} \prec \mathfrak{G}^{(\ell)}(\zeta).$$

- Let

$$\phi(\zeta) = \frac{[\mathbb{A}^{\mu,k}_{\alpha,\beta}\chi(\zeta)]}{\zeta},$$

where $\phi(0) = 1$. If

$$1 + \zeta\eth_q\phi(\zeta) \prec \sqrt{1+\zeta},$$

where q achieves the inequality (15) then in virtue of Theorem 3.11, we obtain

$$\phi(\zeta) \prec \mathfrak{G}^{(\ell)}(\zeta).$$

4. Conclusion

Raina's transformations in $\mathbb{K}$ were generalized utilizing conformable calculus and Jackson calculus in the above investigation. The Raina's convolution operator is acted on the normalized subclass. As an application, we considered the proposed linear convolution operator in fractional differential equation, type wave equation. The solution of a certain type of diffusion differential equation, which is utilized as a case study, is determined by the hypergeometric function.

More investigation is presented by formulating the Raina's convolution operator in a conformable fractional calculus. We studied the main sufficient conditions to get stralike geometry of the operator (see Theorems 3.8 and 3.9).

Finally, the quantum calculus is utilized to recognize the q-starlike function together with the q-parametric Mandelbort function. As an application, we applied the result using the Raina's convolution operator (see Example 3.12).

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Author contributions

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References

[1] Seybold HJ, Hilfer R. Numerical results for the generalized Mittag–Leffler function. Fract Calc Appl Anal. 2005;8(2):127–139.
[2] Mainardi F. Why the Mittag–Leffler function can be considered the Queen function of the fractional calculus? Entropy. 2020;22(12):1359.
[3] Gorenflo R, Mainardi F. The Mittag–Leffler function in the thinning theory for renewal processes. Preprint; 2018. Available from: arXiv:1808.06563

[4] Srivastava HM. Some families of Mittag–Leffler type functions and associated operators of fractional calculus (Survey). TWMS J Pure Appl Math. 2016;7:123–145.

[5] Srivastava HM, Kilicman A, Abdulnaby ZE, et al. Generalized convolution properties based on the modified Mittag–Leffler function; 2017.

[6] Meshram C, Ibrahim RW, Meshram SG, et al. An efficient authentication with key agreement procedure using Mittag–Leffler–Chebyshev summation chaotic map under the multi-server architecture. J Supercomput. 2022;78: 4938–4959.

[7] Ibrahim RW. Maximize minimum utility function of fractional cloud computing system based on search algorithm utilizing the Mittag–Leffler sum. Int J Anal Appl. 2018;16(1):125–136.

[8] Noreen S, Raza M, Liu JL, et al. Geometric properties of normalized Mittag–Leffler functions. Symmetry. 2019;11(1):45.

[9] Liu J, Srivastava R. A linear operator associated with the Mittag–Leffler function and related conformal mappings. J Appl Anal Comput. 2018;8(6):1886–1892.

[10] Ryapolov PA, Postnikov EB. Mittag–Leffler function as an approximant to the concentrated Ferrofluid's magnetization curve. Fractal Fract. 2021;5(4):147.

[11] Shukla AK, Prajapati JC. On a generalization of Mittag–Leffler function and its properties. J Math Anal Appl. 2007;336(2):797–811.

[12] Haubold HJ, Mathai AM, Saxena RK. Mittag–Leffler functions and their applications. J Appl Math. 2011;2011: 1–52.

[13] Chen SB, Rashid S, Hammouch Z, et al. Integral inequalities via Raina's fractional integrals operator with respect to a monotone function. Adv Differ Equ. 2020;2020(1): 1–20.

[14] Chu Y-M, Rashid S, Singh J. A novel comprehensive analysis on generalized harmonically ψ-convex with respect to Raina's function on fractal set with applications. Math Methods Appl Sci. 2021. doi:10.1002/mma.7346.

[15] Rashid S, Hammouch Z, Ashraf R, et al. New computation of unified bounds via a more general fractional operator using generalized Mittag–Leffler function in the kernel. Comput Model Eng Sci. 2021;126(1):359–378.

[16] Mohammed PO, Machado JAT, Guirao JLG, et al. Adomian decomposition and fractional power series solution of a class of nonlinear fractional differential equations. Mathematics. 2021;9(9):1070.

[17] Mohammed PO, Goodrich CS, Brzo AB, et al. New classifications of monotonicity investigation for discrete operators with Mittag–Leffler kernel. Math Biosci Eng. 2022;19:4062–4074.

[18] Mohammed PO, Goodrich CS, Hamasalh FK, et al. On positivity and monotonicity analysis for discrete fractional operators with discrete Mittag–Leffler kernel. Math Methods Appl Sci. 2022: 1–26. https://doi.org/10.1002/mma.7083

[19] Miller SS, Mocanu PT. Differential subordinations: theory and applications. Boca Raton, Florida: CRC Press; 2000.

[20] Ruscheweyh S. Convolutions in geometric function theory. Montreal: Les Presses De L'Universite De Montreal; 1982.

[21] Gil A, Segura J, Temme NM. Numerical methods for special functions. Philadelphia, USA: Society for Industrial and Applied Mathematics; 2007.

[22] Raina RK. On generalized Wright's hypergeometric functions and fractional calculus operators. East Asian Math J. 2015;21:191–203.

[23] Sàlàgean GS. Subclasses of univalent functions. In Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981). Berlin: Springer; 1983. p. 362–372. (Lecture notes in mathematics; vol. 1013).

[24] Carlson BC, Shaffer DB. Starlike and prestarlike hypergeometric functions. SIAM J Math Anal. 1984;15(4):737–745.

[25] Shanmugam TN, Sivasubramanian S, Srivastava HM. Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations. Integral Transforms Spec Funct. 2006;17(12):889–899.

[26] Miller SS, Mocanu PT. Subordinants of differential superordinations. Complex Var. 2003;48(10):815–826.

[27] Cetinkaya A, Yasar PG. q-Harmonic mappings for which analytic part is q-convex functions of complex order. Hacettepe J Math Stat. 2018;47(4):813–820.

[28] Rodriguez RE, Kra I, Gilman JP. Complex analysis: in the spirit of Lipman Bers. Switzerland AG: Springer; 2013.

[29] Ibrahim RW, Jahangiri JM, Cloud Computing Center. Conformable differential operator generalizes the Briot–Bouquet differential equation in a complex domain. AIMS Math. 2019;4(6):1582–1595.

[30] Jackson FH. XI.–On q-functions and a certain difference operator. Earth Environ Sci Trans R Soc Edinb. 1909;46(2):253–281.

[31] Jackson DO, Fukuda T, Dunn O, et al. On q-definite integrals. Quart J Pure Appl Math. 1910;(1):193–203.

[32] Ismail MEH, Merkes E, Styer D. A generalization of starlike functions. Complex Var Theory Appl. 1990;14(1–4): 77–84.

[33] Seoudy TM, Aouf MK. Coefficient estimates of new classes of q-starlike and q-convex functions of complex order. J Math Inequal. 2016;10(1):135–145.

[34] Zainab S, Shakeel A, Imran M, et al. Sufficiency criteria for starlike functions associated with cardioid. J Funct Spaces. 2021;2021:Article ID 9999213.

[35] Hadid SB, Ibrahim RW, Momani S. A new measure of quantum starlike functions connected with Julia functions. J Funct Spaces. 2022;2022:Article ID 4865785.

[36] Ibrahim RW, Elobaid RM, Obaiys SJ. Geometric inequalities via a symmetric differential operator defined by quantum calculus in the open unit disk. J Funct Spaces. 2020;2020:Article ID 6932739.

[37] Ali MA, Budak H, Akkurt A, et al. Quantum Ostrowski-type inequalities for twice quantum differentiable functions in quantum calculus. Open Math. 2021;19(1):440–449.

[38] Ibrahim RW, Elobaid RM, Obaiys SJ. A class of quantum Briot–Bouquet differential equations with complex coefficients. Mathematics. 2020;8(5):794.

[39] Ibrahim RW, Baleanu D. On quantum hybrid fractional conformable differential and integral operators in a complex domain. Rev Real Acad Cienc Exactas Fis Nat Ser A Mat. 2021;115(1):1–13.

[40] Ibrahim RW. Geometric process solving a class of analytic functions using q-convolution differential operator. J Taibah Univ Sci. 2020;14(1):670–677.

[41] Aldawish I, Ibrahim RW. A new parametric differential operator generalized a class of d'Alembert's equations. J Taibah Univ Sci. 2021;15(1):449–457.

[42] Shah R, Khan H, Baleanu D, et al. A semi-analytical method to solve family of Kuramoto–Sivashinsky equations. J Taibah Univ Sci. 2020;14(1):402–411.