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Research article

Coupled fixed point theorems on $C^\star$-algebra valued bipolar metric spaces

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Abstract: In the present paper, we introduce the notion of a $C^\star$-algebra valued bipolar metric space and prove coupled fixed point theorems. Some of the well-known outcomes in the literature are generalized and expanded by the results shown. An example and application to support our result is presented.

Keywords: $C^\star$-algebra; $C^\star$-algebra valued bipolar metric space; coupled fixed point
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1. Introduction

In 2016, Mutlu and Grdal [1] introduced the concepts of bipolar metric space and proved fixed point theorems.

Definition 1.1. [1] Let Γ and Ψ be two non-void sets and $\varphi : \Gamma \times \Psi \rightarrow \mathbb{R}^+$ be a function, such that

(a) $\varphi(\sigma, \zeta) = 0$ iff $\sigma = \zeta$, for all $(\sigma, \zeta) \in \Gamma \times \Psi$.

(b) $\varphi(\sigma, \zeta) = \varphi(\zeta, \sigma)$, for all $\sigma, \zeta \in \Gamma \cap \Psi$.

(c) $\varphi(\sigma, \zeta) \leq \varphi(\sigma, \omega) + \varphi(\sigma_1, \omega) + \varphi(\sigma_1, \zeta)$, for all $\sigma, \sigma_1 \in \Gamma$ and $\omega, \zeta \in \Psi$.

The pair (Γ, Ψ, φ) is called a bipolar metric space.

In bipolar metric spaces, a lot of significant work has been done (see [2–10]). In 2014, Ma, Jiang and Sun [11] introduced the notion of a $C^\star$-algebra-valued metric space and proved fixed point theorem. In 2015, Batul and Kamran [12] proved fixed theorems on $C^\star$-algebra-valued metric space. In the present paper, we introduce a new notion of $C^\star$-algebra valued bipolar metric space and proved coupled fixed point theorems. The details on $C^\star$-algebra are available in [13–17].

An algebra $\mathbb{A}$, together with a conjugate linear involution map $\vartheta \longmapsto \vartheta^\star$, is called a $\star$-algebra if $(\vartheta\varpi)^\star = \varpi^\star\vartheta^\star$ and $(\vartheta^\star)^\star = \vartheta$ for all $\vartheta, \varpi \in \mathbb{A}$. Moreover, the pair $(\mathbb{A}, \star)$ is called a unital $\star$-algebra if $\mathbb{A}$ contains the identity element $1_{\mathbb{A}}$. By a Banach $\star$-algebra we mean a complete normed unital $\star$-algebra $(\mathbb{A}, \star)$ such that the norm on $\mathbb{A}$ is submultiplicative and satisfies $\left\|\vartheta^\star\right\| = \left\|\vartheta\right\|$ for all $\vartheta \in \mathbb{A}$. Further, if for all $\vartheta \in \mathbb{A}$, we have $\left\|\vartheta^\star\vartheta\right\| = \left\|\vartheta\right\|^2$ in a Banach $\star$-algebra $(\mathbb{A}, \star)$, then $\mathbb{A}$ is known as a $C^\star$-algebra. A positive element of $\mathbb{A}$ is an element $\vartheta \in \mathbb{A}$ such that $\vartheta = \vartheta^\star$ and its spectrum $\sigma(\vartheta) \subset \mathbb{R}_+$, where $\sigma(\vartheta) = \{\upsilon \in \mathbb{R} : \upsilon 1_{\mathbb{A}} - \vartheta \text{ is noninvertible}\}$. The set of all positive elements will be denoted by $\mathbb{A}_+$. Such elements allow us to define a partial ordering $\succeq$ on the elements of $\mathbb{A}$. That is,

$$\varpi \succeq \vartheta \text{ if and only if } \varpi - \vartheta \in \mathbb{A}_+.$$

If $\vartheta \in \mathbb{A}$ is positive, then we write $\vartheta \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of $\mathbb{A}$. Each positive element ϑ of a $C^\star$-algebra $\mathbb{A}$ has a unique positive square root. From now on, by $\mathbb{A}$ we mean a unital $C^\star$-algebra with identity element $1_{\mathbb{A}}$. Further, $\mathbb{A}_+ = \{\vartheta \in \mathbb{A} : \vartheta \succeq 0_{\mathbb{A}}\}$ and $(\vartheta^\star\vartheta)^{1/2} = \left|\vartheta\right|$.

2. Preliminaries

In this section, we extend Definition 1.1 to introduce the notion bipolar metric space in the setting of C^*-algebra as follows.

Definition 2.1. Let $\mathbb{A}$ be a C^*-algebra, and Γ, Ψ be two non-void sets. A mapping $\varphi : \Gamma \times \Psi \to \mathbb{A}_+$ be a function such that

(a) $\varphi(\sigma, \zeta) = 0$ iff $\sigma = \zeta$, for all $(\sigma, \zeta) \in \Gamma \times \Psi$.

(b) $\varphi(\sigma, \zeta) = \varphi(\zeta, \sigma)$, for all $\sigma, \zeta \in \Gamma \cap \Psi$.

(c) $\varphi(\sigma, \zeta) \leq \varphi(\sigma, \gamma) + \varphi(\sigma_1, \gamma) + \varphi(\sigma_1, \zeta)$, for all $\sigma, \sigma_1 \in \Gamma$ and $\gamma, \zeta \in \Psi$.

The 4-tuple $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is called a $C^\star$-algebra valued bipolar metric space.

Lemma 2.2 ([14, 17]). *Suppose that $\mathbb{A}$ is a unital $C^\star$-algebra with a unit I.*

(A1) For any $\sigma \in \mathbb{A}_+$, we have $\sigma \preceq I$ iff $\|\sigma\| \leq 1$.

(A2) If $\vartheta \in \mathbb{A}_+$ with $\|\vartheta\| < \frac{1}{2}$, then $(I - \vartheta)$ is invertible and $\|\vartheta(I - a)^{-1}\| < 1$.

(A3) *Suppose that* $\vartheta, \varpi \in \mathbb{A}$ *with* $\vartheta\varpi \succeq 0_{\mathbb{A}}$ *and* $\vartheta\varpi = \varpi\vartheta$, *then* $\varpi\vartheta \succeq 0_{\mathbb{A}}$.

(A4) *By* $\mathbb{A}'$ *we denote the set* $\{\vartheta \in \mathbb{A} : \vartheta\varpi = \varpi\vartheta, \forall \varpi \in \mathbb{A}\}$. *Let* $\vartheta \in \mathbb{A}'$, *if* $\varpi, \mathfrak{c} \in \mathbb{A}$ *with* $\varpi \succeq \mathfrak{c} \succeq 0_{\mathbb{A}}$, *and* $I - \vartheta \in \mathbb{A}'_+$ *is an invertible operator, then*

$$(I - \vartheta)^{-1}\varpi \succeq (I - \vartheta)^{-1}\mathfrak{c}.$$

(A5) *If* $\varpi, \mathfrak{c} \in \mathbb{A}_{\mathfrak{h}} = \{\sigma \in \mathbb{A} : \sigma = \sigma^\star\}$ *and* $\vartheta \in \mathbb{A}$, *then* $\varpi \preceq \mathfrak{c}$ *implies that* $\vartheta^\star\varpi\vartheta \preceq \vartheta^\star\mathfrak{c}\vartheta$.

(A6) $0_{\mathbb{A}} \preceq \vartheta \preceq \varpi$, *then* $\|\vartheta\| \leq \|\varpi\|$.

Notice that in a $C^\star$-algebra, if $0_{\mathbb{A}} \preceq \vartheta, \varpi$, one cannot conclude that $0_{\mathbb{A}} \preceq \vartheta\varpi$.

Definition 2.3. Let $(\Gamma_1, \Psi_1, \mathbb{A}, \varphi)$ and $(\Gamma_2, \Psi_2, \mathbb{A}, \varphi)$ be pairs of sets and given a function $\Phi : \Gamma_1 \cup \Psi_1 \to \Gamma_2 \cup \Psi_2$.

(B1) If $\Phi(\Gamma_1) \subseteq \Gamma_2$ and $\Phi(\Psi_1) \subseteq \Psi_2$, then Φ is called a covariant map, or a map from $(\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1)$ to $(\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2)$ and this is written as

$$\Phi : (\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1) \rightrightarrows (\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2).$$

(B2) If $\Phi(\Gamma_1) \subseteq \Psi_2$ and $\Phi(\Psi_1) \subseteq \Gamma_2$, then Φ is called a contravariant map from $(\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1)$ *to* $(\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2)$ and this is denoted as

$$\Phi : (\Gamma_1, \Psi_1, \mathbb{A}, \varphi_1) \leftrightarrows (\Gamma_2, \Psi_2, \mathbb{A}, \varphi_2).$$

Definition 2.4. Let $(\Gamma, \Psi, \mathbb{A}, \varphi)$ be a $C^\star$-algebra valued bipolar metric space.

(C1) A point $\sigma \in \Gamma \cup \Psi$ is said to be a left point if $\sigma \in \Gamma$, a right point if $\sigma \in \Psi$ and a central point if both hold. Similarly, a sequence$\{\sigma_\alpha\}$ on the set Γ and a sequence $\{\zeta_n\}$ on the set Ψ are called a left and right sequence with respect to $\mathbb{A}$, respectively.

(C2) A sequence $\{\sigma_\alpha\}$ converges to a point ζ with respect to $\mathbb{A}$ iff $\{\sigma_\alpha\}$ is a left sequence, ζ is a right point and $\lim_{\alpha\to\infty} \varphi(\sigma_\alpha, \zeta) = 0_{\mathbb{A}}$ or $\{\sigma_\alpha\}$ is a right sequence, ζ is a left point and $\lim_{\alpha\to\infty} \varphi(\zeta, \sigma_\alpha) = 0_{\mathbb{A}}$.

(C3) A bisequence $(\{\sigma_n\}, \{\zeta_n\})$ is a sequence on the set $\Gamma \times \Psi$. If the sequence $\{\sigma_n\}$ and $\{\zeta_n\}$ are convergent with respect to $\mathbb{A}$, then the bisequence $(\{\sigma_n\}, \{\zeta_n\})$ is said to be convergent with respect to $\mathbb{A}$. $(\{\sigma_n\}, \{\zeta_n\})$ is a Cauchy bisequence with respect to $\mathbb{A}$ if $\lim_{\alpha,\beta\to\infty} \varphi(\sigma_\alpha, \zeta_\beta) = 0_{\mathbb{A}}$, hence biconvergent with respect to $\mathbb{A}$.

(C4) $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is complete, if every Cauchy bisequence with respect to $\mathbb{A}$ is convergent in $\Gamma \times \Psi$.

3. Main results

Theorem 3.1. *Let* $(\Gamma, \Psi, \mathbb{A}, \varphi)$ *be a complete* $C^\star$*-algebra valued bipolar metric space. Suppose*

$$\Phi : (\Gamma^2, \Psi^2, \mathbb{A}, \varphi) \rightrightarrows (\Gamma, \Psi, \mathbb{A}, \varphi)$$

is a covariant mapping such that

$$\varphi(\Phi(\sigma,\zeta),\Phi(\mathfrak{u},\mathfrak{v})) \preceq \upsilon^{\star}\varphi(\sigma,\mathfrak{u})\upsilon + \upsilon^{\star}\varphi(\zeta,\mathfrak{v})\upsilon \text{ for all } \sigma,\zeta \in \Gamma, \mathfrak{u},\mathfrak{v} \in \Psi,$$

where $\upsilon \in \mathbb{A}$ *with* $2\|\upsilon\|^2 < 1$. *Then the function*

$$\Phi : \Gamma^2 \cup \Psi^2 \to \Gamma \cup \Psi$$

has a unique coupled fixed point.

Proof. Let $\sigma_0, \zeta_0 \in \Gamma$ and $\mathfrak{u}_0, \mathfrak{v}_0 \in \Psi$. For each $\alpha \in \mathbb{N}$, define

$$\Phi(\sigma_\alpha,\zeta_\alpha) = \sigma_{\alpha+1},$$

$$\Phi(\zeta_\alpha,\sigma_\alpha) = \zeta_{\alpha+1},$$

$$\Phi(\mathfrak{u}_\alpha,\mathfrak{v}_\alpha) = \mathfrak{u}_{\alpha+1}$$

and

$$\Phi(\mathfrak{v}_\alpha,\mathfrak{u}_\alpha) = \mathfrak{v}_{\alpha+1}.$$

Then $(\{\sigma_\alpha\},\{\zeta_\alpha\})$ and $(\{\mathfrak{u}_\alpha\},\{\mathfrak{v}_\alpha\})$ are bisequences on $(\Gamma,\Psi,\mathbb{A},\varphi)$. Then, for each $\alpha \in \mathbb{N}$,

$$\begin{aligned}\varphi(\sigma_\alpha,\mathfrak{u}_{\alpha+1}) &= \varphi(\Phi(\sigma_{\alpha-1},\zeta_{\alpha-1}),\Phi(\mathfrak{u}_\alpha,\mathfrak{v}_\alpha))\\ &\preceq \upsilon^{\star}\varphi(\sigma_{\alpha-1},\mathfrak{u}_\alpha)\upsilon + \upsilon^{\star}\varphi(\zeta_{\alpha-1},\mathfrak{v}_\alpha)\upsilon\\ &= \upsilon^{\star}\mathcal{M}_\alpha\upsilon,\end{aligned}$$

where

$$\mathcal{M}_\alpha = \varphi(\sigma_{\alpha-1},\mathfrak{u}_\alpha) + \varphi(\zeta_{\alpha-1},\mathfrak{v}_\alpha).$$

Similarly, we get

$$\begin{aligned}\varphi(\zeta_\alpha,\mathfrak{v}_{\alpha+1}) &= \varphi(\Phi(\zeta_{\alpha-1},\sigma_{\alpha-1}),\Phi(\mathfrak{v}_\alpha,\mathfrak{u}_\alpha))\\ &\preceq \upsilon^{\star}\varphi(\zeta_{\alpha-1},\mathfrak{v}_\alpha)\upsilon + \upsilon^{\star}\varphi(\sigma_{\alpha-1},\mathfrak{u}_\alpha)\upsilon\\ &= \upsilon^{\star}\mathcal{M}_\alpha\upsilon.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{M}_{\alpha+1} &= \varphi(\sigma_\alpha,\mathfrak{u}_{\alpha+1}) + \varphi(\zeta_\alpha,\mathfrak{v}_{\alpha+1})\\ &\preceq \upsilon^{\star}[\varphi(\sigma_{\alpha-1},\mathfrak{u}_\alpha) + \varphi(\zeta_{\alpha-1},\mathfrak{v}_\alpha)]\upsilon\\ &\quad + \upsilon^{\star}[\varphi(\zeta_{\alpha-1},\mathfrak{v}_\alpha) + \varphi(\sigma_{\alpha-1},\mathfrak{u}_\alpha)]\upsilon\\ &\preceq (\sqrt{2}\upsilon)^{\star}\mathcal{M}_\alpha(\sqrt{2}\upsilon).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \preceq \mathcal{M}_{\alpha+1} \preceq (\sqrt{2}\upsilon)^{\star}\mathcal{M}_\alpha(\sqrt{2}\upsilon) \preceq \cdots \preceq ((\sqrt{2}\upsilon)^{\star})^{\alpha}\mathcal{M}_1(\sqrt{2}\upsilon)^{\alpha}.$$

On the other hand,

$$\begin{aligned}\varphi(\sigma_{\alpha+1}, \mathfrak{u}_{\alpha}) &= \varphi(\Phi(\sigma_{\alpha}, \zeta_{\alpha}), \Phi(\mathfrak{u}_{\alpha-1}, \mathfrak{v}_{\alpha-1}))\\ &\preceq \upsilon^{\star}\varphi(\sigma_{\alpha}, \mathfrak{u}_{\alpha-1})\upsilon + \upsilon^{\star}\varphi(\zeta_{\alpha}, \mathfrak{v}_{\alpha-1})\upsilon\\ &= \upsilon^{\star}\mathcal{S}_{\alpha}\upsilon,\end{aligned}$$

where

$$\mathcal{S}_{\alpha} = \varphi(\sigma_{\alpha}, \mathfrak{u}_{\alpha-1}) + \varphi(\zeta_{\alpha}, \mathfrak{v}_{\alpha-1}).$$

Similarly, we get

$$\begin{aligned}\varphi(\zeta_{\alpha+1}, \mathfrak{v}_{\alpha}) &= \varphi(\Phi(\zeta_{\alpha}, \sigma_{\alpha}), \Phi(\mathfrak{v}_{\alpha-1}, \mathfrak{u}_{\alpha-1}))\\ &\preceq \upsilon^{\star}\varphi(\zeta_{\alpha}, \mathfrak{v}_{\alpha-1})\upsilon + \upsilon^{\star}\varphi(\sigma_{\alpha}, \mathfrak{u}_{\alpha-1})\upsilon\\ &= \upsilon^{\star}\mathcal{S}_{\alpha}\upsilon.\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{S}_{\alpha+1} &= \varphi(\sigma_{\alpha+1}, \mathfrak{u}_{\alpha}) + \varphi(\zeta_{\alpha+1}, \mathfrak{v}_{\alpha})\\ &\preceq \upsilon^{\star}[\varphi(\sigma_{\alpha}, \mathfrak{u}_{\alpha-1}) + \varphi(\zeta_{\alpha}, \mathfrak{v}_{\alpha-1})]\upsilon\\ &\quad + \upsilon^{\star}[\varphi(\zeta_{\alpha}, \mathfrak{v}_{\alpha-1}) + \varphi(\sigma_{\alpha}, \mathfrak{u}_{\alpha-1})]\upsilon\\ &\preceq (\sqrt{2}\upsilon)^{\star}\mathcal{S}_{\alpha}(\sqrt{2}\upsilon).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \preceq \mathcal{S}_{\alpha+1} \preceq (\sqrt{2}\upsilon)^{\star}\mathcal{S}_{\alpha}(\sqrt{2}\upsilon) \preceq \cdots \preceq ((\sqrt{2}\upsilon)^{\star})^{\alpha}\mathcal{S}_{1}(\sqrt{2}\upsilon)^{\alpha}.$$

Moreover,

$$\begin{aligned}\varphi(\sigma_{\alpha}, \mathfrak{u}_{\alpha}) &= \varphi(\Phi(\sigma_{\alpha-1}, \zeta_{\alpha-1}), \Phi(\mathfrak{u}_{\alpha-1}, \mathfrak{v}_{\alpha-1}))\\ &\preceq \upsilon^{\star}\varphi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1})\upsilon + \upsilon^{\star}\varphi(\zeta_{\alpha-1}, \mathfrak{v}_{\alpha-1})\upsilon\\ &= \upsilon^{\star}\mathcal{R}_{\alpha}\upsilon,\end{aligned}$$

where

$$\mathcal{R}_{\alpha} = \varphi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, \mathfrak{v}_{\alpha-1}).$$

Similarly, we get

$$\begin{aligned}\varphi(\zeta_{\alpha}, \mathfrak{v}_{\alpha}) &= \varphi(\Phi(\zeta_{\alpha-1}, \sigma_{\alpha-1}), \Phi(\mathfrak{v}_{\alpha-1}, \mathfrak{u}_{\alpha-1}))\\ &\preceq \upsilon^{\star}\varphi(\zeta_{\alpha-1}, \mathfrak{v}_{\alpha-1})\upsilon + \upsilon^{\star}\varphi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1})\upsilon\\ &= \upsilon^{\star}\mathcal{R}_{\alpha}\upsilon.\end{aligned}$$

Now,

$$
\begin{aligned}
\mathcal{R}_{\alpha+1} &= \varphi(\sigma_\alpha, \mathfrak{u}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{v}_\alpha) \\
&\preceq \upsilon^\star[\varphi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, \mathfrak{v}_{\alpha-1})]\upsilon \\
&\quad + \upsilon^\star[\varphi(\zeta_{\alpha-1}, \mathfrak{v}_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1})]\upsilon \\
&\preceq (\sqrt{2}\upsilon)^\star \mathcal{R}_\alpha (\sqrt{2}\upsilon).
\end{aligned}
$$

By Lemma 2.2 (A5), we have

$$
0_{\mathbb{A}} \preceq \mathcal{R}_{\alpha+1} \preceq (\sqrt{2}\upsilon)^\star \mathcal{R}_\alpha (\sqrt{2}\upsilon) \preceq \cdots \preceq ((\sqrt{2}\upsilon)^\star)^\alpha \mathcal{R}_1 (\sqrt{2}\upsilon)^\alpha.
$$

Now,

$$
\begin{aligned}
\varphi(\sigma_\alpha, \mathfrak{u}_\beta) &\preceq \varphi(\sigma_\alpha, \mathfrak{u}_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, \mathfrak{u}_{\alpha+1}) + \cdots + \varphi(\sigma_{\beta-1}, \mathfrak{u}_\beta), \\
\varphi(\zeta_\alpha, \mathfrak{v}_\beta) &\preceq \varphi(\zeta_\alpha, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \cdots + \varphi(\zeta_{\beta-1}, \mathfrak{v}_\beta),
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi(\sigma_\beta, \mathfrak{u}_\alpha) &\preceq \varphi(\sigma_\beta, \mathfrak{u}_{\beta-1}) + \varphi(\sigma_{\beta-1}, \mathfrak{u}_{\beta-1}) + \cdots + \varphi(\sigma_{\alpha+1}, \mathfrak{u}_\alpha), \\
\varphi(\zeta_\beta, \mathfrak{v}_\alpha) &\preceq \varphi(\zeta_\beta, \mathfrak{v}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{v}_{\beta-1}) + \cdots + \varphi(\zeta_{\alpha+1}, \mathfrak{v}_\alpha),
\end{aligned}
$$

for each $\alpha, \beta \in \mathbb{N}$, $\alpha < \beta$. Then,

$$
\begin{aligned}
\varphi(\sigma_\alpha, \mathfrak{u}_\beta) + \varphi(\zeta_\alpha, \mathfrak{v}_\beta) &\preceq (\varphi(\sigma_\alpha, \mathfrak{u}_{\alpha+1}) + \varphi(\zeta_\alpha, \mathfrak{v}_{\alpha+1})) + (\varphi(\sigma_{\alpha+1}, \mathfrak{u}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{v}_{\alpha+1})) \\
&\quad + \cdots + (\varphi(\sigma_{\beta-1}, \mathfrak{u}_\beta) + \varphi(\zeta_{\beta-1}, \mathfrak{v}_\beta)) \\
&= \mathcal{M}_{\alpha+1} + \mathcal{R}_{\alpha+2} + \mathcal{M}_{\alpha+2} + \cdots + \mathcal{R}_\beta + \mathcal{M}_\beta \\
&\preceq ((\sqrt{2}\upsilon)^\star)^\alpha \mathcal{M}_1 (\sqrt{2}\upsilon)^\alpha + ((\sqrt{2}\upsilon)^\star)^{\alpha+1} \mathcal{R}_1 (\sqrt{2}\upsilon)^{\alpha+1} + \cdots \\
&\quad + ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{R}_1 (\sqrt{2}\upsilon)^{\beta-1} + ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{M}_1 (\sqrt{2}\upsilon)^{\beta-1} \\
&= \sum_{\mathrm{i}=\alpha}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathrm{i}} \mathcal{M}_1 (\sqrt{2}\upsilon)^{\mathrm{i}} + \sum_{\mathrm{i}=\alpha+1}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathrm{i}} \mathcal{R}_1 (\sqrt{2}\upsilon)^{\mathrm{i}} \\
&= \sum_{\mathrm{i}=\alpha}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathrm{i}} \mathcal{M}_1^{\frac{1}{2}} \mathcal{M}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}} + \sum_{\mathrm{i}=\alpha+1}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathrm{i}} \mathcal{R}_1^{\frac{1}{2}} \mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}} \\
&= \sum_{\mathrm{i}=\alpha}^{\beta-1} (\mathcal{M}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}})^\star (\mathcal{M}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}}) + \sum_{\mathrm{i}=\alpha+1}^{\beta-1} (\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}})^\star (\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}}) \\
&= \sum_{\mathrm{i}=\alpha}^{\beta-1} |\mathcal{M}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}}|^2 + \sum_{\mathrm{i}=\alpha+1}^{\beta-1} |\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}}|^2 \\
&\preceq \| \sum_{\mathrm{i}=\alpha}^{\beta-1} |\mathcal{M}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}}|^2 \| 1_{\mathbb{A}} + \| \sum_{\mathrm{i}=\alpha+1}^{\beta-1} |\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathrm{i}}|^2 \| 1_{\mathbb{A}} \\
&\preceq \sum_{\mathrm{i}=\alpha}^{\beta-1} \|\mathcal{M}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^{\mathrm{i}}\|^2 \| 1_{\mathbb{A}} + \| \sum_{\mathrm{i}=\alpha+1}^{\beta-1} \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^{\mathrm{i}}\|^2 \| 1_{\mathbb{A}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\mathcal{M}_1^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\alpha}^{\beta-1} \|(\sqrt{2}\upsilon)^2\|^{\mathfrak{i}} \|1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \| \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} \|(\sqrt{2}\upsilon)^2\|^{\mathfrak{i}} \|1_{\mathbb{A}} \\
&= \|\mathcal{M}_1^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\alpha}^{\beta-1} (2\|\upsilon\|^2)^{\mathfrak{i}} 1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} (2\|\upsilon\|^2)^{\mathfrak{i}} \|1_{\mathbb{A}} \\
&\to 0_{\mathbb{A}} \ \ (\text{as} \beta, \alpha \to \infty)
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi(\sigma_\beta, \mathfrak{u}_\alpha) + \varphi(\zeta_\beta, \mathfrak{v}_\alpha) &\le (\varphi(\sigma_\beta, \mathfrak{u}_{\beta-1}) + \varphi(\zeta_\beta, \mathfrak{v}_{\beta-1})) + (\varphi(\sigma_{\beta-1}, \mathfrak{u}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{v}_{\beta-1})) \\
&\quad + \cdots + (\varphi(\sigma_{\alpha+1}, \mathfrak{u}_\alpha) + \varphi(\zeta_{\alpha+1}, \mathfrak{v}_\alpha)) \\
&= \mathcal{S}_\beta + \mathcal{R}_\beta + \mathcal{S}_{\beta-1} + \cdots + \mathcal{R}_{\alpha+2} + \mathcal{S}_{\alpha+1} \\
&\le ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{S}_1 (\sqrt{2}\upsilon)^{\beta-1} + ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{R}_1 (\sqrt{2}\upsilon)^{\beta-1} + \cdots \\
&\quad + ((\sqrt{2}\upsilon)^\star)^{\alpha+1} \mathcal{R}_1 (\sqrt{2}\upsilon)^{\alpha+1} + ((\sqrt{2}\upsilon)^\star)^{\alpha} \mathcal{S}_1 (\sqrt{2}\upsilon)^{\alpha} \\
&= \sum_{\mathfrak{i}=\beta}^{\alpha+1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{S}_1 (\sqrt{2}\upsilon)^{\mathfrak{i}} + \sum_{\mathfrak{i}=\beta}^{\alpha+2} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{R}_1 (\sqrt{2}\upsilon)^{\mathfrak{i}} \\
&= \sum_{\mathfrak{i}=\beta}^{\alpha+1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{S}_1^{\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}} + \sum_{\mathfrak{i}=\beta}^{\alpha+2} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{R}_1^{\frac{1}{2}} \mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}} \\
&= \sum_{\mathfrak{i}=\beta}^{\alpha+1} (\mathcal{S}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}})^\star (\mathcal{S}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}) + \sum_{\mathfrak{i}=\beta}^{\alpha+2} (\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}})^\star (\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}) \\
&= \sum_{\mathfrak{i}=\beta}^{\alpha+1} |\mathcal{S}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 + \sum_{\mathfrak{i}=\beta}^{\alpha+2} |\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 \\
&\le \| \sum_{\mathfrak{i}=\beta}^{\alpha+1} |\mathcal{S}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 \| 1_{\mathbb{A}} + \| \sum_{\mathfrak{i}=\beta}^{\alpha+2} |\mathcal{R}_1^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 \| 1_{\mathbb{A}} \\
&\le \sum_{\mathfrak{i}=\beta}^{\alpha+1} \|\mathcal{S}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^{\mathfrak{i}}\|^2 \| 1_{\mathbb{A}} + \| \sum_{\mathfrak{i}=\beta}^{\alpha+2} \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^{\mathfrak{i}}\|^2 \| 1_{\mathbb{A}} \\
&\le \|\mathcal{S}_1^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\beta}^{\alpha+1} \|(\sqrt{2}\upsilon)^2\|^{\mathfrak{i}} \| 1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \| \sum_{\mathfrak{i}=\beta}^{\alpha+2} \|(\sqrt{2}\upsilon)^2\|^{\mathfrak{i}} \| 1_{\mathbb{A}} \\
&= \|\mathcal{S}_1^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\beta}^{\alpha+1} (2\|\upsilon\|^2)^{\mathfrak{i}} 1_{\mathbb{A}} + \|\mathcal{R}_1^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\beta}^{\alpha+2} (2\|\upsilon\|^2)^{\mathfrak{i}} \| 1_{\mathbb{A}} \\
&\to 0_{\mathbb{A}} \ \ (\text{as} \beta, \alpha \to \infty).
\end{aligned}
$$

Therefore, $(\{\sigma_\alpha\}, \{\mathfrak{u}_\alpha\})$ and $(\{\zeta_\alpha\}, \{\mathfrak{v}_\alpha\})$ are Cauchy bisequences in $\Gamma \times \Psi$ with respect to $\mathbb{A}$. By completeness of $(\Gamma, \Psi, \mathbb{A}, \varphi)$, there exist $\sigma, \zeta \in \Gamma$ and $\mathfrak{u}, \mathfrak{v} \in \Psi$ with

$$
\lim_{\alpha\to\infty} \sigma_\alpha = \mathfrak{u}, \ \lim_{\alpha\to\infty} \zeta_\alpha = \mathfrak{v}, \ \lim_{\alpha\to\infty} \mathfrak{u}_\alpha = \sigma \text{ and } \lim_{\alpha\to\infty} \mathfrak{v}_\alpha = \zeta.
$$

Since $(\{\sigma_\alpha\},\{\mathfrak{u}_\alpha\})$ and $(\{\zeta_\alpha\},\{\mathfrak{v}_\alpha\})$ are Cauchy bisequences, we derive that

$$\varphi(\sigma_\alpha,\mathfrak{u}_\alpha)\prec\frac{\epsilon}{2}\text{ and }\varphi(\zeta_\alpha,\mathfrak{v}_\alpha)\prec\frac{\epsilon}{2}.$$

Then,

$$\begin{aligned}\varphi(\Phi(\sigma,\zeta),\mathfrak{u})&\preceq\varphi(\Phi(\sigma,\zeta),\mathfrak{u}_{\alpha+1})+\varphi(\sigma_{\alpha+1},\mathfrak{u}_{\alpha+1})+\varphi(\sigma_{\alpha+1},\mathfrak{u}_\alpha)\\&=\varphi(\Phi(\sigma,\zeta),\Phi(\mathfrak{u}_\alpha,\mathfrak{v}_\alpha))+\varphi(\sigma_{\alpha+1},\mathfrak{u}_{\alpha+1})+\varphi(\sigma_{\alpha+1},\mathfrak{u}_\alpha)\\&\preceq\upsilon^\star\varphi(\sigma,\mathfrak{u}_\alpha)\upsilon+\upsilon^\star\varphi(\zeta,\mathfrak{v}_\alpha)\upsilon+\varphi(\sigma_{\alpha+1},\mathfrak{u}_{\alpha+1})+\varphi(\sigma_{\alpha+1},\mathfrak{u}_\alpha).\end{aligned}$$

As $\alpha\to\infty$, we have

$$\varphi(\Phi(\sigma,\zeta),\mathfrak{u})\prec\epsilon.$$

Then,

$$\varphi(\Phi(\sigma,\zeta),\mathfrak{u})=0.$$

Hence, $\Phi(\sigma,\zeta)=\mathfrak{u}$. Similarly, we can derive $\Phi(\zeta,\sigma)=\mathfrak{v}$, $\Phi(\mathfrak{u},\mathfrak{v})=\sigma$ and $\Phi(\mathfrak{v},\mathfrak{u})=\zeta$. On the other hand, we derive that

$$\varphi(\sigma,\mathfrak{u})=\varphi(\lim_{\alpha\to\infty}\mathfrak{u}_\alpha,\lim_{\alpha\to\infty}\sigma_\alpha)=\lim_{\alpha\to\infty}\varphi(\sigma_\alpha,\mathfrak{u}_\alpha)=0$$

and

$$\varphi(\zeta,\mathfrak{v})=\varphi(\lim_{\alpha\to\infty}\mathfrak{v}_\alpha,\lim_{\alpha\to\infty}\zeta_\alpha)=\lim_{\alpha\to\infty}\varphi(\zeta_\alpha,\mathfrak{v}_\alpha)=0.$$

So, $\sigma=\mathfrak{u}$ and $\zeta=\mathfrak{v}$. Therefore, $(\sigma,\zeta)\in\Gamma^2\cap\Psi^2$ is a coupled fixed point of Φ. Let $(\mathfrak{e},\mathfrak{f})\in\Gamma^2\cup\Psi^2$ is a another coupled fixed point Φ. If $(\mathfrak{e},\mathfrak{f})\in\Gamma^2$, then

$$0_{\mathbb{A}}\preceq\varphi(\mathfrak{e},\sigma)=\varphi(\Phi(\mathfrak{e},\mathfrak{f}),\Phi(\sigma,\zeta))\preceq\upsilon^\star\varphi(\mathfrak{e},\sigma)\upsilon+\upsilon^\star\varphi(\mathfrak{f},\zeta)\upsilon$$

and

$$0_{\mathbb{A}}\preceq\varphi(\mathfrak{f},\zeta)=\varphi(\Phi(\mathfrak{f},\mathfrak{e}),\Phi(\zeta,\sigma))\preceq\upsilon^\star\varphi(\mathfrak{f},\zeta)\upsilon+\upsilon^\star\varphi(\mathfrak{e},\sigma)\upsilon,$$

which implies that

$$0_{\mathbb{A}}\preceq\varphi(\mathfrak{e},\sigma)+\varphi(\mathfrak{f},\zeta)\preceq(\sqrt{2}\upsilon)^\star(\varphi(\mathfrak{f},\zeta)+\varphi(\mathfrak{e},\sigma))(\sqrt{2}\upsilon).$$

Then,

$$0\le\|\varphi(\mathfrak{e},\sigma)+\varphi(\mathfrak{f},\zeta)\|\le\|\sqrt{2}\upsilon\|^2\|\varphi(\mathfrak{f},\zeta)+\varphi(\mathfrak{e},\sigma)\|.$$

Since $2\|\upsilon\|^2<1$, we derive that

$$\varphi(\mathfrak{f},\zeta)+\varphi(\mathfrak{e},\sigma)=0.$$

Therefore, $\mathfrak{e}=\sigma$ and $\mathfrak{f}=\zeta$. Similarly, if $(\mathfrak{e},\mathfrak{f})\in\Psi^2$, then $\mathfrak{e}=\sigma$ and $\mathfrak{f}=\zeta$. Then (σ,ζ) is a unique coupled fixed point of Φ. □

Example 3.2. Let $\Gamma = [0,1]$, $\Psi = \{0\} \cup \mathbb{N} - \{1\}$, $\mathbb{A}_+ = \mathcal{M}_2(\mathbb{C})$ and the map $\varphi : \Gamma \times \Psi \to \mathbb{A}_+$ is defined by

$$\varphi(\sigma, \mathfrak{u}) = \begin{bmatrix} |\sigma - \mathfrak{u}| & 0 \\ 0 & \Bbbk|\sigma - \mathfrak{u}| \end{bmatrix}$$

for all $\sigma \in \Gamma$ and $\mathfrak{u} \in \Psi$, where $\Bbbk \geq 0$ is a constant. Let $\preceq$ be the partial order on $\mathbb{A}$ given by

$$(\vartheta_1, \varpi_1) \preceq (\vartheta_2, \varpi_2) \Leftrightarrow \vartheta_1 \leq \vartheta_2 \text{ and } \varpi_1 \leq \varpi_2.$$

Then $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is a complete $C^\star$-algebra-valued bipolar metric space. Define

$$\Phi : \Gamma^2 \cup \Psi^2 \rightrightarrows \Gamma \cup \Psi$$

by

$$\Phi(\sigma, \zeta) = \frac{\sigma + \zeta}{3},$$

$\forall\ \sigma, \zeta \in \Gamma^2 \cup \Psi^2$. Then

$$\begin{aligned}
\varphi(\Phi(\sigma,\zeta), \Phi(\mathfrak{u},\mathfrak{v})) &= \begin{bmatrix} |\Phi(\sigma,\zeta) - \Phi(\mathfrak{u},\mathfrak{v})| & 0 \\ 0 & \Bbbk|\Phi(\sigma,\zeta) - \Phi(\mathfrak{u},\mathfrak{v})| \end{bmatrix} \\
&= \begin{bmatrix} |\frac{\sigma+\zeta}{3} - \frac{\mathfrak{u}+\mathfrak{v}}{3}| & 0 \\ 0 & \Bbbk|\frac{\sigma+\zeta}{3} - \frac{\mathfrak{u}+\mathfrak{v}}{3}| \end{bmatrix} \\
&\preceq \frac{1}{3}\left(\begin{bmatrix} |\sigma - \mathfrak{u}| & 0 \\ 0 & \Bbbk|\sigma - \mathfrak{u}| \end{bmatrix} + \begin{bmatrix} |\zeta - \mathfrak{v}| & 0 \\ 0 & \Bbbk|\zeta - \mathfrak{v}| \end{bmatrix} \right) \\
&= \upsilon^\star \varphi(\sigma, \mathfrak{u})\upsilon + \upsilon^\star \varphi(\zeta, \mathfrak{v})\upsilon,
\end{aligned}$$

for all $\sigma, \zeta \in \Gamma, \mathfrak{u}, \mathfrak{v} \in \Psi$, where

$$\upsilon = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and $\|\upsilon\| = \frac{1}{3} < \frac{1}{\sqrt{2}}$. All the conditions of Theorem 3.1 are fulfilled and Φ has a unique fixed point $(0, 0)$.

Theorem 3.3. *Let $(\Gamma, \Psi, \mathbb{A}, \varphi)$ be a complete $C^\star$-algebra valued bipolar metric space. Suppose*

$$\Phi : (\Gamma \times \Psi, \Psi \times \Gamma, \mathbb{A}, \varphi) \rightrightarrows (\Gamma, \Psi, \mathbb{A}, \varphi)$$

is a covariant mapping such that

$$\varphi(\Phi(\sigma, \mathfrak{u}), \Phi(\mathfrak{v}, \zeta)) \preceq \upsilon^\star \varphi(\sigma, \mathfrak{v})\upsilon + \upsilon^\star \varphi(\zeta, \mathfrak{u})\upsilon \text{ for all } \sigma, \zeta \in \Gamma, \mathfrak{u}, \mathfrak{v} \in \Psi,$$

where $\upsilon \in \mathbb{A}$ with $2\|\upsilon\|^2 < 1$. Then the function

$$\Phi : (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \to \Gamma \cup \Psi$$

has a unique coupled fixed point.

Proof. Let $\sigma_0, \zeta_0 \in \varGamma$ and $\mathfrak{u}_0, \mathfrak{v}_0 \in \varPsi$. For each $\alpha \in \mathbb{N}$, define $\varPhi(\sigma_\alpha, \mathfrak{u}_\alpha) = \sigma_{\alpha+1}$, $\varPhi(\zeta_\alpha, \mathfrak{v}_\alpha) = \zeta_{\alpha+1}$, $\varPhi(\mathfrak{u}_\alpha, \sigma_\alpha) = \mathfrak{u}_{\alpha+1}$ and $\varPhi(\mathfrak{v}_\alpha, \zeta_\alpha) = \mathfrak{v}_{\alpha+1}$. Then $(\{\sigma_\alpha\}, \{\mathfrak{u}_\alpha\})$ and $(\{\zeta_\alpha\}, \{\mathfrak{v}_\alpha\})$ are bisequences on $(\varGamma, \varPsi, \mathbb{A}, \varphi)$. Then, for each $\alpha \in \mathbb{N}$,

$$\begin{aligned}\varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) &= \varphi(\varPhi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1}), \varPhi(\mathfrak{v}_\alpha, \zeta_\alpha)) \\ &\leq \upsilon^\star \varphi(\sigma_{\alpha-1}, \mathfrak{v}_\alpha)\upsilon + \upsilon^\star \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha-1})\upsilon,\end{aligned}$$

$$\begin{aligned}\varphi(\sigma_{\alpha+1}, \mathfrak{v}_\alpha) &= \varphi(\varPhi(\sigma_\alpha, \mathfrak{u}_\alpha), \varPhi(\mathfrak{v}_{\alpha-1}, \zeta_{\alpha-1})) \\ &\leq \upsilon^\star \varphi(\sigma_\alpha, \mathfrak{v}_{\alpha-1})\upsilon + \upsilon^\star \varphi(\zeta_{\alpha-1}, \mathfrak{u}_\alpha)\upsilon,\end{aligned}$$

$$\begin{aligned}\varphi(\zeta_\alpha, \mathfrak{u}_{\alpha+1}) &= \varphi(\varPhi(\zeta_{\alpha-1}, \mathfrak{v}_{\alpha-1}), \varPhi(\mathfrak{u}_\alpha, \sigma_\alpha)) \\ &\leq \upsilon^\star \varphi(\zeta_{\alpha-1}, \mathfrak{u}_\alpha)\upsilon + \upsilon^\star \varphi(\sigma_\alpha, \mathfrak{v}_{\alpha-1})\upsilon,\end{aligned}$$

$$\begin{aligned}\varphi(\zeta_{\alpha+1}, \mathfrak{u}_\alpha) &= \varphi(\varPhi(\zeta_\alpha, \mathfrak{v}_\alpha), \varPhi(\mathfrak{u}_{\alpha-1}, \sigma_{\alpha-1})) \\ &\leq \upsilon^\star \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha-1})\upsilon + \upsilon^\star \varphi(\sigma_{\alpha-1}, \mathfrak{v}_\alpha)\upsilon.\end{aligned}$$

Let

$$\mathcal{M}_\alpha = \varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_\alpha),$$

for all $\alpha \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{M}_\alpha &= \varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_\alpha) \\ &\leq \upsilon^\star[\varphi(\sigma_{\alpha-1}, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha-1})]\upsilon \\ &\quad + \upsilon^\star[\varphi(\zeta_\alpha, \mathfrak{u}_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, \mathfrak{v}_\alpha)]\upsilon \\ &\leq (\sqrt{2}\upsilon)^\star \mathcal{M}_{\alpha-1}(\sqrt{2}\upsilon).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \leq \mathcal{M}_\alpha \leq (\sqrt{2}\upsilon)^\star \mathcal{M}_{\alpha-1}(\sqrt{2}\upsilon) \leq \cdots \leq ((\sqrt{2}\upsilon)^\star)^\alpha \mathcal{M}_0(\sqrt{2}\upsilon)^\alpha.$$

Let

$$\mathcal{S}_\alpha = \varphi(\sigma_{\alpha+1}, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha+1})$$

for all $\alpha \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{S}_\alpha &= \varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha-1}) + \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha+1}) \\ &\leq \upsilon^\star[\varphi(\sigma_\alpha, \mathfrak{v}_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, \mathfrak{u}_\alpha)]\upsilon \\ &\quad + \upsilon^\star[\varphi(\zeta_{\alpha-1}, \mathfrak{u}_\alpha) + \varphi(\sigma_\alpha, \mathfrak{v}_{\alpha-1})]\upsilon \\ &\leq (\sqrt{2}\upsilon)^\star \mathcal{S}_{\alpha-1}(\sqrt{2}\upsilon).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \preceq \mathcal{S}_\alpha \preceq (\sqrt{2}\upsilon)^\star \mathcal{S}_{\alpha-1}(\sqrt{2}\upsilon) \preceq \cdots \preceq ((\sqrt{2}\upsilon)^\star)^\alpha \mathcal{S}_0(\sqrt{2}\upsilon)^\alpha.$$

On the other hand,

$$\begin{aligned}\varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) &= \varphi(\Phi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1}), \Phi(\mathfrak{v}_\alpha, \zeta_\alpha)) \\ &\preceq \upsilon^\star \varphi(\sigma_{\alpha-1}, \mathfrak{v}_\alpha)\upsilon + \upsilon^\star \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha-1})\upsilon.\end{aligned}$$

$$\begin{aligned}\varphi(\sigma_\alpha, \mathfrak{v}_\alpha) &= \varphi(\Phi(\sigma_{\alpha-1}, \mathfrak{u}_{\alpha-1}), \Phi(\mathfrak{v}_{\alpha-1}, \zeta_{\alpha-1})) \\ &\preceq \upsilon^\star \varphi(\sigma_{\alpha-1}, \mathfrak{v}_{\alpha-1})\upsilon + \upsilon^\star \varphi(\zeta_{\alpha-1}, \mathfrak{u}_{\alpha-1})\upsilon\end{aligned}$$

and

$$\begin{aligned}\varphi(\zeta_\alpha, \mathfrak{u}_\alpha) &= \varphi(\Phi(\zeta_{\alpha-1}, \mathfrak{u}_{\alpha-1}), \Phi(\mathfrak{u}_{\alpha-1}, \sigma_{\alpha-1})) \\ &\preceq \upsilon^\star \varphi(\zeta_{\alpha-1}, \mathfrak{u}_{\alpha-1})\upsilon + \upsilon^\star \varphi(\sigma_{\alpha-1}, \mathfrak{v}_{\alpha-1})\upsilon\end{aligned}$$

for all $\alpha \in \mathbb{N}$. Let

$$\mathcal{R}_\alpha = \varphi(\sigma_\alpha, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{u}_\alpha),$$

for all $\alpha \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{R}_\alpha &= \varphi(\sigma_\alpha, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{u}_\alpha) \\ &\preceq \upsilon^\star[\varphi(\sigma_{\alpha-1}, \mathfrak{v}_{\alpha-1}) + \varphi(\zeta_{\alpha-1}, \mathfrak{u}_{\alpha-1})]\upsilon \\ &\quad + \upsilon^\star[\varphi(\zeta_{\alpha-1}, \mathfrak{u}_{\alpha-1}) + \varphi(\sigma_{\alpha-1}, \mathfrak{v}_{\alpha-1})]\upsilon \\ &\preceq (\sqrt{2}\upsilon)^\star \mathcal{R}_{\alpha-1}(\sqrt{2}\upsilon).\end{aligned}$$

By Lemma 2.2 (A5), we have

$$0_{\mathbb{A}} \preceq \mathcal{R}_\alpha \preceq (\sqrt{2}\upsilon)^\star \mathcal{R}_{\alpha-1}(\sqrt{2}\upsilon) \preceq \cdots \preceq ((\sqrt{2}\upsilon)^\star)^\alpha \mathcal{R}_0(\sqrt{2}\upsilon)^\alpha.$$

Now,

$$\begin{aligned}\varphi(\sigma_\alpha, \mathfrak{v}_\beta) &\preceq \varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \cdots + \varphi(\sigma_{\beta-1}, \mathfrak{v}_\beta), \\ \varphi(\zeta_\alpha, \mathfrak{u}_\beta) &\preceq \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_{\alpha+1}) + \cdots + \varphi(\zeta_{\beta-1}, \mathfrak{u}_\beta),\end{aligned}$$

and

$$\begin{aligned}\varphi(\sigma_\beta, \mathfrak{v}_\alpha) &\preceq \varphi(\sigma_\beta, \mathfrak{v}_{\beta-1}) + \varphi(\sigma_{\beta-1}, \mathfrak{v}_{\beta-1}) + \cdots + \varphi(\sigma_{\alpha+1}, \mathfrak{v}_\alpha), \\ \varphi(\zeta_\beta, \mathfrak{u}_\alpha) &\preceq \varphi(\zeta_\beta, \mathfrak{u}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{u}_{\beta-1}) + \cdots + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_\alpha),\end{aligned}$$

for each $\alpha, \beta \in \mathbb{N}$, $\alpha < \beta$. Then,

$$
\begin{aligned}
\varphi(\sigma_\alpha, \mathfrak{v}_\beta) + \varphi(\zeta_\beta, \mathfrak{u}_\alpha) &\preceq (\varphi(\sigma_\alpha, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_\alpha)) + (\varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \varphi(\zeta_{\alpha+1}, \mathfrak{u}_{\alpha+1})) \\
&\quad + \cdots + (\varphi(\sigma_{\beta-1}, \mathfrak{v}_\beta) + \varphi(\zeta_\beta, \mathfrak{u}_{\beta-1})) \\
&= \mathcal{M}_\alpha + \mathcal{R}_{\alpha+1} + \mathcal{M}_{\alpha+1} + \cdots + \mathcal{R}_{\beta-1} + \mathcal{M}_{\beta-1} \\
&\preceq ((\sqrt{2}\upsilon)^\star)^\alpha \mathcal{M}_0 (\sqrt{2}\upsilon)^\alpha + ((\sqrt{2}\upsilon)^\star)^{\alpha+1} \mathcal{R}_0 (\sqrt{2}\upsilon)^{\alpha+1} + \cdots \\
&\quad + ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{R}_0 (\sqrt{2}\upsilon)^{\beta-1} + ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{M}_0 (\sqrt{2}\upsilon)^{\beta-1} \\
&= \sum_{\mathfrak{i}=\alpha}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{M}_0 (\sqrt{2}\upsilon)^{\mathfrak{i}} + \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{R}_0 (\sqrt{2}\upsilon)^{\mathfrak{i}} \\
&= \sum_{\mathfrak{i}=\alpha}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{M}_0^{\frac{1}{2}} \mathcal{M}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}} + \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{R}_0^{\frac{1}{2}} \mathcal{R}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}} \\
&= \sum_{\mathfrak{i}=\alpha}^{\beta-1} (\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}})^\star (\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}) + \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} (\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}})^\star (\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}) \\
&= \sum_{\mathfrak{i}=\alpha}^{\beta-1} |\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 + \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 \\
&\preceq \| \sum_{\mathfrak{i}=\alpha}^{\beta-1} |\mathcal{M}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 \| 1_{\mathbb{A}} + \| \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}|^2 \| 1_{\mathbb{A}} \\
&\preceq \sum_{\mathfrak{i}=\alpha}^{\beta-1} \|\mathcal{M}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^{\mathfrak{i}}\|^2 \| 1_{\mathbb{A}} + \| \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^{\mathfrak{i}}\|^2 \| 1_{\mathbb{A}} \\
&\preceq \|\mathcal{M}_0^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\alpha}^{\beta-1} \|(\sqrt{2}\upsilon)^2\|^{\mathfrak{i}} \| 1_{\mathbb{A}} + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \| \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} \|(\sqrt{2}\upsilon)^2\|^{\mathfrak{i}} \| 1_{\mathbb{A}} \\
&= \|\mathcal{M}_0^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\alpha}^{\beta-1} (2\|\upsilon\|^2)^{\mathfrak{i}} 1_{\mathbb{A}} + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} (2\|\upsilon\|^2)^{\mathfrak{i}} \| 1_{\mathbb{A}} \\
&\to 0_{\mathbb{A}} \ \ (\text{as}\, \beta, \alpha \to \infty)
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi(\sigma_\beta, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{v}_\beta) &\preceq (\varphi(\sigma_\beta, \mathfrak{v}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{u}_\beta)) + (\varphi(\sigma_{\beta-1}, \mathfrak{v}_{\beta-1}) + \varphi(\zeta_{\beta-1}, \mathfrak{u}_{\beta-1})) \\
&\quad + \cdots + (\varphi(\sigma_{\alpha+1}, \mathfrak{v}_\alpha) + \varphi(\zeta_\alpha, \mathfrak{u}_{\alpha+1})) \\
&= \mathcal{S}_{\beta-1} + \mathcal{R}_{\beta-1} + \mathcal{S}_{\beta-1} + \cdots + \mathcal{R}_{\alpha+1} + \mathcal{S}_\alpha \\
&\preceq ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{S}_0 (\sqrt{2}\upsilon)^{\beta-1} + ((\sqrt{2}\upsilon)^\star)^{\beta-1} \mathcal{R}_0 (\sqrt{2}\upsilon)^{\beta-1} + \cdots \\
&\quad + ((\sqrt{2}\upsilon)^\star)^{\alpha+1} \mathcal{R}_0 (\sqrt{2}\upsilon)^{\alpha+1} + ((\sqrt{2}\upsilon)^\star)^{\alpha} \mathcal{S}_0 (\sqrt{2}\upsilon)^{\alpha} \\
&= \sum_{\mathfrak{i}=\alpha}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{S}_0 (\sqrt{2}\upsilon)^{\mathfrak{i}} + \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{R}_0 (\sqrt{2}\upsilon)^{\mathfrak{i}} \\
&= \sum_{\mathfrak{i}=\alpha}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{S}_0^{\frac{1}{2}} \mathcal{S}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}} + \sum_{\mathfrak{i}=\alpha+1}^{\beta-1} ((\sqrt{2}\upsilon)^\star)^{\mathfrak{i}} \mathcal{R}_0^{\frac{1}{2}} \mathcal{R}_0^{\frac{1}{2}} (\sqrt{2}\upsilon)^{\mathfrak{i}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=\alpha}^{\beta-1} (\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i)^\star (\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i) + \sum_{i=\alpha+1}^{\beta-1} (\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i)^\star (\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i) \\
&= \sum_{i=\alpha}^{\beta-1} |\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i|^2 + \sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i|^2 \\
&\preceq \|\sum_{i=\alpha}^{\beta-1} |\mathcal{S}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i|^2\| 1_{\mathbb{A}} + \|\sum_{i=\alpha+1}^{\beta-1} |\mathcal{R}_0^{\frac{1}{2}}(\sqrt{2}\upsilon)^i|^2\| 1_{\mathbb{A}} \\
&\preceq \sum_{i=\alpha}^{\beta-1} \|\mathcal{S}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^i\|^2 \| 1_{\mathbb{A}} + \|\sum_{i=\alpha+1}^{\beta-1} \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \|(\sqrt{2}\upsilon)^i\|^2 \| 1_{\mathbb{A}} \\
&\preceq \|\mathcal{S}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} \|(\sqrt{2}\upsilon)^2\|^i \| 1_{\mathbb{A}} + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \|\sum_{i=\alpha+1}^{\beta-1} \|(\sqrt{2}\upsilon)^2\|^i \| 1_{\mathbb{A}} \\
&= \|\mathcal{S}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha}^{\beta-1} (2\|\upsilon\|^2)^i 1_{\mathbb{A}} + \|\mathcal{R}_0^{\frac{1}{2}}\|^2 \sum_{i=\alpha+1}^{\beta-1} (2\|\upsilon\|^2)^i \| 1_{\mathbb{A}} \\
&\to 0_{\mathbb{A}} \ \ (\text{as} \beta, \alpha \to \infty).
\end{aligned}
$$

Therefore, $(\{\sigma_\alpha\}, \{\mathfrak{v}_\alpha\})$ and $(\{\zeta_\alpha\}, \{\mathfrak{u}_\alpha\})$ are Cauchy bisequences in $\mathit{\Gamma} \times \mathit{\Psi}$ with respect to $\mathbb{A}$. By completeness of $(\mathit{\Gamma}, \mathit{\Psi}, \mathbb{A}, \varphi)$, there exist $\sigma, \zeta \in \mathit{\Gamma}$ and $\mathfrak{u}, \mathfrak{v} \in \mathit{\Psi}$ with

$$
\lim_{\alpha\to\infty} \sigma_\alpha = \mathfrak{v}, \ \lim_{\alpha\to\infty} \zeta_\alpha = \mathfrak{u}, \ \lim_{\alpha\to\infty} \mathfrak{u}_\alpha = \zeta \text{ and } \lim_{\alpha\to\infty} \mathfrak{v}_\alpha = \sigma.
$$

Then for given $\epsilon > 0$, there exists $\alpha_1 \in \mathbb{N}$ with $\varphi(\sigma_\alpha, \mathfrak{v}) \prec \frac{\epsilon}{2}$ for all $\alpha \geq \alpha_1$. Since $(\{\sigma_\alpha\}, \{\mathfrak{v}_\alpha\})$ and $(\{\zeta_\alpha\}, \{\mathfrak{u}_\alpha\})$ are Cauchy bisequences, we derive that

$$
\varphi(\sigma_\alpha, \mathfrak{v}_\alpha) \prec \frac{\epsilon}{2}.
$$

Then,

$$
\begin{aligned}
\varphi(\mathit{\Phi}(\sigma, \mathfrak{u}), \mathfrak{v}) &\preceq \varphi(\mathit{\Phi}(\sigma, \mathfrak{u}), \mathfrak{v}_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, \mathfrak{v}) \\
&= \varphi(\mathit{\Phi}(\sigma, \mathfrak{u}), \mathit{\Phi}(\mathfrak{u}_\alpha, \zeta_\alpha)) + \varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, \mathfrak{v}) \\
&\preceq \upsilon^\star \varphi(\sigma, \mathfrak{v}_\alpha)\upsilon + \upsilon^\star \varphi(\zeta_\alpha, \mathfrak{u})\upsilon + \varphi(\sigma_{\alpha+1}, \mathfrak{v}_{\alpha+1}) + \varphi(\sigma_{\alpha+1}, \mathfrak{v}).
\end{aligned}
$$

As $\alpha \to \infty$, we have

$$
\varphi(\mathit{\Phi}(\sigma, \mathfrak{u}), \mathfrak{v}) \prec \epsilon.
$$

Then,

$$
\varphi(\mathit{\Phi}(\sigma, \mathfrak{u}), \mathfrak{v}) = 0.
$$

Hence, $\mathit{\Phi}(\sigma, \mathfrak{u}) = \mathfrak{v}$. Similarly, we can derive $\mathit{\Phi}(\mathfrak{u}, \sigma) = \zeta$, $\mathit{\Phi}(\zeta, \mathfrak{v}) = \mathfrak{u}$ and $\mathit{\Phi}(\mathfrak{v}, \zeta) = \sigma$. On the other hand, we derive that

$$
\varphi(\sigma, \mathfrak{v}) = \varphi(\lim_{\alpha\to\infty} \mathfrak{v}_\alpha, \lim_{\alpha\to\infty} \sigma_\alpha) = \lim_{\alpha\to\infty} \varphi(\sigma_\alpha, \mathfrak{v}_\alpha) = 0
$$

and

$$\varphi(\zeta, \mathfrak{u}) = \varphi(\lim_{\alpha\to\infty} \mathfrak{u}_\alpha, \lim_{\alpha\to\infty} \zeta_\alpha) = \lim_{\alpha\to\infty} \varphi(\zeta_\alpha, \mathfrak{u}_\alpha) = 0.$$

So, $\sigma = \mathfrak{v}$ and $\zeta = \mathfrak{u}$. Therefore, $(\sigma, \mathfrak{u}) \in (\Gamma \times \Psi) \cap (\Psi \times \Gamma)$ is a coupled fixed point of Φ. As in the proof of the Theorem 3.1, one can easily prove uniqueness part. □

Example 3.4. Let $\Gamma = \{0, 1, 2, 7\}$, $\Psi = \{0, \frac{1}{4}, \frac{1}{2}, 3\}$, $\mathbb{A}_+ = \mathcal{M}_2(\mathbb{C})$ and the map $\varphi : \Gamma \times \Psi \to \mathbb{A}_+$ is defined by

$$\varphi(\sigma, \mathfrak{u}) = \begin{bmatrix} |\sigma - \mathfrak{u}| & 0 \\ 0 & \Bbbk|\sigma - \mathfrak{u}| \end{bmatrix},$$

for all $\sigma \in \Gamma$ and $\mathfrak{u} \in \Psi$, where $\Bbbk \geq 0$ is a constant. Let $\preceq$ be the partial order on $\mathbb{A}$ given by

$$(\vartheta_1, \varpi_1) \preceq (\vartheta_2, \varpi_2) \Leftrightarrow \vartheta_1 \leq \vartheta_2 \textit{ and } \varpi_1 \leq \varpi_2.$$

Then $(\Gamma, \Psi, \mathbb{A}, \varphi)$ is a complete $C^\star$-algebra-valued bipolar metric space. Define

$$\Phi : (\Gamma \times \Psi) \cup (\Psi \times \Gamma) \to \Gamma \cup \Psi$$

by

$$\Phi(\sigma, \zeta) = \frac{\sigma + \zeta}{5},$$

for all $\sigma, \zeta \in (\Gamma \times \Psi) \cup (\Psi \times \Gamma)$. Then

$$\begin{aligned} \varphi(\Phi(\sigma, \mathfrak{u}), \Phi(\zeta, \mathfrak{v})) &= \begin{bmatrix} |\Phi(\sigma, \mathfrak{u}) - \Phi(\zeta, \mathfrak{v})| & 0 \\ 0 & \Bbbk|\Phi(\sigma, \mathfrak{u}) - \Phi(\zeta, \mathfrak{v})| \end{bmatrix} = \begin{bmatrix} |\frac{\sigma+\mathfrak{u}}{5} - \frac{\zeta+\mathfrak{v}}{5}| & 0 \\ 0 & \Bbbk|\frac{\sigma+\mathfrak{u}}{5} - \frac{\zeta+\mathfrak{v}}{5}| \end{bmatrix} \\ &\preceq \frac{1}{5}\left(\begin{bmatrix} |\sigma - \mathfrak{v}| & 0 \\ 0 & \Bbbk|\sigma - \mathfrak{v}| \end{bmatrix} + \begin{bmatrix} |\zeta - \mathfrak{u}| & 0 \\ 0 & \Bbbk|\zeta - \mathfrak{u}| \end{bmatrix} \right) = \upsilon^\star \varphi(\sigma, \mathfrak{v})\upsilon + \upsilon^\star \varphi(\zeta, \mathfrak{u})\upsilon, \end{aligned}$$

for all $\sigma, \zeta \in \Gamma$ and $\mathfrak{u}, \mathfrak{v} \in \Psi$, where

$$\upsilon = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

and $\|\upsilon\| = \frac{1}{5} < \frac{1}{\sqrt{2}}$. All the conditions of Theorem 3.3 are fulfilled and Φ has a unique fixed point $(0, 0)$.

4. Applications

As an application of Theorem 3.1, we find an existence and uniqueness result for a type of following system of Fredholm integral equations.

Theorem 4.1. *Let us consider the system of Fredholm integral equations*

$$\begin{aligned} \sigma(\mu) &= \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\mu, p, \sigma(p), \zeta(p))dp + \delta(\mu), \ \mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2, \\ \zeta(\mu) &= \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\mu, p, \zeta(p), \sigma(p))dp + \delta(\mu), \ \mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2, \end{aligned} \tag{4.1}$$

where $\mathcal{E}_1 \cup \mathcal{E}_2$ is a Lebesgue measurable set. Suppose

(T1) $\mathcal{G} : (\mathcal{E}_1^2 \cup \mathcal{E}_2^2) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *and* $\delta \in L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$.

(T2) There exists a continuous function $\kappa : \mathcal{E}_1^2 \times \mathcal{E}_2^2 \to \mathbb{R}$ *and* $\theta \in (0, 1)$, *such that*

$$\begin{aligned}&|\mathcal{G}(\mu, p, \sigma(p), \zeta(p)) - \mathcal{G}(\mu, p, \mathfrak{u}(p), \mathfrak{v}(p))|\\ \leq &\theta|\kappa(\mu, p)|(|\sigma(p) - \mathfrak{u}(p)| + |\zeta(p) - \mathfrak{v}(p)| + I - \theta^{-1}I),\end{aligned}$$

for all $\mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2$.

(T3) $\sup_{\mu\in\mathcal{E}_1\cup\mathcal{E}_2} \int_{\mathcal{E}_1\cup\mathcal{E}_2} |\kappa(\mu, p)|dp \leq 1$.

Then the integral equation has a unique solution in $L^\infty(\mathcal{E}_1) \cup L^\infty(\mathcal{E}_2)$.

Proof. Let $\varGamma = L^\infty(\mathcal{E}_1)$ and $\varPsi = L^\infty(\mathcal{E}_2)$ be two normed linear spaces, where $\mathcal{E}_1, \mathcal{E}_2$ are Lebesgue measurable sets and $m(\mathcal{E}_1 \cup \mathcal{E}_2) < \infty$. Let $\mathcal{H} = L^2(\mathcal{E}_1) \cup L^2(\mathcal{E}_2)$. Consider $\varphi : \varGamma \times \varPsi \to L(\mathcal{H})$ defined by $\varphi(\sigma, \zeta) = \pi_{|\sigma-\zeta|}$, where $\pi_{\mathfrak{h}} : \mathcal{H} \to \mathcal{H}$ is the multiplication operator defined by $\pi_{\mathfrak{h}}(\omega) = \mathfrak{h}.\omega$ for $\omega \in \mathcal{H}$. Then $(\varGamma, \varPsi, \mathbb{A}, \varphi)$ is a complete $C^\star$-algebra valued bipolar metric space.

Define the covariant mapping $\varPhi : \varGamma^2 \cup \varPsi^2 \to \varGamma \cup \varPsi$ by

$$\varPhi(\sigma, \zeta)(\mu) = \int_{\mathcal{E}_1\cup\mathcal{E}_2} \mathcal{G}(\mu, p, \sigma(p), \zeta(p))dp + \delta(\mu), \ \forall \mu, p \in \mathcal{E}_1 \cup \mathcal{E}_2.$$

Set $\tau = \theta I$, then $\tau \in L(\mathcal{H})_+$ and $\|\tau\| = \theta < 1$. For any $\omega \in \mathcal{H}$, we have

$$\begin{aligned}\|\varphi(\varPhi(\sigma, \zeta), \varPhi(\mathfrak{u}, \mathfrak{v}))\| &= \sup_{\|\omega\|=1} (\pi_{|\varPhi(\sigma,\zeta)-\varPhi(\mathfrak{u},\mathfrak{v})|+I}\omega, \omega)\\
&= \sup_{\|\omega\|=1} \int_{\mathcal{E}_1\cup\mathcal{E}_2} (|\varPhi(\sigma, \zeta) - \varPhi(\mathfrak{u}, \mathfrak{v})| + I)\omega(\mu)\overline{\omega(\mu)}d\mu\\
&\leq \sup_{\|\omega\|=1} \int_{\mathcal{E}_1\cup\mathcal{E}_2} \int_{\mathcal{E}_1\cup\mathcal{E}_2} |\mathcal{G}(\mu, p, \sigma(p), \zeta(p))\\
&\quad - \mathcal{G}(\mu, p, \mathfrak{u}(p), \mathfrak{v}(p))|dp|\omega(\mu)|^2d\mu\\
&\quad + \sup_{\|\omega\|=1} \int_{\mathcal{E}_1\cup\mathcal{E}_2} \int_{\mathcal{E}_1\cup\mathcal{E}_2} dp|\omega(\mu)|^2d\mu I\\
&\leq \sup_{\|\omega\|=1} \int_{\mathcal{E}_1\cup\mathcal{E}_2} \Big[\int_{\mathcal{E}_1\cup\mathcal{E}_2} \theta|\kappa(\mu, p)|(|\sigma(p) - \mathfrak{u}(p)|\\
&\quad + |\zeta(p) - \mathfrak{v}(p)| + I - \theta^{-1}I)dp\Big]|\omega(\mu)|^2d\mu + I\\
&\leq \theta \sup_{\|\omega\|=1} \int_{\mathcal{E}_1\cup\mathcal{E}_2} \Big[\int_{\mathcal{E}_1\cup\mathcal{E}_2} |\kappa(\mu, p)|dp\Big]|\omega(\mu)|^2d\mu(\|\sigma - \mathfrak{u}\|_\infty\\
&\quad + \|\zeta - \mathfrak{v}\|_\infty)\\
&\leq \theta \sup_{\mu\in\mathcal{E}_1\cup\mathcal{E}_2} \int_{\mathcal{E}_1\cup\mathcal{E}_2} |\kappa(\mu, p)|dp \sup_{\|\omega\|=1} \int_{\mathcal{E}_1\cup\mathcal{E}_2} |\omega(\mu)|^2d\mu(\|\sigma - \mathfrak{u}\|_\infty\\
&\quad + \|\zeta - \mathfrak{v}\|_\infty)\\
&\leq \theta[\|\sigma - \mathfrak{u}\|_\infty + \|\zeta - \mathfrak{v}\|_\infty]\\
&= \|\tau\|[\|\varPhi(\sigma, \mathfrak{u})\| + \|\varPhi(\zeta, \mathfrak{v})\|].\end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are fulfilled. Hence, the integral equation (4.1) has a unique solution. □

5. Conclusions

In this paper, we introduced the notion of a $C^{\star}$-algebra valued bipolar metric space and proved coupled fixed point theorems. An illustrative example is provided that show the validity of the hypothesis and the degree of usefulness of our findings.

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Conflict of interest

The authors declare no conflicts of interest.

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