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ORIGINAL PAPER

Discrete fractional logistic map and its chaos

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Abstract A discrete fractional logistic map is proposed in the left Caputo discrete delta's sense. The new model holds discrete memory. The bifurcation diagrams are given and the chaotic behaviors are numerically illustrated.

Keywords Discrete fractional calculus · Chaos · Time scale · Caputo delta difference

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1 Introduction

For the chaos of continuous fractional differential equations, a lot of fruitful results have been obtained in the past decades. Probably more work has been developed for the fractional chaos and we mainly refer the readers to the following development. Hartley et al. [1] modified the Chua's system to include fractional order elements and reported that by varying the total system order incrementally from 3.6 to 3.7, systems of "order" less than three can exhibit chaos as well as other nonlinear behaviors. From the view of kinetics, Zaslavsky [2] systemically discussed what features of the dynamics have. By using a truncated transfer function method, Li and Chen [3] found that hyperchaos exists in the fractional-order Rossler hyperchaotic equation with order less than 4. Then, Li and Peng [4] gave the predictor–corrector scheme of the fractional Chua's system and proved that the chaos there does exist. The numerical scheme was used to discuss the phase diagrams and attractors of the chaos in later years.

Initially inspired by the discretization of the Riemann–Liouville and the Caputo operators, the field of the fractional difference equations is relatively new. It is developing faster and several new applications were deeply analyzed. For example, Atici and Eloe [5] discussed the discrete initial value problem and gave the existence results. Atici and Senguel [6] extended the variational approach to the fractional discrete case and provided a tool in the mathematical modeling. Holm proposed the Laplace transform [7] for solving discrete fractional equations in the nabla's sense. Abedel [8, 9] systemically discussed the Caputo and the Riemann–Liouville fractional differences as well as their properties. However, less efforts have been contributed to the chaotic behaviors of the dynamical systems.

For the famous logistic map

$$u(n+1)=Ku(n)\big(1-u(n)\big) \tag{1}$$

popularized by May [10] in 1976, the system exhibits chaotic behaviors for most values of the growth coefficient K between 3.57 and 4. Naturally, one question may be proposed: whether there is a discrete fractional logistic map which has a generalized chaos behavior.

The fractional difference provides us a new powerful tool to characterize the dynamics of discrete complex systems more deeply. Several discrete derivatives were introduced recently in the literature [5–8], therefore a deeper analysis must be done in order to see the effects in describing the dynamics. In this paper, we investigate the chaotic behaviors of the following discrete logistic map

$$\begin{aligned}&{}^C\Delta_a^\nu u(t)=\mu u(t+\nu-1)\big(1-u(t+\nu-1)\big),\\ &\quad t\in\mathbb{N}_{a+1-\nu},\ u(a)=c,\ 0<\nu\le 1,\end{aligned} \tag{2}$$

where ${}^C\Delta_a^\nu$ is the left Caputo-like delta difference, $\mathbb{N}_a$ denotes the isolated time scale and $\mathbb{N}_a=\{a,a+1,a+2,\dots\}$ ($a\in\mathbb{R}$ fixed). For the function $f(n)$, the delta difference operator Δ is defined as $\Delta f(n)=f(n+1)-f(n)$.

2 Preliminaries

We start with some necessary definitions from the discrete fractional calculus and revisit the preliminary results.

Definition 2.1 [5] Let $u:\mathbb{N}_a\to\mathbb{R}$ and $0<\nu$ be given. Then the fractional sum of ν order is defined by

$$\begin{aligned}&\Delta_a^{-\nu}u(t):=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}\big(t-\sigma(s)\big)^{(\nu-1)}u(s),\\ &\quad t\in\mathbb{N}_{a+\nu},\end{aligned} \tag{3}$$

where a is the starting point, $\sigma(s)=s+1$ and $t^{(\nu)}$ is the falling function defined as

$$t^{(\nu)}=\frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}. \tag{4}$$

Definition 2.2 [8] For $0<\nu$, $\nu\notin\mathbb{N}$ and $u(t)$ defined on $\mathbb{N}_a$, the Caputo-like delta difference is defined by

$$\begin{aligned}&{}^C\Delta_a^\nu u(t)\\ &\quad:=\Delta_a^{-(m-\nu)}\Delta^m u(t)\\ &\quad=\frac{1}{\Gamma(m-\nu)}\sum_{s=a}^{t-(m-\nu)}\big(t-\sigma(s)\big)^{(m-\nu-1)}\Delta_s^m u(s),\end{aligned} \tag{5}$$

where $t\in\mathbb{N}_{a+m-\nu}$, $m=[\nu]+1$.

Theorem 2.3 [11] *For the delta fractional difference equation*

$$\begin{aligned}&{}^C\Delta_a^\nu u(t)=f\big(t+\nu-1,u(t+\nu-1)\big),\\ &\Delta^k u(a)=u_k,\quad m=[\nu]+1,k=0,\dots,m-1,\end{aligned} \tag{6}$$

the equivalent discrete integral equation can be obtained as

$$\begin{aligned}u(t)&=u_0(t)+\frac{1}{\Gamma(\nu)}\sum_{s=a+m-\nu}^{t-\nu}\big(t-\sigma(s)\big)^{(\nu-1)}\\ &\quad\times f\big(s+\nu-1,u(s+\nu-1)\big),\quad t\in\mathbb{N}_{a+m},\end{aligned} \tag{7}$$

where the initial iteration $u_0(t)$ reads

$$u_0(t)=\sum_{k=0}^{m-1}\frac{(t-a)^{(k)}}{k!}\Delta^k u(a). \tag{8}$$

The existence results for the above nonlinear fractional difference equation have been discussed in [11].

Remark From Eq. (5) to Eq. (7), the domain is changed from $\mathbb{N}_{a+m-\nu}$ to $\mathbb{N}_{a+m}$ and the function $u(t)$ is preserved to define on the isolated time scale $\mathbb{N}_a$ during the summation (see the right sides of (5) and (7)). In view of this point, the discrete fractional calculus is an good tool for the initialization of the fractional difference equations.

3 Chaos of the discrete logistic map

For $a = 0$ and $\nu = 1$, the discrete logistic map (2) can be reduced to the classical one

$$\Delta u(n) = \mu u(n)\big(1 - u(n)\big), \quad u(0) = c \tag{9}$$

or

$$u(n+1) = (1+\mu)u(n) - \mu u^2(n), \quad u(0) = c. \tag{10}$$

Considering the transform

$$u(n) = \frac{1+\mu}{\mu} v(n), \tag{11}$$

Eq. (10) can be written as

$$\begin{aligned} &v(n+1) = (1+\mu)\big(v(n) - v^2(n)\big), \\ &v(0) = \frac{\mu}{1+\mu} u(0) \end{aligned} \tag{12}$$

which exhibits chaos behaviors for most values of μ between 2.57 and 3.

From Theorem 2.3, we can obtain the following equivalent discrete integral form of Eq. (2):

$$\begin{aligned} u(t) = u(0) + \frac{\mu}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} (t-s-1)^{(\nu-1)} \\ \times u(s+\nu-1)\big(1 - u(s+\nu-1)\big), \quad t \in \mathbb{N}_1. \end{aligned} \tag{13}$$

As a result, the numerical formula can be presented accordingly

$$\begin{aligned} u(n) = u(0) + \frac{\mu}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \\ \times u(j-1)\big(1 - u(j-1)\big). \end{aligned} \tag{14}$$

Compared with the map of the integer order (10), the fractionalized one (14) has a discrete kernel function. $u(n)$ depends on the past information $u(0), \ldots, u(n-1)$. As a result, the memory effects of the discrete maps means that their present state of evolution depends on all past states.

Assume $\nu = 0.8$, $u(0) = 0.3$, and $n = 100$. We can derive the numerical solutions $u(n)$ using the Matlab. In what follows, Figs. 1, 2 and 3 show the numerical solutions for different μ. Particularly, the system exhibits a chaotic behavior for $\mu = 2.5$.

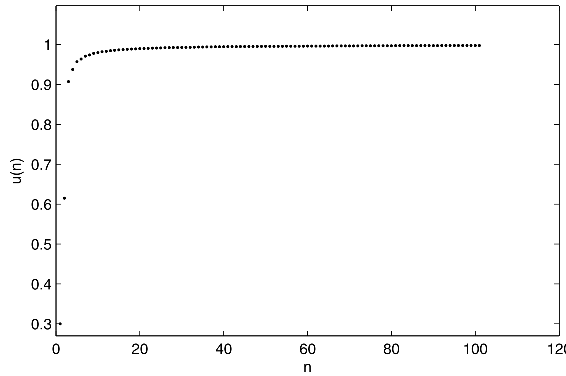

Fig. 1 Stable solution of the fractional discrete logistic map for $\mu = 1.5$ and $\nu = 0.8$

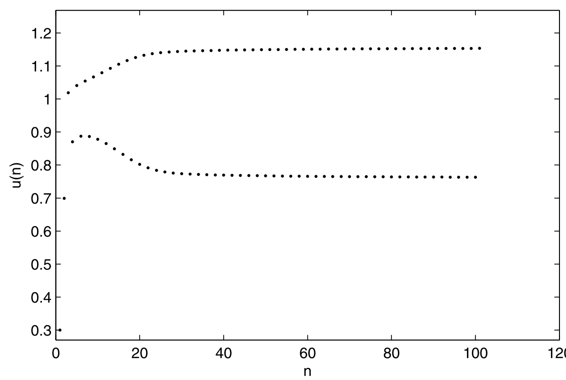

Fig. 2 Vibration in the fractional discrete logistic map for $\mu = 1.9$ and $\nu = 0.8$

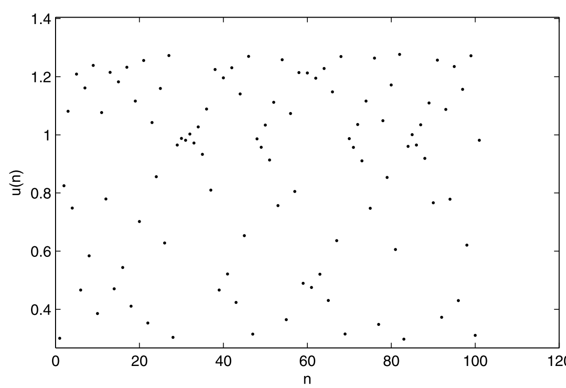

Fig. 3 Chaos of the fractional discrete logistic map for $\mu = 2.5$ and $\nu = 0.8$

Using the numerical formula (14), set the step size of μ as 0.005 and the bifurcation diagrams are plotted in Figs. 4, 5 and 6. The discrete chaos reduces to the

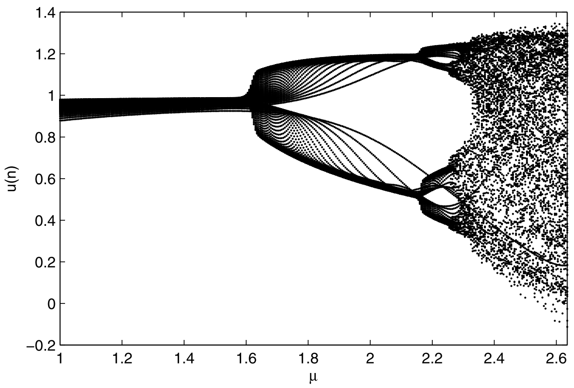

Fig. 4 The bifurcation diagram for $n = 100$, $u(0) = 0.3$, and $\nu = 0.6$

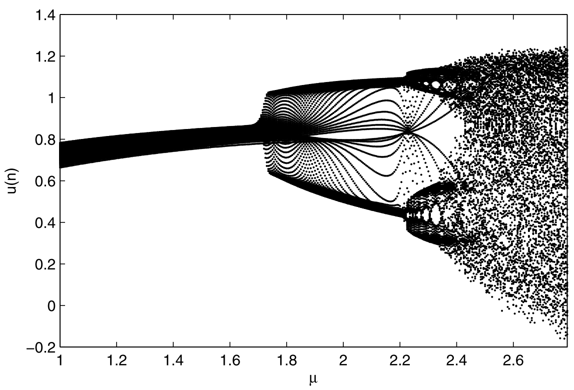

Fig. 5 The bifurcation diagram for $n = 100$, $u(0) = 0.3$, and $\nu = 0.2$

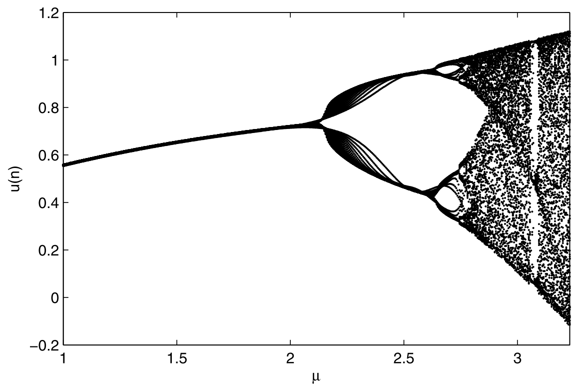

Fig. 6 The bifurcation diagram for $n = 100$, $u(0) = 0.3$, and $\nu = 0.01$

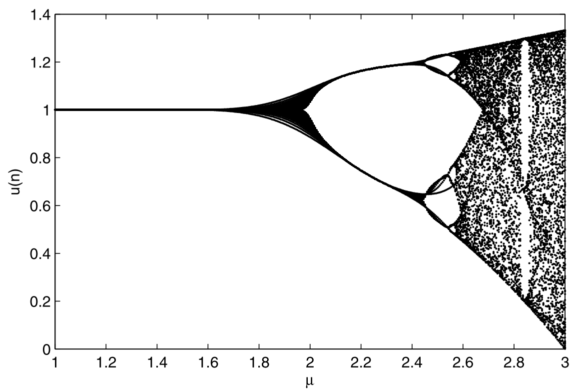

Fig. 7 The bifurcation diagram for $n = 100$, $u(0) = 0.3$, and $\nu = 1$

classical one for $\nu = 1$ in Fig. 7. We can readily obtain the intervals of μ where the chaos happens.

It can be concluded that the chaos zones are clearly different when we change the difference order ν. The bifurcation diagrams (4)–(7) illustrate the evolution and explain this point.

4 Conclusions

This study introduces a fractional discrete logistic map using the delta difference of fractional order. Compared with the one of the integer order, the new model has a discrete memory and a fractional difference order ν. When we change the difference order ν in the numerical results, new chaotic behaviors of the logistic map are observed.

It is interesting to point out that the chaotic zones not only depends on the coefficients μ but the difference order ν. We strongly believe that a combination of the spirit of fractional calculus and the discrete point of view will lead to a better description of fractional dynamics. In this paper, we only consider the left Caputo delta. We will compare with other types of the discrete fractional calculus in future work and show their differences in the applications.

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