



Dynamics and Ulam Stability for Fractional q -Difference Inclusions via Picard Operators Theory

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Abstract

In this manuscript, by using weakly Picard operators we investigate the Ulam type stability of fractional q -difference. An illustrative example is given in the last section.

1 Introduction

Not only fractional differential inclusions (FDIs) but also fractional differential equations (FDEs) have applications in mathematics, and other applied sciences, see e.g. [18, 6, 7, 35, 38, 40, 21, 22, 37, 9, 17, 4, 5]. Fractional q -difference equations received much attention from many authors; see e.g. [12]. Other interesting results about this subject can be found in [24].

Functional differential inclusions and coupled systems of differential inclusions are a generalization of the concept of ordinary differential equation of the form $\frac{d}{dt}x(t) \in F(t, x(t))$, where F is a multivalued map containing one element (single-valued map). Differential inclusions arise in many situations as differential variational inequalities, projected dynamical systems, linear and nonlinear complementarity dynamical systems, discontinuous ordinary differential equations, and fuzzy set arithmetic; see e.g. [14, 19, 36].

Key Words: Fractional q -difference inclusion; multivalued weakly Picard operator; Ulam-Rassias stability; fixed point inclusion.

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Ulam stability for functional differential equations and inclusions has been widely considered; see e.g.[26, 27]. Picard operators [28, 29] seemed to be a powerful method in the processing of Ulam stability theory [10, 27, 16], and ordinary differential inclusions and equations; see e.g.[1, 29, 30, 31].

In this paper we first discuss the stability of the fractional q -difference inclusion below in the sense of Ulam-Rassias

$$({}^c\mathcal{D}_q^\alpha \mathfrak{h})(t) \in F(t, \mathfrak{h}(t)); \quad t \in \mathfrak{J} := [0, T], \quad (1.1)$$

along the initial condition

$$\mathfrak{h}(0) = \mathfrak{h}_0 \in \mathbb{R}, \quad (1.2)$$

with $T > 0$, $\alpha \in (0, 1]$, $q \in (0, 1)$, and $F : \mathfrak{J} \times \mathbb{R} \rightarrow \mathcal{N}(\mathbb{R})$ is a given multi-valued map, $\mathcal{N}(\mathbb{R})$ is the family of all nonempty subsets of \mathbb{R} , and ${}^c\mathcal{D}_q^\alpha$ is the Caputo fractional q -difference derivative of order α .

After getting a solution of (1.1), we shall investigate the coupled fractional q -difference inclusions

$$\begin{cases} ({}^c\mathcal{D}_{q_1}^{\alpha_1} \mathfrak{g}_1)(t) \in F_1(t, \mathfrak{g}_1(t), \mathfrak{g}_2(t)), \\ ({}^c\mathcal{D}_{q_2}^{\alpha_2} \mathfrak{g}_2)(t) \in F_2(t, \mathfrak{g}_1(t), \mathfrak{g}_2(t)) \end{cases} ; \quad t \in \mathfrak{J}, \quad (1.3)$$

with the initial conditions

$$\begin{cases} \mathfrak{g}_1(0) = \mathfrak{i}_1 \\ \mathfrak{g}_2(0) = \mathfrak{i}_2, \end{cases} \quad (1.4)$$

where $T > 0$, $q_i \in (0, 1)$, $\alpha_i \in (0, 1]$, $\mathfrak{i}_i \in \mathbb{R}$, $F_i : \mathfrak{J} \times \mathbb{R} \rightarrow \mathcal{N}(\mathbb{R})$; $i = 1, 2$.

This paper initiates the application of Picard operators for the study of Ulam stability for problems (1.1)-(1.2) and (1.3)-(1.4).

2 Preliminaries

We deal with the following collection

$$C(\mathfrak{J}) := \{ \mathfrak{g} : \mathfrak{J} \rightarrow \mathbb{R} \mid \mathfrak{g} \text{ is continuous} \}.$$

Then, $C(\mathfrak{J})$ forms a Banach space by regarding the supremum (uniform) norm $\|\mathfrak{g}\|_C := \sup_{\tau \in \mathfrak{J}} |\mathfrak{g}(\tau)|$.

$$L^1(\mathfrak{J}) := \{ \mathfrak{g} : \mathfrak{J} \rightarrow \mathbb{R} \mid \mathfrak{g} \text{ is measurable and Lebesgue integrable function.} \}$$

Then, $L^1(\mathfrak{J})$ forms a Banach space by regarding $\|\mathfrak{g}\|_{L^1} = \|\mathfrak{g}\|_1 = \int_{\mathfrak{J}} |\mathfrak{g}(\tau)| d\tau$.

Over a metric space (\mathfrak{M}, δ) , the symbol $\mathfrak{P}(E)$ denotes the family of all nonempty subsets of $E \subset \mathfrak{M}$. Then, we set

$$\mathfrak{P}_\pi(E) = \{F \in \mathfrak{P}(E) : F \text{ fulfills the property } \pi\},$$

where, π can be, for instance, bounded, closed, compact, convex (in short, bd, cl, cp, cv). For clarification, consider, for example $\mathfrak{P}_{bd,cl}(E) = \{F \in \mathfrak{P}(E) : F \text{ is bounded and closed}\}$.

A multivalued function $G : \mathfrak{J} \rightarrow \mathfrak{P}_{cl}(E)$ is called measurable whenever the mapping

$$\tau \rightarrow dist(\mathbf{u}, G(\tau)) = \inf\{\|\mathbf{u} - \nu\| : \nu \in G(\tau)\}$$

is measurable for each $\mathbf{u} \in E$.

A mapping $H_d : \mathfrak{P}(E) \times \mathfrak{P}(E) \rightarrow [0, \infty) \cup \{\infty\}$ described by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

is called Hausdorff metric, where $d(a, B) = \inf_{b \in B} d(a, b)$, $d(A, b) = \inf_{a \in A} d(a, b)$ and $A, B \subset E$. Then, the coupled $(\mathfrak{P}_{bd,cl}(E), H_d)$ is named as Hausdorff metric space.

Definition 2.1. [14] *The set*

$$S_G = \{\mathbf{g} \in L^1(\mathfrak{J}) : \mathbf{g}(\tau) \in G(\tau), \text{ a.e. } \tau \in \mathfrak{J}\},$$

is the selection set of G . Moreover, the set selector $S_{F \circ \mathbf{g}}$, for each $\mathbf{g} \in C(\mathfrak{J})$ from $F \circ \mathbf{g}$ is formulated by $S_{F \circ \mathbf{g}} := \{\mathbf{u} \in L^1(\mathfrak{J}) : \mathbf{u}(\tau) \in F(\tau, \mathbf{g}(\tau)), \text{ a.e. } \tau \in \mathfrak{J}\}$

A selfmapping \mathcal{O} on a metric space (\mathfrak{M}, δ) is called

(*P.o.*) Picard operator (*P.o.*) if $\mathcal{F}i\chi_{\mathcal{O}} = \{\mathfrak{z}^*\}$ for $\mathfrak{z}^* \in \mathfrak{M}$ and $(\mathcal{O}^n(\mathfrak{z}_0))_{n \in \mathbb{N}} \rightarrow \mathfrak{z}^*$ for any $\mathfrak{z}_0 \in \mathfrak{M}$.

(*w.P.o.*) weakly Picard operator (*w.P.o.*) if $(\mathcal{O}^n(\mathfrak{z}))_{n \in \mathbb{N}} \rightarrow \mathfrak{z}^* \in \mathfrak{M}$, in a way that $\mathfrak{z}^* \in \mathcal{F}i\chi_{\mathcal{O}}$, (limit may depend on \mathfrak{z}).

(*k.w.P.o.*) k -weakly Picard operator (*c.w.P.o.*) if it is (*w.P.o.*) and $d(\mathfrak{z}, \mathcal{O}^\infty(\mathfrak{z})) \leq k d(\mathfrak{z}, \mathcal{O}(\mathfrak{z}))$; $\mathfrak{z} \in X$.

where $\mathcal{F}i\chi_{\mathcal{O}} = \{\mathfrak{z} : \mathfrak{z} = \mathcal{O}\mathfrak{z}\}$. Further, for a (*w.P.o.*) \mathcal{O} , we set $\mathcal{O}^\infty = \mathcal{O}^\infty(\mathfrak{z}) = \lim_{n \rightarrow \infty} \mathcal{O}^n(\mathfrak{z})$. Notice that $\mathcal{O}^\infty(\mathfrak{M}) = \mathcal{F}i\chi_{\mathcal{O}}$.

A multivalued mapping $\mathcal{Q} : \mathfrak{M} \rightarrow \mathfrak{P}(\mathfrak{M})$ on (\mathfrak{M}, δ) is called weakly Picard operator (*m.w.P.o.*) [23, 32], if for each $\mathbf{g} \in \mathfrak{M}$ and $\eta \in \mathcal{Q}(x)$, there is $(\mathbf{g}_n)_{n \in \mathbb{N}}$ where

- (i) $\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_1 = \mathfrak{h}$;
- (ii) $\mathfrak{g}_{n+1} \in \mathcal{Q}(\mathfrak{g}_n), n \in \mathbb{N}$;
- (iii) $(\mathfrak{g}_n)_{n \in \mathbb{N}} \rightarrow \mathfrak{g}^*$ so that $\mathfrak{g}^* \in \text{Fix}_{\mathcal{Q}}$.

Set $\mathfrak{A} := \{\varphi : [0, \infty) \rightarrow [0, \infty) \mid \varphi \text{ increasing, and } \lim_{n \rightarrow \infty} \varphi^n(t) \rightarrow 0 \text{ for every } t \in [0, \infty)\}$, where φ^n is the n -th iterate of φ . Here φ is called comparison function [34]. If $\varphi \in \mathfrak{A}$ then φ is continuous at 0 and $\varphi(t) < t$ for all $t > 0$. Furthermore, we set

$$\mathfrak{S} := \{\varphi : [0, \infty) \rightarrow [0, \infty) \mid \varphi \text{ strictly increasing \& } \sum_{n=1}^{\infty} \varphi^n(t) < \infty \text{ for all } t \in [0, \infty)\},$$

Here, $\varphi \in \mathfrak{S}$ is called strictly comparison function and $\mathfrak{S} \subset \mathfrak{A}$.

Definition 2.2. For $\varphi \in \mathfrak{A}$, operator $\mathcal{Q} : \mathfrak{M} \rightarrow \mathfrak{P}_{cl}(\mathfrak{M})$ is called φ -multivalued weakly Picard (briefly φ -m.w.P. operator) if it is a m.w.P. and there is a selection $\mathcal{O}^\infty : \Lambda_{\mathcal{Q}} \rightarrow \text{Fix}_{\mathcal{Q}}$ of \mathcal{Q}^∞ so that

$$d(\theta, \mathcal{O}^\infty(\theta, \nu)) \leq \varphi(d(\theta, \nu)); \text{ for all } (\theta, \nu) \in \Lambda_{\mathcal{Q}}.$$

In particular, if $\varphi(\mathfrak{z}) = k\mathfrak{z}$, for all $\mathfrak{z} \in \mathbb{R}_+$, for some $k > 0$ then \mathcal{Q} is named as k -multivalued weakly Picard operator (k -m.w.P.o.).

Definition 2.3. An operator $\mathcal{Q} : \mathfrak{M} \rightarrow \mathfrak{P}_{cl}(\mathfrak{M})$ is named

- a) multivalued k -Lipschitz if there is $k \geq 0$ with

$$H_\delta(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\nu)) \leq \gamma \delta(\mathfrak{q}, \nu); \text{ for each } \mathfrak{q}, \nu \in \mathfrak{M}, \quad (2.1)$$

- b) a multivalued k -contraction if (2.1) holds for $k \in [0, 1)$,
- c) a multivalued φ -contraction if there is a $\varphi \in \mathfrak{S}$ with

$$H_\delta(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\nu)) \leq \varphi(\delta(\mathfrak{q}, \nu)); \text{ for each } \mathfrak{q}, \nu \in \mathfrak{M}.$$

Definition 2.4. [1]. The inclusion $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ is named generalized Ulam type (g .U.t) stable if there is $\varphi \in \mathfrak{S}$ such that for each $\varepsilon > 0$ and solution $\mathfrak{g} \in C(\mathfrak{J})$ of

$$H_\delta(\mathfrak{g}(\tau), (\mathcal{Q}\mathfrak{g})(\tau)) \leq \varepsilon; \tau \in \mathfrak{J},$$

there is a solution $\mathfrak{u} \in C(\mathfrak{J})$ of $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ (inclusion) so that

$$|\mathfrak{g}(\tau) - \mathfrak{u}(\tau)| \leq \theta_\Omega(\varepsilon); \tau \in \mathfrak{J}.$$

In case of $\varphi(t) = kt; k > 0$, it is called Ulam type stable.

Definition 2.5. [1, 2, 3]. The fixed point inclusion $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ is named generalized Ulam-Rassias type stable with respect to ϕ if there is a real number $c_{N,\phi} > 0$ such that for each solution $\mathfrak{g} \in C_\gamma$ of

$$H_d(\mathfrak{g}(\tau), (\mathcal{Q}\mathfrak{g})(\tau)) \leq \phi(\tau); t \in I,$$

there is a solution $\mathbf{u} \in C(\mathfrak{J})$ of $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ (inclusion) such that

$$|\mathfrak{g}(\tau) - \mathbf{u}(\tau)| \leq c_{N,\phi}\phi(\tau); t \in I.$$

In case of $\phi(t) = kt$; $k > 0$, it is called Ulam-Rassias type stable.

Lemma 2.6. [39] Let \mathcal{Q} be a multivalued mapping from a complete metric space (\mathfrak{M}, δ) to $\mathfrak{P}_{cl}(\mathfrak{M})$, and $\lambda \in \mathfrak{S}$. If \mathcal{Q} is a λ -contraction, then $\text{Fix}_{\mathcal{Q}} \neq \emptyset$. Moreover, $\{\mathcal{Q}$ is (w.P.o).

Theorem 2.7. [20] Let \mathcal{Q} be a multivalued mapping from a complete metric space (\mathfrak{M}, δ) to $\mathfrak{P}_{cl}(\mathfrak{M})$, and $\lambda \in \mathfrak{S}$. If \mathcal{Q} forms a multivalued λ -contraction, then

- (1) h is a m.w.P. operator;
- (2) If additionally $\lambda(\kappa\tau) \leq \kappa\lambda(\tau)$ for every $\tau \in \mathbb{R}^+$ (where $\kappa > 1$), then h is a φ -m.w.P. operator, with $\varphi(\tau) := \tau + \sum_{n=1}^{\infty} \lambda n(\tau)$, for each $\tau \in \mathbb{R}^+$;
- (3) Let $S : \mathfrak{M} \rightarrow \mathfrak{P}_{cl}(\mathfrak{M})$ be a λ -contraction and $\eta > 0$ be such that $H_\delta(S(\tau), \mathcal{Q}(\tau)) \leq \eta$ for each $\tau \in E$. Suppose that $\lambda(\kappa\tau) \leq \kappa\lambda(\tau)$ for every $\tau \in \mathbb{R}^+$ (where $\kappa > 1$). Then,

$$H_\delta(\text{Fix}_S, \text{Fix}_F) \leq \varphi(\eta).$$

For $\varepsilon \in \mathbb{R}$, we set

$$[\varepsilon]_q = \frac{1 - q^\varepsilon}{1 - q}.$$

Definition 2.8. [15] The q -derivative of order $n \in \mathbb{N}$ of $\mathfrak{g} : \mathfrak{J} \rightarrow \mathbb{R}$ is described as $(D_q^0 \mathfrak{g})(\tau) = \mathfrak{g}(\tau)$,

$$(D_q \mathfrak{g})(\tau) := (D_q^1 \mathfrak{g})(\tau) = \frac{\mathfrak{g}(\tau) - \mathfrak{g}(q\tau)}{(1-q)\tau}; \tau \neq 0, \quad (D_q \mathfrak{g})(0) = \lim_{t \rightarrow 0} (D_q \mathfrak{g})(\tau),$$

and

$$(D_q^n \mathfrak{g})(\tau) = (D_q D_q^{n-1} \mathfrak{g})(\tau); \tau \in \mathfrak{J}, n \in \{1, 2, \dots\}.$$

Set $I_s := \{sq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.9. [15] The q -integral of $\mathbf{g} : I_s \rightarrow \mathbb{R}$ is described as

$$(I_q \mathbf{g})(s) = \int_0^s \mathbf{g}(\tau) d_q \tau = \sum_{n=0}^{\infty} t(1-q)q^n \mathbf{g}(sq^n).$$

$(D_q I_q \mathbf{g})(s) = \mathbf{g}(s)$, while if \mathbf{g} is continuous at 0, then

$$(I_q D_q \mathbf{g})(s) = \mathbf{g}(s) - \mathbf{g}(0).$$

Definition 2.10. [8] The Riemann-Liouville fractional q -integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $\mathbf{g} : \mathfrak{J} \rightarrow \mathbb{R}$ is defined by $(I_q^\alpha \mathbf{g})(s) = \mathbf{g}(s)$, and

$$(I_q^\alpha \mathbf{g})(s) = \int_0^t \frac{(s - q\tau)^{(\alpha-1)}}{\Gamma_q(\alpha)} \mathbf{g}(\tau) d_q \tau; \quad t \in \mathfrak{J}.$$

Definition 2.11. [25] The Caputo fractional q -derivative of order $\alpha \in \mathbb{R}_+$ of a function $\mathbf{g} : \mathfrak{J} \rightarrow \mathbb{R}$ is defined by $({}^C D_q^\alpha \mathbf{g})(s) = \mathbf{g}(s)$, and

$$({}^C D_q^\alpha \mathbf{g})(s) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} \mathbf{g})(s); \quad s \in \mathfrak{J}.$$

Lemma 2.12. [25] Let $\alpha \in \mathbb{R}_+$. Then

$$(I_q^\alpha {}^C D_q^\alpha \mathbf{g})(s) = \mathbf{g}(s) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k \mathbf{g})(0).$$

In particular, if $\alpha \in (0, 1)$, then

$$(I_q^\alpha {}^C D_q^\alpha \mathbf{g})(s) = \mathbf{g}(s) - \mathbf{g}(0).$$

Lemma 2.13. Assume that $S_{F \circ \mathbf{g}} \subset C(\mathfrak{J})$ for each $\mathbf{g} \in C(\mathfrak{J})$. Then (1.1)-(1.2) is equivalent to $\mathbf{g} \in \mathcal{Q}(\mathbf{g})$, where $\mathcal{Q} : C(\mathfrak{J}) \rightarrow \mathfrak{P}(C(\mathfrak{J}))$ is the multi-function described as

$$(\mathcal{Q}\mathbf{g})(\tau) = \{\mathbf{g}_0 + (I_q^\alpha \mathbf{g})(\tau) : \mathbf{g} \in S_{F \circ \mathbf{g}}\}.$$

In this manuscript, we launch the study of the Ulam stability for Caputo fractional q -difference inclusions and related coupled systems via Picard operators theory, and it is structured as follows: Section 2 the first main result, existence and stability of (1.1) and (1.2), is expressed. Additionally, in Section 3; we obtain similar results for the coupled system (1.3)-(1.4). Lastly, in Section 4 an example is expressed to indicate the applicability of the derived theorem of the paper.

3 Caputo Fractional q -Difference Inclusions

Definition 3.1. \mathbf{g} is a solution of (1.1)-(1.2) if it achieves the condition (1.2), and the equation $\mathbf{g}(\tau) = \mathbf{g}_0 + (I_q^\alpha g)(\tau)$ on \mathfrak{J} , where $g \in S_{F \circ \mathbf{g}}$.

(H_1) The multifunction $\tau \mapsto F(\tau, \mathbf{g})$ is jointly measurable for each $\mathbf{g} \in \mathbb{R}$

(H_2) The multifunction $\mathbf{g} \mapsto F(\tau, \mathbf{g})$ is l.s.c. for a.a. $\tau \in \mathfrak{J}$;

(H_3) There exists a function $\varrho \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and $\lambda \in \mathfrak{S}$ so that

$$H_\delta(F(\tau, \mathbf{g}), F(\tau, \bar{\mathbf{g}})) \leq \varrho(\tau)\lambda(|\mathbf{g} - \bar{\mathbf{g}}|), \quad (3.1)$$

and

$$\frac{T^\alpha \|\varrho\|_{L^\infty}}{\Gamma_q(1 + \alpha)} \leq 1, \quad (3.2)$$

for almost all $\tau \in \mathfrak{J}$, and each $\mathbf{g}, \bar{\mathbf{g}} \in \mathbb{R}$;

(H_4) There exists $q \in L^1(\mathfrak{J}, \mathbb{R}_+)$ such that

$$F(\tau, \mathbf{g}) \subset q(\tau)B_0,$$

where $B_0 = \{\mathbf{g} \in C(\mathfrak{J}) : \|\mathbf{g}\|_C < 1\}$, for almost all $\tau \in \mathfrak{J}$ and each $\mathbf{g} \in \mathbb{R}$.

A self-mapping λ on $[0, \infty)$ is called quasi-homogenous function if $\lambda(\mathfrak{J}\tau) \leq \mathfrak{d}\lambda(\tau)$ for every $\tau \in \mathbb{R}^+$, where $\mathfrak{d} > 1$.

Theorem 3.2. Assume (H_1) – (H_4) hold. Then:

- (a) Pb. (1.1)-(1.2) admits least one solution and \mathcal{Q} is a m.w.P.o.;
- (b) Furthermore, if λ is quasi-homogenous, then Pb. (1.1)-(1.2) is g.U.t stable, and N is a φ -m.w.P.o., with

$$\varphi(\tau) := t + \sum_{n=1}^{\infty} \lambda^n(\tau), \quad \tau \in \mathbb{R}^+.$$

Proof. First, we assert $\mathcal{Q}(\mathbf{g}) \in \mathfrak{P}_{cp}(C(\mathfrak{J}))$ for each $\mathbf{g} \in C(\mathfrak{J})$. For each $\mathbf{g} \in C(\mathfrak{J})$ there exists $f \in S_{F \circ \mathbf{g}}$, (see [33]). Thus $\nu(\tau) = \mathbf{g}_0 + (I_q^\alpha f)(\tau)$ verify $\nu \in \mathcal{Q}(\mathbf{g})$. From (H_1) and (H_4), via Theorem 8.6.3. in [11], the set $\mathcal{Q}(\mathbf{g})$ is compact, for each $\mathbf{g} \in C(\mathfrak{J})$.

Next, we assert $H_\delta(\mathcal{Q}(\mathbf{g}), \mathcal{Q}(\bar{\mathbf{g}})) \leq \lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C)$ for each $\mathbf{g}, \bar{\mathbf{g}} \in C(\mathfrak{J})$. Let $\mathbf{g}, \bar{\mathbf{g}} \in C(\mathfrak{J})$ and $h \in \mathcal{Q}(\mathbf{g})$. Then, there exists $f \in S_{F \circ \mathbf{g}}$, with

$$h(\tau) = \mathbf{g}_0 + (I_q^\alpha f)(\tau).$$

We get, from (H_3)

$$H_\delta(F(\tau, \mathbf{g}(\tau)), F(\tau, \bar{\mathbf{g}}(\tau))) \leq \varrho(\tau)\lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C).$$

Consequently, there is $w \in S_{F \circ \bar{\mathbf{g}}}$, with

$$|f(\tau) - w(\tau)| \leq \varrho(\tau)\lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C); \tau \in I.$$

Construct a map $\mathfrak{G} : \mathfrak{J} \rightarrow \mathfrak{P}(\mathbb{R})$ by

$$\mathfrak{G}(\tau) = \{\mathbf{b} \in \mathbb{R} : |f(\tau) - \mathbf{b}(\tau)| \leq \varrho(\tau)\lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C)\}.$$

Due to the measurability of $\mathbf{g}(\tau) = \mathfrak{G}(\tau) \cap F(\tau, \bar{\mathbf{g}}(\tau))$ (Proposition III.4 in [13]), then there is \bar{f} which is a measurable selection function for \mathbf{g} . Thus, $\bar{f} \in S_{F \circ \bar{\mathbf{g}}}$, and for each $\tau \in \mathfrak{J}$,

$$|f(\tau) - \bar{f}(\tau)| \leq \varrho(\tau)\lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C).$$

Let the function

$$\bar{h}(\tau) = \mathbf{g}_0 + (I_q^\alpha \bar{f})(\tau).$$

Then

$$\begin{aligned} |h(\tau) - \bar{h}(\tau)| &\leq I_q^\alpha |f(\tau) - \bar{f}(\tau)| \\ &\leq I_q^\alpha (\varrho(\tau)\lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C)) \\ &\leq \|\varrho\|_{L^\infty} \lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C) \left(\int_0^\tau \frac{|\tau - qs|^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs \right) \\ &\leq \frac{T^\alpha \|\varrho\|_{L^\infty}}{\Gamma_q(1 + \alpha)} \lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C). \end{aligned}$$

Thus, from (4.2) yields

$$\|h - \bar{h}\|_C \leq \lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C).$$

By verbatim with changing the roles of \mathbf{g} and $\bar{\mathbf{g}}$, it yields

$$H_\delta(\mathcal{Q}(\mathbf{g}), \mathcal{Q}(\bar{\mathbf{g}})) \leq \lambda(\|\mathbf{g} - \bar{\mathbf{g}}\|_C).$$

Hence, \mathcal{Q} is a λ -contraction.

(a) Lemma 2.6 infer that \mathcal{Q} possesses a fixed point on \mathfrak{J} , and from [Theorem 2.7, (i)] we conclude that \mathcal{Q} is a m.w.P.o.

(b) The problem (1.1)-(1.2) is g.U-H stable. For clarification, for $\varepsilon > 0$ and $\nu \in C(\mathfrak{J})$ there is $\mathbf{g} \in C(\mathfrak{J})$ so that

$$\mathbf{g}(\tau) - \mathbf{g}_0 \in (I_q^\alpha F)(\tau, \nu(\tau)); \tau \in \mathfrak{J},$$

and

$$\|\mathbf{g} - \nu\|_C \leq \varepsilon,$$

where

$$(I_q^\alpha F)(\tau, \nu(\tau)) = \{(I_q^\alpha w)(\tau); w \in S_{F \circ \nu}\}; \tau \in \mathfrak{J}.$$

Then $H_\delta(y, \mathcal{Q}(y)) \leq \varepsilon$. Moreover, the multivalued map \mathcal{Q} is λ -contraction, and from [Theorem 2.7, (i)-(ii)], \mathcal{Q} is a φ -m.w.P.o. Tus, $\mathbf{g} \in \mathcal{Q}(\mathbf{g})$ is g.U-H stable. Hence, our problem (1.1)-(1.2) is g.U-H stable. Theorem 2.7,(iii) concludes the result.

4 Caputo Fractional q -Difference Inclusions

Ulam stability of the problem (1.1)-(1.2) shall be discussed in this section.

Definition 4.1. *If a continuous \mathbf{g} along the initial condition (1.2) achieve $\mathbf{g}(\mathfrak{t}) = \mathbf{g}_0 + (I_q^\alpha g)(\mathfrak{t})$ on \mathfrak{J} , where $g \in S_{F \circ \mathbf{g}}$, then we say that it is a solution of the problem (1.1)-(1.2)*

Now, we present requirements for both Ulam stability of problem (1.1)-(1.2).

The following are the basic requirements for our aim:

- (H₁) The multifunction $\mathfrak{t} \mapsto F(\mathfrak{t}, \mathbf{g})$ is jointly measurable for each $\mathbf{g} \in \mathbb{R}$
- (H₂) The multifunction $\mathbf{g} \mapsto F(\mathfrak{t}, \mathbf{g})$ is lower semi-continuous for almost all $\mathfrak{t} \in \mathfrak{J}$;
- (H₃) There exists $\rho \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and $\varphi \in \mathfrak{S}$ so that for for almost all $\mathfrak{t} \in \mathfrak{J}$, and each $\mathbf{g}, \bar{\mathbf{g}} \in \mathbb{R}$, we have

$$H_d(F(\mathfrak{t}, \mathbf{g}), F(\mathfrak{t}, \bar{\mathbf{g}})) \leq \varrho(\mathfrak{t})\varphi(|\mathbf{g} - \bar{\mathbf{g}}|), \quad (4.1)$$

and

$$\frac{T^\alpha \|\varrho\|_{L^\infty}}{\Gamma_q(1 + \alpha)} \leq 1; \quad (4.2)$$

- (H₄) we have $F(\mathfrak{t}, \mathbf{g}) \subset q(\mathfrak{t})B(0, 1)$, for almost all $\mathfrak{t} \in \mathfrak{J}$ and each $\mathbf{g} \in \mathbb{R}$, where $q : \mathfrak{J} \rightarrow \mathbb{R}$ is integrable and $B(0, 1) = \{\mathbf{g} \in C(\mathfrak{J}) : \|\mathbf{g}\|_C < 1\}$.

Theorem 4.2. *Suppose that (H₁) – (H₄) are achieved by the multifunction $F : \mathfrak{J} \times \mathbb{R} \rightarrow \mathfrak{P}_{cp}(\mathbb{R})$ Then,*

- (a) *Problem (1.1)-(1.2) possesses a solution and \mathcal{Q} is a m.w.P.o;*

(b) In addition, if λ is quasi-homogenous, then the problem defined by (1.1)-(1.2) is *g.U.t.* stable, and \mathfrak{Q} is a λ -m.w.P.o, where λ is described as

$$\lambda(\mathfrak{t}) := t + \sum_{n=1}^{\infty} \varphi^n(\mathfrak{t}), \text{ for each } \mathfrak{t} \in [0, \infty).$$

Remark 4.3. Note that $S_{F \circ \mathfrak{g}}$ is nonempty for all $\mathfrak{g} \in C(\mathfrak{J})$ on account of (H_1) F has a measurable selection (see [13], Theorem III.6)

Proof. Let \mathfrak{Q} a mapping as described in Lemma 2.13. We assert that it achieves the hypothesis of Theorem 2.7. We assert first that $\mathfrak{Q}(x) \in \mathfrak{P}_{cp}(C(\mathfrak{J}))$ for each $\mathfrak{g} \in C(\mathfrak{J})$.

On account of Theorem 2 in [33], for each $\mathfrak{g} \in C(\mathfrak{J})$ there is $f \in S_{F \circ \mathfrak{g}}$, for all $\mathfrak{t} \in \mathfrak{J}$. Then, $\nu(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha f)(\mathfrak{t})$ has the property $\nu \in \mathfrak{Q}(\mathfrak{g})$. In addition, taking (H_1) and (H_4) , together with Theorem 8.6.3. in [11], we find that for each $\mathfrak{g} \in C(\mathfrak{J})$, the set $\mathfrak{Q}(\mathfrak{g})$ is compact.

Next, we assert that $H_d(\mathfrak{Q}(\mathfrak{g}), \mathfrak{Q}(\bar{\mathfrak{g}})) \leq \varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C)$ for each $\mathfrak{g}, \bar{\mathfrak{g}} \in C(\mathfrak{J})$. Let $\mathfrak{g}, \bar{\mathfrak{g}} \in C(\mathfrak{J})$ and $h \in \mathfrak{Q}(\mathfrak{g})$. So, there is $f \in S_{F \circ \mathfrak{g}}$, so that

$$h(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha f)(\mathfrak{t}),$$

for each $\mathfrak{t} \in \mathfrak{J}$. Due to (H_3) , we have

$$H_d(F(\mathfrak{t}, \mathfrak{g}(\mathfrak{t})), F(\mathfrak{t}, \bar{\mathfrak{g}}(\mathfrak{t}))) \leq \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C).$$

Consequently, there is $w \in S_{F \circ \bar{\mathfrak{g}}}$, with

$$|f(\mathfrak{t}) - w(\mathfrak{t})| \leq \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C); \mathfrak{t} \in \mathfrak{J}.$$

We set $\mathfrak{G} : \mathfrak{J} \rightarrow \mathfrak{P}(\mathbb{R})$ as follows

$$\mathfrak{G}(\mathfrak{t}) = \{w \in \mathbb{R} : |f(\mathfrak{t}) - w(\mathfrak{t})| \leq \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C)\}.$$

Note that $\mathfrak{g}(\mathfrak{t}) = \mathfrak{G}(\mathfrak{t}) \cap F(\mathfrak{t}, \bar{\mathfrak{g}}(\mathfrak{t}))$ is a measurable multivalued operator due to Proposition III.4 in [13]. Consequently, there is a measurable selection function \bar{f} for \mathfrak{g} . Thus, $\bar{f} \in S_{F \circ \bar{\mathfrak{g}}}$, and

$$|f(\mathfrak{t}) - \bar{f}(\mathfrak{t})| \leq \varrho(\mathfrak{t})\varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C),$$

for all $\mathfrak{t} \in \mathfrak{J}$. Define

$$\bar{h}(\mathfrak{t}) = \mathfrak{g}_0 + (\mathbb{I}_q^\alpha \bar{f})(\mathfrak{t}),$$

for each $t \in \mathfrak{J}$. Consequently, we find

$$\begin{aligned} |h(t) - \bar{h}(t)| &\leq \mathfrak{J}_q^\alpha |f(t) - \bar{f}(t)| \\ &\leq \mathfrak{J}_q^\alpha (\varrho(t) \varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C)) \\ &\leq \|\varrho\|_{L^\infty} \varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C) \left(\int_0^t \frac{|t - qs|^{(\alpha-1)}}{\Gamma_q(\alpha)} d_qs \right) \\ &\leq \frac{T^\alpha \|\varrho\|_{L^\infty}}{\Gamma_q(1 + \alpha)} \varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C), \end{aligned}$$

for each $t \in \mathfrak{J}$. On account of (4.2), we find

$$\|h - \bar{h}\|_C \leq \varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C).$$

Regarding the analogy, changing the roles of \mathfrak{g} and $\bar{\mathfrak{g}}$, yields

$$H_d(\mathcal{Q}(\mathfrak{g}), \mathcal{Q}(\bar{\mathfrak{g}})) \leq \varphi(\|\mathfrak{g} - \bar{\mathfrak{g}}\|_C).$$

As a result, \mathcal{Q} is a φ -contraction.

By taking Lemma 2.6 into account, we deduce that fixed point of \mathcal{Q} possesses a solution of the inclusion (1.1)-(1.2) on \mathfrak{J} . Further, [Theorem 2.7, (i)] yields that \mathcal{Q} is a m.w.P.o.

Now, we assert that the problem (1.1)-(1.2) is g.U.t stable. For this purpose, take $\epsilon > 0$ and $\nu \in C(\mathfrak{J})$ for which there is $\mathfrak{g} \in C(\mathfrak{J})$ so that

$$\mathfrak{g}(t) - \mathfrak{g}_0 \in (\mathfrak{J}_q^\alpha F)(t, \vartheta(t)); \quad t \in \mathfrak{J},$$

and

$$\|\mathfrak{g} - \nu\|_C \leq \epsilon,$$

with

$$(\mathfrak{J}_q^\alpha F)(t, \nu(t)) = \{(\mathfrak{J}_q^\alpha w)(t); w \in S_{F \circ \nu}\}; \quad t \in \mathfrak{J}.$$

Then $H_d(y, \mathcal{Q}(y)) \leq \epsilon$. In addition, we conclude that \mathcal{Q} is a multivalued φ -contraction. Regarding [Theorem 2.7, (i)-(ii)], we deduce that \mathcal{Q} is a Ψ -m.w.P.o. Thus, the fixed point problem $\mathfrak{g} \in \mathcal{Q}(\mathfrak{g})$ is g.U.t. stable. In conclusion, the problem (1.1)-(1.2) is g.U.t. stable. The rest follows from [Theorem 2.7, (iii)].

5 Coupled System of Caputo Fractional q -Difference Inclusions

This section is devoted for the existence, uniqueness and Ulam stability of (1.3)-(1.4). Here, $\mathcal{C} := C(\mathfrak{J}) \times C(\mathfrak{J})$ denotes the Banach space with the norm

$$\|(\mathfrak{g}, \nu)\|_e = \|\mathfrak{g}\|_C + \|\nu\|_C.$$

Lemma 5.1. *Let $G : \mathcal{C} \rightarrow \mathcal{C}$ described as*

$$(G(\mathbf{g}_1, \mathbf{g}_2))(\mathfrak{t}) = ((G_1 \mathbf{g}_1)(\mathfrak{t}), G_2 \mathbf{g} - 2)(\mathfrak{t}); \quad \mathfrak{t} \in \mathfrak{J}, \quad (5.1)$$

where $G_i : C(\mathfrak{J}) \rightarrow C(\mathfrak{J})$; $i = 1, 2$, are defined by

$$(G_i(\mathbf{g}_1, \mathbf{g}_2))(\mathfrak{t}) = \mathfrak{i}_i + (\mathfrak{J}_{q_i}^{\alpha_i} g_i)(\mathfrak{t}); \quad \mathfrak{t} \in \mathfrak{J}, \quad (5.2)$$

where $g_i \in S_{F_i \circ \mathbf{g}_i}$; $i = 1, 2$. Then, fixed points of G form the solutions of the system (1.3)-(1.4).

Definition 5.2. *By a coupled solutions of problem (1.3)-(1.4) we mean a continuous coupled functions $(\mathbf{g}_1, \mathbf{g}_2)$ those satisfy the initial condition (1.4), and the equations $\mathbf{g}_i(\mathfrak{t}) = \mathfrak{i}_i + (\mathfrak{J}_{q_i}^{\alpha_i} \nu_i)(\mathfrak{t})$ on \mathfrak{J} , where $\nu_i \in S_{F_i \circ \mathbf{g}_i}$; $i = 1, 2$.*

Keeping Lemma 5.1 on mind, we shall investigate the existence and Ulam stability of (1.3)-(1.4), as in Theorem 4.2.

Theorem 5.3. *Assume that the multifunctions $F_i : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{N}_{cp}(\mathbb{R})$ satisfy the following hypotheses*

- (H₀₁) *The multifunctions $\mathfrak{t} \mapsto F_i(\mathfrak{t}, \mathbf{g}_1, \mathbf{g}_2)$ are jointly measurable for each $\mathbf{g}_i \in \mathbb{R}$; $i = 1, 2$,*
- (H₀₂) *The multifunctions $\mathbf{g}_i \mapsto F(\mathfrak{t}, \mathbf{g}_1, \mathbf{g}_2)$ are lower semi-continuous for almost all $\mathfrak{t} \in \mathfrak{J}$;*
- (H₀₃) *There exist $p_i \in L^\infty(\mathfrak{J}, \mathbb{R}_+)$ and $\varphi_i : \mathfrak{S} \rightarrow \mathfrak{S}$ such that*

$$H_d(F_i(\mathfrak{t}, \mathbf{g}_1, \mathbf{g}_2), F_i(\mathfrak{t}, \overline{\mathbf{g}}_1, \overline{\mathbf{g}}_2)) \leq p_i(\mathfrak{t}) \varphi_i(|\mathbf{g}_i - \overline{\mathbf{g}}_i|), \quad (5.3)$$

and

$$\frac{T^{\alpha_i} \|p_i\|_{L^\infty}}{\Gamma_q(1 + \alpha_i)} \leq 1; \quad (5.4)$$

or for almost all $\mathfrak{t} \in \mathfrak{J}$, and each $\mathbf{g}_1, \mathbf{g}_2, \overline{\mathbf{g}}_1, \overline{\mathbf{g}}_2 \in \mathbb{R}$.

- (H₀₄) *There exist integrable functions $q_i : \mathfrak{J} \rightarrow \mathbb{R}$ such that for almost all $\mathfrak{t} \in \mathfrak{J}$ and each $\mathbf{g}_i \in \mathbb{R}$; $i = 1, 2$, we have*

$$F_i(\mathfrak{t}, \mathbf{g}_1, \mathbf{g}_2) \subset q_i(\mathfrak{t})B(0, 1),$$

where $B(0, 1) = \{\nu \in C(\mathfrak{J}) : \|\nu\|_C < 1\}$.

Then, we have

- (a) *Problem (1.3)-(1.4) possess a solution and G is a m.w.P.o;*

- (b) In addition, if each φ_i is quasi-homogenous, ($i = 1, 2,$) then the problem (1.3)-(1.4) is $g. U. t.$ stable, and G is a Ψ -m.w.P.o, with $\Psi = (\Psi_1, \Psi_2)$ and the the functions $\Psi_i; i = 1, 2$ defined by $\Psi_i(t) := t + \sum_{n=1}^{\infty} \varphi_i^n(t)$, for each $t \in [0, \infty)$.

6 An Example

We aim to deal with the Cauchy problem of Caputo fractional $\frac{1}{4}$ -difference inclusion

$$\begin{cases} ({}^C \mathfrak{D}^{\frac{1}{4}} \mathbf{g})(t) \in F(t, \mathbf{g}(t)); t \in [0, 1], \\ \mathbf{g}(t)|_{t=0} = 1, \end{cases} \quad (6.1)$$

for

$$F(t, \mathbf{g}(t)) = \{\nu \in C([0, 1], \mathbb{R}) : |f_1(t, \mathbf{g}(t))| \leq |\nu| \leq |f_2(t, \mathbf{g}(t))|\}; t \in [0, 1],$$

where $f_1, f_2 : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, such that

$$f_1(t, \mathbf{g}(t)) = \frac{t^2 \mathbf{g}(t)}{(1 + |\mathbf{g}(t)|)e^{10+t}}, \quad f_2(t, \mathbf{g}(t)) = \frac{t^2 \mathbf{g}(t)}{e^{10+t}}.$$

Set $\alpha = \frac{1}{2}$ and suppose that F is both convex and closed multivalued function. Notice that the solutions of the problem (6.1) are the solutions

$$\mathbf{g} \in A(\mathbf{g}) \text{ (the fixed point inclusion)}$$

where the multifunction operator $A : C([0, 1], \mathbb{R}) \rightarrow \mathfrak{P}(C([0, 1], \mathbb{R}))$ is described as

$$(A\mathbf{g})(t) = \left\{ 1 + (\mathfrak{J}^{\frac{1}{4}} f)(t); f \in S_{F \circ \mathbf{g}} \right\}; t \in [0, 1].$$

For each $t \in [0, 1]$ and all $z_1, z_2 \in C([0, 1], \mathbb{R})$, we have

$$\|f_2(t, z_2) - f_1(t, z_1)\|_C \leq t^2 e^{-10-t} \|z_2 - z_1\|_C.$$

Consequently, we conclude that all hypotheses $(H_1) - (H_3)$ are achieved with $\varrho(t) = t^2 e^{-10-t}$.

As a next step, we indicate that condition (4.2) is fulfilled for $T = 1$. For clarification, we note that $\|\varrho\|_{L^\infty} = e^{-9}$, $\Gamma_{\frac{1}{4}}(1 + \frac{1}{2}) > \frac{1}{2}$. After an elementary calculation, one can get that

$$\Delta := \frac{T^{\frac{3}{4}} \|\varrho\|_{L^\infty}}{\Gamma_{\frac{1}{4}}(1 + \frac{1}{2})} < 2e^{-9} < 1.$$

Furthermore, (H_4) is fulfilled with $q(t) = \frac{t^2 e^{-10-t}}{\|F\|_{\mathfrak{F}}}$; $t \in [0, 1]$, where

$$\|F\|_{\mathfrak{F}} = \sup\{\|f\|_C : f \in S_{F \circ g}\}; \text{ for all } g \in C([0, 1], \mathbb{R}).$$

As a result, Theorem 4.2 implies that:

- (a) The problem (6.1) possesses a solution and A is a m.w.P.o.
- (b) The function $\varphi(t) = \Delta t$ forms quasi-homogenous. Hence, the problem (6.1) is g.U.t. stable, and A is a Ψ -m.w.P.o, with the function Ψ defined by $\Psi(t) := t + (1 - \Delta t)^{-1}$, for each $t \in [0, \Delta^{-1})$.

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