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ORIGINAL PAPER

Dynamics of multi-point singular fifth-order Lane–Emden system with neuro-evolution heuristics

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Abstract
The objective of the presented communication is to examine and analyze the solutions of nonlinear multi-singular fifth-order Lane–Emden (LE) system for different scenarios by variation of shape factors settled on the equivalent design of the LE equations. The neuro-evolution based stochastic computing is explored for the numerical measures using the artificial neural networks (ANNs) models for the appropriate continuous mapping, while the learning of decision variables is conducted using the integrated meta-heuristic global search of genetic algorithms (GA) hybrid with the local search efficiency of active-set (AS) i.e., ANN-GA-AS scheme. The numerical approach ANN-GA-AS is applied efficiently for the fifth kind of nonlinear LE model and statistical calculations further validate the accuracy, robustness as well as convergence.

Keywords Lane–Emden system · Singular systems · Numerical computing · Active-set method · Genetic algorithms · Artificial neural networks, Statistical assessments

1 Introduction

The Emden–Fowler (EF) equation has a variety of applications in fluid dynamics, pattern formation, chemical reactor systems, population progression, and relativistic mechanics. The mathematical relation of the Emden–Fowler systems in standard form is written as follows (Davis 1962; Chandrasekhar and Chandrasekhar 1957):

$$\begin{aligned}&\omega'' + \frac{k}{t}\omega' + f(t)g(\omega) = 0,\\&\omega(0) = A,\ \ \omega'(0) = 0,\end{aligned} \tag{1}$$

where ω represent the diffusion/reaction index, which appear in many branches of applied science and technology, while $k \geq 1$ represents the shape factor. The Emden–Fowler in (1) is related with Lane–Emden (LE) system by taking the value of $f(t) = 1$ and written as (Zhou and Angelov 2007):

$$\begin{aligned}&\omega'' + \frac{k}{t}\omega' + g(\omega) = 0,\\&\omega(0) = A,\ \ \omega'(0) = 0.\end{aligned} \tag{2}$$

The famous LE equation describes the self-gravitating gas clouds, interior structure of radiative cooling, polytrophic stars, temperature deviation of a spherical gas and used to model the galaxies clusters. The singular models have vast

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applications in physical sciences, dusty fluid models, electromagnetic theory, catalytic diffusion reaction, mathematical physics, classical/quantum mechanics, oscillation based magnetic fields, stellar structure, isotropic continuous media and morphogenesis (Angelov and Zhou 2008; Angelov et al. 2017; Khan et al. 2015; Rach et al. 2014; Bhrawy et al. 2014; Ramos 2003; Dehghan and Shakeri 2008; Taghavi and Pearce 2013; Radulescu and Repovs 2012; Ghergu and Radulescu 2007). The singular models are always very stiff and challenging to solve because of their hard nature and a singular point at the origin. For the Lane–Emden singular system relatively very few existing techniques are reported in the literature including Adomian decomposition (AD) technique, deterministic numerical scheme, analytic approach perturbative methods, power series method along with Pade approximation techniques (Shawagfeh 1993; Wazwaz 2001; Parand and Razzaghi 2004; Liao 2003; Bender et al. 1989; Nouh 2004).

The objective of the presented study is to design and analyze numerically the fifth-order nonlinear LE systems using artificial neural networks (ANNs) trained with the efficacy of global search via genetic algorithms (GAs) combined by active-set (AS), i.e., ANN-GA-AS algorithm, a kind of meta-heuristic computing solver. To construct the fifth-order nonlinear LE systems, the design of the second, third and fourth kinds of singular models based on the delay, pantograph, prediction have been presented (Sabir et al. 2021a, b, c, d, e, f, g; Sabir 2021a, b; Nisar et al. 2021; Abdelkawy et al. 2020; Guirao et al. 2020). Keeping in view of these frameworks, authors are motivated to design a fifth order singular model.

The meta-heuristic intelligent solvers have been broadly implemented to present the analysis of singular/non-singular, linear/nonlinear models using ANN optimized with swarming/ evolutionary-based computing techniques (Lodhi et al. 2019; Raja et al. 2017; Sabir et al. 2020; Umar et al. 2019a). Some recent studies involving evolutionary or swarming based numerical computing are nonlinear Bouc–Wen hysteresis model using supervised neural networks (Naz et al. 2021), HIV infectious disease spread model integrated heuristics (Umar et al. 2020, 2021), exploitation of neuro-heuristics for atomic physics model governed with Thomas–Fermi system (Sabir et al. 2018), heartbeat dynamics model via nonlinear Van der Pol systems (Raja et al. 2018a), nonlinear doubly singular differential systems (Raja et al. 2019), nonlinear singular system for temperature distribution model (Raja et al. 2018b), magnetohydrodynamics flow model over stretchable rotating disk (Mehmood et al. 2018), mathematical models in financial studies via fractional partial differential equations (Ara et al. 2018), controls (Raja et al. 2018c) and power (Shahid et al. 2020). These effective and current submissions motivated the authors to solve the novel fifth-order LE model using numerical computing solver ANN-GA-AS algorithm. Few significant characteristics of the designed methodology ANN-GA-AS algorithm to solving the novel fifth-order LE model in terms of innovative contributions as follows:

1. A novel fifth-order LE equation based mathematical model is presented with the necessary derivation procedure using the standard LE form.
2. A new computing procedure of neuro-evolution heuristics via ANN-GA-AS algorithm is presented to study the dynamics of singular fifth-order multi-point LM system numerically.
3. Two different models of the novel fifth-order LE model of type 1 are implemented effectively by ANN-GA-AS algorithm and comparative study further authenticate its exactness, stability and convergence on single and multiple autonomous runs
4. The design of the soft computing/ machine learning procedures-based ANN-GA-AS algorithm has broad applicability to be implemented for high order LE equation in different engineering and technologies applications.
5. The proposed ANN-GA-AS algorithm performance is recognized through the statistical studies in terms of mean absolute error (MAE), semi interquartile range (S.IR) and Theil's inequality coefficient (TIC).

The reaming parts of the current study is briefly presented as: the construction of fifth-order LE model along with four types is presented in Sect. 2. Method analysis and modeled based examples of each type are given in Sect. 3. The mathematical formulations of the performance measures are described in Sect. 4. The detailed numerical results for solving the designed model-based examples are presented in Sect. 5. The conclusions/research commands are reported in Sect. 6.

2 Construction of fifth-order LE equation

In this section, the construction of LE system of fifth-order is described. The initial conditions are obtained by using the initial conditions sense of the standard LE equation. For the derivation of fifth-order LE equation, the following equation is used as:

$$t^{-k}\frac{d^n}{dt^n}\left(t^k\frac{d^m}{dt^m}\right)\omega+g(\omega)=0, \tag{3}$$

where k is used for positive real number representing the shape factor. To determine the fifth-order differential equation, the values of n and m should be selected as:

$$n+m=5, \quad n,\ m\geq 1, \tag{4}$$

That makes the following four possibilities as:

$$n=4,\ m=1, \tag{5}$$

$$n=3,\ m=2, \tag{6}$$

$$n = 2,\ m = 3, \tag{7}$$

$$n = 1,\ m = 4. \tag{8}$$

3 Type 1

Equation (3) by putting $n = 4,\ m = 1$ takes the form as:

$$t^{-k}\frac{d^4}{dt^4}\left(t^k\frac{d}{dt}\right)\omega + g(\omega) = 0. \tag{9}$$

By calculating the derivatives of

$$\frac{d^4}{dt^4}\left(t^k\frac{d}{dt}\right)\omega = t^k\frac{d^5\omega}{dt^5} + 4kt^{k-1}\frac{d^4\omega}{dt^4} + 6k(k-1)t^{k-2}\frac{d^3\omega}{dt^3} + 4k(k-1)(k-2)t^{k-3}\frac{d^2\omega}{dt^2} + k(k-1)(k-2)(k-3)t^{k-4}\frac{d\omega}{dt}. \tag{10}$$

Using the Eqs. (9) and (10). The multi-singular fifth-order LE equation takes the form as:

$$\begin{cases} \dfrac{d^5\omega}{dt^5} + \dfrac{4k}{t}\dfrac{d^4\omega}{dt^4} + \dfrac{6k(k-1)}{t^2}\dfrac{d^3\omega}{dt^3} + \dfrac{4k(k-1)(k-2)}{t^3}\dfrac{d^2\omega}{dt^2} + \dfrac{k(k-1)(k-2)(k-3)}{t^4}\dfrac{d\omega}{dt} + g(\omega) = 0, \\ \omega(0) = A,\ \dfrac{d\omega(0)}{dt} = 0,\ \dfrac{d^2\omega(0)}{dt^2} = 0,\ \dfrac{d^3\omega(0)}{dt^3} = 0,\ \dfrac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{11}$$

The singular point $t = 0$ represents four times as $t = 0,\ t^2 = 0,\ t^3 = 0$ and $t^4 = 0$ with $4k,\ 6k(k-1),\ 4k(k-2)(k-1)$ and $k(k-3)(k-2)(k-1)$ shape factors, respectively. Furthermore, the 3rd, 4th and 5th terms neglect for $k = 1$ and in this case, the shape factor value transform to 4. However, for $k = 2$, the 4th and 5th terms vanish and accordingly the shape factor for 2nd and 3rd terms are 8 and 12, respectively. Moreover, for $k = 3$, the shape factor values are 12, 36 and 24 for respective 2nd, 3rd and 4th terms.

4 Type 2

Equation (3) by putting $n = 3,\ m = 2$ takes the form as:

$$t^{-k}\frac{d^3}{dt^3}\left(t^k\frac{d^2}{dt^2}\right)\omega + g(\omega) = 0. \tag{12}$$

By calculating the derivatives of

$$\frac{d^3}{dt^3}\left(t^k\frac{d^2}{dt^2}\right)\omega = t^k\frac{d^5\omega}{dt^5} + 3kt^{k-1}\frac{d^4\omega}{dt^4} + 3k(k-1)t^{k-2}\frac{d^3\omega}{dt^3} + k(k-1)(k-2)t^{k-3}\frac{d^2\omega}{dt^2} \tag{13}$$

Using the Eqs. (12) and (13). The multi-singular fifth-order LE equation takes the form as:

$$\begin{cases} \dfrac{d^5\omega}{dt^5} + \dfrac{3k}{t}\dfrac{d^4\omega}{dt^4} + \dfrac{3k(k-1)}{t^2}\dfrac{d^3\omega}{dt^3} + \dfrac{k(k-1)(k-2)}{t^3}\dfrac{d^2\omega}{dt^2} + g(\omega) = 0, \\ \omega(0) = A,\ \dfrac{d\omega(0)}{dt} = B,\ \dfrac{d^2\omega(0)}{dt^2} = 0,\ \dfrac{d^3\omega(0)}{dt^3} = 0,\ \dfrac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{14}$$

The singular point $t = 0$ is noted three times as $t = 0,\ t^2 = 0$ and $t^3 = 0$ along with the shape factors $3k,\ 3k(k-1)$ and $k(k-2)(k-1)$, respectively. Furthermore, the third and fourth factors vanish for $k = 1$ and in this case the shape factor becomes 3. However, for $k = 2$, the fourth term vanishes, whereas, for the second and third terms, the values of the shape factors are found respective as 6 and 6.

5 Type 3

Equation (3) by putting $n = 2,\ m = 3$ takes the form as:

$$t^{-k}\frac{d^2}{dt^2}\left(t^k\frac{d^3}{dt^3}\right)\omega + g(\omega) = 0. \tag{15}$$

By calculating the derivatives of

$$\frac{d^2}{dt^2}\left(t^k\frac{d^3}{dt^3}\right)\omega = t^k\frac{d^5\omega}{dt^5} + 2kt^{k-1}\frac{d^4\omega}{dt^4} + k(k-1)t^{k-2}\frac{d^3\omega}{dt^3} \tag{16}$$

Using the Eqs. (15) and (16). The multi-singular fifth-order LE equation takes the form as:

$$\begin{cases} \dfrac{d^5\omega}{dt^5} + \dfrac{2k}{t}\dfrac{d^4\omega}{dt^4} + \dfrac{k(k-1)}{t^2}\dfrac{d^3\omega}{dt^3} + g(\omega) = 0, \\ \omega(0) = A,\ \dfrac{d\omega(0)}{dt} = B,\ \dfrac{d^2\omega(0)}{dt^2} = C,\ \dfrac{d^3\omega(0)}{dt^3} = \dfrac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{17}$$

In the above case, the singular points at $t = 0$ appear two times as $t = 0,$ and $t^2 = 0$ with $2k$ and $k(k-1)$ shape factors, respectively. Furthermore, the third term for $k = 1$ vanishes and in this case, the shape factor values become 2.

6 Type 4

Equation (3) by putting $n = 1,\ m = 4$ takes the form as:

$$t^{-k}\frac{d}{dt}\left(t^k\frac{d^4}{dt^4}\right)\omega + g(\omega) = 0. \tag{18}$$

By calculating the derivatives of

$$\frac{d}{dt}\left(t^k\frac{d^4}{dt^4}\right)\omega = t^k\frac{d^5\omega}{dt^5} + kt^{k-1}\frac{d^4\omega}{dt^4} \tag{19}$$

Using the Eqs. (18) and (19). The multi-singular fifth-order LE equation takes the form as:

$$\begin{cases} \dfrac{d^5\omega}{dt^5} + \dfrac{k}{t}\dfrac{d^4\omega}{dt^4} + g(\omega) = 0, \\ \omega(0) = A,\ \dfrac{d\omega(0)}{dt} = B,\ \dfrac{d^2\omega(0)}{dt^2} = C,\ \dfrac{d^3\omega(0)}{dt^3} = D,\ \dfrac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{20}$$

In the above case, the singular points at $t=0$ appear as $t = 0$ with shape factor k.

7 Method analysis and numerical examples

Two different examples of singular system of each case have been presented and both problems are solved numerically by using designed ANN-GA-AS algorithm. The structure for the 5th order LE system is divided into two sub-sections.

- To introduce an error-based merit or cost function for the associated differential equations along with their boundary conditions.
- The hybrid integrated intelligent computing via ANN-GA-AS algorithm is portrayed to optimize the cost or fitness function for 5th order nonlinear LE model.

7.1 ANN modeling

The feed-forward ANN models for approximating the results, i.e., solution of differential ω and their lth derivatives, are mathematically formulated as:

$$\hat{\omega} = \sum_{j=1}^{J} a_j f(b_j t + c_j), \tag{21}$$

$$\hat{\omega}^{(l)} = \sum_{j=1}^{J} a_j f^{(l)}(b_j t + c_j), \tag{22}$$

where a_j, b_j and c_j represent the jth components of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, while l represents derivative order. $\hat{\omega}$ is the approximate of ω and $\hat{\omega}^{(l)}$ is the approximate of $\omega^{(l)}$. The log-sigmoid function, i.e., $f(t) = (1+e^{-t})^{-1}$ along with its derivative up to fifth-order used as an activation function. The updated form of the above system for the log-sigmoid function is provided as:

$$\hat{\omega} = \sum_{j=1}^{J} a_j\left(1 + e^{-(b_j t + c_j)}\right)^{-1}, \tag{23}$$

$$\hat{\omega}^{(l)} = \sum_{j=1}^{J} a_j \frac{d^l}{dt^l}\left(\left(1 + e^{-(b_j t + c_j)}\right)^{-1}\right). \tag{24}$$

The fifth order derivative is provided as:

$$\hat{\omega}^{(v)} = \sum_{j=1}^{J} a_j b_j^5 \left(\begin{array}{l} \dfrac{120e^{-5(b_jt+c_j)}}{\left(1+e^{-(b_jt+c_j)}\right)^6} - \dfrac{240e^{-4(b_jt+c_j)}}{\left(1+e^{-(b_jt+c_j)}\right)^5} + \dfrac{150e^{-3(b_jt+c_j)}}{\left(1+e^{-(b_jt+c_j)}\right)^4} \\ -\dfrac{30e^{-2(b_jt+c_j)}}{\left(1+e^{-(b_jt+c_j)}\right)^3} - \dfrac{e^{-(b_jt+c_j)}}{\left(1+e^{-(b_jt+c_j)}\right)^2} \end{array} \right), \tag{25}$$

where the weights are $\boldsymbol{a} = [a_1, a_2, \ldots a_J]$,$\boldsymbol{b} = [b_1, b_2, \ldots, b_J]$ and $\boldsymbol{c} = [c_1, c_2, \ldots, c_J]$, respectively. For solving the fifth-order LE model, the fitness function becomes as:

$$E = E_1 + E_2, \tag{26}$$

where E_1 and E_2 represent the error functions on mean square sense that is related to the differential system and its initial conditions. The LE Eq. (11) for type 1 with $k=4$, these error functions are defined as follows:

$$E = \frac{1}{J}\sum_{j=1}^{J}\left(t_j^4\hat{\omega}_j^{(v)} + 16t_j^3\hat{\omega}_j^{(iv)} + 72t_j^2\hat{\omega}_J''' + 96t_j\hat{\omega}_j'' + 24\hat{\omega}_j' + t_j^4\hat{\omega}_j - 8401t_j^4 - t_j^9)\right)^2,$$

$$E_2 = \frac{1}{5}\left(\left(\hat{\omega}_0 - 1\right)^2 + \left(\hat{\omega}_0'\right)^2 + \left(\hat{\omega}_0''\right)^2 + \left(\hat{\omega}_0'''\right)^2 + \left(\hat{\omega}_0^{iv}\right)^2\right).$$

Similarly, the fitness function for other type of LE equation are constructed.

7.2 Optimization of networks

The ANNs weights are accomplished by functioning the integrated computing meta-heuristic process in terms of GAs enhanced by AS method. The graphical plots of the proposed ANN-GA-AS method for the fifth-order LE model is given in Fig. 1.

GA is a broadly global search scheme introduced by Holand in the last century (Srinivas and Deb 1994) and applied to exploit/explore the weight vector 'W'. The design of the population with applicant individual/solutions is implemented using the restrained real numbers. Each specific applicant has a variety of elements equal to anonymous weights in the ANNs. GAs always operates from its fundamental operatives like 'selection', 'crossover', 'elitism', and 'mutation' that has been used in wide-ranging applications. Few recent applications are face recognition (Zhi and Liu 2019), optimize heterogeneous bin packing (Sridhar et al. 2017) and image classification (Sun et al. 2020).

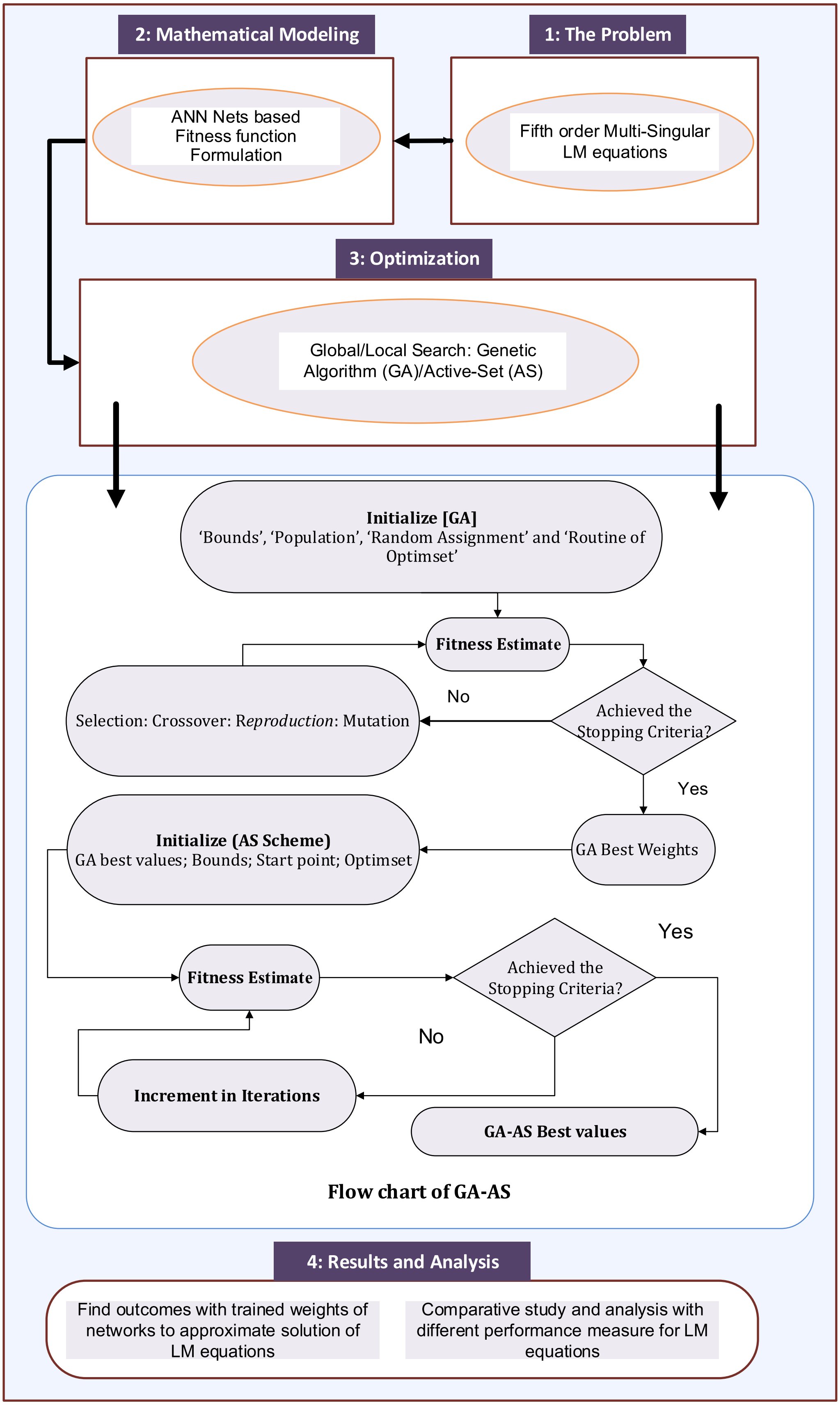

Fig. 1 Framework of the designed ANN-GA-AS scheme for solving the fifth-order EF model

The optimization performance of convergence in GA enhanced considerably by local search algorithm, therefore, an efficient local search with active-set (AS) is used for the parameter's refinements. Recently, AS scheme is applied in extensive applications (Tan et al. 2019; Koehler et al. 2017) and in this study, the hybrid of ANN-GA-AS method is used to get the designed variables for fifth-order LE model. The pseudocode detail of the proposed ANN-GA-AS method is given in Table 1.

7.3 LE equation: type 1

In this form, two different examples based on 5th order Lane–Emden system are discussed. These two examples are obtained by taking $k=4$ in Eq. (11).

Example 1: Consider the multi-singular 5th order Lane–Emden equation as follows:

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{16}{t}\frac{d^4\omega}{dt^4} + \frac{72}{t^2}\frac{d^3\omega}{dt^3} + \frac{96}{t^3}\frac{d^2\omega}{dt^2} + \frac{24}{t^4}\frac{d\omega}{dt} + \omega = 8401 + t^5, \\ \omega(0) = 1, \ \frac{d\omega(0)}{dt} = 0, \ \frac{d^2\omega(0)}{dt^2} = 0, \ \frac{d^3\omega(0)}{dt^3} = 0, \ \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{27}$$

The exact/true solution of (27) is $t^5 + 1$ and the merit function of Eq. (27) is given as:

$$\begin{aligned} E = & \frac{1}{I}\sum_{j=1}^{J}\Big(t_m^4\hat{\omega}_j^{(v)} + 16t_m^3\hat{\omega}_j^{(iv)} + 72t_m^2\hat{\omega}_j''' \\ & +96t_m\hat{\omega}_j'' + 24\hat{\omega}_j' + t_m^4\hat{\omega}_j - 8401t_j^4 - t_j^9)\Big)^2 \\ & + \frac{1}{5}\Big(\big(\hat{\omega}_0 - 1\big)^2 + \big(\hat{\omega}_0'\big)^2 + \big(\hat{\omega}_0''\big)^2 + \big(\hat{\omega}_0'''\big)^2 + \big(\hat{\omega}_0^{iv}\big)^2\Big). \end{aligned} \tag{28}$$

Example 2: Consider the 5th order Lane–Emden multi-singular equation having the trigonometric function as forcing term, we have.

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{16}{t}\frac{d^4\omega}{dt^4} + \frac{72}{t^2}\frac{d^3\omega}{dt^3} + \frac{96}{t^3}\frac{d^2\omega}{dt^2} + \frac{24}{t^4}\frac{d\omega}{dt} + \omega = t^5(\sin t + \cos t) \\ +41t^4\sin t - 592t^3\cos t - 3696t^2\sin t + 9744t\cos t + 8400\sin t + 1, \\ \omega(0) = 1, \ \frac{d\omega(0)}{dt} = 0, \ \frac{d^2\omega(0)}{dt^2} = 0, \ \frac{d^3\omega(0)}{dt^3} = 0, \ \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{29}$$

The reference solution of Eq. (29) is $\left(1 + t^5\right)\sin t$ and the merit function of Eq. (29) is given as:

$$\begin{aligned} E = & \frac{1}{J}\sum_{j=1}^{I}\begin{pmatrix} t_j^4\hat{\omega}_j^{(v)} + 16t_j^3\hat{\omega}_j^{(iv)} + 72t_m^2\hat{\omega}_j''' + 96t_m\hat{\omega}_j'' + 24\hat{\omega}_j' + t_j^4\hat{\omega}_j - t_j^5(\sin t_j + \cos t_j) \\ -41t_j^4\sin t_j + 592t_j^3\cos t_j + 3696t_j^2\sin t_j - 9744t_jcost_j - 8400t_j - 1) \end{pmatrix}^2 \\ & + \frac{1}{5}\Big(\big(\hat{\omega}_0 - 1\big)^2 + \big(\hat{\omega}_0'\big)^2 + \big(\hat{\omega}_0''\big)^2 + \big(\hat{\omega}_0'''\big)^2 + \big(\hat{\omega}_0^{(iv)}\big)^2\Big), \end{aligned} \tag{30}$$

where $N = 1/h, \ \hat{\omega}_J = \hat{\omega}\left(t_j\right), t_j = jh, \ g_k = g\left(t_k\right)$.

7.4 LE equation: type 2

In this form, two examples based on multi-singular 5th order Lane–Emden equation are discussed using $k=3$ in Eq. (14).

Example 1: Consider the multi-singular 5th order Lane–Emden form is given as:

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{9}{t}\frac{d^4\omega}{dt^4} + \frac{18}{t^2}\frac{d^3\omega}{dt^3} + \frac{6}{t^3}\frac{d^2\omega}{dt^2} + \omega = t^5 + t + 1441, \\ \omega(0) = \frac{d\omega(0)}{dt} = 1, \ \frac{d^2\omega(0)}{dt^2} = \frac{d^3\omega(0)}{dt^3} = \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{31}$$

The exact or reference solution of Eq. (31) is given as: $1 + t + t^5$.

Example 2: The multi-singular 5th order Lane–Emden equation having exponential forcing term is taken in the said example as:

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{9}{t}\frac{d^4\omega}{dt^4} + \frac{18}{t^2}\frac{d^3\omega}{dt^3} + \frac{6}{t^3}\frac{d^2\omega}{dt^2} + \omega = 2t^5e^t + 34t^4e^t \\ +398t^3e^t + 1956t^2e^t + 3900te^t + 24002e^t + t + 1, \\ \omega(0) = \frac{d\omega(0)}{dt} = 1, \ \frac{d^2\omega(0)}{dt^2} = \frac{d^3\omega(0)}{dt^3} = \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{32}$$

The true/exact solution of (32) is $t^5e^t + t + 1$.

7.5 LE equation of type 3

In this type, two examples based on multi-singular 5th order Lane–Emden equation are evaluated by taking $k=2$ in Eq. (17).

Example 1: The doubly singular 5th order LE equation is taken in this example as:

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{4}{t}\frac{d^4\omega}{dt^4} + \frac{2}{t^2}\frac{d^3\omega}{dt^3} + \omega = t^5 + t^2 + t + 721, \\ \omega(0) = \frac{d\omega(0)}{dt} = 1, \ \frac{d^2\omega(0)}{dt^2} = 2, \ \frac{d^3\omega(0)}{dt^3} = \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{33}$$

Table 1 Pseudocode of optimization process through ANN-GA-AS method

```
GA process start
      Inputs:
            The chromosome having equal unknown element numbers
            of the Nets as: W = [a, b, c], where a = [a_1, a_2, ..., a_J]
            b = [b_1, b_2, ..., b_J], and c = [c_1, c_2, ..., c_J], respectively.
      Population initialization: The set of chromosomes is
            denoted as:
            'P = [W_1, W_2, W_3, ..., W_n]^t', 'W_i = [a_i, b_i, c_i]'.
      Output:
            The Global Best GAs weights trained are indicated as
            W_BGA
      Initialization
            Construct a 'W' of real numbers to denote a
            chromosome. Set of 'W' to form an preliminary 'P'.
            Modify the 'declarations values' of and 'gaoptimset'
            routine
      Fitness evaluation
            Accomplished the 'fitness E' in 'population P' for
all         the 'weight vectors W' using equation (26).
      Termination
            Stop if any of the standards meet
            •  [TolFun = 10^-23], [Generations = 50],
            •  [Fitness = e = 10^-18], [TolCon = 10^-21],
            •  [StallGenLimit = 80], [PopulationSize = 330]
      Then go to 'storage step'
            Ranked each 'W' of 'P' for the fitness e
      Reproduction
            •  [Selection @selectionuniform].
            •  [Mutations @mutationadaptfeasible]
            •  [Elitism: Best ranked individuals of 'P']
            •  [Crossover @crossoverheuristic].
      Continue from 'fitness formulation' step
      Storage
            Save 'W_BGA values', 'fitness E', 'time',
      'generations' and 'function counts'
GA process Ends

AS scheme Start
      Inputs
            'Start point': W_BGA
      Output
            GA-AS best weights are indicated as W_GAAS
      Initialize
            'Bounded constraints', 'generations',
            'assignments' and 'other values'.
      Terminate
            Stop the algorithm when
            [Fitness = e = 10^-18], [Iterations = 1000],
            [TolX = TolFun =TolCon = 10^-25], [MaxFunEvals = 300000]
            While (Terminate)
      Fitness Assessment
            To achieve the 'fitness E' of the 'weight vector W'
            by using equation (26)
      Adjustments
            For As scheme, invoke 'fmincon'. Adapt 'W', i.e., weight
            vector for the generations of AS scheme. Compute
            fitness value of 'improved W' again by using equations
            (26)
      Accumulate
            Store the W_GA-AS, time, E, generations and function
            counts
AS process End
```

The exact form of Eq. (33) is $t^5 + t^2 + t + 1$.

Example 2: Consider the doubly singular 5th order LE equation having exponential forcing factor as:

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{4}{t}\frac{d^4\omega}{dt^4} + \frac{2}{t^2}\frac{d^3\omega}{dt^3} + \omega = 2t^5e^t + 29t^4e^t + 282t^3e^t \\ +1110t^2e^t + 1680te^t + 612e^t + t^2 + t + 1, \\ \omega(0) = \frac{d\omega(0)}{dt} = 1,\ \frac{d^2\omega(0)}{dt^2} = 2,\ \frac{d^3\omega(0)}{dt^3} = \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{34}$$

The exact result of (34) is $t^5e^t + t^2 + t + 1$.

7.6 LE equation: type 4

In this type, two examples based on 5th order LE are examined by taking $k = 1$ in Eq. (20).

Example 1: Let us take the 5th order LE based singular system as follows:

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{1}{t}\frac{d^4\omega}{dt^4} + \omega = t^5 + t^3 + t^2 + t + 241, \\ \omega(0) = \frac{d\omega(0)}{dt} = 1,\ \frac{d^2\omega(0)}{dt^2} = 2,\ \frac{d^3\omega(0)}{dt^3} = 6,\ \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{35}$$

The exact result of (35) is $t^5 + t^3 + t^2 + t + 1$.

Example 2: Consider the fifth-order LE equation involving exponential function in its forcing factor.

$$\begin{cases} \frac{d^5\omega}{dt^5} + \frac{1}{t}\frac{d^4\omega}{dt^4} + \omega = 2t^5e^t + 26t^4e^t + 220t^3e^t + 720t^2e^t + 840te^t + 240e^t + t^3 + t^2 + t + 1, \\ \omega(0) = \frac{d\omega(0)}{dt} = 1,\ \frac{d^2\omega(0)}{dt^2} = 2,\ \frac{d^3\omega(0)}{dt^3} = 6,\ \frac{d^4\omega(0)}{dt^4} = 0. \end{cases} \tag{36}$$

The true form of Eq. (36) is $1 + t + t^2 + t^3 + t^5e^t$.

8 Performance indices

The current study is to present the statistical measures of Variance Account For (VAF), Theil's inequality coefficient (TIC) and semi interquartile range (SIR) to solve the fifth-order LE differential system base equations. The mathematical illustrations of these performances are given as:

$$\begin{cases} VAF = \left(1 - \frac{\operatorname{var}\left(\omega_i(t) - \hat{\omega}_i(t)\right)}{\operatorname{var}\left(\omega_i(t)\right)}\right) \times 100, \\ EVAF = |VAF - 100|, \end{cases} \tag{37}$$

$$\text{TIC} = \frac{\sqrt{\frac{1}{n}\sum_{m=1}^{n}\left(\omega_m - \hat{\omega}_m\right)^2}}{\left(\sqrt{\frac{1}{n}\sum_{m=1}^{n}\omega_m^2} + \sqrt{\frac{1}{n}\sum_{m=1}^{n}\hat{\omega}_m^2}\right)}, \tag{38}$$

$$\text{SIR} = -0.5\left(Q_1 - Q_3\right), \text{ where } Q_3 \text{ and } Q_1 \text{ represent the third and first quartiles.} \tag{39}$$

9 Result and discussion: LE equation of type 1

The detailed results with necessary elaborative discussions of LE equation of type 1 are presented exhaustively for two examples in this section. The error-based fitness function is presented in Eq. (30) using the networks presented in Eqs. (21–25). The ANN is applied to discretize the LE equation of type 1 to formulate mean squared error-based merit function. The optimization of the networks for the fifth-order LE equation is executed by integrated capability of GA-ASA as per procedure illustrated graphically in Fig. 1 and pseudocode in Table 1. The parameters of global search GAs and local search AS are given in Table 1 for the execution of the both algorithms.

The set of weights of both the examples of type 1 are provided to obtain the approximate solution of the model using the hybrid of GA-ASA and these proposed outcomes are accessible as:

$$\begin{aligned} \hat{\omega}_{Exp-1}(t) = & \frac{-0.5515}{1 + e^{-(-4.386t+4.218)}} + \frac{-3.2507}{1 + e^{-(0.368t-0.914)}} \\ & - \frac{10.9192}{1 + e^{-(-5.977t+9.101)}} + \ldots - \frac{8.1275}{1 + e^{-(-2.340t+3.074)}}, \end{aligned} \tag{40}$$

$$\begin{aligned} \hat{\omega}_{Exp-2}(t) = & \frac{3.6694}{1 + e^{-(-4.294t+4.258)}} - \frac{1.0189}{1 + e^{-(5.309t-6.545)}} \\ & - \frac{1.4086}{1 + e^{-(19.86t+11.659)}} + \ldots + \frac{10.8571}{1 + e^{-(6.153t-10.787)}}. \end{aligned} \tag{41}$$

For the 5th order nonlinear LE multi-singular system, the graphical performance of the present approach for both the examples of type 1 is plotted in Fig. 2. The performance of the optimization based on ANN-GA-AS algorithm is made for 100 independent runs. Figure 2 narrates the set of weights for ten neurons and a comparison of the exact and proposed results performed by ANN-GA-AS algorithm. It is noticed that the obtained results and exact solutions of the multi-singular fifth-order LE equation overlapped to each other for both of the examples of type 1.

The absolute error values (AE) for both the examples of fifth-order LE model of type 1 are drawn in Fig. 3a. It is seen the AE values for type 1 of the LE fifth-order model lie around 10^{-04} to 10^{-05} and 10^{-05} to 10^{-06} for Examples 1 and 2, respectively. Performance indices based on fitness, TIC and EVAF are provided in Fig. 3b. It is seen that the fitness values for both the examples of Type 1 lie about 10^{-05} to 10^{-06}, while the TIC and EVAF values for both the Examples of Type 1 lie about 10^{-08} to 10^{-09}.

For the analysis/evolution of precision for the presented approach ANN-GA-AS, the statistical analysis is conducted/executed for 100 runs (autonomous) on the basis of minimum (MIN), semi interquartile range (SIR) and median (MED) to solve the both examples of multi-point fifth-order LE equation. The results/outcomes of the statistic performance indices for the said are given in Table 2 for solving both the examples of multi-singular fifth-order LE equation based on type 1. The small magnitudes of Min, Median and SIR close to 10^{-6}, 10^{-3} and 10^{-3}, respectively attained by proposed approach both examples of multi-singular fifth-order LE equation, which prove the consistent precision and accuracy.

10 Conclusions

In this study, first time four types of fifth-order generation of LE equations have been presented along with numerical solution with neuro-evolution based heuristics. The first

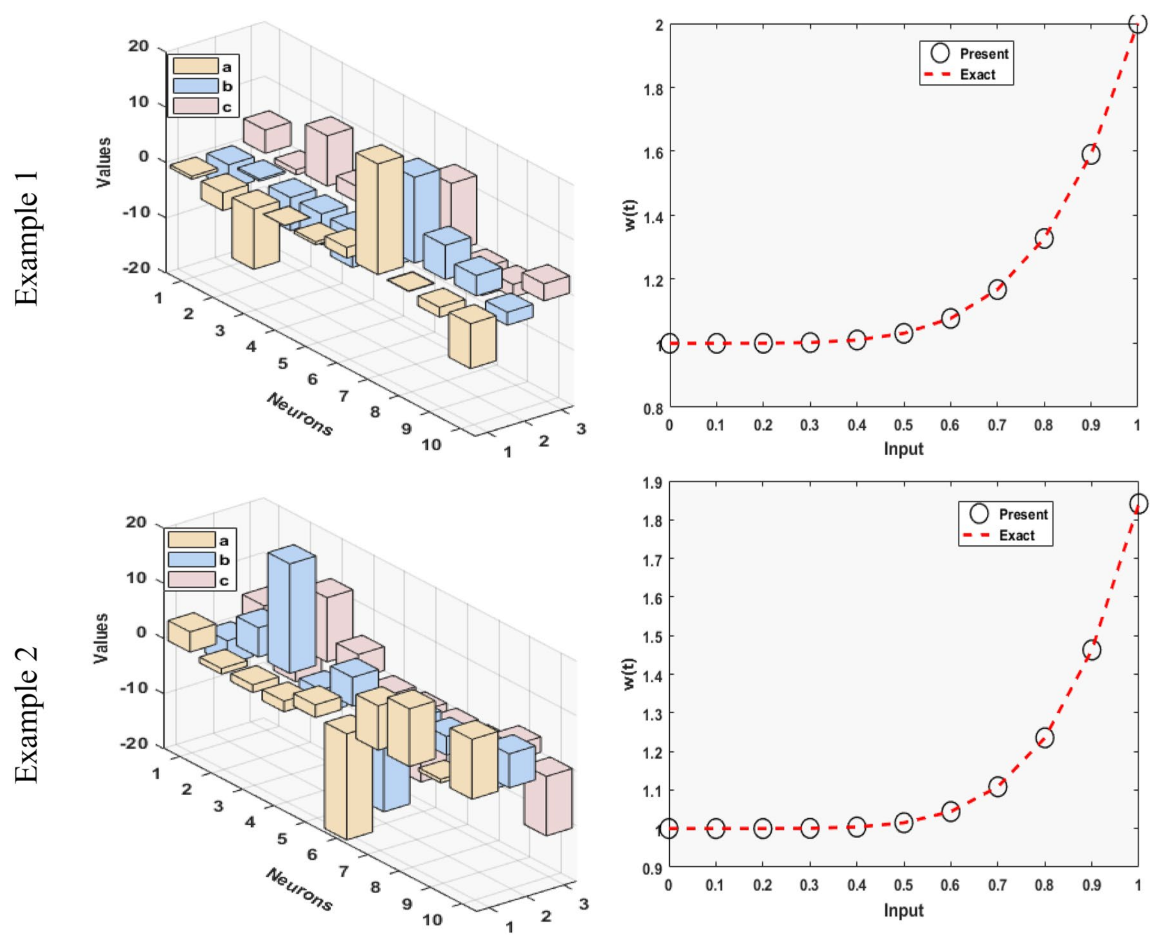

Fig. 2 Set of weights and comparison of exact/proposed results for both the examples of fifth-order LE model of type 1

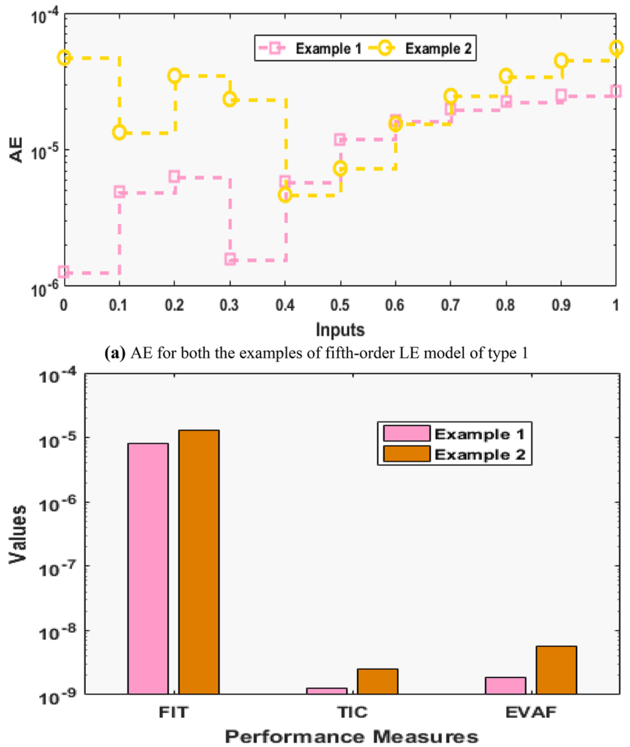

Fig. 3 AE and performance gages for both the examples of fifth-order LE model of type 1

three types show multiple singularity and the last type depicts singularity at one-point only. The shape factor is narrated in each type of the singular LE equation and it appeared thrice in type 1 and type 2, twice in type 3 and once in type 4, whereas the standard form of the LE equation which shows the unique shape factor. Similarly, in the standard form of LE equation, the singular point appears only one time, whereas in this study, the singular point at $t = 0$ appeared multi times in first three types. Moreover, two examples of each type is presented and by considering the length of the paper only two examples of type 1 have been solved numerically by using artificial neural network aided with GA and ASA. The obtained results through statistical assessments validate the reliability and correctness of the present scheme ANN-GA-AS. The only limitation of the presented study is the slowness of global search of GAs, which is handle by the rapid refinement of the results with AS, however, in future one may explore in meta-heuristic algorithm possessing both capability of both global and local search, i.e., differential evolutions, backtracking searrch optimization algorithm, weighted differential evolution and their recently introduced variant for efficient performance. Additionally, the designed ANN-GA-AS computing solver can be extended in future for solution of applications of paramount interest numerically in broad fields such as fractional order models (Ilhan and Kıymaz 2020; Burgos-Simon et al. 2017; Wilczynski 2018), singular periodic nonlinear boundary value problems (Sabir et al. 2021h), Casson nonofluidic flow problems (Umar et al. 2019b; Sabir et al. 2019), fluid mechanics model (Ayub et al. 2021), lonngren-wave problems (Baskonus et al. 2019), quantum differential models (Gençoglu and Agarwal 2021), fractional Harry Dym equation (Yokus and Gülbaha 2019), nonlinear MSEIR model (Qureshi and Yusuf 2019) and soliton dynamics system (Sulaiman et al. 2020).

Table 2 Statistics results for both examples of multi-singular fifth-order LE equation of type 1

t	Example 1			Example 2		
	Min	Med	S.I.R	Min	Med	S.I.R
0	1.262216E−06	8.521872E−04	3.173622E−03	9.932228E−06	8.736360E−04	1.968895E−03
0.1	4.211777E−06	8.094372E−04	3.039249E−03	1.343832E−05	8.760966E−04	2.121608E−03
0.2	3.932942E−06	8.313174E−04	3.050007E−03	4.796893E−06	9.774199E−04	2.071830E−03
0.3	1.570012E−06	7.731050E−04	3.028656E−03	2.180191E−05	1.014125E−03	2.089128E−03
0.4	5.794047E−06	9.343936E−04	2.999918E−03	4.681260E−06	9.504749E−04	2.222949E−03
0.5	1.508801E−06	1.055561E−03	2.979965E−03	3.795466E−07	9.627401E−04	2.116656E−03
0.6	1.622285E−05	1.003785E−03	2.982057E−03	1.008660E−06	9.110853E−04	2.029621E−03
0.7	1.332207E−05	1.066681E−03	3.177961E−03	8.268587E−06	9.134162E−04	2.062615E−03
0.8	1.251250E−06	1.088294E−03	3.378539E−03	7.383179E−06	9.725335E−04	2.088262E−03
0.9	1.996624E−05	1.132833E−03	3.548323E−03	3.203534E−06	1.086592E−03	2.095754E−03
1	2.687347E−05	1.173872E−03	3.647394E−03	8.960921E−07	1.119592E−03	2.092811E−03

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