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Research article

Interpolative contractions and intuitionistic fuzzy set-valued maps with applications

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Abstract: Over time, the interpolative approach in fixed point theory (FPT) has been investigated only in the setting of crisp mathematics, thereby dropping-off a significant amount of useful results. As an attempt to fill up the aforementioned gaps, this paper initiates certain hybrid concepts under the names of interpolative Hardy-Rogers-type (IHRT) and interpolative Reich-Rus-Ciric type (IRRCT) intuitionistic fuzzy contractions in the frame of metric space (MS). Adequate criteria for the existence of intuitionistic fuzzy fixed point (FP) for such contractions are examined. On the basis that FP of a single-valued mapping obeying interpolative type contractive inequality is not always unique, and thereby making the ideas more suitable for FP theorems of multi-valued mappings, a few special cases regarding point-to-point and non-fuzzy set-valued mappings which include the conclusions of some well-known results in the corresponding literature are highlighted and discussed. In addition, comparative examples which dwell on the generality of our obtained results are constructed.

Keywords: interpolative contraction; intuitionistic fuzzy set; fixed point
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1. Introduction and preliminaries

Several existence problems in mathematics and related areas can be analysed using FP techniques. FPT provides useful tools for dealing with problems arising in many branches of mathematical analysis, such as split feasibility, inequality theory, nonlinear optimization, equilibrium, epidemic model and existence theory, complementarity, selection and matching problems, and problems of proving the existence of solutions to integral and differential equation of both integer and non-integer orders. The fixed theorem, commonly termed Banach fixed theorem (see [4]), showed up for the first time in Banach dissertation of 1922. Later after, many authors have come up with its generalizations by bringing up various forms of contractions on metric and quasi MS (see, e.g., [14,17,26]). Along the line, one of the well-known advancements of Banach FP theorem was presented by Hardy-Rogers [7]. The prototype of this result (in [7]) is the following.

Theorem 1.1. *[7] Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and Υ be a single-valued mapping on $\tilde{\Lambda}$ obeying the conditions:*

$$\tilde{\varrho}(\Upsilon\tilde{\jmath}, \Upsilon\tilde{\ell}) \leq a\tilde{\varrho}(\tilde{\jmath}, \Upsilon\tilde{\jmath}) + b\tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\ell}) + c\tilde{\varrho}(\tilde{\jmath}, \Upsilon\tilde{\ell}) + e\tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\jmath}) + l\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell}),$$

where a, b, c, e, l are nonnegative reals. If $a + b + c + e + l < 1$, then Υ has a unique FP in $\tilde{\Lambda}$.

Along the lane, Roldn et al. [21] established some novel FP theorems for a class of contractions depending on two functions and some parameters under the name multiparametric contractions and pointed out a significant number of Hardy-Roger's type contractions in the setting of both metric and b-MS. Related important versions of the Banach contraction mapping principle were independently presented by Ciric [5], Reich [24] and Rus [22].

Definition 1.2. [5,22,24] Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS. A single-valued mapping $\Upsilon : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}$ is named:

(i) Rus contraction if we can find $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that $\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda}$,

$$\tilde{\varrho}(\Upsilon\tilde{\jmath}, \Upsilon\tilde{\ell}) \leq a\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell}) + b\tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\ell}).$$

(ii) Ciric-Reich contraction if we can find $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that $\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda}$,

$$\tilde{\varrho}(\Upsilon\tilde{\jmath}, \Upsilon\tilde{\ell}) \leq a\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell}) + b\tilde{\varrho}(\tilde{\jmath}, \Upsilon\tilde{\jmath}) + c\tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\ell}).$$

A unified form of these results, which is named the Ciric-Reich-Rus FP theorem, is given as follows:

Theorem 1.3. *[5, 22, 24] Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and the single-valued $\Upsilon : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}$ be a Ciric-Reich-Rus contraction, that is,*

$$\tilde{\varrho}(\Upsilon\tilde{\jmath}, \Upsilon\tilde{\ell}) \leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell}) + \tilde{\varrho}(\tilde{\jmath}, \Upsilon\tilde{\jmath}) + \tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\ell})]$$

$\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda}$, where $\tilde{\lambda} \in \left[0, \frac{1}{3}\right)$. Then Υ has a unique FP in $\tilde{\Lambda}$.

Recently, encouraged by the interpolation theory, Karapinar et al. [10] brought up the notion of interpolative Hardy-Rogers type contraction(IHRTC):

Definition 1.4. [10] Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS. The single-valued mapping $\Upsilon : \tilde{\Lambda} \longrightarrow \tilde{\Lambda}$ is named an IHRTC if we can find $\tilde{\lambda} \in [0, 1)$ and $a, b, c \in (0, 1)$ with $a + b + c < 1$ such that

$$\tilde{\varrho}(\Upsilon\tilde{\jmath}, \Upsilon\tilde{\ell}) \leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^b[\tilde{\varrho}(\tilde{\jmath}, \Upsilon\tilde{\jmath})]^a[\tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\ell})]^c\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}, \Upsilon\tilde{\ell}) + \tilde{\varrho}(\tilde{\ell}, \Upsilon\tilde{\jmath}))\right]^{1-a-b-c} \tag{1.1}$$

$\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$, where $\mathcal{F}_{ix}(\Upsilon)$ is the set of all FPs of Υ.

The reader is directed to [11–13] for similar FP results using interpolation theory. An inherent property of the existing FP results via the interpolative type contraction is that the FP of the concerned mapping is not necessarily unique; for example, see [11, Example 1]. This behaviour is an indication that FP theorems using interpolative notions are more suitable for the FPT of point-to-set-valued maps.

On the other hand, one of the challenges in the mathematical problem of practical phenomena has to do with the indeterminacy caused by our inability to sort events with enough precision. It has been realized that crisp mathematics cannot handle effectively the imprecisions pervading our daily life. In an effort to circumvent the aforementioned obstacles, and as an improvement of the ideas of crisp set theory, the evolvement of fuzzy mathematics started with the launching of the concepts of fuzzy sets (F-S) by Zadeh [29] in 1965. F-S theory is now well-known as one of the mathematical tools for managing information with nonprobabilistic uncertainty. Whence, the theory of F-S has enjoyed greater applications in diverse fields. Meanwhile, the basic notions of F-S have been enhanced in several directions; for example, see [16]. In 1981, Heilpern [9] employed the concept of F-S to initiate a class of F-S-valued maps and established a FP theorem for fuzzy contraction mappings which is a fuzzy analogue of FP theorems due to Nadler [20] and Banach [4]. Later on, more than a handful of examiners have investigated the existence of FP in F-S-valued maps, see for example, [6,8,15,18,19,27] and the references therein.

Following Zadeh [29], intuitionistic F-S (IFS) was initiated by Atanassov [3] as an additional refinement of F-S theory. IFS is more applicable than a F-S as it can measure the degrees of both membership and non-membership. Whence, it has enjoyed greater usefulness in many areas. Along the lane, Azam et al. [1] brought up a new technique for examining FP results via intuitionistic F-S-valued map in a MS. Later on, Azam and Tabassum [2] launched novel criteria for existence of coincidence points for three intuitionistic F-S-valued maps and employed their findings to examine existence criteria for solutions of a system of integral equation. On similar vein, Tabassum et al. [28] brought up the view of common FP theorems for a pair of intuitionistic F-S-valued maps in the frame of $(\mathcal{T}, \mathcal{N}, \tilde{\tilde{\beta}})$-cut set for IFS.

It is worth noting that the introduction of interpolation techniques (see [10–12]) witnessed tremendous improvements in metric FPT and applications. However, the corresponding idea of intuitionistic F-S-valued maps is yet to be examined, leaving up gaps in the literature. Hence, this paper bring ups the concepts of IHRT intuitionistic fuzzy contraction and IRRCT intuitionistic fuzzy contraction in the bodywork of MS and examines criteria for the existence of intuitionistic fuzzy FP for such contractions. As earlier observed, FP of a single-valued mapping obeying an interpolative type contractive inequality is not necessarily unique; and whence making the notions more suitable for FP theorems of set-valued maps. In light of the latter remark, novel multivalued analogues of the intuitionistic fuzzy FP results proposed herein are deduced as corollaries. In addition, comparative examples that dwell on the generality of our results are constructed.

Hereafter, we record a few preliminary concepts and notations that are essential to our findings. Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS and $\mathcal{K}(\tilde{\Lambda})$ be the family of nonempty compact(NC) subsets of $\tilde{\Lambda}$. Let $\tilde{\nabla}, \tilde{\triangle} \in \mathcal{K}(\tilde{\Lambda})$ and $\epsilon > 0$ be arbitrary. Then the sets $N_{\tilde{\varrho}}(\epsilon, \tilde{\nabla})$, $N_{\tilde{\varrho}}(\epsilon, \tilde{\triangle})$ and $E^{\tilde{\varrho}}_{(\tilde{\nabla}, \tilde{\triangle})}$ and the distance function $\tilde{\varrho}(\tilde{\nabla}, \tilde{\triangle})$, are respectively defined as follows:

$$N_{\tilde{\varrho}}(\epsilon, \tilde{\nabla}) = \{\tilde{\jmath} \in \tilde{\Lambda} : \tilde{\varrho}(\tilde{\jmath}, a) < \epsilon, \text{ for some } a \in \tilde{\nabla}\}.$$

$$N_{\tilde{\varrho}}(\epsilon, \tilde{\triangle}) = \{\tilde{\jmath} \in \tilde{\Lambda} : \tilde{\varrho}(\tilde{\jmath}, b) < \epsilon, \text{ for some } b \in \tilde{\triangle}\}.$$

$$E^{\tilde{\varrho}}_{(\tilde{\nabla},\tilde{\triangle})} = \{\epsilon > 0 : \tilde{\nabla} \subseteq N_{\tilde{\varrho}}(\epsilon, \tilde{\triangle}),\ \tilde{\triangle} \subseteq N_{\tilde{\varrho}}(\epsilon, \tilde{\nabla})\}.$$

$$\tilde{\varrho}(\tilde{\nabla}, \tilde{\triangle}) = \inf_{\tilde{\jmath}\in\tilde{\nabla},\tilde{\ell}\in\tilde{\triangle}} \tilde{\varrho}(\tilde{\jmath}, \tilde{\ell}).$$

Then, the Hausdorff metric $\aleph$ on $\mathcal{K}(\tilde{\Lambda})$ induced by the metric $\tilde{\varrho}$ is defined as: $\aleph(\tilde{\nabla}, \tilde{\triangle}) = \inf E^{\tilde{\varrho}}_{(\tilde{\nabla},\tilde{\triangle})}$ (see [20, P.3]).

It is familiar that every crisp subset $\tilde{\nabla}$ of $\tilde{\Lambda}$ is determined completely by its indicator function $\tilde{\chi}_{\tilde{\nabla}}$, given by $\tilde{\chi}_{\tilde{\nabla}} : \tilde{\nabla} \longrightarrow \{0, 1\}$:

$$\tilde{\chi}_{\tilde{\nabla}}(\tilde{\jmath}) = \begin{cases} 1, & \text{if } \tilde{\jmath} \in \tilde{\nabla} \\ 0, & \text{if } \tilde{\jmath} \notin \tilde{\nabla}. \end{cases}$$

The quantity $\tilde{\chi}_{\tilde{\nabla}}(\tilde{\jmath})$ shows the inclusiveness or otherwise of an element in $\tilde{\nabla}$. This view is employed to launch F-S by permitting a point $\tilde{\jmath} \in \tilde{\nabla}$ to presume a suitable value in $[0, 1]$. Whence, a F-S in $\tilde{\Lambda}$ is a function with domain $\tilde{\Lambda}$ and values in $[0, 1] = I$.

Definition 1.5. [3] Let $\tilde{\Lambda}$ be a nonempty set (NS). IFS $\tilde{\nabla}$ in $\tilde{\Lambda}$ is a set of ordered 3-tuples given as

$$\tilde{\nabla} = \{\langle \tilde{\jmath}, \mu_{\tilde{\nabla}}(\tilde{\jmath}), \nu_{\tilde{\nabla}}(\tilde{\jmath})\rangle : \tilde{\jmath} \in \tilde{\Lambda}\},$$

where $\mu_{\tilde{\nabla}} : \tilde{\Lambda} \longrightarrow [0, 1]$ and $\nu_{\tilde{\nabla}} : \tilde{\Lambda} \longrightarrow [0, 1]$ represent the degrees of membership and non-membership, respectively of $\tilde{\jmath}$ in $\tilde{\Lambda}$ and obey $0 \leq \mu_{\tilde{\nabla}} + \nu_{\tilde{\nabla}} \leq 1$, for each $\tilde{\jmath} \in \tilde{\Lambda}$. Further, the degree of hesitancy of $\tilde{\jmath} \in \tilde{\nabla}$ is:

$$h_{\tilde{\nabla}}(\tilde{\jmath}) = 1 - \mu_{\tilde{\nabla}}(\tilde{\jmath}) - \nu_{\tilde{\nabla}}(\tilde{\jmath}).$$

Specifically, if $h_{\tilde{\nabla}}(\tilde{\jmath}) = 0\ \forall\ \tilde{\jmath} \in \tilde{\Lambda}$, then an IFS becomes a F-S.

We design the set of all IFS in $\tilde{\Lambda}$ as $(IFS)^{\tilde{\Lambda}}$.

Definition 1.6. [3] Let $\tilde{\nabla}$ be an IFS in $\tilde{\Lambda}$. Then the $\tilde{\tilde{\beta}}$-level set of $\tilde{\nabla}$ is a crisp subset of $\tilde{\Lambda}$ designed as $[\tilde{\nabla}]_{\tilde{\tilde{\beta}}}$ and is defined as

$$[\tilde{\nabla}]_{\tilde{\tilde{\beta}}} = \{\tilde{\jmath} \in \tilde{\Lambda} : \mu_{\tilde{\nabla}}(\tilde{\jmath}) \geq \tilde{\tilde{\beta}} \text{ and } \nu_{\tilde{\nabla}}(\tilde{\jmath}) \leq 1 - \tilde{\tilde{\beta}}\},\ \text{if } \tilde{\tilde{\beta}} \in [0, 1].$$

Definition 1.7. [1] Consider $L = \{(\tilde{\tilde{\beta}},\beta) : \tilde{\tilde{\beta}} + \beta \leq 1,\ (\tilde{\tilde{\beta}},\beta) \in (0, 1] \times [0, 1)\}$ and $\tilde{\nabla}$ is an IFS in $\tilde{\Lambda}$. The $(\tilde{\tilde{\beta}},\beta)$-level set of $\tilde{\nabla}$ is given as

$$[\tilde{\nabla}]_{(\tilde{\tilde{\beta}},\beta)} = \{\tilde{\jmath} \in \tilde{\Lambda} : \mu_{\tilde{\nabla}}(\tilde{\jmath}) \geq \tilde{\tilde{\beta}} \text{ and } \nu_{\tilde{\nabla}}(\tilde{\jmath}) \leq \beta\}.$$

Azam and Tabassum [2] obtained a modification of Definition 1.7 in the following manner.

Definition 1.8. [2] The (M, m)-level set of an intuitionistic F-S $\tilde{\nabla}$ in $\tilde{\Lambda}$ is defined as :

$$[\tilde{\nabla}]_{(M,m)} = \{\varsigma \in \tilde{\Lambda} : \mu_{\tilde{\nabla}}(\varsigma) = M \text{ and } \nu_{\tilde{\nabla}}(\varsigma) = m\},$$

where

$$M = \max_{\varsigma\in\tilde{\Lambda}} \mu_{\tilde{\nabla}}(\varsigma) \text{ and } m = \min_{\varsigma\in\tilde{\Lambda}} \nu_{\tilde{\nabla}}(\varsigma).$$

Definition 1.9. [1] Let $\tilde{\Lambda}$ be a NS. The mapping $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ is named an intuitionistic F-S-valued map. A point $u \in \tilde{\Lambda}$ is an intuitionistic fuzzy FP of Υ if we can find $(\tilde{\tilde{\beta}}, \beta) \in (0,1] \times [0,1)$ such that $u \in [\Upsilon u]_{(\tilde{\tilde{\beta}},\beta)}$.

Consistent with Azam and Tabassum [1], we note down some needed auxiliary concepts: Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS and take $(\tilde{\tilde{\beta}}, \beta) \in (0,1] \times [0,1)$ such that $[\tilde{\nabla}]_{(\tilde{\tilde{\beta}},\beta)}, [\tilde{\triangle}]_{(\tilde{\tilde{\beta}},\beta)} \in \mathcal{K}(\tilde{\Lambda})$. Then,

$$p_{(\tilde{\tilde{\beta}},\beta)}(\tilde{\nabla}, \tilde{\triangle}) = \inf_{\varsigma \in [\tilde{\nabla}]_{(\tilde{\tilde{\beta}},\beta)}, \omega \in [\tilde{\triangle}]_{(\tilde{\tilde{\beta}},\beta)}} \tilde{\varrho}(\varsigma, \omega).$$

$$D_{(\tilde{\tilde{\beta}},\beta)}(\tilde{\nabla}, \tilde{\triangle}) = \inf E^{\tilde{\varrho}}_{(\tilde{\nabla},\tilde{\triangle})}.$$

$$p(\tilde{\nabla}, \tilde{\triangle}) = \sup_{(\tilde{\tilde{\beta}},\beta)} p_{(\tilde{\tilde{\beta}},\beta)}(\tilde{\nabla}, \tilde{\triangle}).$$

$$\tilde{\varrho}_{(\infty,\infty)}(\tilde{\nabla}, \tilde{\triangle}) = \sup_{(\tilde{\tilde{\beta}},\beta)} D_{(\tilde{\tilde{\beta}},\beta)}(\tilde{\nabla}, \tilde{\triangle}).$$

Notice that $p_{(\tilde{\tilde{\beta}},\beta)}$ is a nondecreasing function of $(\tilde{\tilde{\beta}}, \beta)$ (see [9]), $\tilde{\varrho}_{(\infty,\infty)}$ is a metric on $\mathcal{K}(\tilde{\Lambda})$ and the completeness of $(\tilde{\Lambda}, \tilde{\varrho})$ guarantees that of the associated MS $(\mathcal{K}_{\mathcal{IFS}}(\tilde{\Lambda}), \tilde{\varrho}_{(\infty,\infty)})$ (see [9]). Moreover, $(\tilde{\Lambda}, \tilde{\varrho}) \longmapsto (\mathcal{K}(\tilde{\Lambda}), \aleph) \longmapsto (\mathcal{K}_{\mathcal{IFS}}(\tilde{\Lambda}), \tilde{\varrho}_{(\infty,\infty)})$, are embeddings via the relations $\varsigma \longrightarrow \{\varsigma\}$ and $M \longrightarrow \tilde{\chi}_M$, respectively; where

$$\mathcal{K}_{\mathcal{IFS}}(\tilde{\Lambda}) = \{\tilde{\nabla} \in (IFS)^{\tilde{\Lambda}} : [\tilde{\nabla}]_{(\tilde{\tilde{\beta}},\beta)} \in \mathcal{K}(\tilde{\Lambda}),\ (\tilde{\tilde{\beta}}, \beta) \in (0,1] \times [0,1)\}.$$

Lemma 1.10. *[20] Let $\daleth$ and $\urcorner$ be nonempty closed and bounded subsets of a MS $\tilde{\Lambda}$. If $\kappa \in \daleth$, then $\tilde{\varrho}(\kappa, \urcorner) \leq \aleph(\daleth, \urcorner)$.*

2. Main results

We launch this section by bringing up the idea of Hardy Rogers-type intuitionistic fuzzy contraction and establishing the corresponding FP theorem.

Definition 2.1. Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS. Then, the intuitionistic F-S-valued map $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ is named an IHRT intuitionistic fuzzy contraction, if we can find two mappings $\tilde{\tilde{\beta}} : \tilde{\Lambda} \longrightarrow (0,1]$, $\beta : \tilde{\Lambda} \longrightarrow [0,1)$ and constants $\tilde{\lambda}, a, b, c \in (0,1)$ with $a + b + c < 1$ such that $\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$,

$$\begin{aligned}
&\aleph([\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))}, [\Upsilon \tilde{\ell}]_{(\tilde{\tilde{\beta}}(\tilde{\ell}),\beta(\tilde{\ell}))}) \\
&\leq\ \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^b [\tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))})]^a [\tilde{\varrho}(\tilde{\ell}, [\Upsilon \tilde{\ell}]_{(\tilde{\tilde{\beta}}(\tilde{\ell}),\beta(\tilde{\ell}))})]^c \\
&\cdot \left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\ell}]_{(\tilde{\tilde{\beta}}(\tilde{\ell}),\beta(\tilde{\ell}))}) + \tilde{\varrho}(\tilde{\ell}, [\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))}))\right]^{1-a-b-c},
\end{aligned} \tag{2.1}$$

where

$$\mathcal{F}_{ix}(\Upsilon) = \{u \in \tilde{\Lambda} : u \in [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}\}.$$

Theorem 2.2. *Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \to (IFS)^{\tilde{\Lambda}}$ be an IHRT intuitionistic fuzzy contraction. Assume that $[\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))}$ is a NC subset of $\tilde{\Lambda}$ for each $\tilde{\jmath} \in \tilde{\Lambda}$. Then Υ has an intutionistic fuzzy FP in $\tilde{\Lambda}$.*

Proof. Let $\tilde{\jmath}_0 \in \tilde{\Lambda}$ be arbitrary. Then, by hypothesis, $[\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))} \in \mathcal{K}(\tilde{\Lambda})$. Choose $\tilde{\jmath}_1 \in [\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))}$, then for this $\tilde{\jmath}_1 \in \tilde{\Lambda}$, $[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}$ is a NC subset of $\tilde{\Lambda}$. Hence, we can find $\tilde{\jmath}_2 \in [\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}$:

$$\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2) = \tilde{\varrho}(\tilde{\jmath}_1,[\Upsilon\tilde{\jmath}_1]_{\tilde{\tilde{\beta}}(\tilde{\jmath}_1)}) \leq \aleph([\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))},[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}]). \tag{2.2}$$

Setting $\tilde{\jmath} = \tilde{\jmath}_0$ and $\tilde{\ell} = \tilde{\jmath}_1$ in (2.1), we see that

$$\begin{aligned}
&\aleph([\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))},[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))})\\
&\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^b[\tilde{\varrho}(\tilde{\jmath}_0,[\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))})]^a[\tilde{\varrho}(\tilde{\jmath}_1,[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))})]^c\\
&\cdot\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}_0,[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))})+\tilde{\varrho}(\tilde{\jmath}_1,[\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))}))\right]^{1-a-b-c}\\
&\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^b[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^a\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2)]^c\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_2)+\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_1))\right]^{1-a-b-c}\\
&\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^b[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^a\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2)]^c\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)+\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2))\right]^{1-a-b-c}.
\end{aligned} \tag{2.3}$$

Suppose that $\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1) \leq \tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2)$, then (2.3) becomes

$$\begin{aligned}
\aleph([\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))},[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}) &\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2)]^{a+b+c}[\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2)]^{1-a-b-c}\\
&\leq \tilde{\lambda}(\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2))\\
&< \tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2).
\end{aligned} \tag{2.4}$$

Observe that coupling (2.2) and (2.4) gives a contradiction . Whence, $\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2) < \tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)$. Whence, for $\zeta = \sqrt{\tilde{\lambda}}$ and $\omega = \zeta\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)$, (2.3) yields

$$\begin{aligned}
\aleph([\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))},[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}) &\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^{a+b+c}[\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)]^{1-a-b-c}\\
&\leq \tilde{\lambda}\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)\\
&< \omega.
\end{aligned}$$

It follows that $\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2) < \omega$, for some $\tilde{\jmath}_2 \in [\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}$. Thus, $\omega \in E^{\tilde{\varrho}}_{([\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))},[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))})}$. This yields that $[\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))} \subseteq N_{\tilde{\varrho}}(\omega,[\Upsilon\tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))})$ and $\tilde{\jmath}_1 \in N_{\tilde{\varrho}}(\omega,[\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))})$. On similar steps, we can finds $\tilde{\jmath}_2 \in N_{\tilde{\varrho}}(\zeta d(\tilde{\jmath}_0,\tilde{\jmath}_1),[\Upsilon\tilde{\jmath}_2]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_2),\beta(\tilde{\jmath}_2))})$ and $\tilde{\jmath}_3 \in [\Upsilon\tilde{\jmath}_2]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_2),\beta(\tilde{\jmath}_2))}$: for $\omega^2 = \zeta^2\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)$, we see that

$$\begin{aligned}
\tilde{\varrho}(\tilde{\jmath}_2,\tilde{\jmath}_3) &\leq \zeta\tilde{\varrho}(\tilde{\jmath}_1,\tilde{\jmath}_2)\\
&< \omega^2.
\end{aligned}$$

Whence, $\omega^2 \in E^{\tilde{\varrho}}_{([\Upsilon\tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))},[\Upsilon\tilde{\jmath}_2]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_2),\beta(\tilde{\jmath}_2))})}$. Repeatedly, we come up with a sequence $\{\tilde{\jmath}_n\}_{n\geq 1}$ in $\tilde{\Lambda}$ such that $\tilde{\jmath}_{n+1} \in [\Upsilon\tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))}$ and

$$\tilde{\varrho}(\tilde{\jmath}_n,\tilde{\jmath}_{n+1}) \leq \zeta^n\tilde{\varrho}(\tilde{\jmath}_0,\tilde{\jmath}_1)\ \forall\ n \geq 1.$$

Next, by usual arguments, we show that $\{\tilde{\jmath}_n\}_{n\geq 1}$ is a Cauchy sequence in $\tilde{\Lambda}$. By triangular inequality , $\forall\, k \geq 1$,

$$\begin{aligned}\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+k}) &\leq \tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) + \tilde{\varrho}(\tilde{\jmath}_{n+1}, \tilde{\jmath}_{n+2}) + \cdots + \tilde{\varrho}(\tilde{\jmath}_{n+k-1}, \tilde{\jmath}_{n+k}) \\ &\ \vdots \\ &\leq \frac{\zeta^n}{1-\zeta}\tilde{\varrho}(\tilde{\jmath}_0, \tilde{\jmath}_1).\end{aligned} \tag{2.5}$$

Taking limit in (2.5) as $n \longrightarrow \infty$, we see that $\lim_{n\to\infty} \tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+k}) = 0$. Hence, $\{\tilde{\jmath}_n\}_{n\geq 1}$ is a Cauchy sequence in $\tilde{\Lambda}$. By completeness of $\tilde{\Lambda}$, we can find $u \in \tilde{\Lambda} : \tilde{\jmath}_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, to prove that u is an intuitionistic fuzzy FP of Υ, we presume that $u \notin [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}$. Replacing $\tilde{\jmath}$ with $\tilde{\jmath}_n$ and $\tilde{\ell}$ with u in (2.1), leads to

$$\begin{aligned}&\tilde{\varrho}(\tilde{\jmath}_{n+1}, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) \\ &\leq \aleph([\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))}, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) \\ &\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_n, u)]^b[\tilde{\varrho}(\tilde{\jmath}_n, [\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))})]^a[\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))})]^c \\ &\cdot\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}_n, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) + \tilde{\varrho}(u, [\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))})\right]^{1-a-b-c} \\ &\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}_n, u)]^b[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1})]^a[\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))})]^c\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}_n, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) + \tilde{\varrho}(u, \tilde{\jmath}_{n+1})\right]^{1-a-b-c}.\end{aligned} \tag{2.6}$$

Taking limit in (2.6) as $n \longrightarrow \infty$ and employing the continuity of $\tilde{\varrho}$, we see that $\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) = 0$. This proves that we can find $(\tilde{\tilde{\beta}}(u),\beta(u)) \in (0,1] \times [0,1) : u \in [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}$. □

Example 2.3. Let $\tilde{\Lambda} = \{2,3,4,5\}$ be equipped with the usual metric. Then $(\tilde{\Lambda}, \tilde{\varrho})$ is a complete MS. Let $(\eta_1, \eta_2) \in (0,1] \times [0,1)$ and $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ be an intuitionistic F-S-valued map : for each $\tilde{\jmath} \in \tilde{\Lambda}$, $\mu_\Upsilon(\tilde{\jmath}) : \tilde{\Lambda} \longrightarrow (0,1]$ and $\nu_\Upsilon(\tilde{\jmath}) : \tilde{\Lambda} \longrightarrow [0,1)$ are defined as:

$$\mu_\Upsilon(\tilde{\jmath})(t) = \begin{cases} \frac{\eta_1}{3}, & \text{if } t = 2 \\ \left(\frac{\eta_1}{3}\right)^2, & \text{if } t = 3 \\ \left(\frac{\eta_1}{3}\right)^3, & \text{if } t = 4 \\ \left(\frac{\eta_1}{3}\right)^4, & \text{if } t = 5. \end{cases}, \quad \nu_\Upsilon(\tilde{\jmath})(t) = \begin{cases} \frac{\eta_2}{16}, & \text{if } t = 2 \\ \frac{\eta_2}{4}, & \text{if } t = 3 \\ \frac{\eta_2}{2}, & \text{if } t = 4 \\ \eta_2 & \text{if } t = 5. \end{cases},$$

Define the mappings $\tilde{\tilde{\beta}} : \tilde{\Lambda} \longrightarrow (0,1]$ and $\beta : \tilde{\Lambda} \longrightarrow [0,1)$ by the ordered pair $(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath})) := \left(\left(\frac{\eta_1}{3}\right)^2, \frac{\eta_2}{4}\right)$, for each $\tilde{\jmath} \in X$. Then,

$$[\Upsilon \tilde{\jmath}]_{\left((\frac{\eta_1}{3})^2, \frac{\eta_2}{4}\right)} = \{2,3\}.$$

To see that Υ is a Hardy-Rogers-type intuitionistic fuzzy contraction, let $\tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$. Clearly, $\tilde{\jmath}, \tilde{\ell} \in \{4,5\}$. Whence,

$$\aleph([\Upsilon 4]_{(\tilde{\tilde{\beta}}(4),\beta(5))}, [\Upsilon 5]_{(\tilde{\tilde{\beta}}(4),\beta(5))} \;=\; \aleph([\Upsilon 4]_{(\tilde{\tilde{\beta}}(4),\beta(5))}, [\Upsilon 5]_{(\tilde{\tilde{\beta}}(4),\beta(5))}) = 0.$$

Hence, all the claims of Theorem 2.2 are obeyed. In this instance, the set of all intuitionistic fuzzy FPs of Υ is given by $\mathcal{F}_{ix}(\Upsilon) = \{2,3\}$.

Next, motivated by Theorem 1.3 and the result of Karapinar et al. [13, Theorem 4], we bring up the concept of an IRRCT intuitionistic fuzzy contraction and examine the existence of intuitionistic fuzzy FP for such contraction.

Definition 2.4. Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS. An intuitionistic F-S-valued map $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ is named interpolative Reich-Rus-Ciric intuitionistic fuzzy contraction if we can find two mappings $\tilde{\tilde{\beta}} : \tilde{\Lambda} \longrightarrow (0, 1], \beta : \tilde{\Lambda} \longrightarrow [0, 1)$ and constants $\eta \in [0, 1)$, $a, b \in (0, 1)$ with $a + b < 1$:

$$\aleph([\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))}, [\Upsilon \tilde{\ell}]_{(\tilde{\tilde{\beta}}(\tilde{\ell}),\beta(\tilde{\ell}))}) \leq \eta \left[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})\right]^a \left[\tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))})\right]^b \left[\tilde{\varrho}(\tilde{\ell}, [\Upsilon \tilde{\ell}]_{(\tilde{\tilde{\beta}}(\tilde{\ell}),\beta(\tilde{\ell}))})\right]^{1-a-b}, \tag{2.7}$$

$\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$.

Theorem 2.5. *Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ be an interpolative Reich-Rus-Ciric intuitionistic F-S-valued contraction. Assume further that $[\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))}$ is a NC subset of $\tilde{\Lambda}$ for each $\tilde{\jmath} \in \tilde{\Lambda}$. Then Υ has an intuitionistic fuzzy FP in $\tilde{\Lambda}$.*

Proof. Let $\tilde{\jmath}_0 \in \tilde{\Lambda}$ be arbitrary. Then, by hypothesis, we can find $(\tilde{\tilde{\beta}}(\tilde{\jmath}_0), \beta(\tilde{\jmath}_0)) \in (0, 1] \times [0, 1)$: $[\Upsilon \tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))} \in \mathcal{K}(\tilde{\Lambda})$. By compactness of $[\Upsilon \tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))}$, we can find $\tilde{\jmath}_1 \in [\Upsilon \tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))}$ with $\tilde{\varrho}(\tilde{\jmath}_0, \tilde{\jmath}_1) > 0 : \tilde{\varrho}(\tilde{\jmath}_0, \tilde{\jmath}_1) = \tilde{\varrho}(\tilde{\jmath}_0, [\Upsilon \tilde{\jmath}_0]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_0),\beta(\tilde{\jmath}_0))})$. Note that if we can finds no such $\tilde{\jmath}_1$, then $\tilde{\jmath}_0$ is already an intuitionistic fuzzy FP of Υ. Similarly, by hypotheses, we can finds $(\tilde{\tilde{\beta}}(\tilde{\jmath}_1), \beta(\tilde{\jmath}_1)) \in (0, 1] \times [0, 1)$: $[\Upsilon \tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}$ is a NC subset of $\tilde{\Lambda}$. Thus, we can finds $\tilde{\jmath}_2 \in [\Upsilon \tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))}$ with $\tilde{\varrho}(\tilde{\jmath}_1, \tilde{\jmath}_2) > 0$: $\tilde{\varrho}(\tilde{\jmath}_1, \tilde{\jmath}_2) = \tilde{\varrho}(\tilde{\jmath}_1, [\Upsilon \tilde{\jmath}_1]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_1),\beta(\tilde{\jmath}_1))})$. Inductively, we generate a sequence $\{\tilde{\jmath}_n\}_{n\geq 1}$ of points of $\tilde{\Lambda}$ with $\tilde{\jmath}_{n+1} \in [\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))}$, $\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) > 0 : \tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) = \tilde{\varrho}(\tilde{\jmath}_n, [\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))})$. By Lemma 1.10, we see that

$$\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) \leq \aleph([\Upsilon \tilde{\jmath}_{n-1}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_{n-1}),\beta(\tilde{\jmath}_{n-1}))}, [\Upsilon \tilde{\jmath}_{n+1}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_{n+1}),\beta(\tilde{\jmath}_{n+1}))}). \tag{2.8}$$

Now, we establish that $\{\tilde{\jmath}_n\}_{n\geq 1}$ is a Cauchy sequence in $\tilde{\Lambda}$. Setting $\tilde{\jmath} = \tilde{\jmath}_n$ and $\tilde{\ell} = \tilde{\jmath}_{n-1}$ in (2.7), we get

$$\begin{aligned}
\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) &\leq \aleph([\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))}, [\Upsilon \tilde{\jmath}_{n-1}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_{n-1}),\beta(\tilde{\jmath}_{n-1}))}) \\
&\leq \eta[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n-1})]^a[\tilde{\varrho}(\tilde{\jmath}_n, [\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))})]^b[\tilde{\varrho}(\tilde{\jmath}_{n-1}, [\Upsilon \tilde{\jmath}_{n-1}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_{n-1}),\beta(\tilde{\jmath}_{n-1}))})]^{1-a-b} \\
&\leq \eta[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n-1})]^a[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1})]^b[\tilde{\varrho}(\tilde{\jmath}_{n-1}, \tilde{\jmath}_n)]^{1-a-b} \\
&= \eta[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n-1})]^{1-b}[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1})]^b.
\end{aligned} \tag{2.9}$$

From (2.9), we see that

$$\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) \leq \eta^{\frac{1}{1-b}} \tilde{\varrho}(\tilde{\jmath}_{n-1}, \tilde{\jmath}_n) \text{ for all } n \in \mathbb{N}. \tag{2.10}$$

We infer from(2.10) that $\forall\ n \in \mathbb{N}$,

$$\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1}) \leq \eta \tilde{\varrho}(\tilde{\jmath}_{n-1}, \tilde{\jmath}_n) \leq \eta^n \tilde{\varrho}(\tilde{\jmath}_0, \tilde{\jmath}_1). \tag{2.11}$$

From (2.11), adopting the steps in discussing Theorem 2.2, we conclude that $\{\tilde{\jmath}_n\}_{n\geq 1}$ is a Cauchy sequence in $\tilde{\Lambda}$. The completeness of this space yields that we can finds $u \in \tilde{\Lambda} : \tilde{\jmath}_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, we show that u is an intuitionistic fuzzy FP of $\tilde{\Lambda}$. Suppose $u \notin [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}$ so that $\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) > 0$. Then, replacing $\tilde{\jmath}$ and $\tilde{\ell}$ with $\tilde{\jmath}_n$ and u, respectively in (2.7), and using Lemma 1.10, gives

$$\begin{aligned}
\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) &\leq \tilde{\varrho}(u, \tilde{\jmath}_{n+1}) + \tilde{\varrho}(\tilde{\jmath}_{n+1}, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) \\
&\leq \tilde{\varrho}(u, \tilde{\jmath}_{n+1}) + \aleph([\Upsilon \tilde{\jmath}_n]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}_n),\beta(\tilde{\jmath}_n))}, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))}) \\
&\leq \tilde{\varrho}(u, \tilde{\jmath}_{n+1}) + \eta[\tilde{\varrho}(\tilde{\jmath}_n, u)]^a[\tilde{\varrho}(\tilde{\jmath}_n, \tilde{\jmath}_{n+1})]^b[\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))})]^{1-a-b}.
\end{aligned} \tag{2.12}$$

Letting $n \longrightarrow \infty$ in (2.12) and using the continuity of the metric $\tilde{\varrho}$, yields $\tilde{\varrho}(u, [\Upsilon u]_{(\tilde{\tilde{\beta}}(u), \beta(u))}) = 0$, a contradiction. Whence, $u \in [\Upsilon u]_{(\tilde{\tilde{\beta}}(u), \beta(u))}$. □

In what follows, we study the concepts of Hardy-Roger's type intuitionistic fuzzy contraction and Reich-Rus-Ciric intuitionistic fuzzy contraction in connection with $\tilde{\varrho}_{(\infty,\infty)}$-distance for intuitionistic F-S. It is interesting to point out that the investigation of FPs of intuitionistic F-S-valued maps in the setting of $\tilde{\varrho}_{(\infty,\infty)}$-metric is very significant in computing Hausdorff dimensions. These dimensions aid us to comprehend the notion of ε^{∞}-space which is of great importance in higher energy physics.

Theorem 2.6. *Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow \mathcal{K}_{\mathcal{IFS}}(\tilde{\Lambda})$ be intuitionistic F-S valued map. Assume that the following conditions are obeyed: we can find $\tilde{\lambda}, a, b, c \in (0,1)$ with $a + b + c < 1$ such that $\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$,*

$$\tilde{\varrho}_{(\infty,\infty)}(\Upsilon \tilde{\jmath}, \Upsilon \tilde{\ell}) \leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^b [p(\tilde{\jmath}, \Upsilon(\tilde{\jmath})]^a [p(\tilde{\ell}, \Upsilon \tilde{\ell})]^c \left[\frac{1}{2}(p(\tilde{\jmath}, \Upsilon \tilde{\ell}) + p(\tilde{\ell}, \Upsilon \tilde{\jmath}))\right]^{1-a-b-c}. \tag{2.13}$$

Then Υ has an intuitionistic fuzzy FP in $\tilde{\Lambda}$.

Proof. Define the mappings $\tilde{\tilde{\beta}} : \tilde{\Lambda} \longrightarrow (0,1], \beta : \tilde{\Lambda} \longrightarrow [0,1)$ by $\tilde{\tilde{\beta}}(\tilde{\jmath}) = 1$ and $\beta(\tilde{\jmath}) = 0$ for each $\tilde{\jmath} \in \tilde{\Lambda}$. Then, by hypothesis, $[\Upsilon \tilde{\jmath}]_{(1,0)} \in \mathcal{K}(\tilde{\Lambda})$. Now, for every $\tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$,

$$\begin{aligned}
& D_{(1,0)}(\Upsilon \tilde{\jmath}, \Upsilon \tilde{\ell}) \\
\leq\ & \tilde{\varrho}_{(\infty,\infty)}(\Upsilon \tilde{\jmath}, \Upsilon \tilde{\ell}) \\
\leq\ & \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^b [p(\tilde{\jmath}, \Upsilon \tilde{\jmath})]^a [p(\tilde{\ell}, \Upsilon \tilde{\ell})]^c \left[\frac{1}{2}(p(\tilde{\jmath}, \Upsilon \tilde{\ell}) + p(\tilde{\ell}, \Upsilon \tilde{\jmath}))\right]^{1-a-b-c}.
\end{aligned}$$

Since $[\Upsilon \tilde{\jmath}]_1 \subseteq [\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}), \beta(\tilde{\jmath}))} \in \mathcal{K}(\tilde{\Lambda})$, whence, $\tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}), \beta(\tilde{\jmath}))}) \leq \tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\jmath}]_{(1,0)})$ for each $(\tilde{\tilde{\beta}}(\tilde{\jmath}), \beta(\tilde{\jmath})) \in (0,1] \times [0,1)$. It follows that $p(\tilde{\jmath}, \Upsilon \tilde{\jmath}) \leq \tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\jmath}]_{(1,0)})$. Whence,

$$\begin{aligned}
& \aleph([\Upsilon \tilde{\jmath}]_{(1,0)}, [\Upsilon \tilde{\ell}]_{(1,0)}) \\
\leq\ & \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^b [\tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\jmath}]_{(1,0)})]^a [\tilde{\varrho}(\tilde{\ell}, [\Upsilon \tilde{\ell}]_{(1,0)})]^c \left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath}, [\Upsilon \tilde{\ell}]_{(1,0)}), \tilde{\varrho}(\tilde{\ell}, [\Upsilon \tilde{\jmath}]_{(1,0)}))\right]^{1-a-b-c}.
\end{aligned} \tag{2.14}$$

Hence, Theorem 2.2 can be employed to find $u \in \tilde{\Lambda}$ such that $u \in [\Upsilon u]_{(1,0)}$. □

On similar steps as in the proof of Theorem 2.6, we can establish the following result.

Theorem 2.7. *Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $\Upsilon : \tilde{\Lambda} \longrightarrow \mathcal{K}_{\mathcal{IFS}}(\tilde{\Lambda})$ be intuitionistic F-S valued map. Assume that the following conditions are obeyed: we can find $\eta \in [0,1)$ and $a, b \in (0,1)$ with $a + b < 1$ such that $\forall\ \tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$,*

$$\tilde{\varrho}_{(\infty,\infty)}(\Upsilon \tilde{\jmath}, \Upsilon \tilde{\ell}) \leq \eta[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^a [p(\tilde{\jmath}, \Upsilon(\tilde{\jmath})]^b [p(\tilde{\ell}, \Upsilon \tilde{\ell})]^{1-a-b}. \tag{2.15}$$

Then Υ has a fuzzy FP in $\tilde{\Lambda}$.

Example 2.8. Let $\tilde{\Lambda} = \left\{\zeta_n = \frac{n(n+1)}{2} : n = 1, 2, \cdots\right\} \cup \{-1\}$, $\eta_1, \eta_2 \in [0,1) \times [0,1)$ and $\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell}) = |\tilde{\jmath} - \tilde{\ell}|$ $\forall$ $\tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda}$. Then, $(\tilde{\Lambda}, \tilde{\varrho})$ is a complete MS. Define an intuitionistic F-S-valued map $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow \mathcal{K}_{\mathcal{IFS}}(\tilde{\Lambda})$ as follows:
For $\tilde{\jmath} = -1$,

$$\mu_\Upsilon(-1)(t) = \begin{cases} 1 - \frac{\eta_1}{6^2}, & \text{if } t = -1 \\ 1 - \frac{\eta_1}{5^2}, & \text{if } t = \zeta_1 \\ 1 - \frac{\eta_1}{4^2}, & \text{if } t = \zeta_2 \\ 1 - \frac{\eta_1}{3^2}, & \text{if } t = \zeta_n,\ n \geq 3, \end{cases} \qquad \nu_\Upsilon(-1)(t) = \begin{cases} \frac{\eta_2}{25}, & \text{if } t = -1 \\ \frac{\eta_2}{20}, & \text{if } t = \zeta_1 \\ \frac{\eta_2}{15}, & \text{if } t = \zeta_2 \\ \frac{\eta_2}{10}, & \text{if } t = \zeta_n,\ n \geq 3, \end{cases}$$

and for $\tilde{\jmath} \in \tilde{\Lambda} \setminus \{-1\}$,

$$\mu_\Upsilon(\tilde{\jmath})(t) = \begin{cases} 1 - \frac{\eta_1}{7^2}, & \text{if } t = \zeta_1 \\ 1 - \frac{\eta_1}{7^3}, & \text{if } t = \zeta_2 \\ 1 - \frac{\eta_1}{7^4}, & \text{if } t \in \{\zeta_3, \zeta_4, \cdots, \zeta_{n-1}\},\ n \geq 3. \end{cases} , \nu_\Upsilon(\tilde{\jmath})(t) = \begin{cases} \frac{\eta_2}{50}, & \text{if } t = \zeta_1 \\ \frac{\eta_2}{40}, & \text{if } t = \zeta_2 \\ \frac{\eta_2}{30}, & \text{if } t \in \{\zeta_3, \zeta_4, \cdots, \zeta_{n-1}\},\ n \geq 3. \end{cases}$$

Also, define the mappings $\tilde{\tilde{\beta}} : \tilde{\Lambda} \longrightarrow (0,1]$, $\beta : \tilde{\Lambda} \longrightarrow [0,1)$ by the ordered pair $(\tilde{\tilde{\beta}}(\tilde{\jmath}), \beta(\tilde{\jmath})) = \left(1 - \frac{\eta_1}{36}, \frac{\eta_2}{25}\right)$ $\forall$ $\tilde{\jmath} \in \tilde{\Lambda}$. Then,

$$[\Upsilon \tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}), \beta(\tilde{\jmath}))} = \begin{cases} \{-1\}, & \text{if } \tilde{\jmath} = 0 \\ \{\zeta_n\}, & \text{if } \tilde{\jmath} \neq 0,\ n \geq 1. \end{cases}$$

Now, to see that the contractive condition (2.15) holds, let $\tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(\Upsilon)$. Obviously, $\tilde{\jmath}, \tilde{\ell} \in \{-1\}$. Whence,

$$\tilde{\varrho}_{(\infty,\infty)}(\Upsilon(\tilde{\jmath}), \Upsilon(\tilde{\ell})) \quad = \quad 0 \leq \eta[\tilde{\varrho}(\tilde{\jmath}, \tilde{\ell})]^a [p(\tilde{\jmath}, \Upsilon(\tilde{\jmath}))]^b [p(\tilde{\ell}, \Upsilon(\tilde{\ell}))]^{1-a-b},$$

$\forall$ $\eta \in (0,1)$. This proves that (2.15) holds $\forall$ $\tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda}$. Hence, all the conditions of Theorem 2.7 are obeyed. We can see that Υ has many intuitionistic fuzzy FPs in $\tilde{\Lambda}$.

However, Υ is not a fuzzy contraction in the sense of Heilpern [9] , since for $\tilde{\jmath} = -1$ and $\tilde{\ell} = \zeta_{n-1}$, $n \geq 3$, we see that

$$\begin{aligned} \sup_{n \geq 3} \frac{\aleph([\Upsilon(-1)]_{(1-\frac{\eta_1}{36}, \frac{\eta_2}{25})}, [\Upsilon \zeta_{n-1}]_{(1-\frac{\eta_1}{36}, \frac{\eta_2}{25})})}{\tilde{\varrho}(0, \zeta_{n-1})} &= \sup_{n \geq 3} \frac{\zeta_{n-1} - 1}{\zeta_{n-1}} \\ &= \sup_{n \geq 3} \frac{\frac{n(n-1)}{2} - 1}{\frac{n(n-1)}{2}} \\ &= \sup_{n \geq 3} \left[1 - \frac{2}{n(n-1)}\right] = 1. \end{aligned}$$

Whence, the result of Heilpern [9, Theorem 3.1] is unapplicable in this illustration to obtain any fuzzy FP of Υ.

3. Applications in multivalued and single-valued mappings

Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a MS, $CB(\tilde{\Lambda})$ and $\mathcal{N}(\tilde{\Lambda})$ be the family of nonempty closed and bounded and nonempty subsets of $\tilde{\Lambda}$, respectively. Nadler [20, Theorem 5] established that every multivalued contraction on a complete MS has a FP.
Among the well-known generalizations of multivalued contractions due to Nadler related to our focus here are the ones presented by Reich [25] and Rus [23].

Theorem 3.1. *(See Rus [23]) Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $F : \tilde{\Lambda} \longrightarrow CB(\tilde{\Lambda})$ be a multivalued mapping. Assume that we can find $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that $\forall\ \tilde{j}, \tilde{\ell} \in \tilde{\Lambda}$,*

$$\aleph(F\tilde{j}, F\tilde{\ell}) \leq a\tilde{\varrho}(\tilde{j}, \tilde{\ell}) + b\tilde{\varrho}(\tilde{\ell}, F\tilde{\ell}).$$

Then we can find $u \in \tilde{\Lambda}$ such that $u \in Fu$.

Theorem 3.2. *(See Reich [25]) Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $F : \tilde{\Lambda} \longrightarrow CB(\tilde{\Lambda})$ be a multivalued mapping. Assume that we can find $a, b \in \mathbb{R}_+$ with $a + b + c < 1$ such that $\forall\ \tilde{j}, \tilde{\ell} \in \tilde{\Lambda}$,*

$$\aleph(F\tilde{j}, F\tilde{\ell}) \leq a\tilde{\varrho}(\tilde{j}, \tilde{\ell}) + b\tilde{\varrho}(\tilde{j}, F\tilde{j}) + c\tilde{\varrho}(\tilde{\ell}, F\tilde{\ell}).$$

Then we can finds $u \in \tilde{\Lambda} : u \in Fu$.

In this section, we come up with some consequences of our key findings in the frame of both single-valued and multivalued mappings. First, multivalued analogues of Theorems 2.2 and 2.5 are derived. They are also multivalued generalizations of the recently announced FP theorems due to Karapinar et al. [10, Th. 4] and Karapinar et al. [11, Corollary 1], respectively.

Corollary 1. *Let $(\tilde{\Lambda}, \tilde{\varrho})$ be a complete MS and $F : \tilde{\Lambda} \longrightarrow \mathcal{K}(\tilde{\Lambda})$ be a multi-valued mapping. Assume that we can find $\tilde{\lambda}, a, b, c, \in (0, 1]$ with $a + b + c < 1$ such that $\forall\ \tilde{j}, \tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(F)$,*

$$\aleph(F\tilde{j}, F\tilde{\ell}) \leq \tilde{\lambda}[\tilde{\varrho}(\tilde{j}, \tilde{\ell})]^b [\tilde{\varrho}(\tilde{j}, F\tilde{j})]^a [\tilde{\varrho}(\tilde{\ell}, F\tilde{\ell})]^c \left[\frac{1}{2}(\tilde{\varrho}(\tilde{j}, F\tilde{\ell}) + \tilde{\varrho}(\tilde{\ell}, F\tilde{j}))\right]^{1-a-b-c}. \tag{3.1}$$

Then we can finds $u \in \tilde{\Lambda}$ such that $u \in Fu$.

Proof. Consider two mappings $\vartheta : \tilde{\Lambda} \to (0, 1), \pi : \tilde{\Lambda} \longrightarrow [0, 1)$ and an intutionistic F-S valued map $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ defined by

$$\mu_\Upsilon(\tilde{j})(t) = \begin{cases} \vartheta(\tilde{j}), & \text{if } t \in F\tilde{j} \\ 0, & \text{if } t \notin F\tilde{j}, \end{cases} \qquad \nu_\Upsilon(\tilde{j})(t) = \begin{cases} \pi(\tilde{j}), & \text{if } t \in F\tilde{j} \\ 1, & \text{if } t \notin F\tilde{j}. \end{cases}$$

For all $\tilde{j} \in \tilde{\Lambda}$, taking $(\tilde{\tilde{\beta}}(\tilde{j}), \beta(\tilde{j})) = (\vartheta(\tilde{j}), \pi(\tilde{j})) \in (0, 1) \times [0, 1) \subset (0, 1] \times [0, 1)$, we see that

$$[\Upsilon\tilde{j}]_{(\tilde{\tilde{\beta}}(\tilde{j}), \beta(\tilde{j}))} = \{t \in \tilde{\Lambda} : \mu_\Upsilon(\tilde{j})(t) \geq \tilde{\tilde{\beta}}(\tilde{j}) \text{ and } \nu_\Upsilon(\tilde{j})(t) \leq \beta(\tilde{j})\} = F\tilde{j}.$$

Whence , Theorem 2.2 can be applied to find $u \in \tilde{\Lambda}$ such that $u \in Fu = [\Upsilon u]_{(\tilde{\tilde{\beta}}(u), \beta(u))}$. □

Example 3.3. Let $\tilde{\Lambda} = [2,7]$ and $\tilde{\varrho}(\tilde{\jmath},\tilde{\ell}) = |\tilde{\jmath}-\tilde{\ell}|$ $\forall$ $\tilde{\jmath},\tilde{\ell} \in \tilde{\Lambda}$. Then, $(\tilde{\Lambda},\tilde{\varrho})$ is a complete MS. Define $F : \tilde{\Lambda} \longrightarrow \mathcal{K}(\tilde{\Lambda})$ by

$$F\tilde{\jmath} = \begin{cases} [2,3], & \text{if } 2 \leq \tilde{\jmath} < 3 \\ [4,7], & \text{if } 3 \leq \tilde{\jmath} \leq 7. \end{cases}$$

Let $\tilde{\jmath},\tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(F)$. Clearly, $\tilde{\jmath},\tilde{\ell} \in (2,3)$ and

$$\begin{aligned} \aleph(F\tilde{\jmath},F\tilde{\ell}) &= \aleph([2,3],[2,3]) = 0 \\ &\leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath},\tilde{\ell})]^b[\tilde{\varrho}(\tilde{\jmath},F\tilde{\jmath})]^a[\tilde{\varrho}(\tilde{\ell},F\tilde{\ell})]^c\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath},F\tilde{\ell})+\tilde{\varrho}(\tilde{\ell},F\tilde{\jmath}))\right]^{1-a-b-c}. \end{aligned}$$

Hence, all the claims of Corollary 1 are obeyed. We see that F has many FPs in $\tilde{\Lambda}$.

On the other hand, F is not a multivalued contraction, since for $\tilde{\jmath} = 2$ and $\tilde{\ell} = 3$, we see that

$$\begin{aligned} \aleph(F2,F3) &= \aleph([2,3],[4,7]) \\ &= 4 > \tilde{\lambda}(1) = \tilde{\lambda}\tilde{\varrho}(2,3), \end{aligned}$$

$\forall$ $\tilde{\lambda} \in [0,1)$. Hence, the result of Nadler [20, Theorem 5] is unapplicable in this illustration to locate a FP of F.

Similarly, since $F2 = [2,3]$ and $F3 = [4,7]$, we see that

$$\tilde{\varrho}(2,F2) = \inf_{\omega\in[2,3]} \tilde{\varrho}(2,\omega) = 0,$$

$$\tilde{\varrho}(3,F3) = \inf_{\xi\in[4,7]} \tilde{\varrho}(3,\xi) = 1.$$

Hence,

$$\begin{aligned} \aleph(F2,F3) &= \aleph([2,3],[4,7]) \\ &= 4 > a(1)+b(1) \\ &= a\tilde{\varrho}(2,3)+b\tilde{\varrho}(3,F3), \end{aligned}$$

$\forall$ $a,b \in \mathbb{R}_+$ obeying $a+b < 1$. This means that Theorem 3.1 due to Rus [23] is unapplicable in this illustration to find a FP of F.

In like manner,

$$\begin{aligned} \aleph(F2,F3) &= \aleph([2,3],[4,7]) \\ &= 4 > a+c = a(1)+b(0)+c(1) \\ &= a\tilde{\varrho}(2,3)+b\tilde{\varrho}(2,F2)+c\tilde{\varrho}(3,F3), \end{aligned}$$

$\forall$ $a,b,c \in \mathbb{R}_+$ with $a+b+c < 1$. Hence, Theorem 3.2 due to Reich [25] is unapplicable in this case to locate any FP of F.

Corollary 2. *(See Karapinar et al. [10, Theorem 4]) Let $(\tilde{\Lambda},\tilde{\varrho})$ be a complete MS and $f : \tilde{\Lambda} \rightarrow \tilde{\Lambda}$ be a single-valued mapping. Assume that we can find $\tilde{\lambda},a,b,c \in (0,1)$ with $a+b+c < 1$ such that $\forall$ $\tilde{\jmath},\tilde{\ell} \in \tilde{\Lambda} \setminus \mathcal{F}_{ix}(f)$,*

$$\tilde{\varrho}(f\tilde{\jmath},f\tilde{\ell}) \leq \tilde{\lambda}[\tilde{\varrho}(\tilde{\jmath},\tilde{\ell})]^b[\tilde{\varrho}(\tilde{\jmath},f\tilde{\jmath})]^a[\tilde{\varrho}(\tilde{\ell},f\tilde{\ell})]^c\left[\frac{1}{2}(\tilde{\varrho}(\tilde{\jmath},f\tilde{\ell})+\tilde{\varrho}(\tilde{\ell},f\tilde{\jmath}))\right]^{1-a-b-c}. \tag{3.2}$$

Then we can find $u \in \tilde{\Lambda}$ such that $fu = u$.

Proof. Let $\vartheta : \tilde{\Lambda} \longrightarrow (0,1), \pi : \tilde{\Lambda} \longrightarrow [0,1)$ be two mappings, and define an intuitionistic F-S-valued map $\Upsilon = \langle \mu_\Upsilon, \nu_\Upsilon \rangle : \tilde{\Lambda} \longrightarrow (IFS)^{\tilde{\Lambda}}$ as follows:

$$\mu_\Upsilon(\tilde{\jmath})(\tilde{t}) = \begin{cases} \vartheta(\tilde{\jmath}), & \text{if } \tilde{t} = f\tilde{\jmath} \\ 0, & \text{if } \tilde{t} \neq f\tilde{\jmath}, \end{cases} \quad \nu_\Upsilon(\tilde{\jmath})(\tilde{t}) = \begin{cases} \pi(\tilde{\jmath}), & \text{if } \tilde{t} = f\tilde{\jmath} \\ 0, & \text{if } \tilde{t} \neq f\tilde{\jmath}. \end{cases}$$

Then, for $\tilde{\jmath} \in \tilde{\Lambda}$, letting $(\tilde{\tilde{\beta}}(\tilde{\jmath}), \beta(\tilde{\jmath})) = (\vartheta(\tilde{\jmath}), \pi(\tilde{\jmath})) \in (0,1] \times [0,1)$, gives

$$[\Upsilon\tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))} = \{t \in \tilde{\Lambda} : \mu_\Upsilon(\tilde{\jmath})(t) \geq \tilde{\tilde{\beta}}(\tilde{\jmath}) \text{ and } \nu_\Upsilon(\tilde{\jmath})(t) \leq \beta(\tilde{\jmath})\} = \{f\tilde{\jmath}\}.$$

Obviously, $\{f\tilde{\jmath}\} \in \mathcal{K}(\tilde{\Lambda})\ \forall\ \tilde{\jmath} \in \tilde{\Lambda}$. Note that in this case, $\aleph([\Upsilon\tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\jmath}),\beta(\tilde{\jmath}))}, [\Upsilon\tilde{\jmath}]_{(\tilde{\tilde{\beta}}(\tilde{\ell}),\beta(\tilde{\ell}))}) = \tilde{\varrho}(f\tilde{\jmath}, f\tilde{\ell})\ \forall$ $\tilde{\jmath}, \tilde{\ell} \in \tilde{\Lambda}$. Whence, Theorem 2.2 can be employed to find $u \in \tilde{\Lambda}$ such that $u \in [\Upsilon u]_{(\tilde{\tilde{\beta}}(u),\beta(u))} = \{fu\}$; which further implies that $u = fu$. □

4. Conclusions

This note proposed a new advancement in fuzzy mathematics and FPT of crisp mappings by bringing up the interpolative approaches. To achieve this, IHRT intuitionistic fuzzy contraction and IRRCT intuitionistic fuzzy contraction are launched and the corresponding FP results are proved, with examples supporting the hypotheses of our main results. The findings in this work, being discussed in the frame of MS, are fundamental. Whence, they can be fine-tuned when examined in the bodywork of generalized/quasi or pseudo MS such as b-MS, G-MS, F-MS, fuzzy MS, and related domains. From application viewpoint, the contractive inequalities put up herein can be used to study solvability criteria of some classes of differential/integral inclusions of either integer or non-integer orders.

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Conflict of interest

The authors declare that they have no competing interests.

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