mathematics

Article

Interpolative Meir–Keeler Mappings in Modular Metric Spaces

Erdal Karapınar [1,2,3,*], Andreea Fulga [4] and Seher Sultan Yeşilkaya [1]

1 Division of Applied Mathematics, Thu Dau Mot University, Thu Dau Mot 75000, Vietnam
2 Department of Mathematics, Çankaya University, Etimesgut, Ankara 06790, Turkey
3 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
4 Department of Mathematics and Computer Science, Transilvania University of Brasov, 500123 Brasov, Romania
* Correspondence: erdalkarapinar@tdmu.edu.vn or erdalkarapinar@yahoo.com

Abstract: Modular metric space is one of the most interesting spaces in the framework of the metric fixed point theory. The main goal of the paper is to provide some certain fixed point results in the context of modular metric spaces and non-Archimedean modular metric spaces. In particular, we examine the existence of interpolative Meir–Keeler contraction types via admissible mappings for fixed point theory. Our results bring together several results available in the current corresponding literature.

Keywords: modular metric spaces; interpolative contraction; fixed point; Meir–Keeler contraction

MSC: 47H10; 54H25; 46J10

Citation: Karapınar, E.; Andreea, F.; Yeşilkaya, S.S. Interpolative Meir–Keeler Mappings in Modular Metric Spaces. *Mathematics* **2022**, *10*, 2986. https://doi.org/10.3390/math10162986

Academic Editor: Salvatore Sessa

Received: 28 July 2022
Accepted: 16 August 2022
Published: 18 August 2022

1. Introduction

The idea of the theory of modular spaces was first put forward by Nakano [1] and was later reconsidered in detail by Musielak and Orlicz [2]. In 2010, Chistyakov [3,4] generalized modular spaces and complete metric spaces by introducing modular metric spaces. In the last two decades, the modular metric space has been an interesting abstract space for nonlinear functional analysis, and hence, it has been investigated densely by many researchers. For more features of the concepts of modular metric space, see e.g., [5–7].

The metric fixed point theory is a very rich area for research that was initiated by Banach. One of the most interesting and early characterizations of the Banach theorem was given by Kannan [8]. It was later understood that Kannan contraction is independent from the Banach contractions [9]. Another crucial contraction, weakly uniformly strict contraction, was observed by Meir–Keeler [10]. It was later called the Meir–Keeler contraction. As it is expected, it is a proper generalization of Banach's principle, see e.g., [11,12]. In addition to all such linear extensions of the contraction mapping, we underline the notion of admissible auxiliary function that plays one of the key roles in initiating many interesting contractions. In particular, Samet et al. [13] used admissible functions to extend the renowned Banach fixed point theorem. Following this paper, a huge number of papers appeared in which the admissible functions are used to improve well-known existing results in the metric fixed point theory.

In addition to all these advances on the metric fixed point theory, we need to mention the interpolative contractions. Very recently, the concept of interpolative contraction was suggested by the first author [14] by revising the Kannan contraction [8]. Following this paper [14], many research papers have been published on interpolative contractions in the setting of distinct abstract spaces [15–25] and for differently combined well-known contractions [26–32].

The aim of the paper is to examine the existence and uniqueness of interpolative Meir–Keeler contraction types via admissible mappings for fixed point theory in the context of the modular metric spaces. For this purpose, we reserve the first section for the introduction. The aim of the second section is to collect and clarify the mentioned notions above as well as to give the fundamentals of the metric fixed point theory. In the third section, we shall highlight the main results for interpolative Meir–Keeler contraction types in modular metric spaces and non-Archimedean modular metric spaces.

2. Preliminaries

We start this section by presenting well-known notations, collecting the basic definitions and fundamental results.

2.1. Concepts Related to Modular Metric Spaces

In this subsection, we shall present some basic concepts and properties in modular metric spaces. First of all, we recall the definition of the modular space:

Definition 1 ([2]). *Let $\mathcal{M}$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho\colon \mathcal{M} \to [0,\infty]$ is called a modular if for any y and z in $\mathcal{M}$, it satisfies the following conditions:*

(n_1) $\rho(y) = 0$ iff $y = 0$;
(n_2) $\rho(\alpha y) = \rho(y)$ for every scalar α with $|\alpha| = 1$;
(n_3) $\rho(\alpha y + \beta z) \leq \rho(y) + \rho(z)$, for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Let $\mathcal{M}$ be a nonempty set, $\lambda \in (0,\infty)$ and because of the disparity of the arguments, function $w\colon (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ will be written by $w_\lambda(y,z) = w(\lambda, y, z)$ for every $\lambda > 0$ and $y, z \in \mathcal{M}$.

In what follows, we state the definition of modular metric spaces (hereinafter referred to as "MMS").

Definition 2 ([3,4]). *Let $\mathcal{M}_w \neq \emptyset$ be a set. A function $w\colon (0,\infty)\times \mathcal{M}_w \times \mathcal{M}_w \to [0,\infty]$ is said to be a metric modular on $\mathcal{M}_w$ if it satisfies the following, for all $y, z, \mathrm{t} \in \mathcal{M}_w$,*

(w_1) $w_\lambda(y,z) = 0$ for all $\lambda > 0 \Leftrightarrow y = z$;
(w_2) $w_\lambda(y,z) = w_\lambda(z,y)$ for every $\lambda > 0$;
(w_3) $w_{\lambda+\mu}(y,z) \leq w_\lambda(y,\mathrm{t}) + w_\mu(\mathrm{t},z)$ for every $\lambda, \mu > 0$.

If instead of (w_1), we have only the condition
(w_1') $w_\lambda(y,y) = 0$, for each $\lambda > 0$, then w is said to be a (metric) pseudomodular on $\mathcal{M}_w$.

If we replace (w_3) by

$$(w_4)\ w_{\max\{\lambda,\mu\}}(y,z) \leq w_\lambda(y,\mathrm{t}) + w_\mu(\mathrm{t},z),$$

for all $\lambda, \mu > 0$ and all $y, z, \mathrm{t} \in \mathcal{M}_w$, then $\mathcal{M}_w$ is called a non-Archimedean modular metric space (hereinafter referred to "non-AMMS") [33]. Since (w_4) implies (w_3), each non-AMMS is an MMS.

Remark 1. *If w is a pseudomodular metric on a set $\mathcal{M}_w$, then the function $\lambda \to w_\lambda(y,z)$ is nonincreasing on $(0,+\infty)$ for all $y, z \in \mathcal{M}$. Indeed, if $0 < \mu < \lambda$, then*

$$w_\lambda(y,z) \leq w_{\lambda-\mu}(y,y) + w_\mu(y,z) = w_\mu(y,z).$$

Definition 3 ([34]). *A pseudomodular w on $\mathcal{M}$ is said to satisfy the Δ_2-condition (on $\mathcal{M}_w$) if the following condition holds:*

(Δ_2) Given a sequence $\{y_\mathrm{k}\} \subset \mathcal{M}_w$ and $y \in \mathcal{M}_w$, if there exists a number $\lambda > 0$, possibly depending on $\{y_\mathrm{k}\}$ and y, such that if $\lim_{\mathrm{k}\to\infty} w_\lambda(y_\mathrm{k}, y) = 0$, then $\lim_{\mathrm{k}\to\infty} w_{\frac{\lambda}{2}}(y_\mathrm{k}, y) = 0$.

Next, we recollect the basic topological notions in the context of modular spaces.

Definition 4 ([3,35]). *Let X_w be an MMS.*

(i) *The sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}_w$ is notified to be convergent to $y \in \mathcal{M}_w$ if $w_\lambda(y_n, y) \to 0$, as $n \to \infty$ for every $\lambda > 0$.*
(ii) *The sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}_w$ is notified to be Cauchy if $w_\lambda(y_m, y_n) \to 0$, as $m, n \to \infty$ for all $\lambda > 0$.*
(iii) *A subset K of $\mathcal{M}_w$ is notified to be closed if each limit of a convergent sequence of K is contained in K.*
(iv) *A subset K of $\mathcal{M}_w$ is notified to be complete if any Cauchy sequence in K is a convergent sequence and its limit is in K.*
(v) *A subset K of $\mathcal{M}_w$ is notified to be bounded if for every $\lambda > 0$,*

$$\delta_w(\mathsf{K}) = \sup\{w_\lambda(y, z) \colon y, z \in \mathsf{K}\} < \infty.$$

2.2. Basic Definitions and Theorems

We shall start this section by stating the renowned Meir–Keeler fixed point theorem.

Theorem 1 ([10]). *Let $(\mathcal{M}, \mathsf{d})$ be a complete metric space. If P forms a Meir–Keeler contraction on $\mathcal{M}$, that is,*
"for any given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\varepsilon \leq \mathsf{d}(y, z) < \varepsilon + \delta \quad \textit{implies} \quad \mathsf{d}(\mathsf{P}y, \mathsf{P}z) < \varepsilon,$$

for every $y, z \in \mathcal{M}$",
then P possesses a unique fixed point.

Let us recall the α-admissible functions:

Definition 5 ([13]). *Let $\mathcal{M}$ be a nonempty set, $\mathsf{P}\colon \mathcal{M} \to \mathcal{M}$ and $\alpha\colon \mathcal{M} \times \mathcal{M} \to [0, +\infty)$. We notify that P is α-admissible if $y, z \in \mathcal{M}$,*

$$\alpha(y, z) \geq 1 \Longrightarrow \alpha(\mathsf{P}y, \mathsf{P}z) \geq 1. \tag{1}$$

Karapınar et al. [36] defined the concept of triangular α-admissible mapping as follows.

Definition 6 ([36]). *Let $\mathcal{M}$ be a nonempty set, P be a self mapping defined on $\mathcal{M}$ and $\alpha\colon \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be a function. P is said to be a triangular α -admissible mapping if the following conditions:*

(1) $\alpha(y, z) \geq 1$ *implies* $\alpha(\mathsf{P}y, \mathsf{P}z)) \geq 1$, $y, z \in \mathcal{M}$;
(2) $\alpha(y, \mathsf{t}) \geq 1, \alpha(\mathsf{t}, z) \geq 1$, *implies* $\alpha(y, z) \geq 1$;

hold for all $y, z, \mathsf{t} \in \mathcal{M}$.

Lemma 1 ([13]). *Let $\mathcal{M}$ be a nonempty set and $\mathsf{P}\colon \mathcal{M} \to \mathcal{M}$ be a triangular α-admissible mapping. Suppose that there exists $y_0 \in \mathcal{M}$ such that $\alpha(y_0, \mathsf{P}y_0) \geq 1$. If $y_\mathsf{k} = \mathsf{P}^\mathsf{k} y_0$, then*

$$\alpha(y_\mathsf{k}, y_\mathsf{t}) \geq 1,$$

for all $\mathsf{k}, \mathsf{t} \in \mathbb{N}$ with $\mathsf{k} < \mathsf{t}$.

By Ψ, we denote the family of altering distance functions [37], that is, function $\psi\colon [0, +\infty) \to [0, +\infty)$, such that the following conditions fulfill:

(1) ψ is continuous and nondecreasing;
(2) $\psi(\mathsf{h}) = 0$ if and only if $\mathsf{h} = 0$.

Lemma 2 ([37–39]). *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and continuous function. Then, the following two conditions are equivalent:*

(1) $\psi^n(h) \to 0, n \to \infty$ *for all* $h \geq 0$;

(2) $\psi(h) < h$ *for all* $h > 0$.

Next, we state the definition of the interpolative contraction:

Definition 7 ([14])**.** *Let* $(\mathcal{M}, \mathsf{d})$ *be a metric space. A mapping* $\mathsf{P}\colon \mathcal{M} \to \mathcal{M}$ *is said to be an interpolative Kannan-type contraction if we have two constants* $\lambda \in [0,1]$ *and* $\alpha \in [0,1]$ *such that*

$$\mathsf{d}(\mathsf{P}y, \mathsf{P}z) \le \lambda(\mathsf{d}(y, \mathsf{P}y))^{\alpha}(\mathsf{d}(z, \mathsf{P}z))^{1-\alpha},$$

for every $y, z \in \mathcal{M}$ *with* $y \neq \mathsf{P}y$ *and* $z \neq \mathsf{P}z$.

Theorem 2 ([14])**.** *Let* $(\mathcal{M}, \mathsf{d})$ *be a complete metric space and* $\mathsf{P}\colon \mathcal{M} \to \mathcal{M}$ *be an interpolative Kannan-type contraction mapping. Then,* P *possesses a fixed point.*

Inspired by interpolative and the Meir–Keeler, the notion of the interpolative Kannan–Meir–Keeler [30] was defined in 2021:

Definition 8 ([30])**.** *Let* $(\mathcal{M}, \mathsf{d})$ *be a complete metric space. A mapping* $\mathsf{P}\colon \mathcal{M} \to \mathcal{M}$ *is said to be an interpolative Kannan–Meir–Keeler contraction on* $\mathcal{M}$ *if there exists* $\gamma \in (0,1)$ *such that given* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for every* $y, z \in \mathcal{M} \setminus \mathrm{Fix}(\mathsf{P})$,

$$\varepsilon < [\mathsf{d}(y, \mathsf{P}y)]^{\gamma}[\mathsf{d}(z, \mathsf{P}z)]^{1-\gamma} < \varepsilon + \delta \Longrightarrow \mathsf{d}(\mathsf{P}y, \mathsf{P}z) \le \varepsilon,$$

$$\mathsf{d}(\mathsf{P}y, \mathsf{P}z) < \mathsf{d}(y, \mathsf{P}y)]^{\gamma}[\mathsf{d}(z, \mathsf{P}z)]^{1-\gamma},$$

where $Fix_{\mathcal{M}}(\mathsf{P}) = \{y \in \mathcal{M} \colon \mathsf{P}y = y\}$.

Let $Fix_{\mathcal{M}_w}(\mathsf{P}) = \{y \in \mathcal{M}_w \colon \mathsf{P}y = y\}$.

The aim of this paper is to introduce a new contraction, namely, interpolative Meir–Keeler contraction, in the context of modular metric space. Consequently, we shall examine the existence and uniqueness of the fixed point for such mapping in the mentioned setting. In order to indicate the validity, an illustrative example is considered.

3. Main Results

We start this section by stating the definition of the interpolative Meir–Keeler contraction in MMS via admissible mappings:

Definition 9. *Let* $(\mathcal{M}_w, w)$ *be an MMS. A self-mapping* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *is defined as an* (α, ψ) interpolative Meir–Keeler contraction of type I *if there exist the functions* $\alpha\colon \mathcal{M}_w \times \mathcal{M}_w \to [0, \infty)$, $\psi \in \Psi$ *and constants* $\vartheta \in [0,1)$, λ_0 *whenever for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \le \psi\Big((w_{2\lambda}(y, \mathsf{P}y))^{\vartheta}(w_{2\lambda}(z, \mathsf{P}z))^{1-\vartheta}\Big) < \varepsilon + \delta \implies \alpha(y,z)w_{\lambda}(\mathsf{P}y, \mathsf{P}z) < \varepsilon, \tag{2}$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(\mathsf{P})$ *and* $0 < \lambda < \lambda_0$.

The following lemma is an immediate consequence of Definition 9.

Lemma 3. *If* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *is an* (α, ψ) interpolative Meir–Keeler contraction of type I mapping, *then for every* $y, z \in \mathcal{M}_w$.

$$\alpha(y,z)w_{\lambda}(\mathsf{P}y, \mathsf{P}z) < \psi\Big((w_{2\lambda}(y, \mathsf{P}y))^{\vartheta}(w_{2\lambda}(z, \mathsf{P}z))^{1-\vartheta}\Big). \tag{3}$$

Theorem 3. *Let* $(\mathcal{M}_w, w)$ *be a complete MMS, where the modular* w *satisfies the* (Δ_2) *condition. Let* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *be an* (α, ψ) *interpolative Meir–Keeler contraction of type I mapping and assume that*

(m_1) P *is a triangular* α *-admissible mapping;*
(m_2) *there exists* $y_0 \in \mathcal{M}_w$ *such that* $\alpha(y_0, \mathsf{P}y_0) \geq 1$*;*
(m_3) *either* P *is continuous; or,*

($\mathcal{R}$) if $\{y_\mathsf{k}\}$ *is a sequence in* $\mathcal{M}_w$ *such that* $\alpha(y_\mathsf{k}, y_{\mathsf{k}+1}) \geq 1$ *for each* k *and* $y_\mathsf{k} \to y^* \in \mathcal{M}_w$ *as* $\mathsf{k} \to \infty$*, then* $\alpha(y_\mathsf{k}, y^*) \geq 1$ *for all* k*.*

Then, there exists $y^* \in \mathcal{M}_w$ *such that* $y^* = \mathsf{P}y^*$*.*

Proof. Let $\{y_0\}$ in $\mathcal{M}_w$ be such that $\alpha(y_0, \mathsf{P}y_0) \geq 1$. Define the Picard sequence $\{y_\mathsf{k}\}$, starting at $\{y_0\}$, that is, $y_\mathsf{k} = \mathsf{P}^\mathsf{k}(y_0) = \mathsf{P}y_{\mathsf{k}-1}$ for $\mathsf{k} = 1, 2, \ldots$. Using the conditions (m_1) and (m_2), we obtain $\alpha(y_0, y_1) = \alpha(y_0, \mathsf{P}y_0) \geq 1$, which implies $\alpha(\mathsf{P}y_0, \mathsf{P}y_1) = \alpha(y_1, y_2) \geq 1$, and also, using Lemma 1 $\alpha(y_m, y_n) \geq 1$ for every $m, n \in \mathbb{N}$, $(m < n)$.

Furthermore, clearly if $y_{\mathsf{k}_0+1} = y_{\mathsf{k}_0}$ for some $\mathsf{k}_0 \in \mathbb{N}$, then evidently, P has a fixed point. Thus, we assume that $y_{\mathsf{k}+1} \neq y_\mathsf{k}$ $(\mathsf{k} \geq 0)$. Hence, we have

$$\alpha(y_\mathsf{k}, y_{\mathsf{k}+1}) \geq 1. \tag{4}$$

Using Lemma 3, it follows that for every $\mathsf{k} \in \mathbb{N}$, we have

$$\begin{aligned} w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) &= w_\lambda(\mathsf{P}y_{\mathsf{k}-1}, \mathsf{P}y_\mathsf{k}) \\ &\leq \alpha(y_{\mathsf{k}-1}, y_\mathsf{k}) w_\lambda(\mathsf{P}y_{\mathsf{k}-1}, \mathsf{P}y_\mathsf{k}) \\ &< \psi\Big((w_{2\lambda}(y_{\mathsf{k}-1}, \mathsf{P}y_{\mathsf{k}-1}))^\vartheta (w_{2\lambda}(y_\mathsf{k}, \mathsf{P}y_\mathsf{k}))^{1-\vartheta}\Big) \\ &= \psi\Big((w_{2\lambda}(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_{2\lambda}(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta}\Big), \end{aligned}$$

and using the property of function ψ,

$$\begin{aligned} w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) &< (w_{2\lambda}(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_{2\lambda}(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta} \\ &\leq (w_\lambda(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta}. \end{aligned}$$

Thus, we obtain

$$w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1})^\vartheta < w_\lambda(y_{\mathsf{k}-1}, y_\mathsf{k})^\vartheta.$$

Consequently, we obtain that the sequence $\{w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1})\}$ is strictly decreasing. In addition, from $w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) > 0$, for all $\mathsf{k} \in \mathbb{N} \cup \{0\}$, we obtain that the sequence $\{w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1})\}$ is convergent and so there exists a point $\wp \geq 0$ such that $\lim\limits_{\mathsf{k}\to\infty} w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) = \wp$. We claim that $\wp = 0$. On the contrary, if $\wp > 0$, then taking $\varepsilon = \wp$, by (2), we deduce that there exists $\delta(\wp) > 0$ such that

$$\begin{gathered} \wp < \psi((w_{2\lambda}(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_{2\lambda}(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta}) \leq \wp + \delta(\wp) \text{ implies,} \\ w_\lambda(\mathsf{P}y_{\mathsf{k}-1}, \mathsf{P}y_\mathsf{k}) < \wp. \end{gathered} \tag{5}$$

Moreover, using $\lim\limits_{\mathsf{k}\to\infty} w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) = \wp$ (for $\delta(\wp) > 0$), we write $N \in \mathbb{N}$ such that for each $\mathsf{k} \geq N$, we have

$$\wp < w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) < \psi\Big((w_{2\lambda}(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_{2\lambda}(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta}\Big),$$

or, using the properties of the function ψ,

$$\begin{aligned} \wp &< w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) < (w_{2\lambda}(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_{2\lambda}(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta} \\ &\leq (w_\lambda(y_{\mathsf{k}-1}, y_\mathsf{k}))^\vartheta (w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}))^{1-\vartheta} \\ &\leq w_\lambda(y_{\mathsf{k}-1}, y_\mathsf{k}) < \wp + \delta(\wp) \end{aligned}$$

for any $\mathsf{k} \geq N$. Therefore, by (5), we obtain that

$$w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) < \wp, \text{ for any } \mathsf{k} \geq N,$$

which is a contradiction. Accordingly, we prove that

$$\lim_{\mathsf{k}\to\infty} w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) = 0.$$

Now, we will indicate that $\{y_\mathsf{k}\}$ is a Cauchy sequence. Let $\varepsilon > 0$ and we consider that $\delta(\varepsilon) > 0$, with $\delta(\varepsilon) < \varepsilon$. As we have $\lim\limits_{\mathsf{k}\to\infty} w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) = 0$, and from (Δ_2) $\lim\limits_{\mathsf{k}\to\infty} w_{\frac{\lambda}{2}}(y_\mathsf{k}, y_{\mathsf{k}+1}) = 0$, we choose $\mathsf{r} \in \mathbb{N}$ such that $w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) < \frac{\varepsilon}{2}$ and $w_{\frac{\lambda}{2}}(y_\mathsf{k}, y_{\mathsf{k}+1}) < \frac{\varepsilon}{2}$ for $\mathsf{k} \geq \mathsf{r}$, and we assert that

$$w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+\mathsf{t}}) < \varepsilon, \tag{6}$$

for any $\mathsf{t} \in \mathbb{N}$. Indeed for $\mathsf{t} = 1$,

$$w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) < \frac{\varepsilon}{2} < \varepsilon,$$

and (6) holds. Assume that the above inequality holds for t. Then, we show that the above inequality holds for $\mathsf{t}+1$. Indeed, by property (w_3), Lemmas 3 and 1, we obtain

$$\begin{aligned}
w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+\mathsf{t}+1}) &\leq w_{\frac{\lambda}{2}}(y_\mathsf{k}, y_{\mathsf{k}+1}) + w_{\frac{\lambda}{2}}(y_{\mathsf{k}+1}, y_{\mathsf{k}+\mathsf{t}+1}) \\
&\leq w_{\frac{\lambda}{2}}(y_\mathsf{k}, y_{\mathsf{k}+1}) + \alpha(y_\mathsf{k}, y_{\mathsf{k}+\mathsf{t}}) w_{\frac{\lambda}{2}}(\mathsf{P}y_\mathsf{k}, \mathsf{P}y_{\mathsf{k}+\mathsf{t}}) \\
&\leq w_{\frac{\lambda}{2}}(y_\mathsf{k}, y_{\mathsf{k}+1}) + \psi\Big((w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}))^\vartheta (w_\lambda(y_{\mathsf{k}+\mathsf{t}}, y_{\mathsf{k}+\mathsf{t}+1}))^{1-\vartheta}\Big) \\
&< \frac{\varepsilon}{2} + \psi\Big((\frac{\varepsilon}{2})^\vartheta (\frac{\varepsilon}{2})^{1-\vartheta}\Big) \\
&= \frac{\varepsilon}{2} + \psi\Big(\frac{\varepsilon}{2}\Big) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \text{ (using property of function } \psi).
\end{aligned} \tag{7}$$

Consequently, the sequence $\{y_\mathsf{k}\}$ is Cauchy. As the completeness of the space $\mathcal{M}_w$, there exists $y^* \in \mathcal{M}_w$ such that $\lim\limits_{\mathsf{k}\longrightarrow\infty} y_\mathsf{k} = y^*$. Since P is continuous,

$$y^* = \lim_{\mathsf{k}\longrightarrow\infty} y_{\mathsf{k}+1} = \mathsf{P}(\lim_{\mathsf{k}\longrightarrow\infty} y_\mathsf{k}) = \mathsf{P}y^*,$$

so $y^* = \mathsf{P}(y^*)$; that is, $y^* \in \mathcal{M}_w$ is a fixed point of P.

We can obtain that there is a fixed point of P without any continuity assumption for the mapping P by property $(\mathcal{R})$. Using the condition $(\mathcal{R})$ and (4), we have $\alpha(y_\mathsf{k}, y^*) \geq 1$ for every $\mathsf{k} \in \mathbb{N}$. We claim that $y^* = \mathsf{P}y^*$. On the contrary, if $y^* \neq \mathsf{P}y^*$, by Picard sequence, we have $\lim\limits_{\mathsf{k}\to\infty} y_{\mathsf{k}+1} = \lim\limits_{\mathsf{k}\to\infty} \mathsf{P}y_\mathsf{k} = y^*$, and then

$$\begin{aligned}
w_\lambda(y^*, \mathsf{P}y^*) &\leq w_{\frac{\lambda}{2}}(y^*, \mathsf{P}y_\mathsf{k}) + w_{\frac{\lambda}{2}}(\mathsf{P}y_\mathsf{k}, \mathsf{P}y^*) \\
&< w_{\frac{\lambda}{2}}(y^*, y_{\mathsf{k}+1}) + \alpha(y_\mathsf{k}, y^*) w_{\frac{\lambda}{2}}(\mathsf{P}y_\mathsf{k}, \mathsf{P}y^*) \\
&< w_{\frac{\lambda}{2}}(y^*, y_{\mathsf{k}+1}) + \psi\Big((w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}))^\vartheta (w_\lambda(y^*, \mathsf{P}y^*))^{1-\vartheta}\Big).
\end{aligned} \tag{8}$$

Letting $\mathsf{k} \longrightarrow \infty$ in the above inequality and by the right continuity of ψ at 0, we obtain that $w_\lambda(y^*, \mathsf{P}y^*) = 0$, so $y^* = \mathsf{P}y^*$. Consequently, $y^* \in \mathcal{M}_w$ is a fixed point of P. $\square$

If in Theorem 3, we take $\psi(y) = \kappa y$ where $\kappa \in (0,1)$, we have the following corollary:

Corollary 1. *Let* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *be a continuous self mapping on a complete MMS* $(\mathcal{M}_w, w)$. *Suppose that there exist a function* $\alpha\colon \mathcal{M}_w \times \mathcal{M}_w \to [0,\infty)$ *and constants* $\kappa \in (0,1)$, $\vartheta \in [0,1)$, $\lambda_0 > 0$ *such that for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \kappa \cdot (w_{2\lambda}(y, \mathsf{P}y))^{\vartheta}(w_{2\lambda}(z, \mathsf{P}z))^{1-\vartheta} < \varepsilon + \delta \implies \alpha(y,z)w_{\lambda}(\mathsf{P}y, \mathsf{P}z) < \varepsilon,$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(\mathsf{P})$ *and* $0 < \lambda < \lambda_0$.

Moreover,

(m_1) P *is a triangular* α*-admissible mapping,*
(m_2) *there exists* $y_0 \in \mathcal{M}_w$ *such that* $\alpha(y_0, \mathsf{P}y_0) \geq 1$;

then there exist $y^* \in \mathcal{M}_w$ *such that* $y^* = \mathsf{P}y^*$, *that is,* P *possesses a fixed point.*

If in Theorem 3, we obtain $\alpha(y,z) = 1$ for every $y, z \in \mathcal{M}_w \in Fix(\mathsf{P})$, we obtain the following corollary:

Corollary 2. *Let* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *be a continuous self mapping on a complete MMS* $(\mathcal{M}_w, w)$. *If there exist a function* $\psi \in \Psi$, $\lambda_0 > 0$ *and a constant* $\vartheta \in [0,1)$, *such that for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \psi((w_{2\lambda}(y, \mathsf{P}y))^{\vartheta}(w_{2\lambda}(z, \mathsf{P}z))^{1-\vartheta}) < \varepsilon + \delta \implies w_{\lambda}(\mathsf{P}y, \mathsf{P}z) < \varepsilon,$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(\mathsf{P})$ *and* $0 < \lambda < \lambda_0$. *Then,* P *has a fixed point.*

If in Corollary 2, we consider $\psi(y) = \kappa y$ where $\kappa \in (0,1)$, we have the following corollary:

Corollary 3. *Let* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *be a continuous self mapping on a complete MMS* $(\mathcal{M}_w, w)$. *If there exist* $\kappa \in (0,1)$ *and* $\vartheta \in [0,1)$, *such that for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \kappa \cdot (w_{2\lambda}(y, \mathsf{P}y))^{\vartheta}(w_{2\lambda}(z, \mathsf{P}z))^{1-\vartheta} < \varepsilon + \delta \implies w_{\lambda}(\mathsf{P}y, \mathsf{P}z) < \varepsilon, \tag{9}$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(\mathsf{P})$, *and* $0 < \lambda < \lambda_0$. *Then,* P *has a fixed point.*

Example 1. *Let* $\mathcal{M} = M_1 \cup M_2$, *where* $M_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R} \right\}$, $M_2 = \left\{ \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$, *and* $w_{\lambda}\colon (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty)$, *be a metric modular on* $\mathcal{M}$, *with* $w_{\lambda}(\mathcal{Y}, \mathcal{Z}) = \frac{1}{\lambda} \max_{1 \leq i \leq 2} |y_i - z_i|$, *where* $\mathcal{Y} = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_1 \end{pmatrix}$, $\mathcal{Z} = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_1 \end{pmatrix}$, *and* $y_1, y_2, z_1, z_2 \in \mathbb{R}$. *Let* $\mathsf{P}\colon \mathcal{M} \to \mathcal{M}$, *be defined by*

$$\mathsf{P}\mathcal{Y} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{for } \mathcal{Y} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\ A \cdot \mathcal{Y}, & \text{otherwise} \end{cases},$$

where $A = \begin{pmatrix} 0 & 1/4 \\ 1/4 & 0 \end{pmatrix}$. *Let also* $\psi \in \Psi$, $\psi(\mathsf{h}) = \frac{3\mathsf{h}}{4}$, $\vartheta = \frac{1}{2}$ *and* $\alpha\colon \mathcal{M} \times \mathcal{M} \to [0,\infty)$,

$$\alpha(\mathcal{Y}, \mathcal{Z}) = \begin{cases} det(\mathcal{Y} \cdot \mathcal{Z}) + 1, & \text{if } \mathcal{Y} = \mathcal{Z} \\ 1, & \text{if } \mathcal{Y} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \\ 2, & \text{if } \mathcal{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0, & \text{otherwise} \end{cases}.$$

Since it is easy to check that (m_1)–(m_3) *hold, we will verify that the mapping* P *is an* (α, ψ) *interpolative Meir–Keeler contraction of type I, by choosing* $\delta(\varepsilon) = \frac{\varepsilon}{2}$ *for any* $\varepsilon > 0$.

For $\mathcal{Y}, \mathcal{Z} \in \mathcal{M}$, $\mathcal{Y} = \mathcal{Z}$, *we have* $w_\lambda(\mathcal{Y}, \mathcal{Z}) = 0$, *and then* (2) *holds for every* $\varepsilon > 0$.

For $\mathcal{Y} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \in M_1$ *and* $\mathcal{Z} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \in M_2$, *we have* $\mathsf{P}\mathcal{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *respectively* $\mathsf{P}\mathcal{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *We obtain*

$$w_\lambda(\mathsf{P}\mathcal{Y}, \mathsf{P}\mathcal{Z}) = \frac{1}{\lambda},\ w_{2\lambda}(\mathcal{Y}, \mathsf{P}\mathcal{Y}) = \frac{4}{2\lambda} = \frac{2}{\lambda},\ w_{2\lambda}(\mathcal{Z}, \mathsf{P}\mathcal{Z}) = \frac{4}{2\lambda} = \frac{2}{\lambda}$$

and

$$\psi\big((w_{2\lambda}(\mathcal{Y}, \mathsf{P}\mathcal{Y}))^{\vartheta}(w_{2\lambda}(\mathcal{Z}, \mathsf{P}\mathcal{Z}))^{1-\vartheta}\big) = \tfrac{3}{2\lambda}$$

$$\alpha(\mathcal{Y}, \mathcal{Z}) w_\lambda(\mathsf{P}\mathcal{Y}, \mathsf{P}\mathcal{Z}) = \tfrac{1}{\lambda}.$$

Thus,

$$\varepsilon \le \frac{3}{2\lambda} < \varepsilon + \frac{\varepsilon}{2} \Longrightarrow \frac{2}{3}\varepsilon \le \frac{1}{\lambda} < \varepsilon \Longrightarrow \alpha(\mathcal{Y}, \mathcal{Z}) w_\lambda(\mathsf{P}\mathcal{Y}, \mathsf{P}\mathcal{Z}) < \varepsilon.$$

Thereupon, (2) *holds for every* $\mathcal{Y}, \mathcal{Z} \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(\mathsf{P})$. *(The other cases are not interesting, taking in account the definition of the function* α.*)*

Consequently, all the assumptions of Theorem 3 *being satisfied, it follows that the mapping* P *has fixed points; there are* $\mathcal{Z}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *respectively* $\mathcal{Y}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Now, we investigate fixed point results for interpolative Meir–Keeler contraction in non-AMMS via admissible mappings.

Definition 10. *Let* $(\mathcal{M}_w, w)$ *be a non-AMMS. A mapping* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *is called an* (α, ψ) *interpolative Meir–Keeler contraction of type II if there exist two functions* $\alpha\colon \mathcal{M}_w \times \mathcal{M}_w \to [0, \infty)$, $\psi \in \Psi$ *and constants* $\lambda_0 > 0$, $\vartheta_1, \vartheta_2 \in [0, 1)$ *with* $\vartheta_1 + \vartheta_2 < 1$, *such that for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\begin{aligned} \varepsilon \le& \psi\Big((w_\lambda(y, z))^{\vartheta_1}(w_\lambda(y, \mathsf{P}y))^{\vartheta_2}(w_\lambda(z, \mathsf{P}z))^{1-\vartheta_1-\vartheta_2}\Big) < \varepsilon + \delta \\ &\Longrightarrow \alpha(y, z) w_\lambda(\mathsf{P}y, \mathsf{P}z) < \varepsilon, \end{aligned} \tag{10}$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(\mathsf{P})$ *and* $0 < \lambda < \lambda_0$.

Lemma 4. *If* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *is an* (α, ψ) *interpolative Meir–Keeler contraction type II mapping, then for every* $y, z \in \mathcal{M}_w$.

$$\alpha(y, z) w_\lambda(\mathsf{P}y, \mathsf{P}z) < \psi((w_\lambda(y, z))^{\vartheta_1}(w_\lambda(y, \mathsf{P}y))^{\vartheta_2}(w_\lambda(z, \mathsf{P}z))^{1-\vartheta_1-\vartheta_2}). \tag{11}$$

Theorem 4. *Let* $(\mathcal{M}_w, w)$ *be a complete non-AMMS and* $\mathsf{P}\colon \mathcal{M}_w \to \mathcal{M}_w$ *be an* (α, ψ) *interpolative Meir–Keeler contraction of type II. Suppose that*

(m_1) P *is a triangular* α *-admissible mapping;*
(m_2) *there exists* $y_0 \in \mathcal{M}_w$ *such that* $\alpha(y_0; \mathsf{P}y_0) \ge 1$,
(m_3) *if* $\{y_k\}$ *is a sequence in* $\mathcal{M}_w$ *such that* $\alpha(y_k, y_{k+1}) \ge 1$ *for each* k *and* $y_k \to y^* \in \mathcal{M}_w$ *as* $k \to \infty$, *then* $\alpha(y_k, y^*) \ge 1$ *for all* k.

Then there exists $y^* \in \mathcal{M}_w$ *such that* $y^* = \mathsf{P}y^*$.

Proof. Let $y_0 \in \mathcal{M}_w$ be such that $\alpha(y_0, \mathsf{P}y_0) \ge 1$. Let $\{y_k\}$ be a Picard sequence starting at $\{y_0\}$, that is, $y_k = \mathsf{P}^k(y_0) = \mathsf{P}y_{k-1}$ for $k = 1, 2, \ldots$. By (m_1) and (m_2), we obtain $\alpha(y_0, y_1) =$

$\alpha(y_0, \mathsf{P}y_0) \geq 1$ implies $\alpha(\mathsf{P}y_0, \mathsf{P}y_1) = \alpha(y_1, y_2) \geq 1$, and taking Lemma 1 into account, we obtain $\alpha(y_m, y_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.

In addition, understandably, if there exists $\mathsf{k}_0 \in \mathbb{N}$ such that $y_{\mathsf{k}_0+1} = y_{\mathsf{k}_0}$, then $\mathsf{P}y_{\mathsf{k}_0} = y_{\mathsf{k}_0}$ and openly P has a fixed point. So, we assume that $y_{\mathsf{k}+1} \neq y_{\mathsf{k}}$ $(\mathsf{k} \geq 0)$. So, we have

$$\alpha(y_{\mathsf{k}}, y_{\mathsf{k}+1}) \geq 1. \tag{12}$$

Keeping Lemma 3 in mind, it follows that for every $\mathsf{k} \in \mathbb{N}$, we write

$$\begin{aligned}
w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) &= w_\lambda(\mathsf{P}y_{\mathsf{k}-1}, \mathsf{P}y_{\mathsf{k}}) \\
&\leq \alpha(y_{\mathsf{k}-1}, y_{\mathsf{k}}) w_\lambda(\mathsf{P}y_{\mathsf{k}-1}, \mathsf{P}y_{\mathsf{k}}) \\
&< \psi\Big((w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_1} (w_\lambda(y_{\mathsf{k}-1}, \mathsf{P}y_{\mathsf{k}-1}))^{\vartheta_2} (w_\lambda(y_{\mathsf{k}}, \mathsf{P}y_{\mathsf{k}}))^{1-\vartheta_1-\vartheta_2}\Big) \\
&= \psi\Big((w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_1} (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_2} (w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}))^{1-\vartheta_1-\vartheta_2}\Big) \\
&\text{(using property of } \psi) \\
&< (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_1} (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_2} (w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}))^{1-\vartheta_1-\vartheta_2}
\end{aligned} \tag{13}$$

and thus, we obtain

$$w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1})^{\vartheta_1+\vartheta_2} < w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}})^{\vartheta_1+\vartheta_2}.$$

Accordingly, we show that sequence $\{w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1})\}$ is strictly decreasing. From $w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) > 0$, for every $\mathsf{k} \in \mathbb{N} \cup \{0\}$, we provide that the sequence $\{w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1})\}$ is convergent; at the time, we have a point $\wp \geq 0$ such that $\lim_{\mathsf{k}\to\infty} w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) = \wp$. We pretense that $\wp = 0$. If not $\wp > 0$, then letting $\varepsilon = \wp$, from (10), we deduce that there exists $\delta(\wp) > 0$ such that

$$\begin{gathered}
\wp \leq \psi((w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_1} (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_2} (w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}))^{1-\vartheta_1-\vartheta_2}) < \wp + \delta(\wp) \\
\text{implies, } \alpha(y_{\mathsf{k}-1}, y_{\mathsf{k}}) w_\lambda(\mathsf{P}y_{\mathsf{k}-1}, \mathsf{P}y_{\mathsf{k}}) < \wp.
\end{gathered} \tag{14}$$

Again, since $\lim_{\mathsf{k}\to\infty} w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) = \wp$, (for $\delta(\wp) > 0$) we can find $N \in \mathbb{N}$ such that

$$\begin{aligned}
\wp < w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) &< \psi((w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_1} (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_2} (w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}))^{1-\vartheta_1-\vartheta_2}) \\
&\text{(using property of function } \psi) \\
&< (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_1} (w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}))^{\vartheta_2} (w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}))^{1-\vartheta_1-\vartheta_2} \\
&\leq w_\lambda(y_{\mathsf{k}-1}, y_{\mathsf{k}}) < \wp + \delta(\wp),
\end{aligned}$$

for any $\mathsf{k} \geq N$. By using (14), we obtain that

$$w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) < \wp, \text{ for any } \mathsf{k} \geq N,$$

which is a contradiction. Correspondingly, we show that

$$\lim_{\mathsf{k}\to\infty} w_\lambda(y_{\mathsf{k}}, y_{\mathsf{k}+1}) = 0. \tag{15}$$

Now, we indicate that $\{y_{\mathsf{k}}\}$ is a Cauchy sequence. Let $\wp > 0$ and choose $\varepsilon > 0$ with $4\wp < \varepsilon$, there exists $\delta(\wp) > 0$ so that

$$\begin{aligned}
\wp \leq{}& \psi\Big((w_\lambda(y,z))^{\vartheta_1} (w_\lambda(y, \mathsf{P}y))^{\vartheta_2} (w_\lambda(z, \mathsf{P}z))^{1-\vartheta_1-\vartheta_2}\Big) < \wp + \delta(\wp) \\
&\Longrightarrow \alpha(y,z) w_\lambda(\mathsf{P}y, \mathsf{P}z) < \wp.
\end{aligned} \tag{16}$$

Consider

$$\delta^1 = \min\{1, \wp, \delta(\wp)\}. \tag{17}$$

Evidently, condition (16) is true with $\delta(\wp)$ replaced by δ^1. Moreover, by using (15), we have $\mathsf{t} \in \mathbb{N}$ so that

$$w_\lambda(y_\mathsf{k}, y_{\mathsf{k}+1}) < \delta^1, \tag{18}$$

for all $\mathsf{k} \geq \mathsf{t}$. Let

$$\Pi = \{\mathsf{h} \in \mathbb{N} \colon \mathsf{h} \geq \mathsf{t}, \;\; w_\lambda(y_\mathsf{h}, y_\mathsf{t}) < \wp + \delta^1\}.$$

Apparently, $\Pi \neq \emptyset$, since $\mathsf{t} \in \Pi$. We will prove that

$$\mathsf{h} \in \Pi \Rightarrow \mathsf{h} + 1 \in \Pi. \tag{19}$$

If $\mathsf{h} = \mathsf{t}$, then $\mathsf{h} + 1 \in \Pi$ in (18). If $\mathsf{h} > \mathsf{t}$, then we obtain the following two cases:
Case 1: Presume that

$$\wp \leq w_\lambda(y_\mathsf{h}, y_\mathsf{t}) < \wp + \delta^1.$$

Then from (17)–(19), we obtain

$$\begin{aligned}
\wp \leq & \psi((w_\lambda(y_\mathsf{h}, y_\mathsf{t}))^{\vartheta_1}(w_\lambda(y_\mathsf{h}, \mathsf{P}y_\mathsf{h}))^{\vartheta_2}(w_\lambda(y_\mathsf{t}, \mathsf{P}y_\mathsf{t}))^{1-\vartheta_1-\vartheta_2}) \\
\leq & \psi\Big((w_\lambda(y_\mathsf{h}, y_\mathsf{t}))^{\vartheta_1}(w_\lambda(y_\mathsf{h}, y_{\mathsf{h}+1}))^{\vartheta_2}(w_\lambda(y_\mathsf{t}, y_{\mathsf{t}+1}))^{1-\vartheta_1-\vartheta_2}\Big) \\
\leq & \psi((\wp + \delta^1)^{\vartheta_1}(\delta^1)^{1-\vartheta_1}) \\
\leq & \psi((\wp + \delta^1)^{\vartheta_1}(\wp + \delta^1)^{1-\vartheta_1}) \\
= & \psi(\wp + \delta^1) \\
< & \wp + \delta^1.
\end{aligned}$$

Then, using Lemma 1 we obtain

$$\begin{aligned}
\wp & \leq w_\lambda(y_\mathsf{h}, y_\mathsf{t}) \\
& \leq \alpha(y_\mathsf{h}, y_\mathsf{t}) w_\lambda(y_\mathsf{h}, y_\mathsf{t}) \\
& < \psi((w_\lambda(y_\mathsf{h}, y_\mathsf{t}))^{\vartheta_1}(w_\lambda(y_\mathsf{h}, y_{\mathsf{h}+1}))^{\vartheta_2}(w_\lambda(y_\mathsf{t}, y_{\mathsf{t}+1}))^{1-\vartheta_1-\vartheta_2}) < \wp + \delta^1 \\
& \Rightarrow \alpha(y_\mathsf{h}, y_\mathsf{t}) w_\lambda(\mathsf{P}y_\mathsf{h}, \mathsf{P}y_\mathsf{t}) < \wp,
\end{aligned}$$

so, $w_\lambda(y_{\mathsf{h}+1}, y_{\mathsf{t}+1}) < \wp$ and then, we deduce that

$$w_\lambda(y_{\mathsf{h}+1}, y_\mathsf{t}) = w_{\max\{\lambda,\lambda\}}(y_{\mathsf{h}+1}, y_\mathsf{t}) \leq w_\lambda(y_{\mathsf{h}+1}, y_{\mathsf{t}+1}) + w_\lambda(y_{\mathsf{t}+1}, y_\mathsf{t}) < \wp + \delta^1.$$

In this way, we obtain $\mathsf{h} + 1 \in \Pi$.
Case 2: Consider

$$w_\lambda(y_\mathsf{h}, y_\mathsf{t}) < \wp.$$

Then, we write

$$w_\lambda(y_{\mathsf{h}+1}, y_\mathsf{t}) = w_{\max\{\lambda,\lambda\}}(y_{\mathsf{h}+1}, y_\mathsf{t}) \leq w_\lambda(y_{\mathsf{h}+1}, y_\mathsf{h}) + w_\lambda(y_\mathsf{h}, y_\mathsf{t}) < \wp + \delta^1,$$

which shows that $\mathsf{h} + 1 \in \Pi$. Therefore, by Case 1 and Case 2, we show that

$$w_\lambda(y_\mathsf{h}, y_\mathsf{t}) < \wp + \delta^1, \tag{20}$$

for all $\mathsf{h} \geq \mathsf{t}$. Now, for $\mathsf{h}, \mathsf{k} \in \mathbb{N}, \;\; (\mathsf{h} \geq \mathsf{k} \geq \mathsf{t})$ by (20), we show that

$$\begin{aligned}
w_\lambda(y_\mathsf{h}, y_\mathsf{k}) = w_{\max\{\lambda,\lambda\}}(y_\mathsf{h}, y_\mathsf{k}) \leq & w_\lambda(y_\mathsf{h}, y_\mathsf{t}) + w_\lambda(y_\mathsf{t}, y_\mathsf{k}) \\
< & 2\wp + 2\delta^1 \leq 4\wp < \varepsilon,
\end{aligned}$$

which indicates that $\{y_k\}$ is Cauchy. As the completeness of the space $\mathcal{M}_w$, there exists $y^* \in \mathcal{M}_w$ such that $\lim_{k \longrightarrow \infty} y_k = y^*$.

Using condition (m_3) and (12), we have $\alpha(y_k, y^*) \geq 1$ for every $k \in \mathbb{N}$. We show that $y^* = Py^*$. Inversely, let $y^* \neq Py^*$, using the Picard sequence, we obtain $\lim_{k\to\infty} Py_k = y^*$. Moreover,

$$\begin{aligned}
0 < w_\lambda(y^*, Py^*) = w_{\max\{\lambda,\lambda\}}(y^*, Py^*) &\leq w_\lambda(y^*, Py_k) + w_\lambda(Py_k, Py^*) \\
&\leq w_\lambda(y_{k+1}, y^*) + \alpha(y_k, y^*) w_\lambda(Py_k, Py^*) \\
&< w_\lambda(y_{k+1}, y^*) + \psi\Big((w_\lambda(y_k, y^*))^{\vartheta_1}(w_\lambda(y_k, y_{k+1}))^{\vartheta_2}(w_\lambda(y^*, Py^*))^{1-\vartheta_1-\vartheta_2}\Big) \\
&\quad \text{(using property of } \psi) \\
&< w_\lambda(y_{k+1}, y^*) + (w_\lambda(y_k, y^*))^{\vartheta_1}(w_\lambda(y_k, y_{k+1}))^{\vartheta_2}(w_\lambda(y^*, Py^*))^{1-\vartheta_1-\vartheta_2}.
\end{aligned} \tag{21}$$

Taking $k \longrightarrow \infty$, we obtain the above inequality, $w_\lambda(y^*, P(y^*)) = 0$, so $y^* = P(y^*)$. Thus, $y^* \in \mathcal{M}_w$ is a fixed point of P. □

If in Theorem 4, we obtain $\psi(y) = \kappa y$ where $\kappa \in (0, 1)$, then we have the following corollary:

Corollary 4. *Let $P\colon \mathcal{M}_w \to \mathcal{M}_w$ be a continuous self-mapping on a complete non-AMMS $(\mathcal{M}_w, w)$. Assume that there exists a function $\alpha\colon \mathcal{M}_w \times \mathcal{M}_w \to [0, \infty)$, $\kappa \in (0, 1)$ and constants $\vartheta_1, \vartheta_2 \in [0, 1)$ with $\vartheta_1 + \vartheta_2 < 1$, such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned}
\varepsilon \leq &\kappa \cdot (w_\lambda(y, z))^{\vartheta_1}(w_\lambda(y, Py))^{\vartheta_2}(w_\lambda(z, Pz))^{1-\vartheta_1-\vartheta_2} < \varepsilon + \delta \\
&\Longrightarrow \alpha(y, z) w_\lambda(Py, Pz) < \varepsilon,
\end{aligned}$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(P)$. Furthermore, assume that

(m_1) *P is a triangular α -admissible mapping;*

(m_2) *there exists $y_0 \in \mathcal{M}_w$ such that $\alpha(y_0; Py_0) \geq 1$,*

Then, there exists $y^* \in \mathcal{M}_w$ such that $y^* = Py^*$, that is, P possesses a fixed point.

If in Theorem 4, we obtain $\alpha(y, z) = 1$ for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(P)$, then we have the following corollary:

Corollary 5. *Let $P\colon \mathcal{M}_w \to \mathcal{M}_w$ be a continuous self mapping on a complete non-AMMS $(\mathcal{M}_w, w)$. Suppose that there exist a function $\psi \in \Psi$ and constants $\vartheta_1, \vartheta_2 \in [0, 1)$ with $\vartheta_1 + \vartheta_2 < 1$, such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned}
\varepsilon \leq &\psi((w_\lambda(y, z))^{\vartheta_1}(w_\lambda(y, Py))^{\vartheta_2}(w_\lambda(z, Pz))^{1-\vartheta_1-\vartheta_2}) < \varepsilon + \delta \\
&\Longrightarrow w_\lambda(Py, Pz) < \varepsilon,
\end{aligned}$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(P)$. Then, there exists $y^* \in \mathcal{M}_w$ such that $y^* = Py^*$; that is, P possesses a fixed point.

If in Corollary 5, we obtain $\psi(y) = \kappa y$ where $\kappa \in (0, 1)$, then we have the following corollary:

Corollary 6. *Let $P\colon \mathcal{M}_w \to \mathcal{M}_w$ be a continuous self-mapping on a complete non-AMMS $(\mathcal{M}_w, w)$. Suppose that there exist $\kappa \in (0, 1)$ and constants $\vartheta_1, \vartheta_2 \in [0, 1)$ with $\vartheta_1 + \vartheta_2 < 1$, such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned}
\varepsilon \leq &\kappa \cdot (w_\lambda(y, z))^{\vartheta_1}(w_\lambda(y, Py))^{\vartheta_2}(w_\lambda(z, Pz))^{1-\vartheta_1-\vartheta_2} < \varepsilon + \delta \\
&\Longrightarrow w_\lambda(Py, Pz) < \varepsilon,
\end{aligned}$$

for every $y, z \in \mathcal{M}_w \setminus Fix_{\mathcal{M}_w}(P)$. Then, the mapping P possesses a fixed point.

4. Conclusions

In this paper, we aim to characterize the interpolative Meir–Keeler contraction in modular metric spaces and non-Archimedean modular metric spaces by involving the interesting auxiliary functions, that is, α-admissible mappings. Utilizing the definition of interpolative Meir–Keeler contraction, we proved the existence of fixed point in our theorems. We also present an interesting example along with our fixed point results. At the same time, we aim to underline the importance of the concept of interpolative contractions and to indicate that there is much more to be done in this regard.

Author Contributions: All authors contributed equally and significantly in writing this paper. All authors have read and agreed to publish the present version of the manuscript

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to thank the referees for their careful reading of the manuscript and valuable suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Nakano, H. *Modular Semi-Ordered Spaces*; Maruzen Company, Tokyo, Japan, 1950.
2. Musielak, J.; Orlicz, W. On modular spaces. *Stud. Math.* **1959**, 18, 591–597. [CrossRef]
3. Chistyakov, V.V. Modular metric spaces, I: Basic concepts. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 1–14. [CrossRef]
4. Chistyakov, V.V. Modular metric spaces, II: Application to superposition operators. *Nonlinear Anal. Theory Methods Appl.* **2010**, *72*, 15–30. [CrossRef]
5. Chistyakov, V. *Metric Modular Spaces: Theory and Applications*; Springer: Cham, Switzerland, 2015.
6. Aksoy, U.; Karapınar, E.; Erhan, I.M. Fixed point theorems in complete modular metric spaces and an application to anti-periodic boundary value problems. *Filomat* **2017**, *31*, 5475–5488. [CrossRef]
7. Hosseinzadeh, H.; Parvaneh, V. Meir–Keeler type contractive mappings in modular and partial modular metric spaces. *Asian-Eur. J. Math.* **2020**, *13*, 2050087. [CrossRef]
8. Kannan, R. Some results on fixed points. *Bull. Calcutta Math. Soc.* **1968**, *60*, 71–76.
9. Rhoades, B.E. A comparison of various definitions of contractive mappings. *Trans. Am. Math. Soc.* **1977**, *226*, 257–290. [CrossRef]
10. Meir, A.; Keeler, E. A theorem on contraction mappings. *J. Math. Anal. Appl.* **1969**, *28*, 326–329. [CrossRef]
11. Aksoy, U.; Karapınar, E.; Erhan, I.M.; Rakoćević, V. Meir-Keeler type contractions on modular metric spaces. *Filomat* **2018**, *32*, 3697–3707. [CrossRef]
12. Redjel, N.; Dehici, A.; Karapınar, E.; Erhan, I. Fixed point theorems for (α, ψ)-Meir-Keeler-Khan. *J. Nonlinear Sci. Appl.* **2015**, *8*, 955–964. [CrossRef]
13. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for $\alpha - \psi$-contractive type mappings. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 2154–2165. [CrossRef]
14. Karapınar, E. Revisiting the Kannan type contractions via interpolative. *Adv. Theory Nonlinear Anal. Its App.* **2018**, *2*, 85–87.
15. Aydi, H.; Karapınar, E.; Roldán López de Hierro, A.F. ω-interpolative Ćirić-Reich-Rus-type contractions. *Mathematics* **2019**, *7*, 57. [CrossRef]
16. Gaba, Y.U.; Karapınar, E. A new approach to the interpolative contractions. *Axioms* **2019**, *8*, 110. [CrossRef]
17. Karapınar, E.; Agarwal, R.P. Interpolative Rus-Reich-Ćirić- type contractions via simulation functions. *Analele Univ. Ovidius-Constanta-Ser. Mat.* **2019**, *27*, 137–152. [CrossRef]
18. Karapınar, E. Revisiting simulation functions via interpolative contractions. *Appl. Anal. Discret. Math.* **2019**, *13*, 859–870. [CrossRef]
19. Karapınar, E.; Fulga, A.; Roldán López de Hierro, A.F. Fixed point theory in the setting of $(\alpha, \beta, \psi, \phi)$-interpolative contractions. *Adv. Differ. Equations* **2021**, *2021*, 339. [CrossRef]
20. Karapınar, E.; Fulga, A.; Yeşilkaya, S.S. New results on Perov-Interpolative contractions of Suzuki type mappings. *J. Funct. Spaces* **2021**, *2021*, 9587604. [CrossRef]
21. Khan, M.S.; Singh, Y.M.; Karapınar, E. On the interpolative (ϕ, ψ)-type Z-contraction. *U.P.B. Sci. Bull. Series A* **2021**, *83*, 25–38.
22. Karapınar, E.; Agarwal, R.; Aydi, H. Interpolative Reich–Rus–Ćirić type contractions on partial metric spaces. *Mathematics* **2018**, *6*, 256. [CrossRef]
23. Karapınar, E.; Alqahtani, O.; Aydi, H. On interpolative Hardy-Rogers type contractions. *Symmetry* **2019**, *11*, 8. [CrossRef]

24. Aydi, H.; Chen, C.M.; Karapınar, E. Interpolative Ćirić-Reich-Rus type contractions via the Branciari distance. *Mathematics* **2019**, *7*, 84. [CrossRef]
25. Öztürk, M. On interpolative contractions in modular spaces. *J. Nonlinear Convex Anal.* **2022**, *23*, 1495–1507.
26. Yeşilkaya, S.S. On interpolative Hardy-Rogers contractive of Suzuki type mappings. *Topol. Algebra Its Appl.* **2021**, *9*, 13–19. [CrossRef]
27. Fulga, A.; Yeşilkaya, S.S. On some interpolative contractions of Suzuki type mappings. *J. Funct. Spaces* **2021**, *2021*, 659609. [CrossRef]
28. Karapınar, E.; Aydi, H.; Mitrovic, Z.D. On interpolative Boyd–Wong and Matkowski type contractions. *TWMS J. Pure Appl. Math* **2020**, *11*, 204–212.
29. Noorwali, M.; Yeşilkaya, S.S. New fixed point theorems for admissible hybrid maps. *J. Funct. Spaces* **2022**, *2022*, 5800790. [CrossRef]
30. Karapınar, E. Interpolative Kannan-Meir-Keeler type contraction. *Adv. Theory Nonlinear Anal. Its App.* **2021**, *5*, 611–614. [CrossRef]
31. Öztürk, A.; Yeşilkaya, S.S. On interpolative Q-Meir-Keeler contractions of rational forms. *J. Nonlinear Convex Anal.* **2022**, *23*, 1049–106.
32. Karapınar, E. A Survey on interpolative and hybrid contractions. In *Mathematical Analysis in Interdisciplinary Research*; Springer: Cham, Switzerland, 2021; pp. 431–475.
33. Paknazar, M.; Kutbi, M.A.; Demma, M.; Salimi, P. On non-Archimedean modular metric spaces and some nonlinear contraction mappings. *J. Nonlinear Sci. Appl. in press*.
34. Chistyakov, V.V. A fixed point theorem for contractions in modular metric spaces. *arXiv* **2011**, arXiv:1112.5561.
35. Mongkolkeha, C.; Sintunavarat, W.; Kumam, P. Fixed point theorems for contraction mappings in modular metric spaces. *Fixed Point Theory Appl.* **2011**, *2011*, 93. [CrossRef]
36. Karapınar, E.; Kumam, P.; Salimi, P. On α-ψ-Meir-Keeler contractive mappings. *Fixed Point Theory Appl.* **2013**, *2013*, 1–12. [CrossRef]
37. Khamsi, M.A.; Kozlowski, W.M.; Reich, S. Fixed point theory in modular function spaces. *Nonlinear Anal. Theory, Methods Appl.* **1990**, *14*, 935–953. [CrossRef]
38. Deep, A.; Rezaei Roshan, J.; Nisar, K.S.; Abdeljawad, T. An extension of Darbo's fixed point theorem for a class of system of nonlinear integral equations. *Adv. Differ. Equations* **2020**, *2020*, 1–17. [CrossRef]
39. Roshan, J.R. Existence of solutions for a class of system of functional integral equation via measure of noncompactness. *J. Comput. Appl. Math.* **2017**, *313*, 129–141. [CrossRef]