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Estimates for Coefficients of Bi-Univalent Functions Associated with a Fractional q-Difference Operator

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Abstract: In the present paper, we discuss a class of bi-univalent analytic functions by applying a principle of differential subordinations and convolutions. We also formulate a class of bi-univalent functions influenced by a definition of a fractional q-derivative operator in an open symmetric unit disc. Further, we provide an estimate for the function coefficients $|a_2|$ and $|a_3|$ of the new classes. Over and above, we study an interesting Fekete–Szego inequality for each function in the newly defined classes.

Keywords: differential subordination; q-analogue; difference operator; coefficient estimates

1. Preliminaries

In mathematics, functions and symmetric functions are very common in theory and applications. They are applied to various fields including group theory, Lie algebras and the algebraic geometry and may others, to mention but a few. In the concept of geometric functions, the set $\mathcal{A}$ has has been introduced as the set of all analytic class of normalized functions defined on the open symmetric disc $D = \{z \in \mathbb{C}, |z| < 1\}$ such that they possess the following formula (see [1])

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The univalent class of functions in $\mathcal{A}$ is denoted by S. The best known subclasses of S containing starlike, close to convex and convex functions, respectively, are denoted by ST, C and CV. If f is a function satisfying (1) and g is another function defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then the Hadamard product or the convolution of the functions f and g is offered by Ruscheweyh [2] as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Assume $\mathcal{B}$ represents the class of Schwartz functions $w(z)$ in the set $\mathcal{A}$ such that

$$\mathcal{B} = \{w : w \in \mathcal{A}, w(0) = 0, |w(z)| < 1, (\forall z \in D)\}.$$

Then, for two arbitrary analytic functions f and g in $\mathcal{A}$, we say f is the subordinate function to the function g, expressed as $f \prec g$ or $f(z) \prec g(z)$, if there can be found a function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$ (see [1]). A unified treatment of the familiar subclass of univalent functions has been considered by Ma and Minda [3], following a principle of differential sub-ordinations. Here, we present $\mathcal{P}$ as the class of functions p which are analytic in D provided $p(0) = 1$ and $Re\, p(z) > 0$ and, for each $z \in D$, the functions $p(z)$ can be written in the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{2}$$

Such a class of functions is the so-called Caratheodory class of functions (see [1]).

Lemma 1 ([1]). *Let the function $p \in P$ satisfy (2). Then, we have*

$$|p_n| \leq 2, (n \in \mathbb{N}) \tag{3}$$

and the bounds are sharp.

Due to the one-quarter theorem of Koebe (see [4]), it is shown that every function f in S has an inverse function f^{-1} such that

$$f^{-1}(f(z)) = z \text{ and } (z \in D), f(f^{-1}(w)) = w \text{ provided } |w| < r_0(f), r_0(f) \geqslant \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots$$

We say a function f in the set $\mathcal{A}$ is bi-univalent in D if $D \subseteq f(D)$ and both of the functions f and its inverse f^{-1} are univalent functions in D. The class of such bi-univalent functions defined in D is denoted by Σ. For a useful summary and applicable examples in the class of bi-univalent functions we refer to [5]. The class Σ of bi-univalent functions was discussed by Lewin [6] to derive the second coefficient bound. Srivastava et al. [7] have introduced and investigated a new class of analytic and bi-univalent functions. In recent years, some researchers have studied a number of different subclasses of Σ in the context of theory of geometric functions [8–11].

In recent decades, the fractional q-calculus was applied in the approximation theory which has a new generalization of the classical operator. The concept of q-calculus operator has been broadly been applied in various fields including optimal control, q-difference, hypergeometric series, quantum physics, fractional subdiffusion equations and q-integral equations [12–16]. The concept of p-calculus was first applied in this context by Lupas [17]. Jackson in [18] defined the q-analogues of the ordinary derivative. He also studied the q-integral operator and investigated some applications of this theory.

Nowadays, a number of researchers have studied q-analogues of an analytic class of bi-univalent functions. We may refer readers to Darus [19] who investigated a generalized q-differential operator by utilizing the q-hypergeometric functions. Furthermore, he worked on an operator to derive some useful applications. Zhang et al. [20] discussed a q-Starlike functions associated with generalized conic domain $\Omega_{k,\alpha}$. Kanas et al. [21] defined the operator and later, Arif et al. [22] extended this operator for multivalent functions. Srivastava et al. [23] defined the q-starlike class of functions over conic regions. Furthermore, they presented some of their applications. Srivastava et al. [24] studied q-Noor integral operators and some of their applications. The q-calculus theory in a fractional sense and its real applications in the geometric class of functions of complex analysis and related fields are investigated in [25–41].

Let q be a fixed number with $0 < q < 1$ and $n \in \mathbb{N}$. Then, we recall the following useful definitions and notations (see [18])

$$(a;q)_n = \prod_{i=0}^{n-1}(1-q^i a), \quad (a;q)_0 = 1, \quad (a;q)_\infty = \prod_{i=0}^{\infty}(1-q^i a).$$

The q-analogue of the Γ-function is defined by

$$\Gamma_q(1-\alpha) = \frac{(q;q)_\infty}{(q^{1-\alpha};q)_\infty}(1-q)^\alpha, \quad 0 < \alpha < 1.$$

The q-analogue of the difference operator of non-integer order α is defined by

$$D_q^\alpha f(z) = \frac{1}{(1-q)^\alpha z^\alpha} \sum_{n=0}^{\infty} \frac{(q^{-\alpha};q)_n}{(q;q)_n} q^n f(q^n z),$$

where $\alpha \neq 0$ and $q \in (0,1)$.

Note that $D_q^\alpha f(z) \longrightarrow D_q f(z)$ when $\alpha \longrightarrow 1$, where $D_f(z)$ is the q-derivative of f, introduced in [5] as

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in D, 'z \neq 0, |q| < 1.$$

For $n \in \mathbb{N}$, the following equation is satisfied

$$D_q^\alpha z^n = \frac{z^{n-\alpha}}{\Gamma_q(1-\alpha)} \frac{(q;q)_n}{(q^{1-\alpha};q)_n}, \quad 0 < \alpha < 1.$$

Therefore for $f \in \mathcal{A}$, we obtain that

$$\frac{\Gamma_q(1-\alpha)}{z^{-\alpha}} \frac{1-q^{1-\alpha}}{1-q} D_q^\alpha f(z) = f(z) * I_{\alpha,q}^{\ddagger}(z), \quad 0 < \alpha < 1,$$

provided

$$I_{\alpha,q}^{\ddagger}(z) = z + \sum_{n=2}^{\infty} \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n(1-q)} z^n, \quad 0 < \alpha < 1.$$

Stimulated by the previous result in the paper, we estimate the initial coefficients for a new subclass of bi-univalent functions defined by q-analogue difference operator with non-integer order α. Additionally, we establish corresponding Fekete–Szego type fractional inequality for this functions. Furthermore, we estimate bounds for $|a_2|$ and $|a_3|$.

2. Coefficient Estimates for the Function Class Σ_q^α

In this section, the coefficients $|a_2|$ and $|a_3|$ for a function are estimated in the class Σ_q^α in some details.

Definition 1. *Let a function f be estimated by (1) and $g(w) = f^{-1}(w)$. Then $f \in \Sigma_q^\alpha$, $0 < \alpha < 1$, provided the following subsequent subordination conditions hold:*

$$\frac{1}{z}\left[qf(z) + (1-q)f(z) * I_{\alpha,q}^{\ddagger}(z)\right] \prec \hat{\psi}(z), (\alpha > 0, z \in D), \tag{4}$$

and

$$\frac{1}{w}\left[qg(w) + (1-q)g(w) * I_{\alpha,q}^{\ddagger}(w)\right] \prec \hat{\psi}(w), (\alpha > 0, z \in D), \tag{5}$$

where the functions $\hat{\psi}(z) : D \to \mathbb{C}$ are analytic of positive real parts. Consequently, we have the appropriate form:

$$\hat{\psi}(z) = 1 + \hat{\psi}_1(z) + \hat{\psi}_2(z) + \hat{\psi}_3(z) + \cdots, (\hat{\psi}_1 > 0). \tag{6}$$

Theorem 1. *Let a function f be estimated by* (1) *and $g = f^{-1}$. If $f \in \Sigma_q^\alpha, 0 < \alpha < 1$, then*

$$|a_2| \leq 2\sqrt{\frac{\hat{\psi}_1^3}{4\Lambda_3\hat{\psi}_1^2 + \Lambda_2^2(\hat{\psi}_1 - \hat{\psi}_2)}}, \tag{7}$$

and

$$|a_3| \leq \hat{\psi}_1\left(\frac{4\hat{\psi}_1}{\Lambda_2^2} + \frac{1}{\Lambda_3}\right), \tag{8}$$

where $\Lambda_n = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n}, n = 2, 3.$

Proof. Since $g = f^{-1}$, the right-hand side of (4) and (5) have the expansions

$$\frac{1}{z}\left[qf(z) + (1-q)f(z) * I_{\alpha,q}^{\ddagger}(z)\right] = 1 + \Lambda_2 a_2 z + \Lambda_3 a_3 z^2 + \dots \tag{9}$$

and

$$\begin{aligned}\frac{1}{w}\left[qg(w) + (1-q)g(w) * I_{\alpha,q}^{\ddagger}(w)\right] &= 1 - \Lambda_2 a_2 w + \Lambda_3(2a_2^2 - a_3)w^2 \\ &\quad - \Lambda_4(5 2 a_2^2 - 5a_2 a_3 + a_4)w^3 + \dots \end{aligned} \tag{10}$$

By using the definition of subordination, we derive

$$\frac{1}{z}\left[qf(z) + (1-q)f(z) * I_{\alpha,q}^{\ddagger}(z)\right] = \hat{\psi}(\phi(z)), \tag{11}$$

$$\frac{1}{w}\left[qg(w) + (1-q)g(w) * I_{\alpha,q}^{\ddagger}(w)\right] = \hat{\psi}(\hat{\phi}(w)), \tag{12}$$

where $\phi(z), \hat{\phi}(z) \in \mathcal{B}$. We define two functions u_1, u_2 as the subsequent form

$$u_1(z) = \frac{1 + \phi(z)}{1 - \phi(z)}, \quad u_2(w) = \frac{1 + \hat{\phi}(w)}{1 - \hat{\phi}(w)}, \quad (z, w \in D).$$

It can be easily shown that $u_1, u_2 \in \mathcal{P}$ and write the relations of $\phi(z)$ and $\hat{\phi}(z)$ of the restated subsequent form

$$\phi(z) = \frac{u_1(z) - 1}{u_1(z) + 1} = \frac{1}{2}\left(\phi_1 z + \left(\phi_2 - \frac{\phi_1^2}{2}\right)z^2 + \dots\right), \tag{13}$$

and

$$\hat{\phi}(w) = \frac{u_2(w) - 1}{u_2(w) + 1} = \frac{1}{2}\left(\hat{\phi}_1 w + \left(\hat{\phi}_2 - \frac{\hat{\phi}_1^2}{2}\right)w^2 + \dots\right). \tag{14}$$

Therefore, from Equations (11), (13) and (14), we obtain

$$\hat{\psi}(\phi(z)) = \hat{\psi}\left(\frac{u_1(z) - 1}{u_1(z) + 1}\right) = 1 + \frac{1}{2}\hat{\psi}_1\phi_1 z + \left(\frac{1}{2}\hat{\psi}_1\left(\phi_2 - \frac{\phi_1^2}{4}\right) + \frac{1}{4}\hat{\psi}_2\phi_1^2\right)z^2 + \dots \tag{15}$$

and

$$\hat{\psi}(\hat{\phi}(w)) = \hat{\psi}\left(\frac{u_2(w)-1}{u_2(w)+1}\right) = 1 + \frac{1}{2}\hat{\psi}_1\hat{\phi}_1 w + \left(\frac{1}{2}\hat{\psi}_1\left(\hat{\phi}_2 - \frac{\hat{\phi}_1^2}{4}\right) + \frac{1}{4}\hat{\psi}_2\hat{\phi}_1^2\right)w^2 + \dots \quad (16)$$

Equating the corresponding coefficients (9) and (15) reveals

$$a_2 = \frac{1}{\Lambda_2}\hat{\psi}_1\phi_1, \quad (17)$$

$$a_3\Lambda_3 = \frac{1}{2}\hat{\psi}_1(\phi_2 - \frac{\phi_1^2}{2}) + \frac{1}{4}\hat{\psi}_2\phi_1^2. \quad (18)$$

Once again, equating the corresponding coefficients (10) and (16) implies

$$a_2 = -\frac{1}{\Lambda_2}\hat{\psi}_1\hat{\phi}_1, \quad (19)$$

$$(2a_2^2 - a_3)\Lambda_3 = \frac{1}{2}\hat{\psi}_1(\hat{\phi}_2 - \frac{\hat{\phi}_1^2}{2}) + \frac{1}{4}\hat{\psi}_2\hat{\phi}_1^2. \quad (20)$$

Comparing the coefficients (17) and (19) yields

$$\phi_1 = -\hat{\phi}_1 \quad (21)$$

and

$$a_2^2 = \frac{\hat{\psi}_1^2}{2\Lambda_2^2}(\phi_1^2 + \hat{\phi}_1^2). \quad (22)$$

Moreover, by using (20)–(22), the above reveals

$$a_2^2 = \frac{\hat{\psi}_1^3(\phi_2 + \hat{\phi}_2)}{4\Lambda_3\hat{\psi}_1^2 + \Lambda_2^2(\hat{\psi}_1 - \hat{\psi}_2)},$$

which, by applying Lemma 1, we derive the desired estimate on $|a_2|$ as presented in (7).

To complete the proof of this theorem, we find the estimate on $|a_3|$. Subtracting Equation (20) from Equation (18) and using (21), we get

$$a_3 = a_2^2 + \frac{\hat{\psi}_1(\phi_2 - \hat{\phi}_2)}{4\Lambda_3}. \quad (23)$$

Hence, following Equations (22) and (23) gives

$$a_3 = \frac{\hat{\psi}_1^2}{2\Lambda_2^2}(\phi_1^2 + \hat{\phi}_1^2) + \frac{\hat{\psi}_1(\phi_2 - \hat{\phi}_2)}{4\Lambda_3}.$$

Finally, by applying Lemma 1 on the coefficients of ϕ_2 and $\hat{\phi}_2$ in the last equation, we derive the desired estimate on $|a_3|$ as presented in (8). □

In the next theorem, we calculate the Fekete–Szego result for the class Σ_q^α.

Theorem 2. *Let a function f be given by (1), $g = f^{-1}$ and $\lambda \in \mathbb{C}$. If $f \in \Sigma_q^\alpha, 0 < \alpha < 1$ then*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} \frac{p_1}{4\Lambda_3}, & 0 \leq \psi(\lambda) < \frac{1}{4\Lambda_3}, \\ 4\hat{\psi}_1\psi(\lambda), & \psi(\lambda) \geq \frac{1}{4\Lambda_3}, \end{cases}$$

and

$$\psi(\lambda) = \frac{\hat{\psi}_1^2(1-\lambda)}{4\Lambda_3\hat{\psi}_1^2 + \Lambda_2^2(\hat{\psi}_1 - \hat{\psi}_2)},$$

(24)

where $\Lambda_n = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n}, n = 2, 3.$

Proof. By using Equations (22) and (23), we get

$$a_3 - \lambda a_2^2 = \frac{\hat{\psi}_1(r_2 - s_2)}{4\Lambda_3} + (1-\lambda)a_2^2.$$

The preceding equation is indeed equivalent to

$$a_3 - \lambda a_2^2 = \frac{\hat{\psi}_1(r_2 - s_2)}{4\Lambda_3} + \frac{(1-\lambda)\hat{\psi}_1^3(r_2 + s_2)}{4\Lambda_3\hat{\psi}_1^2 + \Lambda_2^2(\hat{\psi}_1 - \hat{\psi}_2)}.$$

Therefore, we have

$$a_3 - \eta a_2^2 = \hat{\psi}_1\left[\left(\psi(\lambda) + \frac{1}{4\Lambda_3}\right)r_2 + \left(\psi(\lambda) - \frac{1}{4\Lambda_3}\right)s_2\right],$$

where $\psi(\lambda)$ *is defined in* (48). *Since* $\hat{\psi}_1 > 0$ *and the coefficient* $\hat{\psi}_2$ *is a real number, then* we have

$$|a_3 - \eta a_2^2| = \hat{\psi}_1\left|\left(\psi(\lambda) + \frac{1}{4\Lambda_3}\right)r_2 + \left(\psi(\lambda) - \frac{1}{4\Lambda_3}\right)s_2\right|.$$

This completes the proof of Theorem 4. $\square$

Remark 1. *Let* $\lambda = 1$ *then, if* $f \in \Sigma_q^\alpha$, $0 < \alpha < 1$, *then*

$$|a_3 - a_2^2| \le \frac{\hat{\psi}_1}{\Lambda_3},$$

where $\Lambda_3 = q + \frac{(q;q)_3(1-q^{1-\alpha})}{(q^{1-\alpha};q)_3}.$

This remark is a straightforward consequence of Theorem 4.

Applications of Coefficient Estimates for Σ_q^α

Definition 2. *Let a function* f *be estimated by* (1), $g = f^{-1}$ *and* $0 \le \zeta < 1$. *Then* $f \in \Sigma_q^\alpha(\zeta)$, $0 < \alpha < 1$ *if the following subsequent subordination conditions hold:*

$$\frac{1}{z}\left[qf(z) + (1-q)f(z) * I_{\alpha,q}^{\ddagger}(z)\right] \prec \hat{v}_1(z), (z \in D),$$

and

$$\frac{1}{w}\left[qg(w) + (1-q)g(w) * I_{\alpha,q}^{\ddagger}(w)\right] \prec \hat{v}_1(w), (z \in D),$$

where $\hat{v}_1$ *is function estimated as*

$$\hat{v}_1(z) = \frac{1 + (1-(1+q)\zeta)z}{1 - qz} = 1 + (1+q)(1-\zeta)z + q(1+q)(1-\zeta)z^2 + \cdots$$

(25)

Hence, it is evident that

$$\hat{\psi}_1 = (1+q)(1-\zeta), \quad \hat{\psi}_2 = q(1+q)(1-\zeta).$$

Corollary 1. *Let a function f be estimated by* (1), $g = f^{-1}$ *and* $0 \leq \zeta < 1$. *If* $f \in \Sigma_q^\alpha(\zeta)$, $0 < \alpha < 1$, *then*

$$|a_2| \leq \sqrt{\frac{(1+q)^3(1-\zeta)^3}{4\Lambda_3(1+q)^2(1-\alpha)^2 + \Lambda_2^2(1-q^2)(1-\zeta)}}, \tag{26}$$

and

$$|a_3| \leq (1+q)(1-\alpha)\left(\frac{4(1+q)(1-\zeta)}{\Lambda_2^2} + \frac{1}{\Lambda_3}\right), \tag{27}$$

where $\Lambda_n = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n}, n = 2, 3$.

Corollary 2. *Let f be a function estimated by* (1) *and* $0 \leq \zeta < 1$. *If* $f \in \Sigma_q^\alpha(\zeta)$, $0 < \alpha < 1$, *then*

$$|a_3 - a_2^2| \leq \frac{(1+q)(1-\zeta)}{\Lambda_3}, \tag{28}$$

where $\Lambda_3 = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_3}$.

Definition 3. *Let a function f be estimated by* (1), $g = f^{-1}$ *and* $0 < \gamma \leq 1$. *The function* $f \in \Sigma_q^\alpha(\gamma)$, $0 < \alpha < 1$ *if the following subsequent subordination conditions hold:*

$$\frac{1}{z}\left[qf(z) + (1-q)f(z) * I_{\alpha,q}^\ddagger(z)\right] \prec \hat{v}_2(z), (z \in D),$$

and

$$\frac{1}{w}\left[qg(w) + (1-q)g(w) * I_{\alpha,q}^\ddagger(w)\right] \prec \hat{v}_2(w), (z \in D),$$

where the function $\hat{v}_2$ *is given by*

$$\hat{v}_2(z) = \left(\frac{1+z}{1-qz}\right)^\gamma = 1 + (1+q)\gamma z + \frac{(1+q)((1+q)\gamma + q - 1)\gamma}{2}z^2 + \cdots. \tag{29}$$

Hence, it is evident that

$$\hat{\psi}_1 = (1+q)\gamma, \quad \hat{\psi}_2 = \frac{[(1+q)\gamma + q - 1](1+q)\gamma}{2}. \tag{30}$$

Corollary 3. *Let f be a function estimated by* (1), $g = f^{-1}$ *and* $0 < \gamma \leq 1$. *If* $f \in \Sigma_q^\alpha(\gamma)$, $0 < \alpha < 1$, *then*

$$|a_2| \leq \sqrt{\frac{2(1+q)^3\gamma^3}{8\Lambda_3(1+q)^2\gamma^2 + \Lambda_2^2[\gamma(1+q)(3 - (1-q)\gamma - q)]}}, \tag{}$$

and

$$|a_3| \leq \left(\frac{4(1+q)\gamma}{\Lambda_2^2} + \frac{1}{\Lambda_3}\right),$$

where $\Lambda_n = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n}, n = 2, 3.$

Corollary 4. *Let $f(z)$ be a function estimated by (1) and $0 < \gamma \le 1$. If $f \in \Sigma_q^\alpha(\gamma), 0 < \alpha < 1$, then*

$$|a_3 - a_2^2| \le \frac{(1+q)\gamma}{\Lambda_3}, \tag{31}$$

where $\Lambda_3 = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_3}.$

3. Coefficient Estimates for the Function Class $\Sigma_q^{\alpha,\beta}$

In this section, we define the class $\Sigma_q^{\alpha,\beta}$ and obtain coefficient estimates for $|a_2|$ and $|a_3|$ for the function in this class.

Definition 4. *Let a function f be given by (1), $g = f^{-1}$ and $0 \le \beta < 1$. The function $f \in \Sigma$ belongs to the class $\Sigma_q^{\alpha,\beta}$, $0 < \alpha < 1$ if the subsequent subordination conditions hold:*

$$Rel\left\{\frac{1}{z}\left(f(z) * I_{\alpha,q}^\ddagger(z)\right)\right\} > \beta, \quad (z \in D, \ 0 \le \beta < 1), \tag{32}$$

and

$$Rel\left\{\frac{1}{w}\left(g(w) * I_{\alpha,q}^\ddagger(w)\right)\right\} > \beta, \quad (w \in D, \ 0 \le \beta < 1). \tag{33}$$

Theorem 3. *Let f be a function estimated by (1), $g = f^{-1}$ and $0 \le \beta < 1$. If $f \in \Sigma$ belongs to the class $\Sigma_q^{\alpha,\beta}$, $0 < \alpha < 1$, then*

$$|a_2| \le (1-\beta)\min\left\{\sqrt{\frac{\hat{\psi}_1^2}{2\Omega_2^2}(|\phi_1|^2 + |\hat{\phi}_1|^2)}, \sqrt{\frac{\hat{\psi}_1^3(|\phi_2| + |\hat{\phi}_2|)}{|4(1-\beta)\hat{\psi}_1^2\Omega_3 + (\hat{\psi}_1 - \hat{\psi}_2)\Omega_2^2|}}\right\} \tag{34}$$

and

$$|a_3| \le \frac{1}{4}(1-\beta)\hat{\psi}_1(|\phi_2| + |\hat{\phi}_2|)\min\left\{\begin{array}{c} \frac{(1-\beta)^2\hat{\psi}_1^2}{2\Omega_2^2}(|\phi_1|^2 + |\hat{\phi}_1|^2), \\ \frac{(1-\beta)^2\hat{\psi}_1^3(|\phi_2| + |\hat{\phi}_2|)}{|4(1-\beta)\hat{\psi}_1^2\Omega_3 + (\hat{\psi}_1 - \hat{\psi}_2)\Omega_2^2|} \end{array}\right\}, \tag{35}$$

where $\Omega_n = \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n(1-q)}, n = 2, 3.$

Proof. First, for the argument inequalities in (32) and (33), there exist $\phi(z), \hat{\phi}(z) \in \mathcal{B}$ such that

$$\frac{1}{z}\left(f(z) * I_{\alpha,q}^\ddagger(z)\right) = \beta + (1-\beta)\hat{\psi}(\phi(z)), \tag{36}$$

and

$$\frac{1}{w}\left(g(w) * I_{\alpha,q}^\ddagger(w)\right) = \beta + (1-\beta)\hat{\psi}(\hat{\phi}(z)). \tag{37}$$

We expand the right-hand side of (36) and (37) as

$$\frac{1}{z}\left(f(z) * I_{\alpha,q}^\ddagger(z)\right) = 1 + \Omega_2 a_2 z + \Omega_3 a_3 z^2 + \dots \tag{38}$$

and

$$\frac{1}{w}\left(g(w) * I^{\ddagger}_{\alpha,q}(w)\right) = 1 - \Omega_2 a_2 w + \Omega_3(2a_2^2 - a_3)w^2$$
$$- \Omega_4(5a_2^2 - 5a_2 a_3 + a_4)w^3 + \dots. \quad (39)$$

Now, by equating the coefficients (15) and (38), we establish

$$a_2 = (1 - \beta)\frac{\hat{\psi}_1 \phi_1}{\Omega_2}, \quad (40)$$

$$a_3 \Omega_3 = \frac{1}{2}(1 - \beta)\hat{\psi}_1\left(r_2 - \frac{\phi_1^2}{2}\right) + \frac{1}{4}(1 - \beta)\hat{\psi}_2^2 \phi_1^2. \quad (41)$$

In the same manner, by equating the coefficients (16) and (39), we obtain

$$a_2 = -(1 - \beta)\frac{\hat{\psi}_1 \hat{\phi}_1}{\Omega_2}, \quad (42)$$

$$(2a_2^2 - a_3)\Omega_3 = \frac{1}{2}(1 - \beta)\hat{\psi}_1\left(\hat{\phi}_2 - \frac{\hat{\phi}_1^2}{2}\right) + \frac{1}{4}(1 - \beta)\hat{\psi}_2^2 \hat{\phi}_1^2. \quad (43)$$

By omparing the coefficients (40) and (42), we get

$$\phi_1 = -\hat{\phi}_1, \quad (44)$$

$$a_2^2 = \frac{(1 - \beta)^2 \hat{\psi}_1^2}{2\Omega_2^2}(\phi_1^2 + \hat{\phi}_1^2). \quad (45)$$

By using (42) and (45), it follows that

$$a_2^2 = \frac{(1 - \beta)^2 \hat{\psi}_1^3(\phi_2 + \hat{\phi}_2)}{4(1 - \beta)\hat{\psi}_1^2 \Omega_3 + (\hat{\psi}_1 - \hat{\psi}_2)\Omega_2^2}. \quad (46)$$

So, following (45) and (46), we find

$$|a_2|^2 \leq \frac{(1 - \beta)^2 \hat{\psi}_1^2}{2\Omega_2^2}(|\phi_1|^2 + |\hat{\phi}_1|^2),$$

$$|a_2|^2 \leq \frac{(1 - \beta)^2 \hat{\psi}_1^3(|\phi_2| + |\hat{\phi}_2|)}{|4(1 - \beta)\hat{\psi}_1^2 \Omega_3 + (\hat{\psi}_1 - \hat{\psi}_2)\Omega_2^2|}.$$

Hence, the desired bounds on $|a_2|$ are obtained. The proof will be completed by finding the bound for the coefficient. Therefore, subtracting Equation (43) from Equation (41), we obtain

$$a_2^2 - a_3 = \frac{1}{4\Omega_3}(1 - \beta)\hat{\psi}_1(\hat{\phi}_2 - \phi_2). \quad (47)$$

Put the value of a_2^2, given in (45) into (47) to get

$$a_3 = \frac{(1 - \beta)^2 \hat{\psi}_1^2}{2\Omega_2^2}(\phi_1^2 + \hat{\phi}_1^2) + \frac{1}{4\Omega_3}(1 - \beta)\hat{\psi}_1(\phi_2 - \hat{\phi}_2).$$

Therefore, we conclude that

$$|a_3| \leq \frac{(1-\beta)^2 \hat{\psi}_1^2}{2\Omega_2^2} (|\phi_1|^2 + |\hat{\phi}_1|^2) + \frac{1}{4\Omega_3} (1-\beta)\hat{\psi}_1 (|r_2| + |\hat{\phi}_2|).$$

Similarly, by putting the value of a_2^2 of (46) into (47), we conclude the following bound:

$$|a_3| \leq \frac{(1-\beta)^2 \hat{\psi}_1^3 (|\phi_2| + |s_2|)}{|4(1-\beta)\hat{\psi}_1^2 \Omega_3 + (\hat{\psi}_1 - \hat{\psi}_2)\Omega_2^2|} + \frac{1}{4\Omega_3} (1-\beta)\hat{\psi}_1 (|\phi_2| + |\hat{\phi}_2|).$$

This finishes the proof of Theorem 3. $\square$

Similarly, from Theorem 4, we get the Fekete–Szego result for the class $\Sigma_q^{\alpha,\beta}$.

Theorem 4. *Let $f(z)$ be a function estimated by* (1), *$g = f^{-1}$, $0 \leq \beta < 1$ and $\lambda \in \mathbb{C}$. If $f \in \Sigma$ belongs to the class $\Sigma_q^{\alpha,\beta}$, $0 < \alpha < 1$, then*

$$|a_3 - \lambda a_2^2| \leq (1-\beta)\hat{\psi}_1 \begin{cases} \frac{1}{\Omega_3}, & 0 \leq \psi_1(\lambda) < \frac{1}{4\Omega_3}, \\ 4\psi_1(\lambda), & \psi_1(\lambda) \geq \frac{1}{4\Omega_3}, \end{cases}$$

where

$$\psi_1(\lambda) = \frac{(1-\lambda)(1-\beta)\hat{\psi}_1^2}{(1-\beta)\hat{\psi}_1^2 \Omega_3 + (\hat{\psi}_1 - \hat{\psi}_2)\Omega_2^2}, \tag{48}$$

and $\Omega_n = \frac{(q;q)_n (1-q^{1-\alpha})}{(q^{1-\alpha};q)_n (1-q)}$, $n = 2, 3$.

Applications of Coefficient Estimates for $\Sigma_q^{\alpha,\beta}$

Substituting $\hat{\psi}(z)$ in Theorem 3 by $\hat{v}_1(z)$, we obtain the following definition.

Definition 5. *Let a function $f(z)$ be estimated by* (1), *$g = f^{-1}$, $0 \leq \beta < 1$ and $0 \leq \zeta < 1$. Then $f \in \Sigma_q^{\alpha,\beta}(\zeta)$, $0 < \alpha < 1$ if the subsequent subordination conditions hold:*

$$Rel\left\{ \frac{1}{z} \left(f(z) * I_{\alpha,q}^{\ddagger}(z) \right) \right\} \prec \beta + (1-\beta)\hat{v}_1(z), \quad (z \in D, \ 0 \leq \beta < 1),$$

and

$$Rel\left\{ \frac{1}{w} \left(g(w) * I_{\alpha,q}^{\ddagger}(w) \right) \right\} \prec \beta + (1-\beta)\hat{v}_1(w), \quad (w \in D, \ 0 \leq \beta < 1).$$

where the function $\hat{v}_1$ is given by (25).

Corollary 5. *Let f be a function estimated by* (1), *$g = f^{-1}$, $0 \leq \beta < 1$ and $0 \leq \zeta < 1$. If $f \in \Sigma_q^{\alpha,\beta}(\zeta)$, $0 < \alpha < 1$, then*

$$|a_2| \leq \sqrt{\frac{(1+q)^3 (1-\zeta)^3}{4\Lambda_3 (1+q)^2 (1-\alpha)^2 + \Lambda_2^2 (1-q^2)(1-\zeta)}}, \tag{49}$$

and

$$|a_3| \leq (1+q)(1-\alpha)\left(\frac{4(1+q)(1-\zeta)}{\Lambda_2^2} + \frac{1}{\Lambda_3} \right), \tag{50}$$

where $\Lambda_n = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n}, n = 2, 3.$

Corollary 6. *Let f be a function estimated by* (1), $0 \le \beta < 1$ *and* $0 \le \zeta < 1$. *If* $f \in \Sigma_q^{\alpha,\beta}(\zeta)$, $0 < \alpha < 1$, *then*

$$|a_3 - a_2^2| \le \frac{(1+q)(1-\zeta)}{\Lambda_3}, \tag{51}$$

where $\Lambda_3 = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_3}.$

Substituting $\hat{\psi}(z)$ in Theorem 3 by $\hat{v}_2(z)$, we obtain the following definition.

Definition 6. *Let f be a function estimated by* (1), $g = f^{-1}$, $0 \le \beta < 1$ *and* $0 < \gamma \le 1$. *Then* $f \in \Sigma_q^{\alpha,\beta}(\gamma)$, $0 < \alpha < 1$ *if the subsequent subordination conditions hold*

$$Rel\left\{ \frac{1}{z}\left(f(z) * I_{\alpha,q}^{\ddagger}(z) \right) \right\} \prec \beta + (1-\beta)\hat{v}_2(z), \quad (z \in D, \ 0 \le \beta < 1),$$

and

$$Rel\left\{ \frac{1}{w}\left(g(w) * I_{\alpha,q}^{\ddagger}(w) \right) \right\} \prec \beta + (1-\beta)\hat{v}_2(w), \quad (w \in D, \ 0 \le \beta < 1),$$

where the function $\hat{v}_2$ *is given by* (29).

Corollary 7. *Let f be a function estimated by* (1), $g = f^{-1}$, $0 \le \beta < 1$ *and* $0 < \gamma \le 1$. *If* $f \in \Sigma_q^{\alpha,\beta}(\gamma)$, $0 < \alpha < 1$, *then*

$$|a_2| \le \sqrt{\frac{2(1+q)^3\gamma^3}{8\Lambda_3(1+q)^2\gamma^2 + \Lambda_2^2[\gamma(1+q)(3 - (1-q)\gamma - q)]}},$$

and

$$|a_3| \le \left(\frac{4(1+q)\gamma}{\Lambda_2^2} + \frac{1}{\Lambda_3} \right),$$

where $\Lambda_n = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_n}, n = 2, 3.$

Corollary 8. *Let f be a function estimated by* (1), $g = f^{-1}$, $0 \le \beta < 1$ *and* $0 < \gamma \le 1$. *If* $f \in \Sigma_q^{\alpha,\beta}(\gamma)$, $0 < \alpha < 1$, *then*

$$|a_3 - a_2^2| \le \frac{(1+q)\gamma}{\Lambda_3}, \tag{52}$$

where $\Lambda_3 = q + \frac{(q;q)_n(1-q^{1-\alpha})}{(q^{1-\alpha};q)_3}.$

4. Conclusions

In this article, a new analytic subclass of bi-univalent functions was studied on aid of the principle of differential subordination and convolution. Certain set of bi-univalent functions influenced by a q-derivative operator with non-integer order α in the open unit disc were also formulated. Moreover, the Fekete–Szego inequality for each function in the newly defined classes was also obtained.

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