EXACT SOLUTIONS OF STOCHASTIC KdV EQUATION WITH CONFORMABLE DERIVATIVES IN WHITE NOISE ENVIRONMENT

by

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In this article, we have considered Wick-type stochastic Korteweg de Vries (KdV) equation with conformable derivatives. By the help of white noise analysis, Hermit transform and extended G'/G- expansion method, we have obtained exact travelling wave solutions of KdV equation with conformable derivatives. We have applied the inverse Hermit transform for stochastic soliton solutions and then we have shown how stochastic solutions can be presented as Brownian motion functional solutions by an application example.

Key words: *stochastic KdV equation, Wick product, conformable derivative, Hermit transforms, extended G'/G-expansion method*

Introduction

Let us consider the Wick-type stochastic KdV equation with conformable derivatives in the form:

$$\mathrm{D}_t^\alpha Q + U(t)\lozenge Q\lozenge \mathrm{D}_x^\alpha Q + V(t)\lozenge \mathrm{D}_x^{3\alpha} Q = 0 \tag{1}$$

which is a perturbation of the KdV equation with conformable derivatives of the form:

$$\mathrm{D}_t^\alpha q + u(t)q\mathrm{D}_x^\alpha q + v(t)\mathrm{D}_x^{3\alpha} q = 0 \tag{2}$$

where u and v are non-zero integrable function on $\mathbb{R}^+$. In eq. (1), $\lozenge$ denotes the Wick product on the Kondratiev distribution space $(S)_{-1}$, $U(t)$, and $V(t)$ are $(S)_{-1}$-valued functions. Please see [1] for more details about stochastic Kondratiev spaces and Wick product.

In recent years, stochastic non-linear PDE have drawn great interest in many fields. The exact solutions of the stochastic non-linear PDE reveal the internal mechanism of physical events. However solving stochastic equations is more complicated by virtue of the additional random terms when compared to deterministic equations. Wadati [2] firstly introduced and

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studied the stochastic partial differential KdV equation and obtained deformation of the soliton during the propagation in the white noise environment. Liu [3] found general formal solutions of Jacobi elliptic function for stochastic non-linear KdV equation, Liu *et al.* [4] obtained three types of exact solutions to generalized stochastic KdV equation using Riccati equation mapping method. There are also other studies on various stochastic wave equations [5-7].

In this work, we consider eq. (1) in a white noise environment, namely we will deal with the Wick-type stochastic KdV equation.

Exact solutions of eq. (1)

Using the Hermit transform for eq. (1), (please see [1] for details about Hermit transforms) we acquire the deterministic equation:

$$\mathrm{D}_t^{\alpha}\tilde{Q}(x,t,z)+\tilde{U}(t,z)\tilde{Q}(x,t,z)\mathrm{D}_x^{\alpha}\tilde{Q}(x,t,z)+\tilde{V}(t,z)\mathrm{D}_x^{3\alpha}\tilde{Q}(x,t,z)=0 \tag{3}$$

where $z = (z_1, z_2,\ldots) \in (\mathbb{C})$ is a vector parameter. For the sake of simplicity we take $\tilde{U}(t,z)=u(t,z), \tilde{V}(t,z)=v(t,z),$ and $\tilde{Q}(x,t,z)=q(x,t,z).$ We use the transformation:

$$q=q(\xi), \quad \xi(x,t,z)=k\left(\frac{x^{\alpha}}{\alpha}\right)+w\int_0^t \frac{\theta(\tau,z)}{\tau^{1-\alpha}}\mathrm{d}\tau \tag{4}$$

where k and w are free constants while θ is a non-zero function to be determined. So, eq. (3) reduces to following non-linear ordinary differential equation (NODE):

$$w\theta\frac{\mathrm{d}q}{\mathrm{d}\xi}+ku(t,z)q\frac{\mathrm{d}q}{\mathrm{d}\xi}+k^3v(t,z)\frac{\mathrm{d}^3q}{\mathrm{d}\xi^3}=0 \tag{5}$$

The solution of NODE (5) can be given by a polynomial in (G'/G):

$$q(\xi)=a_0(t,z)+\sum_{i=1}^{n}a_i(t,z)\left(\frac{G'}{G}\right)^i+b_i(t,z)\left(\frac{G'}{G}\right)^{-i} \tag{6}$$

where $a_0, a_i, b_i (i=1,2,\ldots,n)$ are functions to be determined later. The $G=G(\xi)$ satisfies the second order linear differential equation in the form:

$$G''+\lambda G'+\mu G=0 \tag{7}$$

where λ and μ are arbitrary constants. From the balancing between $q(\mathrm{d}q/\mathrm{d}\xi)$ and $\mathrm{d}^3q/\mathrm{d}\xi^3$ appearing in NODE (5), we obtain the positive integer $n = 2$. Then solution of eq. (1) can be written in the form:

$$q(\xi)=a_0(t,z)+a_1(t,z)\left(\frac{G'}{G}\right)+a_2(t,z)\left(\frac{G'}{G}\right)^2+b_1(t,z)\left(\frac{G'}{G}\right)^{-1}+b_2(t,z)\left(\frac{G'}{G}\right)^{-2} \tag{8}$$

Substituting eqs. (7) and (8) into eq. (5), collecting all terms with the same power of (G'/G) and setting each coefficient to zero, we have the following solution sets.

Case 1

$$a_1(t,z) = -\frac{12k^2 v(t,z)\lambda}{u(t,z)}, \quad a_2(t,z) = -\frac{12k^2 v(t,z)}{u(t,z)}, \quad b_1(t,z) = 0$$

$$b_2(t,z) = 0, \quad a_0(t,z) = a_0(t,z), \quad a_1(t,z)w \neq 0$$

$$ku(t,z)v(t,z) \neq 0, \quad \theta = -\frac{a_0(t,z)ku(t,z) + k^3 v(t,z)\lambda^2 + 8k^3 v(t,z)\mu}{w}$$

Case 2

$$a_1(t,z) = 0, \quad a_2(t,z) = 0, \quad u \neq 0, \quad b_1(t,z) = -\frac{12k^2 v(t,z)\lambda\mu}{u(t,z)}$$

$$b_2(t,z) = -\frac{12k^2 v(t,z)\mu^2}{u(t,z)}, \quad a_0(t,z) = a_0(t,z), \quad \lambda \neq 0, \quad w \neq 0$$

$$kq(t,z)\mu \neq 0, \quad \theta = -\frac{a_0(t,z)ku(t,z) + k^3 v(t,z)\lambda^2 + 8k^3 v(t,z)\mu}{w}$$

The values in the previous cases are substituted in eq. (8) and if the solutions of second order linear ordinary differential eq. (7) are used, soliton solutions of eq. (5) are obtained as following.

For case 1:

When $\lambda^2 - 4\mu > 0$, the hyperbolic function travelling wave solution is obtained:

$$q_1(x,t,z) = a_0 + \frac{3k^2 v(t,z)}{u(t,z)} \left\{ 4\mu - \frac{(A-B)(A+B)(\lambda^2 - 4\mu)}{\left[B\cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi \right) + A\sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi \right) \right]^2} \right\} \tag{9}$$

When $\lambda^2 - 4\mu < 0$, we acquire the following trigonometric function travelling wave solution:

$$q_2(x,t,z) = a_0 + \frac{3k^2 v(t,z)}{u(t,z)} \left\{ 4\mu - \frac{(A^2 + B^2)(\lambda^2 - 4\mu)}{\left[A\cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi \right) + B\sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi \right) \right]^2} \right\} \tag{10}$$

When $\lambda^2 - 4\mu = 0$, we get following rational solution:

$$q_3(x,t,z) = a_0 + \frac{3k^2 v(t,z)[-4A^2 + \lambda^2 (A\xi + B)^2]}{u(t,z)(A\xi + B)^2} \tag{11}$$

where A and B are arbitrary constants and:

$$\xi = k\left(\frac{x^\alpha}{\alpha}\right) + \int_0^t \frac{a_0 k u(\tau,z) + k^3 v(\tau,z)\lambda^2 + 8k^3 v(\tau,z)\mu}{\tau^{1-\alpha}} \mathrm{d}\tau \tag{12}$$

in equations (9)-(11).

For case 2:

When $\lambda^2 - 4\mu > 0$:

$$q_4(x,t,z) = a_0 - \frac{24k^2 \mu v(t,z)}{u(t,z)} \left\{ \frac{2\mu}{[\lambda - \sqrt{\lambda^2 - 4\mu}\Gamma(x,t)]^2} + \frac{\lambda}{-\lambda + \sqrt{\lambda^2 - 4\mu}\Gamma(x,t)} \right\} \tag{13}$$

where

$$\Gamma(x,t) = \frac{B}{A + B\coth\left(\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} + \frac{A}{B + A\tanh\left(\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} \tag{14}$$

When $\lambda^2 - 4\mu < 0$:

$$q_5(x,t,z) = a_0 + \frac{24k^2 v(t,z)\mu}{u(t,z)} \left\{ \frac{1}{\lambda - \sqrt{4\mu - \lambda^2}\Gamma(x,t)} + \frac{2\mu}{\left[\lambda - \sqrt{4\mu - \lambda^2}\Gamma(x,t)\right]^2} \right\} \tag{15}$$

where

$$\Gamma(x,t) = \frac{B\cos\left(\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) - A\sin\left(\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{A\cos\left(\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + B\sin\left(\dfrac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)} \tag{16}$$

When $\lambda^2 - 4\mu = 0$:

$$q_6(x,t,z) = a_0 + \frac{24k^2 v(t,z)\mu(A\xi + B)[-2A + A\xi(\lambda - 2\mu)]}{u(t,z)[-2A + \lambda(A\xi + B)]^2} \tag{17}$$

where in (14), (16), and (17), ξ is as given in (11).

Exact stochastic solutions of eq. (1)

In this part, we have used the *Theorem 4.1.1* from Holden *et al*. [1]. By applying the inverse Hermite transform to the above solutions we have exact stochastic hyperbolic, trigonometric and rational solutions of eq. (1) respectively:

$$Q_1(x,t) = a_0 + \frac{3k^2 V(t)}{U(t)} \Diamond \left\{ 4\mu - \frac{(A-B)(A+B)(\lambda^2 - 4\mu)}{\left[B\cosh^\Diamond\left(\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + A\sinh^\Diamond\left(\dfrac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)\right]^{\Diamond 2}} \right\} \tag{18}$$

$$Q_2(x,t) = a_0 + \frac{3k^2V(t)}{U(t)}\Diamond\left\{4\mu - \frac{(A^2+B^2)(\lambda^2-4\mu)}{\left[A\cos^{\Diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+B\sin^{\Diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right]^{\Diamond 2}}\right\} \tag{19}$$

$$Q_3(x,t) = a_0 + \frac{3k^2V(t)\Diamond[-4A^2+\lambda^2(A\xi+B)^{\Diamond 2}]}{U(t)\Diamond(A\xi+B)^{\Diamond 2}} \tag{20}$$

where

$$\xi = k\left(\frac{x^\alpha}{\alpha}\right) + \int_0^t \frac{a_0kU(\tau)+k^3\lambda^2V(\tau)+8k^3\mu V(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau \tag{21}$$

in eqs. (18)-(20).

$$Q_4(x,t) = a_0 - \frac{24k^2\mu V(t)}{U(t)}\Diamond\left\{\frac{2\mu}{[\lambda-\sqrt{\lambda^2-4\mu}\Gamma(x,t)]^{\Diamond 2}} + \frac{\lambda}{-\lambda+\sqrt{\lambda^2-4\mu}\Gamma(x,t)}\right\} \tag{22}$$

where

$$\Gamma(x,t) = \frac{B}{A+B\coth^{\Diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} + \frac{A}{B+A\tanh^{\Diamond}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \tag{23}$$

$$Q_5(x,t) = a_0 + \frac{24k^2\mu V(t)}{U(t)}\Diamond\left\{\frac{1}{\lambda-\sqrt{4\mu-\lambda^2}\Gamma(x,t)} + \frac{2\mu}{\left[\lambda-\sqrt{4\mu-\lambda^2}\Gamma(x,t)\right]^{\Diamond 2}}\right\} \tag{24}$$

where

$$\Gamma(x,t) = \frac{B\cos^{\Diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) - A\sin^{\Diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{A\cos^{\Diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + B\sin^{\Diamond}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \tag{25}$$

$$Q_6(x,t) = a_0 + \frac{24k^2\mu V(t)\Diamond(A\xi+B)[-2A+A\xi(\lambda-2\mu)]}{U(t)\Diamond[-2A+\lambda(A\xi+B)]^{\Diamond 2}} \tag{26}$$

in eqs. (23), (25), and (26), ξ is as given in (21).

Example

Assume that $\alpha = 1$, $V(t) = \gamma U(t)$, and $U(t) = f(t) + \sigma W_t$, where γ and σ are free constants, $f(t)$ is bounded or integrable function on R_+ and W_t is the Gaussian white noise that satisfies $W_t = B_t$, B_t is a Brownian motion. The Hermit transform of W_t is given by

$\tilde{W}_t(z)=\sum_{i=1}^{\infty}z_i\int_0^t\eta_i(\tau)d\tau$. Using the definiton of $\tilde{W}_t(z)$ we get following white noise functional solutions.

$$Q_{B_1}(x,t)=a_0+3k^2\gamma\left\{4\mu-\frac{(A-B)(A+B)(\lambda^2-4\mu)}{\left[B\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+A\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)\right]^2}\right\} \tag{27}$$

$$Q_{B_2}(x,t)=a_0+3k^2\gamma\left\{4\mu-\frac{(A^2+B^2)(\lambda^2-4\mu)}{\left[A\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+B\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\right]^2}\right\} \tag{28}$$

$$Q_{B_3}(x,t)=a_0+3k^2\gamma\lambda^2-\frac{12k^2\gamma A^2}{(A\xi+B)^2} \tag{29}$$

where

$$\xi=kx+(a_0k+k^3\gamma\lambda^2+8k^3\gamma\mu)\left[\int_0^t f(\tau)\mathrm{d}\tau+\sigma\left(B_t-\frac{t^2}{2}\right)\right] \tag{30}$$

in eqs. (27)-(29).

$$Q_{B_4}(x,t)=a_0-24k^2\mu\gamma\left\{\frac{2\mu}{\left[\lambda-\sqrt{\lambda^2-4\mu}\Gamma(x,t)\right]^2}+\frac{\lambda}{-\lambda+\sqrt{\lambda^2-4\mu}\Gamma(x,t)}\right\} \tag{31}$$

where

$$\Gamma(x,t)=\frac{B}{A+B\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}+\frac{A}{B+A\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \tag{32}$$

$$Q_{B_5}(x,t)=a_0-24k^2\mu\gamma\left(\frac{1}{-\lambda+\sqrt{4\mu-\lambda^2}\Gamma(x,t)}+\frac{2\mu}{-\left[\lambda-\sqrt{4\mu-\lambda^2}\Gamma(x,t)\right]^2}\right) \tag{33}$$

where

$$\Gamma(x,t)=\frac{B\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)-A\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{A\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+B\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \tag{34}$$

$$Q_{B_6}(x,t) = a_0 + \frac{24k^2\mu\gamma[f(t)+\sigma W_t](A\xi+B)[-2A+A\xi(\lambda-2\mu)]}{[f(t)+\sigma W_t][-2A+\lambda(A\xi+B)]^2} \tag{35}$$

in eqs. (32), (34), and (35), ξ is as given in (30).

Conclusion

In this study, we have constructed the exact solutions of stochastic non-linear partial differential KdV equation with conformable derivative driven by Gaussian white noise. With the help of Hermit transform Wick products converted to the ordinary products and eventually the Wick-type stochastic equation reduced into the deterministic model. We have used the extended G'/G-expansion method for determining various exact soliton solutions. Then by applying the inverse Hermite transform to these obtained solutions we have acquired exact stochastic hyperbolic, trigonometric and rational solutions of eq. (1). Additionally we have demonstrated how the obtained stochastic solutions are presented as Brownian motion functions solutions giving an example.

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