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Investigation of wave solutions and conservation laws of generalized Calogero–Bogoyavlenskii–Schiff equation by group theoretic method

Fahd Jarad [a,b,c], Adil Jhangeer [d], Jan Awrejcewicz [e], Muhammad Bilal Riaz [e,f,*], M. Junaid-U-Rehman [g]

[a] Department of Mathematics, Cankaya University, 06530 Ankara, Turkey
[b] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[c] Department of Medical Research, China Medical University, Taichung 40402, Taiwan
[d] Department of Mathematics, Namal University, 30 Km Talagang Road, Mianwali 42250, Pakistan
[e] Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowskiego St., 90-924 Lodz, Poland
[f] Department of Mathematics, University of Management and Technology, 54770, Lahore, Pakistan
[g] Department of Mathematics, Quaid Azam University, 15320, Islamabad , Pakistan

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ABSTRACT

This work is focused to analyze the generalized Calogero–Bogoyavlenskii–Schiff equation (GCBSE) by the Lie symmetry method. GCBS equation has been utilized to explain the wave profiles in soliton theory. GCBSE was constructed by Bogoyavlenskii and Schiff in different ways (explained in the introduction section). With the aid of Lie symmetry analysis, we have computed the symmetry generators of the GCBSE and commutation relation. We observed from the commutator table, translational symmetries make an Abelian algebra. Then by using the theory of Lie, we have discovered the similarity variables, which are used to convert the supposed nonlinear partial differential equation (NLPDE) into a nonlinear ordinary differential equation (NLODE). Using the new auxiliary method (NAM), we have to discover some new wave profiles of GCBSE in the type of few trigonometric functions. These exits some parameters which we give to some suitable values to attain the different diagrams of some obtained solutions. Further, the GCBSE is presented by non-linear self-adjointness, and conserved vectors are discovered corresponding to each generator.

Introduction

Uses of NLEEs [1–6] in designing, applied science, and physical science are imperative and starting not very far in the past have drawn great mindfulness concerning various researchers, scientists, and specialists. The most extreme number of complex issues or physical problems can be displayed by evolution equations. NLEEs are utilized in electrical designing, optics, high-energy material science, astronomy, dense matter physical science, optical fiber, biomechanics, synthetic kinematics, gas elements, electrodynamics, plasma physical science, sea, and quantum designing, and so on. The investigation of traveling wave patterns to NLPDEs takes part a useful role in physics, fluid mechanics, and many other areas of engineering and science. In this task, the new auxiliary technique [7] is used to work out a few solitary wave structures of the GCBSE. The new auxiliary method (NAM) is strongly applied to attain the solitary wave structures of the GCBSE in the type of trigonometric and hyperbolic function solutions. NAM is very skilled and practically developed for obtaining new analytical solutions to non-linear differential equations. Computing the exact solution in a well-understood manner. Easier and faster employing a symbolic computation system. Effective frameworks to tackle the various aspects of analytical solitary solutions. Outcomes are more comprehensively generalized. We have drawn some of these solutions by giving suitable values to the parameters to get useful results.

Optical solitons have shaped their way through fiber optic innovation in a quick way. Now a day, all electronic sources for exchanging information are just conceivable due to soliton technology. Internet websites, Facebook correspondence, twitter remarks are all surely impressions of soliton innovation. It is a form of pulse that safeguards its shape when going at a steady speed. The assortment of models that review this innovation has given a wide scope of scholarly action in this direction. Nonlinear differential equations are used to describe

* Corresponding author at: Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowskiego St., 90-924 Lodz, Poland.

E-mail addresses: fahdjarad@gmail.com (F. Jarad), adil.jhangeer@namal.edu.pk (A. Jhangeer), jan.awrejcewicz@p.lodz.pl (J. Awrejcewicz), muhammad.riaz@p.lodz.pl, bilal.riaz@umt.edu.pk (M.B. Riaz), mjrehman@math.qau.edu.pk (M. Junaid-U-Rehman).

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the many physical models and help us to compute the solutions of that nonlinear physical models. These results are used in the many branches of science and engineering. The importance of wave patterns has much value and role in mathematical physics. It is very difficult to compute the exact analytical solutions to non-linear models. As we know, there are many number of schemes are constructed to find the exact analytical solutions of NLPDEs, for example, extended algebraic scheme [8], the variational scheme [9], G'/G-expansion method [10], soliton perturbation theory [11], the inverse scattering method [12], the integral scheme [13], and different schemes are showed in [14–22].

We will discuss the generalized Calogero–Bogoyavlenskii–Schiff equation (GCBSE) [23,24] of the following form

$$Q_{xt} + Q_{xxxy} + 3Q_xQ_{xy} + 3Q_{xx}Q_y + \delta_1 Q_{xy} + \delta_2 Q_{yy} = 0, \tag{1}$$

where $Q = Q(t, x, y)$ is continuous function and x, y are spatial components where t is temporal component, δ_1 and δ_2 both are non-zero parameters. GCBS equation has been utilized to explain the wave profiles in soliton theory. GCBSE was constructed by Bogoyavlenskii and Schiff in different ways. In [25], Yu and Toda used the KdV equation to construct the CBS equation. Using the Hirota bilinear method, Ma ad Chen constructed the different lump type solutions, with the help of Mathematica symbolic computations by the quadratic system of equations, to a GCBSE. Different types of singular soliton and multiple-soliton solutions were computed by Wazwaz for the (2 + 1) and (3 + 1)-dimensional CBSE by using the Hirota bilinear scheme and the Cole–Hopf transformation.

As referred to, there exists no integration scheme to manage a big scope of nonlinear PDEs. Maybe the significant technique to assemble solitary wave profiles for nonlinear PDEs is the new auxiliary strategy [7]. This technique is getting affirmation among the exploration local area for its short estimation strategy soon after its initiation. In this article, we construct the exact soliton structures of the GCBSE with the help of the powerful method NAM. Moreover, Lie symmetry investigation technique [26–31] is utilizing to consider the alleged nonlinear model. This procedure is utilized to investigate the various kinds of NLEEs. The symmetry method is utilized to study the GCBSE and exercise the symmetry generators. By the Lie hypothesis, we obtain the various types of important answers for the alleged nonlinear PDE. Over the latest years, various researchers have constructed the methodology of the Lie hypothesis. Not many of the specialists who do an extraordinary commitment in this space are auto-Bäcklund transformation [32], A.F Cheviakov [33], Ibragimov [34], Olver [35], and Bluman [36]. Various sorts of techniques have been created to investigate the NLPDEs, for example, Hirota's bilinear strategy [37], Lie symmetry [38] method, and so on.

Conservation laws have a lot of significance in the analytical solutions for PDEs. Many conserved quantities for PDE sure that the PDE is strongly integrable [39]. Some different methods in which conserved quantities have shown in the form of characters [40]. In this work, by applying the nonlinear self-adjointness scheme [41] to compute the conserved vectors for the discussed problem. In this research, the results are presented here are new and contains the trigonometric, hyperbolic functions, an exponential function. These results show the traveling wave profiles in different fields of nonlinear science. These results are not present in the available literature. The design of this study is elaborated as in Section "Preliminaries", the basics of this work are presented. In Section "Classical symmetries", classical symmetries are computed. In Section "Traveling wave profiles by abelian algebra", similarity reduction by abelian algebra and wave solutions of the supposed model are presented. Nonlinear self-adjointness classification and conservation laws of the GCBSE are investigated in Section "Nonlinear self-adjointness classification". In the end, the conclusion is presented.

Preliminaries

The new auxiliary method

Suppose, we have a general NLPDE of integer order:

$$F(Q, Q_t, Q_x, Q_y, Q_{xx}, Q_{xt}, Q_{yy}, \ldots) = 0, \tag{2}$$

where $Q = Q(x, y, t)$ and x, y, and t are space and temporal components. Some steps of the technique are given below.

Step:1 Assuming the transformation of the form

$$Q(x, y, t) = U(\rho), \qquad \rho = k(x + y) + ct. \tag{3}$$

Here ρ is new independent variable, where c and k are real parameter for Eq. (2). Using Eq. (3) into Eq. (2) and gives the following ODE is of the form

$$P(U, U', U'', \ldots) = 0. \tag{4}$$

Step:2 Suppose the solution for Eq. (4) of the type

$$U(\rho) = \sum_{i=0}^{N} g_i g^{if(\rho)}, \tag{5}$$

where g_i's are constants which are to be found later, and also first order ODE satisfy $f(\rho)$.

$$f'(\rho) = \frac{1}{\ln(g)}\{\nu g^{-f(\rho)} + \mu + \gamma g^{f(\rho)}\}, \quad g > 0, \quad g \neq 1. \tag{6}$$

Step:3 To find the value of N in Eq. (5), we use the balancing procedures i.e., the highest order derivative is balanced by the linear and nonlinear terms.

Step: 4 Putting Eqs. (5) and (6) into Eq. (2) and collecting the like term in the powers of $g^{f(\rho)}$ $(i = 0, 1, 2, 3.)$. After collecting the like term, we put them equal to zero and we have a collection of equations and then solving these by computer algebra system (CAS) i.e., $Mathematica$.

Step:5 Using all the values of $g^{f(\rho)}$ into Eq. (5), we get the results for Eq. (2).

Nonlinear self-adjointness

Suppose the mth order PDE:

$$G = G(x, Q, Q_1, Q_2, \ldots, Q_m), \tag{7}$$

where $Q = Q(x)$ and $x = x(x_1, x_2, \ldots, x_n)$. Suppose the Lagrangian $\mathcal{L} = \upsilon G$ for Eq. (7), and we obtain adjoint equation below

$$G^\star \equiv \frac{\delta}{\delta Q}(\upsilon G), \tag{8}$$

where

$$\frac{\delta}{\delta Q} = \frac{\partial}{\partial Q} + \sum_{i=1}^{\infty}(-1)^s D_{i^1} \ldots D_{i^s} \frac{\partial}{\partial Q_{i^1 \ldots i^s}}, \tag{9}$$

is the Euler–Lagrange operator, where D_i is defined as

$$D_i = \frac{\partial}{\partial x_i} + Q_i \frac{\partial}{\partial Q} + Q_{ij}\frac{\partial}{\partial Q_j} + \cdots \ . \tag{10}$$

Definition 1. Eq. (7) is called a strictly self-adjoint if the equation obtained from its adjoint equation with the help of the transformation $\upsilon = Q$, such that

$$G^\star|_{\upsilon = Q} = \mu(x, Q, \ldots)G, \tag{11}$$

for some $\mu \in \mathcal{D}$.

Definition 2. Eq. (7) is said as quasi-self-adjoint if the equation obtained from its adjoint equation with the help of the transformation $v = V(Q) \neq 0$, such that

$$G^{\star}|_{v=V(Q)} = \mu(x, Q, \ldots)G, \tag{12}$$

where $\mu \in \mathcal{D}$.

Definition 3. Eq. (7) is said to be weak self-adjoint if the equation acquired from its adjoint equation with the help of the transformation $v = V(x, Q) \neq 0$ for a particular function V such that $V_Q \neq 0$ and $V_{x^i} \neq 0$ for some x^i, such that

$$G^{\star}|_{v=V(x,Q)} = \mu(x, Q, \ldots)G, \tag{13}$$

for some $\mu \in \mathcal{D}$.

Definition 4. If the equation is obtained from its adjoint equation then Eq. (1) is said to be nonlinearly self-adjoint with the help of the substitution $v = V(x, Q)$, with a some function such that $V(x, Q) \neq 0$, (7) satisfy the condition,

$$G^{\star}|_{v=V(x,Q)} = \mu(x, Q, \ldots)G, \tag{14}$$

for some $\mu \in \mathcal{D}$.

It is great to specify that Ibragimov [42] gave the idea of stated Defs. 1, 2, 4 and Gandarias [41] gives the concept of Def. 3.

Theorem 1. *Let us suppose Lie point, Lie–Backlund, or nonlocal symmetry of (7):*

$$Z = \phi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial Q}, \tag{15}$$

with a formal Lagrangian $\mathcal{L}$. We define the Conservation laws for Eqs. (1) and *(8):*

$$\begin{aligned} C^{x^i} =& \phi^i \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial Q_i} - D_j\left(\frac{\partial \mathcal{L}}{\partial Q_{ij}}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial Q_{ijk}}\right)\right] \\ &+ D_j(W)\left[\frac{\partial \mathcal{L}}{\partial Q_{ij}} - D_k\left(\frac{\partial \mathcal{L}}{\partial Q_{ijk}}\right)\right] + D_j D_k(W)\frac{\partial \mathcal{L}}{\partial Q_{ijk}}, \end{aligned} \tag{16}$$

here W is named as the Lie characteristic function which can be obtained from

$$W = \eta - \phi^i Q_i \ , \tag{17}$$

while $D_i\ (C^{x^i}) = 0$.

Classical symmetries

Here, we will compute the whole vector field of Eq. (1). For this, assume Lie algebra of Eq. (1) below:

$$\Phi = \xi^1(x, y, t, Q)\frac{\partial}{\partial x} + \xi^2(x, y, t, Q)\frac{\partial}{\partial y} + \xi^3(x, y, t, Q)\frac{\partial}{\partial t} + \eta(x, y, t, Q)\frac{\partial}{\partial Q}, \tag{18}$$

we define the fourth prolongation of Φ which can be written as:

$$\begin{aligned} \Phi^{[4]} =& \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial Q} + \Theta + \eta^{xt}\frac{\partial}{\partial Q_{xt}} + \eta^{xxxy}\frac{\partial}{\partial Q_{xxxy}} + \eta^x\frac{\partial}{\partial Q_x} \\ &+ \eta^y\frac{\partial}{\partial Q_y} + \eta^{xx}\frac{\partial}{\partial Q_{xx}} + \eta^{xy}\frac{\partial}{\partial Q_{xy}} + \eta^{yy}\frac{\partial}{\partial Q_{yy}}, \end{aligned} \tag{19}$$

applying $\Phi^{[4]}$ to Eq. (1) and we get

$$\Phi^{[4]}\left(Q_{xt} + Q_{xxxy} + 3Q_xQ_{xy} + 3Q_{xx}Q_y + \delta_1 Q_{xy} + \delta_2 Q_{yy}\right)|_{\text{Eq. (1)}} = 0, \tag{20}$$

Table 1
Commutator table.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	0	$3X_2$
X_3	0	0	0	$3X_3$
X_4	$-X_1$	$-3X_2$	$-3X_3$	0

furthermore, we have

$$\begin{cases} \eta^x = D_x(\eta) - Q_x D_x(\xi^1) - Q_y D_x(\xi^2) - Q_t D_x(\xi^3), \\ \eta^{xx} = D_x(\eta^x) - Q_{xx} D_x(\xi^1) - Q_{yx} D_x(\xi^2) - Q_{tx} D_x(\xi^3), \\ \eta^{yy} = D_y(\eta^{yy}) - Q_{xy} D_y(\xi^1) - Q_{yy} D_y(\xi^2) - Q_{ty} D_y(\xi^3), \\ \eta^{xt} = D_t(\eta^t) - Q_{xx} D_t(\xi^1) - Q_{xy} D_t(\xi^2) - Q_{xt} D_t(\xi^3), \\ \eta^{xxx} = D_x(\eta^{xx}) - Q_{xxx} D_x(\xi^1) - Q_{xxy} D_x(\xi^2) - Q_{xxt} D_x(\xi^3), \\ \eta^{xxxx} = D_y(\eta^{xxx}) - Q_{xxxx} D_y(\xi^1) - Q_{xxxy} D_y(\xi^2) - U_{xxxt} D_y(\xi^3). \end{cases} \tag{21}$$

Let $(x^1, x^2, x^3) = (x, y, t)$, where D_i is written as:

$$D_i = \frac{\partial}{\partial x^i} + U_i\frac{\partial}{\partial U} + U_{ij}\frac{\partial}{\partial U_j} + \cdots, \qquad i = 1, 2.$$

Eq. (20) gives the following vector field of Eq. (1): (see Table 1).

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y}, \\ &X_4 = \frac{1}{5}x\frac{\partial}{\partial x} + \frac{3}{5}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - \left(\frac{2}{15}\alpha x + \frac{1}{5}u\right)\frac{\partial}{\partial u}. \end{aligned} \tag{22}$$

Traveling wave profiles by abelian algebra

It can be seen that $\Phi = \{X_1, X_2, X_3\}$ forms an abelian subalgebra [29]. Here we will find a traveling wave solution for the supposed model corresponding to the linear combination of the translation symmetries of the form

$$\mathfrak{B} = k\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + c\frac{\partial}{\partial t}.$$

Using the above linear combination $\mathfrak{B}$ of symmetries, we define the transformation below

$$Q(x, y, t) = U(\rho), \qquad \rho = k(x + y) + ct, \tag{23}$$

using Eq. (23) in Eq. (2) we have

$$k^3U'''' + 6k^2U'U'' + (c + (\delta_1 + \delta_2)k)U'' = 0, \tag{24}$$

integrating once Eq. (24) and we have

$$k^3U''' + 3k^2(U')^2 + (c + (\delta_1 + \delta_2)k)U' = 0. \tag{25}$$

The next task is to calculate the traveling wave profiles for Eq. (1) from Eq. (25).

Application of new auxiliary method

Here we construct the traveling wave solutions for the supposing model from Eq. (25) by using the new auxiliary scheme. By using the balancing procedure, we have $N = 1$; so, the solution is of the kind

$$U(\gamma) = g_0 + g_1 g^{f(\rho)}. \tag{26}$$

By putting Eq. (26) and its derivatives into Eq. (25). Collecting the coefficients of $g^{f(\rho)}$ gives the system of equations. Computing the obtained system, which is stated below.

$$g_0 = g_0, \quad g_1 = -2k\gamma, \quad k = k, \quad c = (4\nu\gamma - \mu^2)k^3 - (\delta_1 + \delta_2)k. \tag{27}$$

Now by substituting Eq. (27) into Eq. (26) the wave solutions of the given equation for the obtained result

$$Q(x, y, t) = g_0 - 2k\gamma g^{f(\rho)}. \tag{28}$$

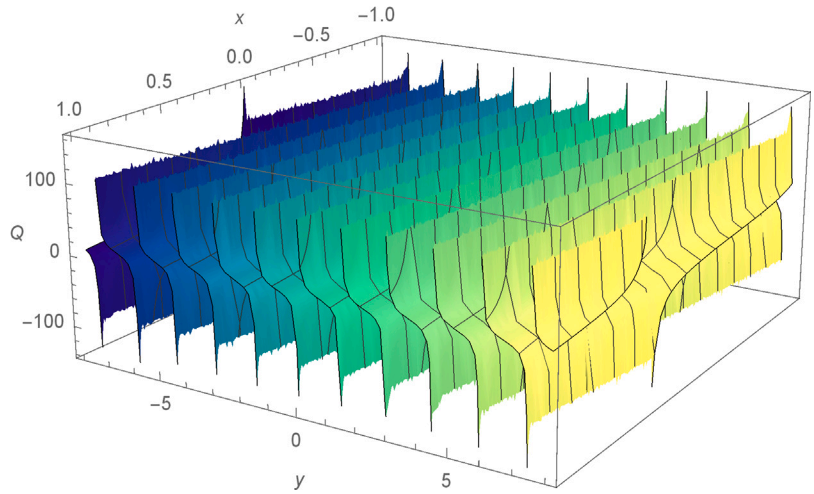


Fig. 1. 3D diagram of (29) for $\mu = 1$, $\gamma = 1.2$, $\nu = 0.75$, $k = 1$, $t = 1$, $\delta_2 = 0.50$, and $c = 1.50$.

Case: 1 When $\mu^2 - \nu\gamma < 0$ and $\gamma \neq 0$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{-(\mu^2 - \nu\gamma)}}{\gamma}\tan\left(\frac{\sqrt{-(\mu^2 - \nu\gamma)}}{2}\right)(k(x+y)+ct), \tag{29}$$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{-(\mu^2 - \nu\gamma)}}{\gamma}\cot\left(\frac{\sqrt{-(\mu^2 - \nu\gamma)}}{2}\right)(k(x+y)+ct). \tag{30}$$

Case: 2 When $\mu^2 + \nu\gamma > 0$ and $\gamma \neq 0$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{(\mu^2 - \nu\gamma)}}{\gamma}\tanh\left(\frac{\sqrt{(\mu^2 - \nu\gamma)}}{2}\right)(k(x+y)+ct), \tag{31}$$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} - \frac{\sqrt{(\mu^2 - \nu\gamma)}}{\gamma}\coth\left(\frac{\sqrt{(\mu^2 - \nu\gamma)}}{2}\right)(k(x+y)+ct). \tag{32}$$

Case: 3 When $\mu^2 + \nu\gamma > 0$ and $\gamma \neq 0$ and $\gamma \neq -\nu$

$$g^{f(\rho)} = \frac{\mu}{\gamma} + \frac{\sqrt{(\mu^2 + \nu^2)}}{\gamma}\tanh\left(\frac{\sqrt{(\mu^2 + \nu^2)}}{2}\right)(k(x+y)+ct). \tag{33}$$

$$g^{f(\rho)} = \frac{\mu}{\gamma} + \frac{\sqrt{(\mu^2 + \nu^2)}}{\gamma}\coth\left(\frac{\sqrt{(\mu^2 + \nu^2)}}{2}\right)(k(x+y)+ct), \tag{34}$$

Case: 4 When $\mu^2 + \nu\gamma < 0$, $\gamma \neq 0$ and $\gamma \neq -\nu$

$$g^{f(\rho)} = \frac{\mu}{\gamma} + \frac{\sqrt{-(\mu^2 + \nu^2)}}{\gamma}\tan\left(\frac{\sqrt{-(\mu^2 + \nu^2)}}{2}\right)(k(x+y)+ct), \tag{35}$$

$$g^{f(\rho)} = \frac{\mu}{\gamma} + \frac{\sqrt{-(\mu^2 + \nu^2)}}{\gamma}\cot\left(\frac{\sqrt{-(\mu^2 + \nu^2)}}{2}\right)(k(x+y)+ct). \tag{36}$$

Case: 5 When $\mu^2 - \nu^2 < 0$ and $\gamma \neq -\nu$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{-(\mu^2 - \nu^2)}}{\gamma}\tan\left(\frac{\sqrt{-(\mu^2 - \nu^2)}}{2}\right)(k(x+y)+ct), \tag{37}$$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{-(\mu^2 - \nu^2)}}{\gamma}\cot\left(\frac{\sqrt{-(\mu^2 - \nu^2)}}{2}\right)(k(x+y)+ct). \tag{38}$$

Case: 6 When $\mu^2 - \nu^2 > 0$ and $\gamma \neq -\nu$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{(\mu^2 - \nu^2)}}{\gamma}\tanh\left(\frac{\sqrt{(\mu^2 - \nu^2)}}{2}\right)(k(x+y)+ct), \tag{39}$$

$$g^{f(\rho)} = \frac{-\mu}{\gamma} + \frac{\sqrt{(\mu^2 - \nu^2)}}{\gamma}\coth\left(\frac{\sqrt{(\mu^2 - \nu^2)}}{2}\right)(k(x+y)+ct). \tag{40}$$

Case: 7 When $\nu\gamma > 0$, $\gamma \neq 0$ and $\mu = 0$

$$g^{f(\rho)} = \sqrt{\frac{-\nu}{\gamma}}\tanh\left(\frac{\sqrt{-\nu\gamma}}{2}\right)(k(x+y)+ct), \tag{41}$$

$$g^{f(\rho)} = \sqrt{\frac{-\nu}{\gamma}}\coth\left(\frac{\sqrt{-\nu\gamma}}{2}\right)(k(x+y)+ct). \tag{42}$$

Case: 8 When $\mu = 0$ and $\nu = -\gamma$

$$g^{f(\rho)} = \frac{-(1+e^{2\nu(k(x+y)+ct)}) \pm \sqrt{2(1+e^{2\nu(k(x+y)+ct)})}}{e^{2\nu(k(x+y)+ct)} - 1}. \tag{43}$$

Case: 9 When $\mu^2 = \nu\gamma$

$$g^{f(\rho)} = \frac{-\nu(\mu(k(x+y)+ct)+2)}{\mu^2(k(x+y)+ct)}. \tag{44}$$

Case: 10 When $\mu = k$, $\nu = 2k$ and $\gamma = 0$

$$g^{f(\rho)} = e^{(k(x+y)+ct)} - 1. \tag{45}$$

Case: 11 When $\mu = k$, $\gamma = 2k$ and $\nu = 0$

$$g^{f(\rho)} = \frac{e^{(k(x+y)+ct)}}{1 - e^{(k(x+y)+ct)}}. \tag{46}$$

Case: 12 When $2\mu = \nu + \gamma$

$$g^{f(\rho)} = \frac{1 + \nu e^{\frac{1}{2}(\nu-\gamma)(k(x+y)+ct)}}{1 + \gamma e^{\frac{1}{2}(\nu-\gamma)(k(x+y)+ct)}}. \tag{47}$$

Case: 13 When $-2\mu = \nu + \gamma$

$$g^{f(\rho)} = \frac{\nu + \nu e^{\frac{1}{2}(\nu-\gamma)(k(x+y)+ct)}}{\gamma + \gamma e^{\frac{1}{2}(\nu-\gamma)(k(x+y)+ct)}}. \tag{48}$$

Case: 14 When $\nu = 0$

$$g^{f(\rho)} = \frac{\mu e^{\mu(k(x+y)+ct)}}{1 + \frac{\gamma}{2} e^{\mu(k(x+y)+ct)}}. \tag{49}$$

Case: 15 When $\nu = \mu = \gamma \neq 0$

$$g^{f(\rho)} = \frac{-(\nu(k(x+y)+ct)+2)}{\nu(k(x+y)+ct)}. \tag{50}$$

Case: 16 When $\nu = \gamma$, $\mu = 0$

$$g^{f(\rho)} = tan\left(\frac{\nu(k(x+y)+ct)+c}{2}\right). \tag{51}$$

Case: 17 When $\gamma = 0$

$$g^{f(\rho)} = e^{\mu(k(x+y)+ct)} - \frac{\nu}{2\mu}. \tag{52}$$

Graphical representation of the results

This section is devoted to reflecting on the physical interpretation of some of the exact outcomes obtained in this paper. We have used a NAM to get the summarized required results.

Figs. 1, 2a and 2b demonstrate the 3D, 2D, and contour graphical behaviors in wave propagation. This behavior can be more clearly

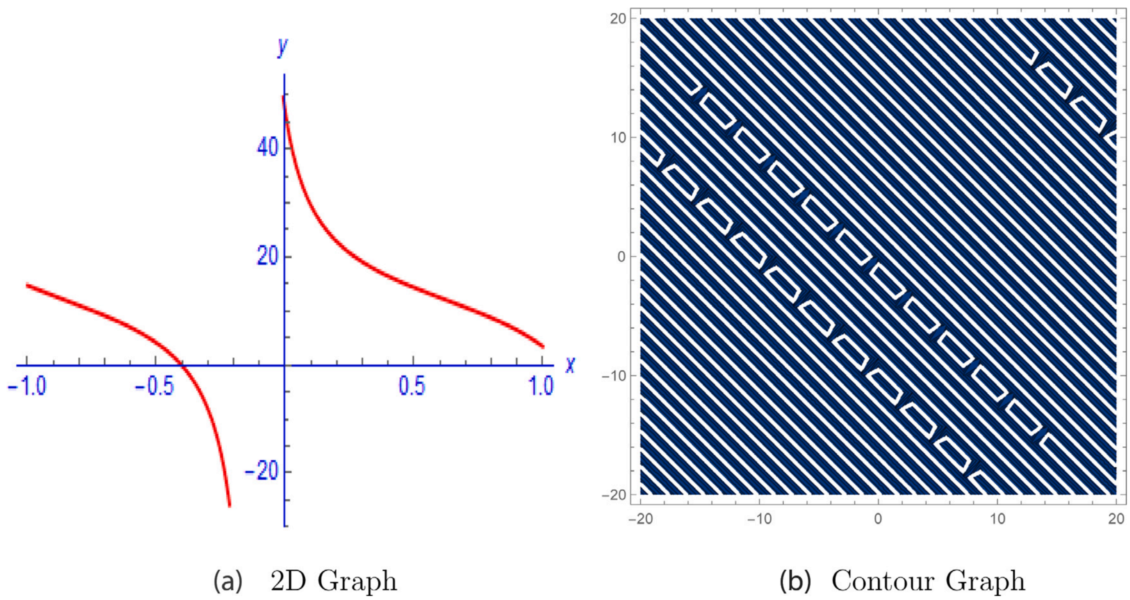


Fig. 2. Diagrams 2a and 2b represent the $2D$ and contour diagram of Eq. (29) with values $\mu = 1$, $\gamma = 1.2$, $\nu = 0.75$, $k = 1$, $t = 1$, $\delta_2 = 0.50$, and $c = 1.50$.

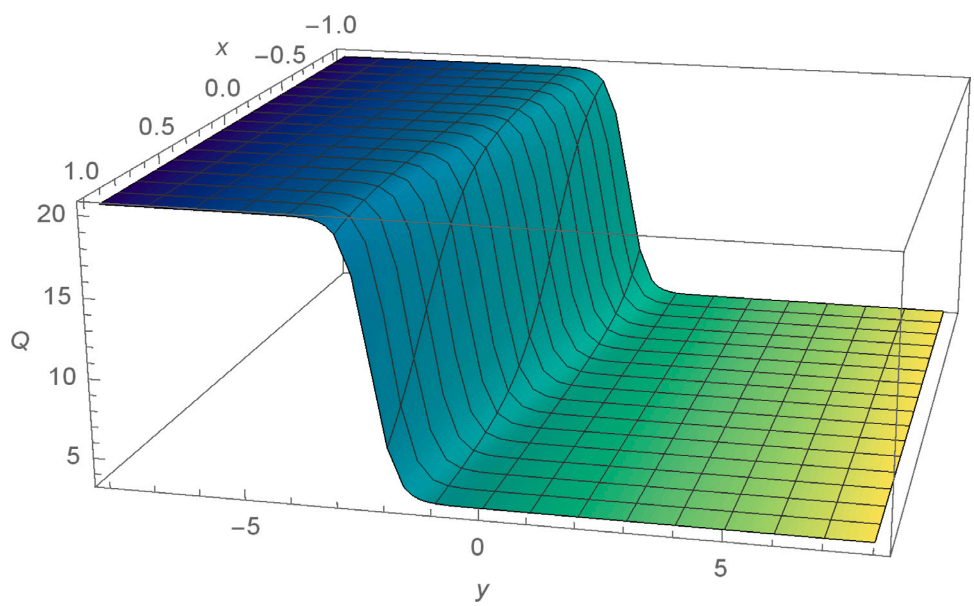


Fig. 3. 3D Graphical behavior of (31) for $\mu = 1.5$, $\gamma = 0.5$, $\nu = 1$, $k = 0.5$, $t = 1.5$, $\delta_2 = 1$, and $c = 1.2$.

understood for taking the unique values of parameters for solution (29) with $\mu = 1$, $\gamma = 1.2$, $\nu = 0.75$, $k = 1$, $t = 1$, $\delta_2 = 0.50$, and $c = 1.50$.

Figs. 3, 4a and 4b represent the 3D, 2D and contour diagrams of (31) by choosing the unique values of involving parameters $\mu = 1.5$, $\gamma = 0.5$, $\nu = 1$, $k = 0.5$, $t = 1.5$, $\delta_2 = 1$, and $c = 1.2$.

Graphical behavior of solution (43) with parameters values $\mu = 1.25$, $\gamma = 1.5$, $\nu = 1$, $k = 1.2$, $t = 1$, $\delta_2 = 0.7$, and $c = 0.75$ is represented in Figs. 7, 8a and 8b.

Figs. 9, 10a and 10b represent the different diagrams of (44) with parameters values $\mu = 1.5$, $\gamma = 1.2$, $\nu = 2.1$, $k = 0.5$, $t = 1$, $\delta_2 = 0.75$, and $c = 1$.

Figs. 5, 6a and 6b represent the 3D, 2D and contour diagrams of traveling wave solution (49) with special values of involving parameters $\mu = 1$, $\gamma = 1.75$, $\nu = 1.50$, $k = 1$, $t = 1$, $\delta_2 = 0.50$, and $c = 1$.

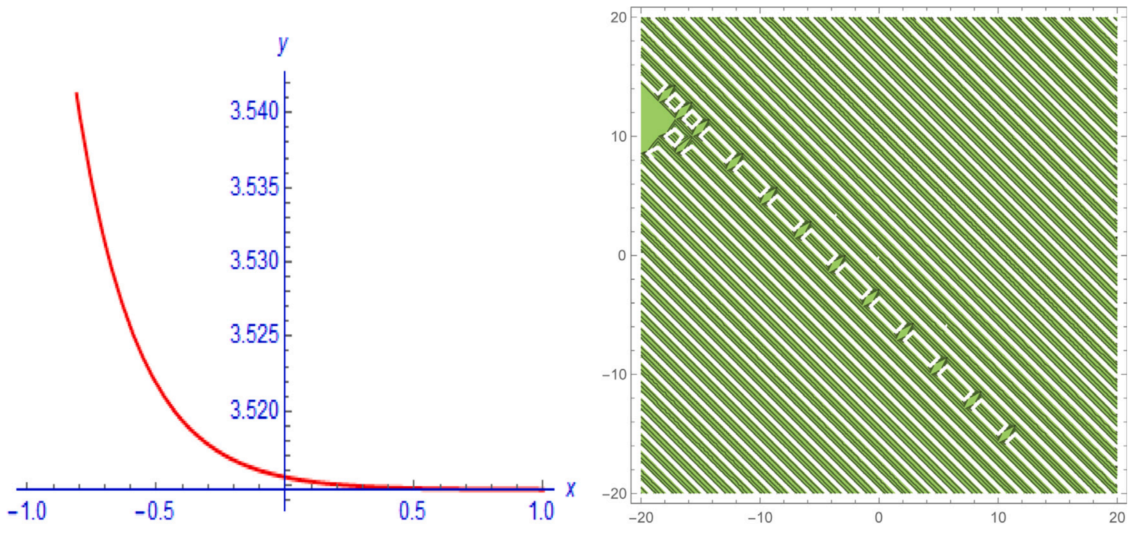
Nonlinear self-adjointness classification

Classification of nonlinear self-adjointness of Eq. (1) is presented in this portion. Suppose the formal Lagrangian $\mathcal{L}$ is:

$$\mathcal{L} = v\left[Q_{xt} + Q_{xxxy} + 3Q_xQ_{xy} + 3Q_{xx}Q_y + \delta_1 Q_{xy} + \delta_2 Q_{yy}\right]. \tag{53}$$

Eq. (8) yields:

$$G^{\star} \equiv \frac{\delta}{\delta Q}\left[v\left(Q_{xt} + Q_{xxxy} + 3Q_xQ_{xy} + 3Q_{xx}Q_y + \delta_1 Q_{xy} + \delta_2 Q_{yy}\right)\right], \tag{54}$$



(a) 2D Graph (b) Contour Graph

Fig. 4. Figs. 4a and 4b represent the $2D$ and contour diagram of Eq. (31) with values $\mu = 1.5$, $\gamma = 0.5$, $\nu = 1$, $k = 0.5$, $t = 1.5$, $\delta_2 = 1$, and $c = 1.2$.

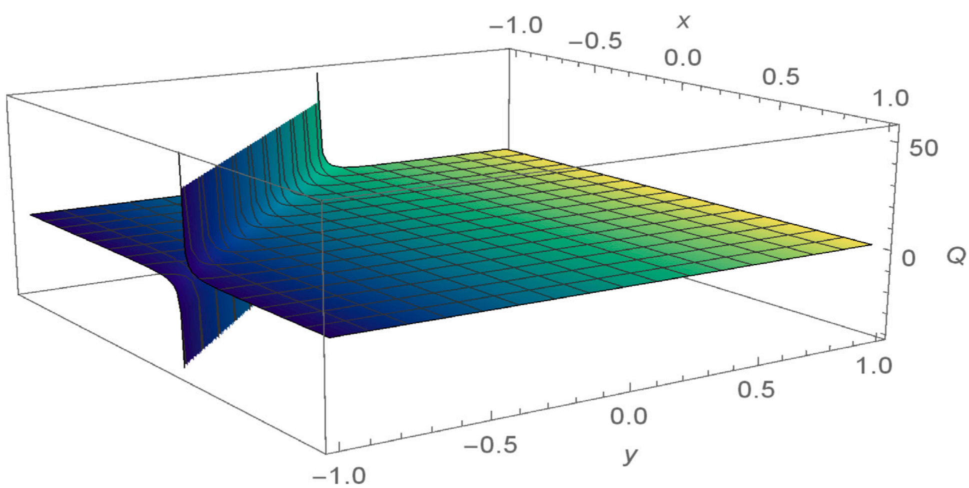


Fig. 5. 3D Graphical representations of (43) for $\mu = 1.25$, $\gamma = 1.5$, $\nu = 1$, $k = 1.2$, $t = 1$, $\delta_2 = 0.7$, and $c = 0.75$.

which gives as follows

$$\begin{aligned} G^{\star} = {} & 2vQ_{xxy} + 2v_yQ_{xx} + v_{xx}Q_y + 4v_xQ_{xy} + \delta_2 v_{yy} \\ & + v_{xt} + \delta_1 v_{xy} + 3v_{xy}Q_{xx} + v_{xxxy}. \end{aligned} \tag{55}$$

Using the Definitions 1 – 4, We have reached this conclusion to state the theorem below:

Theorem 2. *Eq. (1) is not weak self-adjoint, quasi-self-adjoint, and strictly self-adjoint. After all, Eq. (1) is nonlinearly self-adjoint for $v = \Phi$, while $\Phi(\tau, \xi)$ is satisfy the equation given below:*

$$2\Phi\Phi_{xxy} + 4\Phi_x\Phi_{xy} = 0. \tag{56}$$

Conservation laws

Conservation laws are calculated in this section by using the symmetries. The entire vector field of Eq. (1) is

$$\begin{aligned} & X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial y} + \frac{1}{3}x\frac{\partial}{\partial u}, \\ & X_5 = \frac{1}{5}x\frac{\partial}{\partial x} + \frac{3}{5}y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - \left(\frac{2}{15}\alpha x + \frac{1}{5}u\right)\frac{\partial}{\partial u}. \end{aligned} \tag{57}$$

(I) The conservation laws for X_1 are

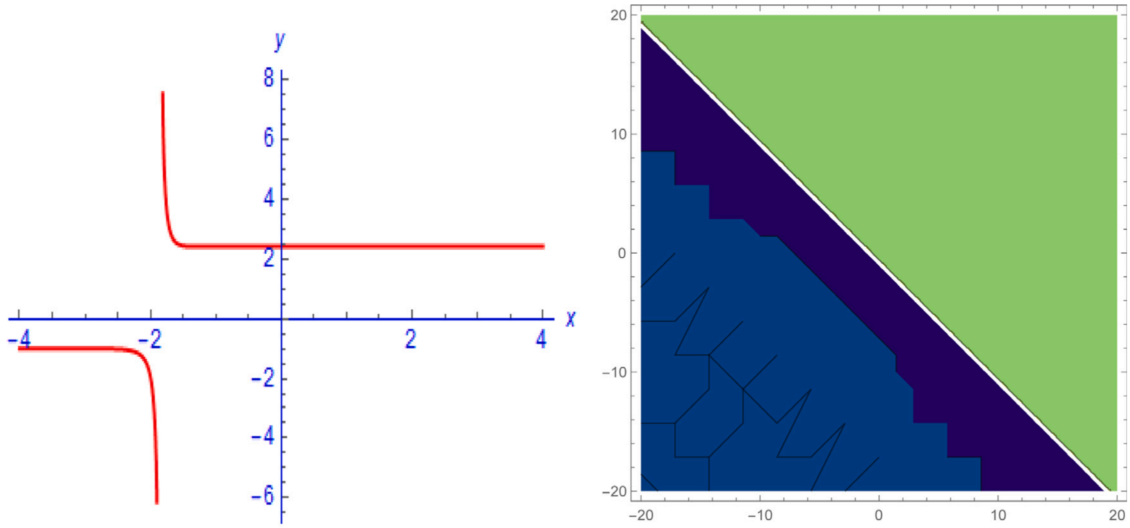
$$\begin{aligned} C^t = {} & \mathcal{L} + \Phi_t Q_t, \\ C^x = {} & Q_t\left[3\Phi_x Q_y + 3\Phi Q_{xy} + \delta_1\Phi_x + \delta_2\Phi_y + \Phi_{xxx}\right] \\ & + Q_{xt}\left[3\Phi Q_y + 3\Phi Q + \delta_1\Phi\right] + Q_{yt}\delta_2\phi + Q_{xxt}\Phi_x, \\ C^y = {} & -Q_t\left[3\Phi Q_{xx} - 3\Phi_y Q_x - \delta_1\Phi_y + \delta_2\Phi_y - 3\Phi Q_{xy}\right] \\ & + Q_{yt}\left[\delta_2\Phi + 3\Phi Q_x + \delta_1\Phi\right], \end{aligned}$$

where Φ fulfills the Eq. (56).

(II) The conserved vectors for X_2 are

$$\begin{aligned} C^t = {} & \mathcal{L} + \Phi_t Q_y, \\ C^x = {} & Q_y\left[3\Phi_x Q_y + 3\Phi Q_{xy} + \delta_1\Phi_x + \delta_2\Phi_y + \Phi_{xxx}\right] \\ & + Q_{xy}\left[3\Phi Q_y + 3\Phi Q + \delta_1\Phi\right] + Q_{yy}\delta_2\phi + Q_{xxy}\Phi_x, \\ C^y = {} & -Q_y\left[3\Phi Q_{xx} - 3\Phi_y Q_x - \delta_1\Phi_y + \delta_2\Phi_y - 3\Phi Q_{xy}\right] \\ & + Q_{yy}\left[\delta_2\Phi + 3\Phi Q_x + \delta_1\Phi\right], \end{aligned}$$

where Φ satisfies Eq. (56).



(a) 2D Graph (b) Contour Graph

Fig. 6. Figs. 8a and 8b represent the $2D$ and contour diagram of Eq. (43) with values $\mu = 1.25$, $\gamma = 1.5$, $v = 1$, $k = 1.2$, $t = 1$, $\delta_2 = 0.7$, and $c = 0.75$.

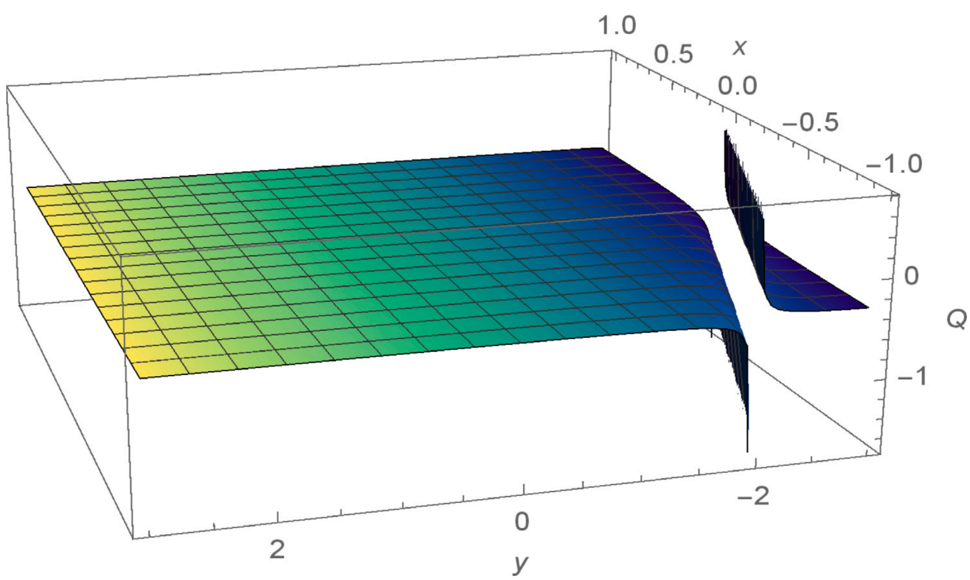


Fig. 7. 3D Graphical behavior of (44) for $\mu = 1.5$, $\gamma = 1.2$, $v = 2.1$, $k = 0.5$, $t = 1$, $\delta_2 = 0.75$, and $c = 1$.

(III) The conservation laws for X_3 are

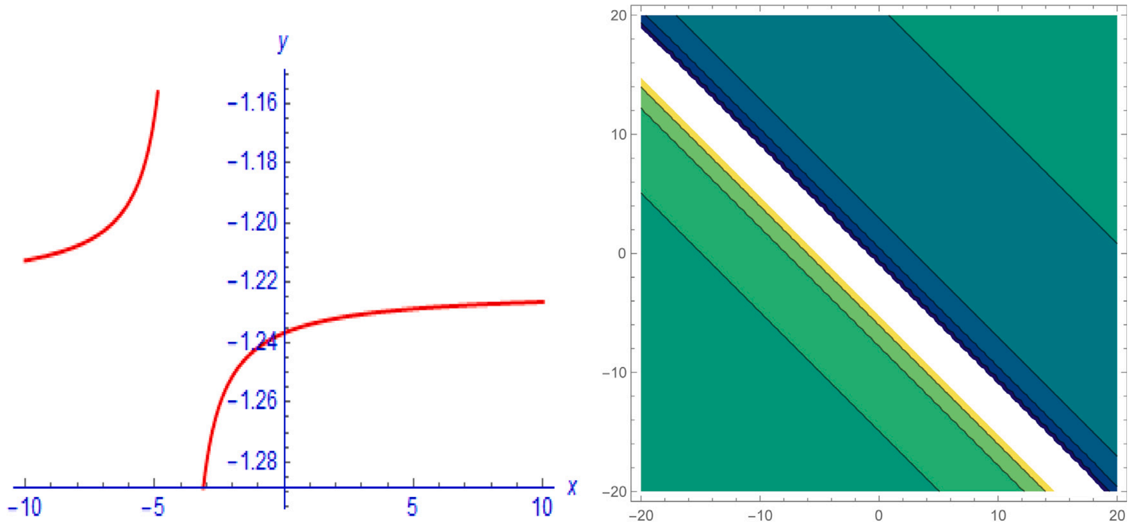
$$\begin{aligned}
C^t =& \mathcal{L} + \Phi_t Q_x,\\
C^x =& Q_x\left[3\Phi_x Q_y + 3\Phi Q_{xy} + \delta_1\Phi_x + \delta_2\Phi_y + \Phi_{xxx}\right]\\
&+ Q_{xx}\left[3\Phi Q_y + 3\Phi Q + \delta_1\Phi\right] + Q_{xy}\delta_2\phi + Q_{xxx}\Phi_x,\\
C^y =& -Q_x\left[3\Phi Q_{xx} - 3\Phi_y Q_x - \delta_1\Phi_y + \delta_2\Phi_y - 3\Phi Q_{xy}\right]\\
&+ Q_{xy}\left[\delta_2\Phi + 3\Phi Q_x + \delta_1\Phi\right],
\end{aligned}$$

where Φ satisfies Eq. (56).

Similarly, we can find conserved vectors corresponding to the remaining two operators.

Conclusion

In this work, GCBSE was analyzed by the Lie symmetry method and further nonlinear self-adjointness. The Lie theory is considered to examine the GCBSE and evaluate the symmetry operators. Conservation laws are constructed by nonlinear self-adjointness theory for the assumed equation. NAM is used to get some new wave solutions for NLPDEs. We get more exact wave profiles by the new auxiliary scheme for the generalized Calogero–Bogoyavlenskii–Schiff equation (GCBSE) compared with some other methods as compared to other methods. We have used a NAM to get the required results and we get, bright solutions, soliton-like solutions, singular bright solutions, periodic soliton solutions, and combined soliton are produced. The proposed method is victoriously employed to evaluate the wave patterns of a discussed nonlinear model in the type of trigonometric and hyperbolic function



(a) 2D Graph (b) Contour Graph

Fig. 8. Figs. 10a and 10b represent the $2D$ and contour diagram of Eq. (44) with values $\mu = 1.5$, $\gamma = 1.2$, $\nu = 2.1$, $k = 0.5$, $t = 1$, $\delta_2 = 0.75$, and $c = 1$.

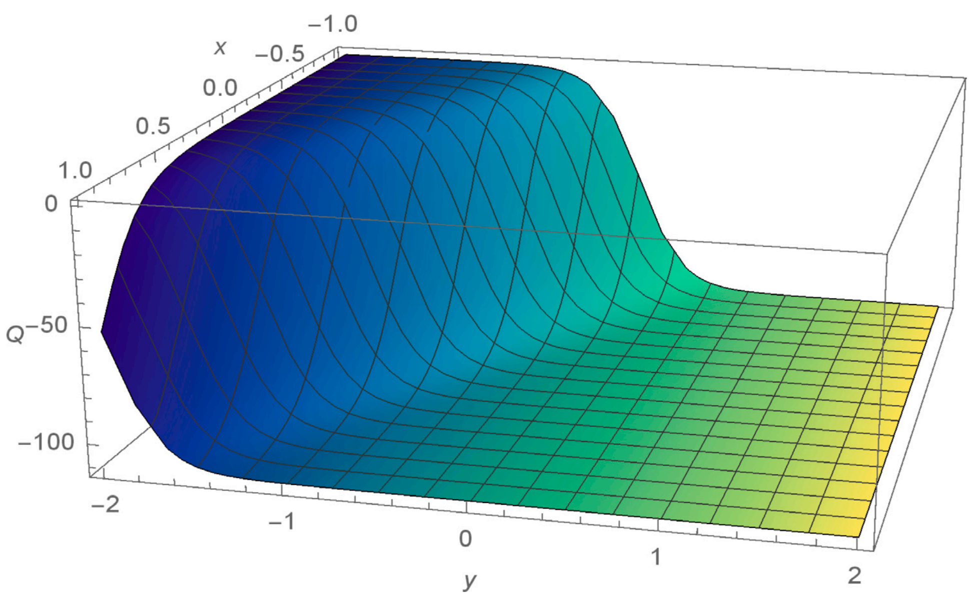


Fig. 9. 3D Graphical representations of (49) for $\mu = 1$, $\gamma = 1.75$, $\nu = 1.50$, $k = 1$, $t = 1$, $\delta_2 = 0.50$, and $c = 1$.

results. The obtained results consist of trigonometric function results which are more useful to break down the GCBSE. As far as we could know, the outcomes represented in this paper generally have not been described in the literature. We have drawn some 2D, 3D, and density diagrams of wave profiles of some of these solutions by giving suitable values to the parameters to get useful results.

CRediT authorship contribution statement

Fahd Jarad: Investigation, Software, Formulation, Review and checking results. **Adil Jhangeer:** Software, Visualization, Supervision, Formal analysis. **Jan Awrejcewicz:** Data curation, Data analysis, Project administration, Final checking, Validation. **Muhammad Bilal Riaz:** Conceptualization, Investigation, Methodology, Initial writing. **M. Junaid-U-Rehman:** Methodology, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

My Manuscript has no associated data.

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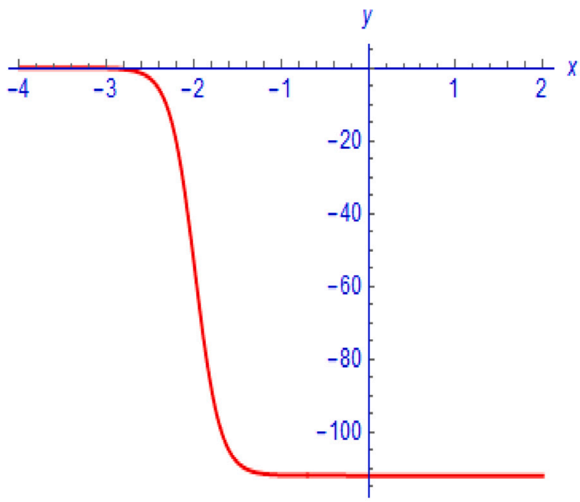
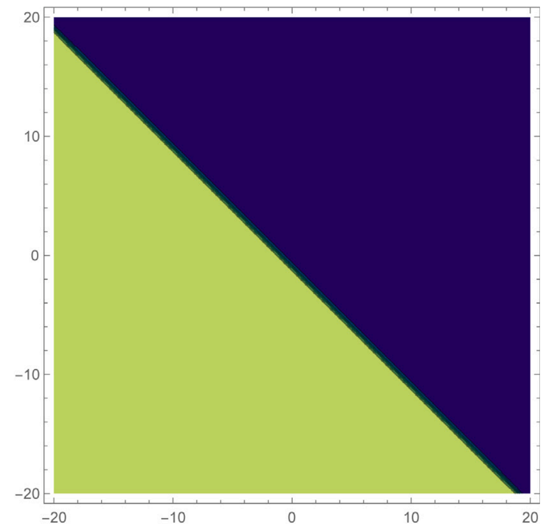


(a) 2D Graph



(b) Contour Graph

Fig. 10. Figs. 6a and 6b represent the $2D$ and contour diagram of Eq. (49) with values $\mu = 1$, $\gamma = 1.75$, $\nu = 1.50$, $k = 1$, $t = 1$, $\delta_2 = 0.50$, and $c = 1$.

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