FRACTIONAL HEAT EQUATION OPTIMIZED BY A CHAOTIC FUNCTION

by

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In this effort, we propose a new fractional differential operator in the open unit disk. The operator is an extension of the Atangana-Baleanu differential operator without singular kernel. We suggest it for a normalized class of analytic functions in the open unit disk. By employing the extended operator, we study the time-2-D space heat equation and optimizing its solution by a chaotic function.

Key words: *fractional calculus, thermal, heat equation, subordination, chaotic, univalent function, analytic function*

Introduction

The class of fractional heat equations is investigated by many researchers. They modeled different physical environments, involving time-space, random walks, non-local transport theory and delayed flux-force associations [1-4]. Moreover, some investigators introduced a general physical introduction to fractional diffusion equations, motivated by Atangana-Baleanu differential operator [5] to simulate heat transfer processes. Optimization by using chaotic functions is used in financial studies. Chaotic functions play a significant role in improving diffusion, symmetry ergodicity and stochasticity of chaos.

In this work, we shall optimize the fractional heat equation type time-2-D space in terms of a special class of chaotic functions, which is used to define a chaotic map [6]. Our method is based on the majorization and subordination theory in the open unit disk [7, 8]. For two analytic functions φ and ψ in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$, we say that φ is majorized by $\psi(\varphi\ll\psi)$ if there is an analytic function $\varpi,|\varpi|<1$ such that $\varphi(z)=\varpi(z)\psi(z)$. Moreover, φ is subordinated to ψ if $\varphi(z)=\psi[\varpi(z)]$, [9].

Preparation

A fractional differential operator for the complex Atangana and Baleanu is defined [10]:

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$$^{C}\Delta^{\nu} g(z) = \frac{\beta(\nu)}{2\pi i(1-\nu)} \int_{\mathbb{D}} f'(\zeta)\Xi_{\nu}[-\mu_{\nu}(z-\zeta)^{\nu}]\mathrm{d}\zeta \tag{1}$$

where $\beta(\nu)$ is normalized by $\beta(0)=\beta(1)=1$ and $\Xi_{\nu}(\omega)$ is the Mittag-Leffler function. Moreover, they introduced the following fractional differential operator:

$$^{R}\Delta^{\nu} g(z) = \frac{\beta(\nu)}{2\pi i(1-\nu)} \frac{\mathrm{d}}{\mathrm{d}z} \int_{\mathbb{D}} f(\zeta)\Xi_{\nu}[-\mu_{\nu}(z-\zeta)^{\nu}]\mathrm{d}\zeta \tag{2}$$

$$\mu_{\nu} = \frac{\nu}{1-\nu}, \quad \nu \in (0,1), \quad \mathbb{D} = [z + re^{i\pi}(z-\ell) : 0 < r < 1]$$

To modify the previous operators, we define a class of analytic functions by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{U}.$ This class is denoted by Λ calling the class of univalent functions and normalized by $f(0) = f'(0) - 1 = 0.$

Definition 1 Let $f \in \Lambda.$ Then the modified operators of (1) and (2) are given by the integrals, respectively:

$$^{C}\Delta_{z}^{\nu} f(z) = \frac{\beta(\nu)}{1-\nu} \int_{0}^{z} f'(\zeta)\Xi_{\nu,\upsilon}(-\mu_{\nu}\zeta^{\nu})\Xi_{\nu}[-\mu_{\nu}(z-\zeta)^{\nu}]\mathrm{d}\zeta \tag{3}$$

and

$$^{R}\Delta_{z}^{\nu} f(z) = \frac{\beta(\nu)}{1-\nu} \frac{\mathrm{d}}{\mathrm{d}z} \int_{0}^{z} f(\zeta)\Xi_{\nu,\upsilon}(-\mu_{\nu}\zeta^{\nu})\Xi_{\nu}[-\mu_{\nu}(z-\zeta)^{\nu}]\mathrm{d}\zeta \tag{4}$$

where υ indicates the power of z.

For example, let $f(z) = z$, then by [11], Theorem 2.4 or [12], Theorem 11.2, we have:

$$\begin{aligned}
^{C}\Delta_{z}^{\nu}(z) &= [\beta(\nu)/1-\nu]\int_{0}^{z} \Xi_{\nu}(-\mu_{\nu}\zeta^{\nu})\Xi_{\nu}[-\mu_{\nu}(z-\zeta)^{\nu}]\mathrm{d}\zeta = \\
&= [\beta(\nu)/1-\nu]z\Xi_{\nu,2}^{2}[-\mu_{\nu}(z)^{\nu}] = \\
&= [\beta(\nu)/1-\nu]z\sum_{k=0}^{\infty}\frac{(2)_k z^k}{k!\Gamma(k\nu+2)}, \quad (\wp)_0 = 1, \quad (\wp)_n = \wp(\wp+1)...(\wp+n-1)
\end{aligned}$$

And in view of [11], Theorem 2.2, we have:

$$\begin{aligned}
^{R}\Delta_{z}^{\nu}(z) &= [\beta(\nu)/1-\nu]\frac{\mathrm{d}}{\mathrm{d}z}\int_{0}^{z} \Xi_{\nu}(-\mu_{\nu}\zeta^{\nu})\Xi_{\nu}[-\mu_{\nu}(z-\zeta)^{\nu}]\zeta\,\mathrm{d}\zeta = \\
&= [\beta(\nu)/1-\nu]\{z^2\Xi_{\nu,3}^{2}[-\mu_{\nu}(z)^{\nu}]\} = \\
&= [\beta(\nu)/1-\nu]\{z\Xi_{\nu,2}^{2}[-\mu_{\nu}(z)^{\nu}]\}
\end{aligned}$$

It is clear that $^{C}\Delta_{z}^{\nu}(z) = {}^{R}\Delta_{z}^{\nu}(z).$ In general, we have:

$$^{C}\Delta_z^{\nu}(z^n) = [\beta(\nu)/1-\nu](n-1)z^n\{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]\}, \quad n>1$$

$$^{R}\Delta_z^{\nu}(z^n) = [\beta(\nu)/1-\nu]z^n\{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]\}$$

We have the following property.

Proposition 1 Consider the operators (3) and (4) for $f \in \Lambda$. Then by letting $\flat(\nu) := [\beta(\nu)/1-\nu]$:

– $^{\mathfrak{C}}\Delta_z^{\nu} f(z) := \dfrac{^{C}\Delta_z^{\nu} f(z)}{\flat(\nu)\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]} \in \Lambda$ and $^{\mathfrak{R}}\Delta_z^{\nu} f(z) := \dfrac{^{R}\Delta_z^{\nu} f(z)}{\flat(\nu)\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]} \in \Lambda$

– $^{\mathfrak{R}}\Delta_z^{\nu} f(z) \ll {}^{\mathfrak{C}}\Delta_z^{\nu} f(z)$

– $^{\mathfrak{R}}\Delta_z^{\nu} f(z) \prec {}^{\mathfrak{C}}\Delta_z^{\nu} f(z)$, provided that $^{\mathfrak{C}}\Delta_z^{\nu} f(z)$, is locally univalent of the first order (like convex function [13]) when $|z| \in (0.28, \sqrt{2}-1)$ or locally univalent of the second order (like the class of univalent functions [13]) when $|z| \in (0.21, 0.3)$.

Proof 1 Let $f \in \Lambda$. Then a direct computation yields:

$$\frac{^{C}\Delta_z^{\nu} f(z)}{\flat(\nu)\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]} = \frac{\flat(\nu)\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]z + \sum_{n=2}^{\infty} a_n \flat(\nu)(n-1)\{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]\}z^n}{\flat(\nu)\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]} =$$

$$= z + \sum_{n=2}^{\infty} a_n (n-1)\left\{\frac{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]}{\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]}\right\} z^n \Rightarrow {}^{\mathfrak{C}}\Delta_z^{\nu} f(z) \in \Lambda$$

Similarly, we obtain $^{\mathfrak{R}}\Delta_z^{\nu} f(z) \in \Lambda$. This completes the first part. For the second part, it is sufficient to prove that, [14], $|^{\mathfrak{R}}\Delta_z^{\nu} f(z)| \le |^{\mathfrak{C}}\Delta_z^{\nu} f(z)|$. A computation yields:

$$\begin{aligned} |^{\mathfrak{R}}\Delta_z^{\nu} f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n \left\{\frac{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]}{\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]}\right\} z^n \right| \le \\ &\le \left| z + \sum_{n=2}^{\infty} a_n (n-1) \left\{\frac{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]}{\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]}\right\} z^n \right| = \\ &= |^{\mathfrak{C}}\Delta_z^{\nu} f(z)| \end{aligned}$$

The last part immediately comes from [14] Corollary 1 and 2, respectively.

Based on *Proposition 1*, we shall focus on $^{\mathfrak{C}}\Delta_z^{\nu} f(z)$.

Heat equation associated with $^{\mathfrak{C}}\Delta_z^{\nu}$

The Koebe function is an extreme function in the field of geometric function theory. To determine the heat equation associated with $^{\mathfrak{C}}\Delta_z^{\nu}$, we deal with the parametric Koebe function of the form:

$$f_\sigma(t,z) = z/(1-tz)^\sigma = \\ = z + \sum_{n=2}^{\infty} \frac{(\sigma)_{n-1}}{(n-1)!} t^{n-1} z^n, \quad t < |z| < 1$$

Then the generalized heat equation is given by:

$$\Psi(t,z) = [{}^{\mathfrak{C}}\Delta_z^\nu f_\sigma(t,z)]_t - [{}^{\mathfrak{C}}\Delta_z^\nu f_\sigma(t,z)]_{zz}, \quad z \in \mathbb{U} \tag{5}$$

Our aim is to optimize the solution of (5) by the chaotic function, fig. 1:

$$\begin{aligned}\sin[z/(1-tz)^\sigma] &= z + \sigma t z^2 + z^3[1/2\sigma(\sigma+1)t^2 - 1/6] + 1/6\sigma t z^4[(\sigma+1)(\sigma+2)t^2 - 3] + \\ &+ 1/120 z^5[-60\sigma^2 t^2 + 5\sigma(\sigma+1)(\sigma+2)(\sigma+3)t^4 - 30\sigma(\sigma+1)t^2 + 1] + \\ &+ 1/120\sigma t z^6[-10(9\sigma^2 + 9\sigma + 2)t^2 + \\ &+ (\sigma^4 + 10\sigma^3 + 35\sigma^2 + 50\sigma + 24)t^4 + 5] + O(z^7) \\ &:= z + \sum_{n=2}^{\infty} \varsigma_n(\sigma,t) z^n, \quad t \le |z| < 1\end{aligned} \tag{6}$$

Note that sin (ω) is univalent in the disk $|z| < \pi/2$, see [15].

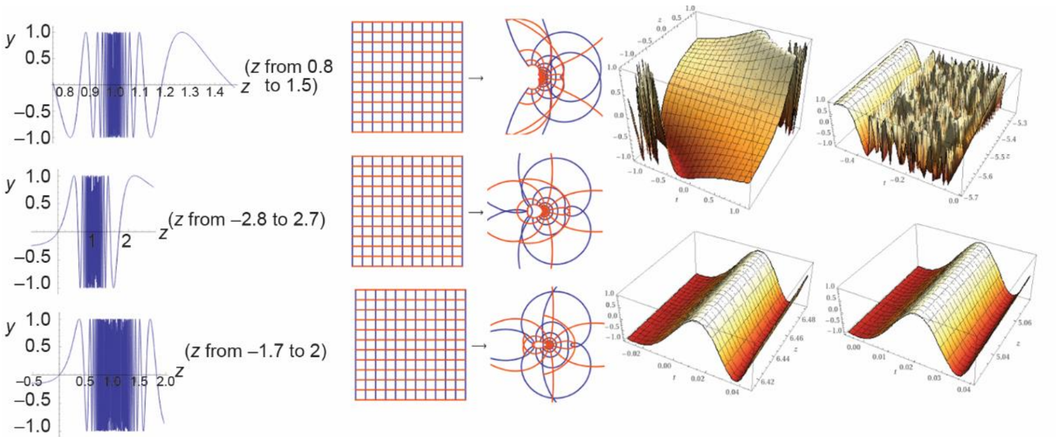

Figure 1. The plot of sin[*z*/(1 – *tz*)$^\sigma$], when *t* = 1, σ = 1, 2, 3; the last two columns are 2-D plot for σ = 1, 2, 3, 4 (*for color image see journal web site*)

Theorem 1 Consider the heat eq. (5). For a small value of $\nu \in [0,1]$, the solution of (5) is optimized by the chaotic function sin[z/(1 – tz)$^\sigma$].

Proof 2 By *Proposition 1*, we indicate that $\Psi(t,0) = 0$. Also, $\nu \to 0$, implies:

$$\left(\frac{\Xi_{\nu,1+n}^2[-\mu_\nu(z)^\nu]}{\Xi_{\nu,2}^2[-\mu_\nu(z)^\nu]}\right) \approx 1, \text{ then}$$

$$\begin{aligned} {}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z) &= z+\sum_{n=2}^{\infty}\frac{(\sigma)_{n-1}}{(n-1)!}(n-1)\left\{\frac{\Xi^2_{\nu,1+n}[-\mu_\nu(z)^\nu]}{\Xi^2_{\nu,2}[-\mu_\nu(z)^\nu]}\right\}t^{n-1}z^n \approx \\ &\approx z+\sum_{n=2}^{\infty}\frac{(\sigma)_{n-1}}{(n-1)!}(n-1)t^{n-1}z^n = \\ &= z+\sum_{n=2}^{\infty}\frac{(\sigma)_{n-1}}{(n-2)!}t^{n-1}z^n = \\ &:= z+\sum_{n=2}^{\infty}\kappa_n(\sigma,t)z^n \end{aligned} \tag{7}$$

To optimize the solution of (5), it is sufficient to show that $|\kappa_n(\sigma,t)|\le|\varsigma_n(\sigma,t)|$. This means that we must find the value of σ whenever $t<1$. A comparison between the coefficients $|\kappa_n(\sigma,t)|$ and $|\varsigma_n(\sigma,t)|$, we obtain the value $0<\sigma\le 1/\sqrt{3}\approx 0.57735...$ This completes the proof.

Corollary 1 Consider the heat equation (5). Then for $\nu,t\to 1$:

$${}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z)\prec\sin[z/(1-tz)^{\sigma}],\quad 0.21<|z|<0.3$$

Proof 3 In view of *Theorem 1*, we have ${}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z)\ll\sin[z/(1-tz)^{\sigma}]$. Since $\sin(\omega)$ is univalent and $[{}^{\mathfrak{C}}\Delta_z^{\nu} f(t,0)]_z=1>0$, then in view of [14] Corollary 2, we conclude that ${}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z)\prec\sin[z/(1-tz)^{\sigma}]$.

Corollary 2 Consider the heat eq. (5). Then for $t\to 1$:

$$[{}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z)]_z\ll\{\sin[z/(1-tz)^{\sigma}]\}_z,\quad |z|\le 0.26794$$

Proof 4 In view of *Theorem 1*, we obtain ${}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z)\ll\sin[z/(1-tz)^{\sigma}]$. According to [14] Theorem 1, where $\sin(\omega)$ is of the second kind of locally univalent function, we get the require assertion.

Remark 1 In view of *Proposition 1* (C) and *Corollary 1*, we confirm that:

$${}^{\mathfrak{R}}\Delta_z^{\nu} f(t,z)\prec{}^{\mathfrak{C}}\Delta_z^{\nu} f(t,z)\prec\sin[z/(1-tz)^{\sigma}],\quad |z|\in(0.21,0.3)$$

Conclusion

We formulated a modified Atangana-Baleanu differential operator of a class of normalized analytic functions in the open unit disk. We presented a new generalization of time-2-D heat equations based on the suggested operator. Analytic solution is indicated by using the chaotic function $\sin[z/(1-tz)^{\sigma}]$. The optimal solution is appeared when $\sigma=0.57735$ (see fig. 2). For future works, one may suggest another class of analytic function in the open unit disk such as meromorhpic, multivalent and harmonic functions.

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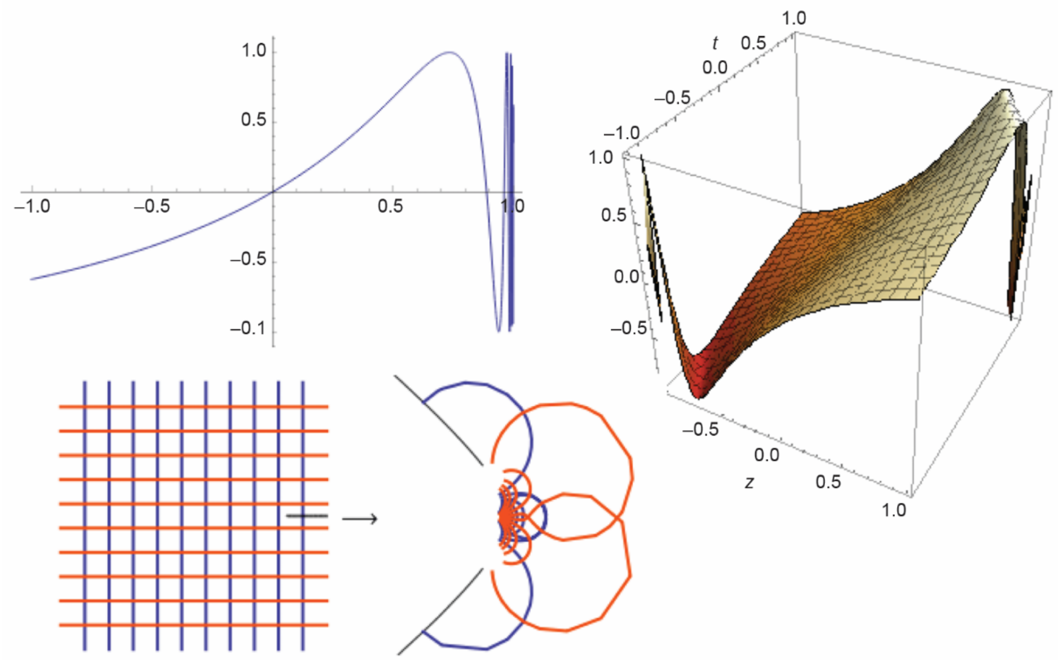

Figure 2. The plot of sin[$z/(1 - tz)^{0.577}$], which is the optimal solution of heat eq. (5)

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