FRACTIONAL MODEL OF SECOND GRADE FLUID INDUCED BY GENERALIZED THERMAL AND MOLECULAR FLUXES WITH CONSTANT PROPORTIONAL CAPUTO

by

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In this research article, the constant proportional Caputo approach of fractional derivative is applied to derive the generalized thermal and molecular profiles for flow of second grade fluid over a vertical plate. The governing equations of the prescribed flow model are reduced to dimensionless form and then solved for temperature, concentration, and velocity via Laplace transform. Further graphs of field variables are sketched for parameter of interest. Comparison between present result and the existing results is also presented graphically.

Key words: *constant proportional Caputo fractional derivative, natural convection, heat transfer, vertical geometry, analytical solution*

Introduction

The fractional calculus is the study of differential operators of the arbitrary order and become a potent too to describe the viscoelastic behavior of the fluids. There are several approaches of fractional differentiation but the most important are Caputo, Caputo-Fabrizio, and constant proportional Caputo (CPC) approaches [1-8]. Hristov [9] investigated the results for transient flow of a non-Newtonian fluid with time space derivative. Hristov [10] discussed the transient heat diffusion with a non-singular fading memory by Cattaneo constitutive equation with Caputo-Fabrizio time fractional derivative. In this research our aims is to find results for second grade fluid flow for generalized thermal and molecular diffusion by applying the CPC fractional derivative [2]. The governing equations of flow model are solve analytically with help of Laplace transform.

Mathematical formulation

Let us consider a flow of an incompressible second grade fluid past a flat surface by subject to the Newtonian heating and constant concentration level at boundary. The flat surface,

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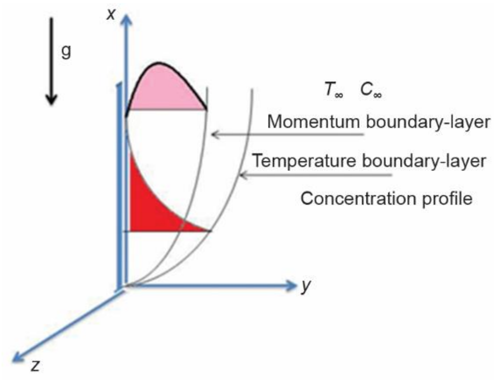

Figure 1. Flow geometry and co-ordinate system

fig. 1, is oriented in co-ordinates system that the y-axis pointed normal to the plane of surface. Initially fluid and its boundary was in equilibrium with all respect at temperature T_0 and at concentration level C_0. For $t = 0^+$ heat transfers from surface to fluid is proportional to the surface temperature T with concentration level C_w and consequently fluid flow along the x-axis only under the bouncy effects of temperature and concentration gradients.

The governing equations under Boussinesq's approximation are reduced to the following PDE [9-11]:

$$\rho Cp\frac{\partial u(\xi,t)}{\partial t}=\mu\frac{\partial^2 u(\xi,t)}{\partial \xi^2}+\mu_1\frac{\partial^3 u(\xi,t)}{\partial t\partial \xi^2}+$$

$$+\rho g\beta_1[T(\xi,t)-T_o]+\rho g\beta_2[C(\xi,t)-C_o] \tag{1}$$

$$\rho C_P\frac{\partial T(\xi,t)}{\partial t}=-\frac{\partial J}{\partial \xi},\quad \xi,t>0 \tag{2}$$

where J is the heat flux, and is given by following classical Fourier's law:

$$J=-k\frac{\partial T(\xi,t)}{\partial \xi},\quad \xi,t>0 \tag{3}$$

where k is the classical thermal conductivity:

$$\frac{\partial C(\xi,t)}{\partial t}=-\frac{\partial q}{\partial \xi},\quad \xi,t>0 \tag{4}$$

where q is the molecular flux, and is given by following classical Fick's law:

$$q=-D\frac{\partial C(\xi,t)}{\partial \xi},\quad \xi,t>0 \tag{5}$$

where D is the classical molecular diffusion. Relevant initial and boundary conditions:

$$u(\xi,0)=0,\quad T(\xi,0)=T_0,\quad C(\xi,0)=C_0,\quad \xi\ge 0 \tag{6}$$

$$u(0,t)=0,\quad \frac{\partial T(\xi,t)}{\partial \xi}|_{\xi=0}=-\frac{h}{k}T(0,t),\quad C(0,t)=C_w,\quad t\ge 0 \tag{7}$$

$$u(\xi,t)\to 0,\quad T(\xi,t)\to T_o,\quad C(\xi,t)\to C_o\quad \text{as}\quad \xi\to\infty \tag{8}$$

Modeling with constant proportional Caputo fractional derivative

To obtain the geometry free model, the following dimensionless relations:

$$\xi^* = \frac{\xi h}{k}, \quad t^* = \frac{tv}{g\left(\frac{k}{h}\right)^2}, \quad u^* = \frac{uv}{g\left(\frac{k}{h}\right)^2}, \quad T^* = \frac{T - T_0}{T_0}, \quad C^* = \frac{C - C_0}{C_w - C_0},$$

$$\mathrm{Sc} = \frac{v}{D}, \quad \mathrm{Pr} = \frac{\mu C_P}{k} \tag{9}$$

are introduced to eq. (1), and obtain the following non-dimensional momentum balance:

$$\frac{\partial}{\partial t}\left[u(\xi,t) - \mu_2 \frac{\partial^2 u(\xi,t)}{\partial \xi^2}\right] = \frac{\partial^2 u(\xi,t)}{\partial \xi^2} + \mathrm{Gr}\,T(\xi,t) + \mathrm{Gm}\,C(\xi,t), \quad t,\xi > 0 \tag{10}$$

where $\mathrm{Gr} = \beta_1 T_\circ$ is thermal Grashof number, $\mathrm{Gm} = \beta_2 C_o$ is mass Grashof number and $\mu_2 = [(\mu_1/g)/(v^2/\rho)](k/h)$ is the dimensionless material parameter for second grade fluid.

Fractional thermal diffusion

Thermal conservation is stated:

$$\rho C_P \frac{\partial T(\xi,t)}{\partial t} = -\frac{\partial J}{\partial \xi}, \quad \xi,t > 0 \tag{11}$$

where J is the thermal flux and C_P – the specific heat of fluid at constant pressure. The generalized thermal flux is stated by fractional form of Fourier's law [10, 11]:

$$J = -k\,{}^{\mathrm{CPC}}D_t^{\beta}\left(\frac{\partial T}{\partial \xi}\right), \quad \xi,t > 0 \tag{12}$$

Plugging the eq. (12) into eq. (11) and using the dimensionless relation from eq. (11) we obtain:

$$\mathrm{Pr}\frac{\partial T(\xi,t)}{\partial t} = {}^{\mathrm{CPC}}D_t^{\beta}\left[\frac{\partial^2 T(\xi,t)}{\partial \xi^2}\right], \quad \xi,t > 0 \tag{13}$$

Fractional molecular diffusion

Molecular conservation is stated:

$$\frac{\partial C(\xi,t)}{\partial t} = -\frac{\partial q}{\partial y}, \quad \xi,t > 0 \tag{14}$$

where q is the molecular flux. The generalized molecular flux is stated by fractional form of Fick's law [11]:

$$q = -D\,{}^{\mathrm{CPC}}D_t^{\gamma}\left(\frac{\partial C}{\partial \xi}\right), \quad \xi,t > 0 \tag{15}$$

Plugging the eq. (15) into eq. (14) and using the dimensionless relation from eq. (11) we obtain:

$$\mathrm{Sc}\frac{\partial C(\xi,t)}{\partial t}={}^{\mathrm{CPC}}D_t^{\gamma}\left[\frac{\partial^2 C(\xi,t)}{\partial \xi^2}\right],\quad \xi,t>0 \tag{16}$$

Associated dimensionless conditions are:

$$u(\xi,0)=0,\quad T(\xi,0)=0,\quad C(\xi,0)=0,\quad \xi\geq 0 \tag{17}$$

$$u(0,t)=0,\quad \frac{\partial T(\xi,t)}{\partial \xi}|_{\xi=0}=-[T(0,t)+1],\quad C(0,t)=1,\quad t>0 \tag{18}$$

$$u(\xi,t)\to 0,\quad T(\xi,t)\to 0,\quad C(\xi,t)\to 0,\quad \text{as}\quad \xi\to\infty \tag{19}$$

Solution of problem

The governing eqs. (10), (13), and (16) of flow model are solved subject to the conditions stated in eqs. (17)-(19) via Laplace transform method and after inverting the Laplace transform the analytical result only for velocity field expressed in terms of series.

Velocity field

$$u(\xi,t)=\mathrm{Gr}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\frac{(-\xi)^i(-\mu_2)^j(\mathrm{Pr}_1)^{\frac{m}{2}}[-L_1(\beta)]^n(\mu_2)^c}{i!\,j!\,m!\,n!\,b!\,c![L_o(\beta)]^{\frac{m}{2}+n+a+b}}\times$$

$$\times\frac{\Gamma\left(\frac{i}{2}+j\right)\Gamma\left(\frac{m}{2}+n\right)\Gamma(a+b)\Gamma(a+1)}{\Gamma\frac{i}{2}\Gamma\frac{m}{2}\Gamma(a)\Gamma(a+1-c)}t^{-\frac{i}{2}-j-\frac{\beta m}{2}+n+a-\beta a+b-c}+$$

$$+\mathrm{Gr}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\frac{(-\xi\sqrt{\mathrm{Pr}_1})^k[-L_1(\beta)]^{l+n+b}(\mathrm{Pr}_1)^{\frac{m}{2}}(\mu_2)^c}{k!\,l!\,m!\,n!\,b!\,c![L_o(\beta)]^{\frac{k}{2}+l+\frac{m}{2}+n+a+b}}\times$$

$$\times\frac{\Gamma\frac{k}{2}\Gamma\left(\frac{m}{2}+n\right)\Gamma(a+b)\Gamma(a+1)}{\Gamma\frac{k}{2}\Gamma\frac{m}{2}\Gamma(a)\Gamma(a+1-c)}t^{-\frac{\beta k}{2}+l-\frac{\beta m}{2}+n+a-\beta a-c}+$$

$$+\mathrm{Gm}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\frac{(-\xi)^i(-\mu_2)^j(\mathrm{Pr}_2)\frac{m}{2}[-L_1(\gamma)]^{n+b}(\mu_2)^c}{i!\,j!\,m!\,n!\,b!\,c![L_o(\gamma)]^{\frac{m}{2}+n+a+b}}\times$$

$$\times\frac{\Gamma\left(\frac{i}{2}+j\right)\Gamma\left(\frac{m}{2}+n\right)\Gamma(a+b)\Gamma(a+1)}{\Gamma\frac{i}{2}\Gamma\frac{m}{2}\Gamma(a)\Gamma(a+1-c)}t^{-\frac{i}{2}-j-\frac{\gamma m}{2}+n+a-\gamma a+b-c}+$$

$$+\mathrm{Gm}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\frac{(-\xi\sqrt{\mathrm{Pr}_2})^k[-L_1(\gamma)]^{l+n+b}(\mathrm{Pr}_2)^{\frac{m}{2}}(\mu_2)^c}{k!l!m!n!b!c![L_o(\gamma)]^{\frac{k}{2}+l+\frac{m}{2}+n+a+b}}\times$$

$$\times\frac{\Gamma\left(\frac{k}{2}+l\right)\Gamma\left(\frac{m}{2}+n\right)\Gamma(a+b)\Gamma(a+1)}{\Gamma\frac{k}{2}\Gamma\frac{m}{2}\Gamma(a)\Gamma(a+1-c)}t^{-\frac{\gamma k}{2}+l-\frac{\gamma m}{2}+n+a-\gamma a-c} \tag{20}$$

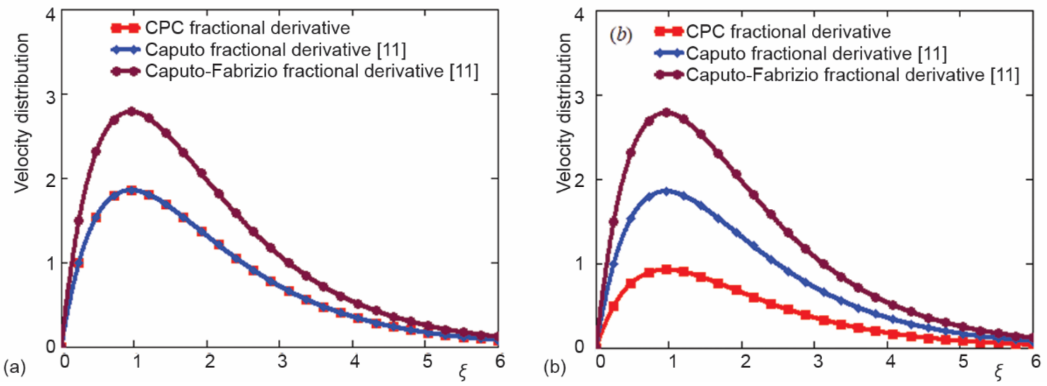

Figure 2. (a) Comparison velocity profile for $L_1 \to 0$ $L_0 \to 1$ and (b) $\beta = \gamma = 0.5$
(for color image see journal web site)

Conclusions

Some useful outcomes of this research in the form of application in transport phenomena are the following.

– Fractional parameter can be used to control the boundary layer of the fluid properties due to constants L_1 and L_0 appearing in CPC lies between 0 and 1.
– For $L_1 \to 0$; $L_0 \to 1$, $\beta \to 1$, and $\gamma \to 1$, CPC reduces to Caputo and presented in fig. 2(a) it validate the present and again depicted better decay nature than Caputo-Fabrizio.
– New fractional operator constant proportional Caputo present a better memory than Caputo and Caputo-Fabrizio for different fractional parameter values and presented in figs. 2(a) and 2(b).

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