fractal and fractional

Article

Fractional Order Mathematical Model of Serial Killing with Different Choices of Control Strategy

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Abstract: The current manuscript describes the dynamics of a fractional mathematical model of serial killing under the Mittag–Leffler kernel. Using the fixed point theory approach, we present a qualitative analysis of the problem and establish a result that ensures the existence of at least one solution. Ulam's stability of the given model is presented by using nonlinear concepts. The iterative fractional-order Adams–Bashforth approach is being used to find the approximate solution. The suggested method is numerically simulated at various fractional orders. The simulation is carried out for various control strategies. Over time, all of the compartments demonstrate convergence and stability. Different fractional orders have produced an excellent comparison outcome, with low fractional orders achieving stability sooner.

Keywords: Adams–Bashforth method; fixed point theory; serial killing; Mittag–Leffler kernel

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1. Introduction

All governments throughout the globe have focused their attention on crimes, specifically serial killings. All the crime is a dignified sociological problem that has been broadly studied in the scientific literature [1]. Every year, billions of dollars are spent around the world to control crime. Different imprisonment and restoration centers are made to handle the crimes of addicted persons [2]. Crime has five types, violent crime, white-collar crime, property crime, organized crime, and consensual crime [3]. One type of violent crime is a serial killer. Serial killer refers to one who commits the crime (murder) three times or more with a cooling-off period between the murders [4,5]. These killers murder their targets and create a very negative impact on society. The killers may or may not be strangers to the target. The killers prefer the areas where they have a recognizable base and the area is being circumscribed by their offenses [6,7].

Gangs cannot be ignored in the history of crimes. Gangs are expanding worldwide and are involved in different types of crimes in rural and urban areas, especially violent crimes. Globally, 48 percent of violent crimes are reportedly due to gangs. Further, 27 percent of crimes are general in small cities, while 66 percent of general crimes in big cities due to gangs. In order to securely carry out their perverted urges and bloodlust, a serial killer joins a criminal organization. As such, the existence of gangs has emerged as the most sensitive public matter [8,9].

The mathematical model provides an examination of the development of criminality and its different effects, including sociological and economic factors. Such types of research suggest possible strategies for decreasing and controlling crimes [10–13]. Recently, several researchers have studied different models for crimes by using ordinary and partial differential equations [2,14–17]. The model used is expressed in the form of the following DEs:

$$
\begin{aligned}
\frac{d\mathbf{S}_h}{dt} &= \Lambda - \frac{b(\beta_1+\beta_2)\mathbf{S}_h\mathbf{G}_h}{\mathbf{N}} - \pi\mathbf{S}_h + \beta_4 u_4 \mathbf{J}_h, \\
\frac{d\mathbf{W}_h}{dt} &= \frac{b\beta_1\mathbf{S}_h\mathbf{G}_h}{\mathbf{N}} - (u_1+\pi)\mathbf{W}_h, \\
\frac{d\mathbf{C}_h}{dt} &= u_1\mathbf{W}_h + \frac{b\beta_2\mathbf{S}_h\mathbf{G}_h}{\mathbf{N}} - (u_2+u_3+\pi)\mathbf{C}_h, \\
\frac{d\mathbf{G}_h}{dt} &= u_2\mathbf{C}_h - (\pi+\beta_3)\mathbf{G}_h + \beta_4(1-u_4)\mathbf{J}_h, \\
\frac{d\mathbf{J}_h}{dt} &= u_3\mathbf{C}_h + \beta_3\mathbf{G}_h - (\beta_4+\pi)\mathbf{J}_h,
\end{aligned}
\tag{1}
$$

In Model (1), the total population is divided into five classes based on stages of addiction to crimes (serial killing). The first class is the susceptible humans $\mathbf{S}_h$ in the population that may or may not have contact with serial killers. Most individuals are in the age of 14 years or greater. The $\mathbf{W}_h$ is the weaponized class, $\mathbf{C}_h$ is the active class of serial killers, $\mathbf{G}_h$ is the class of gang and $\mathbf{J}_h$ is the serial killers detained in jail. The parameters in Model (1) are given with complete descriptions below (Table 1).

Table 1. Meaning of parameters of the model (1).

Notation	Description of the Parameter
Λ	Rate of recruitment in the susceptible class
b	Rate of contact of susceptible and gangs members
u_1	Rate of fraction of weaponized individuals opting to serial killing
β_1	Probability rate of weaponized in susceptible class
β_2	Rate of probability of susceptible to be serial killer
β_3	Rate of arrest members of gang
β_4	Rate of sentence of jail
u_2	Rate of fraction of serial killers moving to gang
u_3	Rate of arrest of serial killer
π	Rate of natural death

Researchers have given fractional calculus a lot of attention, and it has been applied in a variety of disciplines. Researchers have developed mathematical models for a variety of diseases, such as in [18–20].The majority of mathematical models are based on integer-order differential and integral equations. Fractional differential equations (FDEs) have been extensively utilized for the last twenty years to construct models of real processes with a higher degree of precision and accuracy [21,22]. Many scholars have utilized a variety of approaches to analyze fractional order (FO) mathematical models qualitatively; see, for example, [23,24]. Non-linear FDEs are notoriously difficult to solve. To deal with this problem, many mathematicians have constructed a variety of approaches for computing approximate solutions for nonlinear systems [25–28].

To address the shortcomings of the ordinary operator, a variety of fractional order derivatives have been designed [29]. Riemann–Liouville constructed the definition of the fractional derivative (FD). Later on, Caputo subsequently redefined and enhanced the definition of FD. The definition of Caputo FD is based on the singular power-law kernel. The study of real problems using FDs frequently results in singularities that are unsatisfactory for mathematical model dynamics. After many decades, a new FD known as the Caputo–Fabrizio (CF) operator was defined through a non-singular kernel to avoid such a problem [30]. In this operator, there is the kernel's locality problem. To address these limitations, Atangana and Baleanu (AB) [31] introduced a novel type of FD through the nonsingular and nonlocal kernel, which we call the Mittag–Leffler kernel. The new derivative operator was also employed, ensuring that the kernel has neither singularity nor localization. The AB operator has many applications in the applied sciences. For instance, Rahman et al. used the AB operator to analyze the TB disease with incomplete treatment [32]. Ahmad et al. investigated the tumor-immune-vitamins model using the AB fractional operator [33]. The Kawahara equation has been studied under the AB operator by Rahman et al. in [34].

In this paper, we examine a novel fractional mathematical model of serial killing. This model, reported in [35], has not been studied for particular crimes in the sense of ABC FO. This new work provides qualitative and quantitative results regarding the dynamics of serial killing in terms of different strategies. The proposed model under the $\mathbb{ABC}$ operator is expressed as follows:

$$\begin{aligned} {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{S}_h &= \Lambda - \frac{b(\beta_1+\beta_2)\mathbf{S}_h\mathbf{G}_h}{\mathbf{T}} - \pi\mathbf{S}_h + \beta_4 u_4 \mathbf{J}_h, \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{W}_h &= \frac{b\beta_1\mathbf{S}_h\mathbf{G}_h}{\mathbf{T}} - (u_1+\pi)\mathbf{W}_h, \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{C}_h &= u_1\mathbf{W}_h + \frac{b\beta_2\mathbf{S}_h\mathbf{G}_h}{\mathbf{T}} - (u_2+u_3+\pi)\mathbf{C}_h, \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{G}_h &= u_2\mathbf{C}_h - (\pi+\beta_3)\mathbf{G}_h + \beta_4(1-u_4)\mathbf{J}_h, \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{J}_h &= u_3\mathbf{C}_h + \beta_3\mathbf{G}_h - (\beta_4+\pi)\mathbf{J}_h, \end{aligned} \tag{2}$$

along with initial conditions:

$$\begin{aligned} &\mathbf{S}_h(0) = \mathbf{S}_h(0) \geq 0,\ \mathbf{W}_h(0) = \mathbf{W}_h(0) \geq 0,\ \mathbf{C}_h(0) = \mathbf{C}_h(0) \geq 0,\ \mathbf{G}_h(0) = \mathbf{G}_h(0) \geq 0, \\ &\mathbf{J}_h(0) = \mathbf{J}_h(0) \geq 0. \end{aligned} \tag{3}$$

2. Preliminaries

In this section, few fundamental results and definitions are given, which may be useful for readers [31]. Let FD and FI denote the fractional derivative and integral, respectively.

Definition 1. *The $\mathbb{ABC}$ FD of order $0 < \wp \leq 1$ for a function $\mathscr{X}(t) \in \mathcal{H}^1[0,T]$ is given as:*

$$ {}^{\mathbb{ABC}}D_t^{\wp}(\mathscr{X}(t)) = \frac{\mathbb{N}(\wp)}{1-\wp}\int_0^t \mathbf{E}_{\wp}\left[\frac{-\wp}{1-\wp}\Big(t-\eta\Big)^{\wp}\right]\frac{d}{d\eta}\mathscr{X}(\eta)d\eta, \tag{4}$$

where $\mathbb{N}(\wp)$ denotes the normalization function such that $\mathbb{N}(0) = \mathbb{N}(1) = 1$, and $\mathbf{E}_{\wp}$ is given by:

$$\mathbf{E}_{\wp}(y) = \sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\wp k+1)},$$

where $\Gamma(.)$ denotes the Gamma function and $Re(\wp) > 0$.

Definition 2. *The $\mathbb{AB}$ fractional integration of $\mathscr{X} \in L^1(0,T)$ is defined as:*

$$ {}^{\mathbb{ABC}}I_t^{\wp}\mathscr{X}(t) = \frac{1-\wp}{\mathbb{N}(\wp)}\mathscr{X}(t) + \frac{\wp}{\mathbb{N}(\wp)}\frac{1}{\Gamma(\wp)}\int_0^t (t-\eta)^{(\wp-1)}\mathscr{X}(\eta)d\eta, \quad t > 0. \tag{5}$$

Lemma 1. *Let us consider:*

$$\begin{aligned} {}^{\mathbb{ABC}}D_t^{\wp}\mathscr{X}(t) &= \Psi(t), \\ \mathscr{X}(0) &= \mathscr{X}_0, \end{aligned} \tag{6}$$

then:

$$\mathscr{X}(t) = \mathscr{X}_0 + \frac{1-\wp}{\mathbb{N}(\wp)}\Psi(t) + \frac{\wp}{\mathbb{N}(\wp)}\frac{1}{\Gamma(\wp)}\int_0^t (t-\eta)^{\wp-1}\Psi(\eta)d\eta. \tag{7}$$

Proof. Applying fractional integration to both side of Equation (6), we have:

$$ {}_0^{\mathbb{ABC}}\mathbf{I}_t\left[{}^{\mathbb{ABC}}D_t^{\wp}\mathscr{X}(t)\right] = {}_0^{\mathbb{ABC}}\mathbf{I}_t\Big[\Psi(t)\Big],$$

we get:

$$\mathscr{X}(t) - \mathscr{X}(0) = \frac{1-\wp}{\mathbb{N}(\wp)}\Psi(t) + \frac{\wp}{\mathbb{N}(\wp)}\frac{1}{\Gamma(\wp)}\int_0^t (t-\eta)^{(\wp-1)}\Psi(\eta)d\eta,$$

or:

$$\mathscr{X}(t) = \mathscr{X}(0) + \frac{1-\wp}{\mathbb{N}(\wp)}\Psi(t) + \frac{\wp}{\mathbb{N}(\wp)}\frac{1}{\Gamma(\wp)}\int_0^t (t-\eta)^{(\wp-1)}\Psi(\eta)d\eta.$$

Hence proved. □

Theorem 1. *Assume O denotes a Banach space and $\mathtt{Q} \subset O$ be a bounded and convex closed set. Let $\psi : \mathtt{Q} \to \mathtt{Q}$ be a continuous mapping. If $\psi\mathtt{D} \subset O$ and $\psi\mathtt{D}$ is relatively compact, then at least one fixed point will be exists of the operator in $\mathtt{D}$.*

3. Qualitative Study of the Proposed Model

3.1. Existence Theory

Utilizing the popular theorems of fixed point theory, we present the existence, uniqueness of solution, and stability results of the studied Model (2) in this portion of the article. We rewrite the model under consideration in the following way to get the required results.

$$\begin{cases} {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{S}_h(t) = \mathbb{G}_1(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h), \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{W}_h(t) = \mathbb{G}_2(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h), \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{C}_h(t) = \mathbb{G}_3(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h), \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{G}_h(t) = \mathbb{G}_4(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h), \\ {}^{\mathbb{ABC}}D_t^{\wp}\mathbf{J}_h(t) = \mathbb{G}_5(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h), \end{cases} \tag{8}$$

where:

$$\begin{cases} \mathbb{G}_1(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h) = \Lambda - \dfrac{b(\beta_1+\beta_2)\mathbf{S}_h\mathbf{G}_h}{\mathbf{T}} - \pi\mathbf{S}_h + \beta_4 u_4 \mathbf{J}_h, \\ \mathbb{G}_2(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h) = \dfrac{b\beta_1\mathbf{S}_h\mathbf{G}_h}{\mathbf{T}} - (u_1+\pi)\mathbf{W}_h, \\ \mathbb{G}_3(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h) = u_1\mathbf{W}_h + \dfrac{b\beta_2\mathbf{S}_h\mathbf{G}_h}{\mathbf{T}} - (u_2+u_3+\pi)\mathbf{C}_h, \\ \mathbb{G}_4(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h) = u_2\mathbf{C}_h - (\pi+\beta_3)\mathbf{G}_h + \beta_4(1-u_4)\mathbf{J}_h, \\ \mathbb{G}_5(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h) == u_3\mathbf{C}_h + \beta_3\mathbf{G}_h - (\beta_4+\pi)\mathbf{J}_h. \end{cases} \tag{9}$$

Next, we express Models (2) and (3) as:

$$\begin{aligned} {}^{\mathbb{ABC}}D_t^{\wp}\Psi(t) &= \Omega(t, \Psi(t)), \\ \Psi(0) &= \Psi_0, \end{aligned} \tag{10}$$

where:

$$\begin{cases} \Psi := \left(\mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h\right)^T, \\ \Psi(0) := \left(\mathbf{S}_h(0), \mathbf{W}_h(0), \mathbf{C}_h(0), \mathbf{G}_h(0), \mathbf{J}_h(0)\right)^T, \\ \Omega(t, \Psi(t)) := \mathbb{G}_i\left(t, \mathbf{S}_h, \mathbf{W}_h, \mathbf{C}_h, \mathbf{G}_h, \mathbf{J}_h\right)^T, \quad i = 1,2,3,4,5. \end{cases} \tag{11}$$

Note that $(.)^T$ presents the transpose of vector. Using Lemma 1, the system (10) converts to:

$$\Psi(t) = \Psi_0 + \frac{1-\wp}{\mathbb{N}(\wp)}\Omega(t, \Psi(t)) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\int_0^t (t-\eta)^{\wp-1}\Omega(\eta, \Psi(\eta))d\eta. \tag{12}$$

Let $\Upsilon = C([0,T], \mathbf{R})$. Let us define a Banach space $\digamma = (\Upsilon^5, \|\Psi\|)$ with norm $\|\Psi\| = \sup_{t\in[0,T]}\left(|\mathbf{S}_h| + |\mathbf{W}_h| + |\mathbf{C}_h| + |\mathbf{G}_H| + |\mathbf{J}_h|\right)$.

Now, we explore the existence results for the proposed Models (2) and (3) with the help of the "Schauder's fixed point theorem".

Theorem 2. *Let $\Omega \in F$ be a continuous function and $\exists\ \mathcal{M} > 0, \ni |\Omega(t, \Psi(t))| \leq \mathcal{M}(1 + |\Psi|), \forall$ $0 \leq t \leq T$ and $\Psi \in F$. If:*

$$\nabla_1 = \left(\frac{(1-\wp)\Gamma(\wp)\mathcal{M} + \mathcal{M}T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp)} \right) < 1, \tag{13}$$

then the model under consideration has at least one solution.

Proof. We define $\mathbf{Y} : F \to F$ as:

$$(\mathbf{Y}\Psi)(t) = \Psi_0 + \frac{1-\wp}{\mathbb{N}(\wp)}\Omega(t, \Psi(t)) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)} \int_0^t (t-\eta)^{\wp-1}\Omega(\eta, \Psi(\eta))d\eta. \tag{14}$$

Let $B_{\varpi} = \{\Psi \in \Omega : \|\Psi \leq \varpi,\ \ \varpi > 0\|\}$ be convex and closed ball with $\varpi \geq \frac{\nabla_2}{1-\nabla_1}$, where:

$$\nabla_2 = |\Psi_0| + \frac{1-\wp}{\mathbb{N}(\wp)}\mathcal{M} + \frac{T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp)}\mathcal{M}. \tag{15}$$

First, we have to show that $(\mathbf{Y}B_{\varpi}) \subset B_{\varpi}, \forall\ 0 \leq t \leq T$. One can get:

$$\begin{aligned} |(\mathbf{Y}\Psi)(t)| &\leq |\Psi_0| + \frac{1-\wp}{\mathbb{N}(\wp)}|\Omega(t, \Psi(t))| + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)} \int_0^t (t-\eta)^{\wp-1}|\Omega(\eta, \Psi(\eta))|d\eta, \\ &\leq |\Psi_0| + \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{N}(1 + |\Psi(t)|) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)} \int_0^t (t-\eta)^{\wp-1}\mathbb{N}(1 + |\Psi(t)|)d\eta. \end{aligned} \tag{16}$$

Again since $\Psi \in B_{\varpi}$, one may write:

$$\begin{aligned} \|(\mathbf{Y}\Psi)(t)\| &\leq |\Psi_0| + \frac{1-\wp}{\mathbb{N}(\wp)}\mathcal{M}(1 + \|\Psi(t)\|) + \frac{T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp)}\mathcal{M}(1 + \|\Psi(t)\|), \\ &\leq |\Psi_0| + \frac{1-\wp}{\mathbb{N}(\wp)}\mathcal{M} + \frac{T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp)}\mathcal{M} + \left[\frac{1-\wp}{\mathbb{N}(\wp)}\mathcal{M} + \frac{T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp)}\mathcal{M}\right]\rho, \\ &\leq \nabla_2 + \nabla_1\varpi \leq \varpi. \end{aligned}$$

Hence $(\mathbf{Y}B_{\varpi}) \subset B_{\varpi}$. Now, our next task is to verify that $\mathbf{Y}$ is continuous. Let $\{\Psi_n\}$ be a sequence $\ni \Psi_n \to \Psi$ in B_{ϖ} as $n \to \infty$. Now, one may get:

$$\begin{aligned} \left|(\mathbf{Y}\Psi_n)(t) - (\mathbf{Y}\Psi)(t)\right| &\leq \frac{-\wp+1}{\mathbb{N}(\wp)}\left|\Omega(t, \Psi_n(t)) - \Omega(t, \Psi(t))\right| + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)} \times \\ &\quad \int_0^t (t-\eta)^{\wp-1}\left|\Omega(\eta, \Psi_n(\eta)) - \Omega(\eta, \Psi(\eta))\right|d\eta \\ &\leq \frac{1-\wp}{\mathbb{N}(\wp)}\|\Omega(t, \Psi_n(t)) - \Omega(t, \Psi(t))\| + \\ &\quad \frac{T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp)}\|\Omega(\eta, \Psi_n(\eta)) - \Omega(\eta, \Psi(\eta))\|. \end{aligned}$$

It follows that:

$$\|(\mathbf{Y}\Psi_n) - (\mathbf{Y}\Psi)\| \to 0 \quad \text{as} \quad n \to \infty.$$

This verifies the continuity of $\mathbf{Y}$ in B_{ϖ}. Now, we have to show that $\mathbf{Y}B_{\varpi}$ is a relatively compact. Since, we have proved that $(\mathbf{Y}B_{\varpi}) \subset B_{\varpi}$, its easy to show the uniform boundedness of $(\mathbf{Y}B_{\varpi})$. Finally, we show the equi-continuity of operator $\mathbf{Y}$ on B_{ϖ}. To do so let $\Psi \in B_{\varpi}$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have:

$$\begin{aligned} \|(\mathbf{Y}\Psi)(t_2) - (\mathbf{Y}\Psi)(t_1)\| &\leq \frac{1-\wp}{\mathbb{N}(\wp)}|\Omega(t_2, \Psi(t_2)) - \Omega(t_1, \Psi(t_1))| \\ &+ \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\left|\left[\int_0^{t_2}(t_2-\eta)^{\wp-1} - \int_0^{t_1}(t_1-\eta)^{\wp-1}\right]\Omega(\eta, \Psi(\eta))d\eta\right| \\ &\leq \frac{1-\wp}{\mathbb{N}(\wp)}|\Omega(t_2, \Psi(t_2)) - \Omega(t_1, \Psi(t_1))| + \frac{\wp}{\mathbb{N}(\wp)}\frac{\mathcal{M}(1 + \|\Psi\|)}{\Gamma(\wp+1)}(t_2^{\wp} - t_1^{\wp}). \end{aligned}$$

Apparently, the right side $\|\mathbf{Y}\Psi(t_2) - \mathbf{Y}\Psi(t_1)\| \to 0$ as $t_2 \to t_1$. By "Arzela-Ascoli theorem", $(\mathbf{Y}B_{\varpi})$ is a relatively compact operator and so $\mathbf{Y}$ is completely continuous. By Theorem 1, at least one fixed point of the operator will exists. Consequently, the fractional order considered model has at least one solution. □

3.2. Stability Result

In this portion, we discuss the Ulam's type stability of the given Model (2) for the concept of stability through Ulam–Hyer's techniques by considering a perturbation term $\Omega(t)$, which mostly depend on the solution of the system along with initial condition $\Omega(0) = 0$ as follows:

- $|\Omega(t)| \leq \epsilon$ for $\epsilon > 0$.
- ${}_0^{\mathbb{ABC}}\mathbf{D}_t^{\wp}\Psi(t) = y(t, \Psi) + \Omega(t)$.

Lemma 2. *The solution of:*

$$\begin{aligned} {}_0^{\mathbb{ABC}}\mathbf{D}_t^{\wp}\Psi(t) &= y(t, \Psi(t)) + \Omega(t),\ 0 < \wp \leq 1 \quad t \in [0, T], \\ \Psi(0) &= \Psi_0, \end{aligned} \tag{17}$$

satisfies:

$$\begin{aligned} \Bigg|\Psi(t) &- \left(\Psi_0(t) + [y(t, \Psi(t)) + \Omega(t)]\frac{(1-\wp)}{\mathbb{N}(\wp)} + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\int_0^t (t-\eta)^{\wp-1}[y(\eta, \Psi(\eta)) + \Omega(\eta)]d\eta\right)\Bigg|, \\ &\leq \frac{(1-\wp)\Gamma(\wp+1) + \wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}\epsilon = \Lambda\epsilon, \end{aligned} \tag{18}$$

where:

$$\Lambda = \frac{(1-\wp)\Gamma(\wp+1) + \wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}.$$

Theorem 3. *Let $\Omega \in F$ and $\exists\ \mathbf{X} > 0 \ni \big|\Omega(t, \Psi) - \Omega(t, \widetilde{\Psi})\big| \leq \mathbf{X}\big|\Psi - \widetilde{\Psi}\big|, \forall\ t \in [0, T]$ and $\Psi \in F$ with:*

$$1 > \frac{(1-\wp)\Gamma(\wp+1)\mathbf{X} + \wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}.$$

Let Ψ and $\widetilde{\Psi}$ be the solutions for model (10) and:

$$\begin{cases} {}^{\mathbb{ABC}}\mathbb{D}^{\wp}\widetilde{\Psi}(t) = \Omega(t, \widetilde{\Psi}(t)), \\ \widetilde{\Psi}(0) = \Psi_0 \geq 0, \end{cases} \tag{19}$$

where:

$$\begin{cases} \widetilde{\Psi} = (\widetilde{\mathbf{S}}_H, \widetilde{\mathbf{W}}_H, \widetilde{\mathbf{C}}_H, \widetilde{\mathbf{G}}_H, \widetilde{\mathbf{X}}_H)^T \\ \Psi_0 = \Big(\mathbf{S}_h(0), \mathbf{W}_h(0), \mathbf{C}_h(0), \mathbf{G}_h(0), \mathbf{X}_h(0)\Big)^T \\ \Omega(t, \widetilde{\Psi}(t)) = \mathbb{G}_i(\widetilde{\mathbf{S}}_H, \widetilde{\mathbf{W}}_H, \widetilde{\mathbf{C}}_H, \widetilde{\mathbf{G}}_H, \widetilde{\mathbf{X}}_H)^T, \quad i = 1, 2, 3, 4, 5. \end{cases} \tag{20}$$

Then,

$$\|\Psi - \widetilde{\Psi}\| \leq \left[1 - \frac{(1-\wp)\Gamma(\wp+1)\mathbf{X} + \wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}\right]^{-1}. \tag{21}$$

Proof. Since the equivalent form of the system (19) is:

$$\widetilde{\Psi}(t) = \Psi_0 + \epsilon + \frac{1-\wp}{\mathbb{N}(\wp)}\Omega(t, \widetilde{\Psi}(t)) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\int_0^t (t-\eta)^{\wp-1}\Omega(\eta, \widetilde{\Psi}(\eta))d\eta, \tag{22}$$

Now, $\forall\ t \in [0, T]$, consider:

$$\begin{aligned}|\Psi(t)-\widetilde{\Psi}(t)| &\leq |\varepsilon|+\frac{1-\wp}{\mathbb{N}(\wp)}\big|\Omega(t,\Psi(t))-\Omega(t,\widetilde{\Psi}(t))\big|+\frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\\ &\times \int_0^t (t-\eta)^{\wp-1}\big|\Omega(\eta,\Psi(\eta))-\Omega(\eta,\widetilde{\Psi}(\eta))\big|d\eta,\\ &\leq |\varepsilon|+\frac{1-\wp}{\mathbb{N}(\wp)}\mathbf{J}\big|\Psi(t)-\widetilde{\Psi}(t)\big|+\frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\\ &\times \int_0^t (t-\eta)^{\wp-1}\mathbf{X}\big|\Psi(\eta)-\widetilde{\Psi}(\eta)\big|d\eta,\\ &\leq |\varepsilon|+\left[\frac{1-\wp}{\mathbb{N}(\wp)}+\frac{\wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}\right]\mathbf{X}\|\Psi-\widetilde{\Psi}\|.\end{aligned}$$

We have:

$$\|\Psi-\widetilde{\Psi}\| \leq |\varepsilon|+\left[\frac{(1-\wp)\Gamma(\wp)+\wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}\right]\mathcal{G}\|\Psi-\widetilde{\Psi}\|.$$

Hence:

$$\|\Psi-\widetilde{\Psi}\| \leq \left[1-\frac{(1-\wp)\Gamma(\wp+1)\mathbf{X}+\wp T^{\wp}}{\mathbb{N}(\wp)\Gamma(\wp+1)}\right]^{-1}|\varepsilon|. \tag{23}$$

This finishes the proof. □

4. Numerical Scheme

The numerical solutions of the proposed systems (2) and (3) are investigated in this part of the paper. The numerical results are produced using the suggested technique. One may write the considered model by using FI as:

$$\begin{cases}\mathbf{S}_h(t)-\mathbf{S}_h(0)={}^{\mathbb{AB}}\ \mathbb{I}_0^{\wp}\mathbb{K}_1(t,\mathbf{S}_h(t)),\\ \mathbf{W}_h(t)-\mathbf{W}_h(0)={}^{\mathbb{AB}}\ \mathbb{I}_0^{\wp}\mathbb{K}_2(t,\mathbf{W}_h(t)),\\ \mathbf{C}_h(t)-\mathbf{C}_h(0)={}^{\mathbb{AB}}\ \mathbb{I}_0^{\wp}\mathbb{K}_3(t,\mathbf{C}_h(t)),\\ \mathbf{G}_h(t)-\mathbf{G}_h(0)={}^{\mathbb{AB}}\ \mathbb{I}_0^{\wp}\mathbb{K}_4(t,\mathbf{G}_h(t)),\\ \mathbf{J}_h(t)-\mathbf{J}_h(0)={}^{\mathbb{AB}}\ \mathbb{I}_0^{\wp}\mathbb{K}_5(t,\mathbf{J}_J(t)).\end{cases} \tag{24}$$

We have:

$$\begin{aligned}\mathbf{S}_h(t)-\mathbf{S}_h(0)&=\frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_1(\mathbf{S}_h(t),t)+\frac{\wp}{\Gamma(\wp)\mathbb{N}(\wp)}\int_0^t(t-\eta)^{\wp-1}\mathbb{K}_1(\mathbf{S}_h(\eta),\eta)d\eta,\\ \mathbf{W}_h(t)-\mathbf{W}_h(0)&=\frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_2(\mathbf{W}_h(t),t)+\frac{\wp}{\Gamma(\wp)\mathbb{N}(\wp)}\int_0^t(t-\eta)^{\wp-1}\mathbb{K}_2(\mathbf{W}_h(\eta),\eta)d\eta,\\ \mathbf{C}_h(t)-\mathbf{C}_h(0)&=\frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_3(\mathbf{C}_h(t),t)+\frac{\wp}{\Gamma(\wp)\mathbb{N}(\wp)}\int_0^t(t-\eta)^{\wp-1}\mathbb{K}_3(\mathbf{C}_h(\eta),\eta)d\eta,\\ \mathbf{G}_h(t)-\mathbf{G}_h(0)&=\frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_4(\mathbf{G}_h(t),t)+\frac{\wp}{\Gamma(\wp)\mathbb{N}(\wp)}\int_0^t(t-\eta)^{\wp-1}\mathbb{K}_4(\mathbf{G}_h(\eta),\eta)d\eta,\\ \mathbf{J}_h(t)-\mathbf{J}_h(0)&=\frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_5(\mathbf{J}_h(t),t)+\frac{\wp}{\Gamma(\wp)\mathbb{N}(\wp)}\int_0^t(t-\eta)^{\wp-1}\mathbb{K}_5(\mathbf{J}_h(\eta),\eta)d\eta.\end{aligned} \tag{25}$$

To derive numerical results, put $t=t_{\wp+1}$ for $\wp=0,1,2,\ldots,$ into the system (25), we have:

$$\begin{aligned}
&\mathbf{S}_h(t_{\wp+1}) - \mathbf{S}_h(0) = \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_1(\mathbf{S}_h(t_\wp), t_\wp) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{\wp}\int_{t_\varkappa}^{t_\varkappa+1}(t_{\wp+1}-\eta)^{\wp-1}\mathbb{K}_1(\mathbf{S}_h(\eta),\eta)d\eta,\\
&\mathbf{W}_h(t_{\wp+1}) - \mathbf{W}_h(0) = \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_2(\mathbf{W}_h(t_\wp), t_\wp) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{\wp}\int_{t_\varkappa}^{t_\varkappa+1}(t_{\wp+1}-\eta)^{\wp-1}\mathbb{K}_2(\mathbf{W}_h(\eta),\eta)d\eta,\\
&\mathbf{C}_h(t_{\wp+1}) - \mathbf{C}_h(0) = \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_3(\mathbf{C}_h(t_\wp), t_\wp) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{\wp}\int_{t_\varkappa}^{t_\varkappa+1}(t_{\wp+1}-\eta)^{\wp-1}\mathbb{K}_3(\mathbf{C}_h(\eta),\eta)d\eta, \quad (26)\\
&\mathbf{G}_h(t_{\wp+1}) - \mathbf{G}_h(0) = \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_4(\mathbf{G}_h(t_\wp), t_\wp) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{\wp}\int_{t_\varkappa}^{t_\varkappa+1}(t_{\wp+1}-\eta)^{\wp-1}\mathbb{K}_4(\mathbf{G}_h(\eta),\eta)d\eta,\\
&\mathbf{J}_h(t_{\wp+1}) - \mathbf{J}_h(0) = \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_5(\mathbf{J}_h(t_\wp), t_\wp) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{\wp}\int_{t_\varkappa}^{t_\varkappa+1}(t_{\wp+1}-\eta)^{\wp-1}\mathbb{K}_5(\mathbf{J}_h(\eta),\eta)d\eta.
\end{aligned}$$

Now, we approximate the functions $\mathbb{K}_1\big(\mathbf{S}_h(\eta),\eta\big), \mathbb{K}_2\big(\mathbf{W}_h(\eta),\eta\big)$, $\mathbb{K}_3\big(\mathbf{C}_h(\eta),\eta\big), \mathbb{K}_4\big(\mathbf{G}_h(\eta),\eta\big)$ and $\mathbb{K}_5\big(\mathbf{J}_h(\eta),\eta\big)$ on the interval $[t_\varkappa, t_{\varkappa+1}]$ by using two points interpolation, we have:

$$\begin{cases}
\mathbb{K}_1(\mathbf{S}_h(\eta),\eta) \cong \frac{\mathbb{K}_1(\mathbf{S}_h(t_\varkappa),t_\varkappa)}{\hbar}(t-t_{\varkappa-1}) + \frac{\mathbb{K}_1(\mathbf{S}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}(t-t_\varkappa),\\
\mathbb{K}_2(\mathbf{W}_h(\eta),\eta) \cong \frac{\mathbb{K}_2(\mathbf{W}_h(t_\varkappa),t_\varkappa)}{\hbar}(t-t_{\varkappa-1}) + \frac{\mathbb{K}_2(\mathbf{W}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}(t-t_\varkappa),\\
\mathbb{K}_3(\mathbf{C}_h(\eta),\eta) \cong \frac{\mathbb{K}_3(\mathbf{C}_h(t_\varkappa),t_\varkappa)}{\hbar}(t-t_{\varkappa-1}) + \frac{\mathbb{K}_3(\mathbf{C}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}(t-t_\varkappa),\\
\mathbb{K}_4(\mathbf{G}_h(\eta),\eta) \cong \frac{\mathbb{K}_4(\mathbf{G}_h(t_\varkappa),t_\varkappa)}{\hbar}(t-t_{\varkappa-1}) + \frac{\mathbb{K}_4(\mathbf{G}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}(t-t_\varkappa),\\
\mathbb{K}_5(\mathbf{J}_h(\eta),\eta) \cong \frac{\mathbb{K}_5(\mathbf{J}_h(t_\varkappa),t_\varkappa)}{\hbar}(t-t_{\varkappa-1}) + \frac{\mathbb{K}_5(\mathbf{J}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}(t-t_\varkappa).
\end{cases} \quad (27)$$

We get:

$$\begin{aligned}
\mathbf{S}_h(t_{g+1}) &= \mathbf{S}_h(0) + \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_1(\mathbf{S}_h(t_g),t_g) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{g}\bigg(\frac{\mathbb{K}_1(\mathbf{S}_h(t_\varkappa),t_\varkappa)}{\hbar}I_{\varkappa-1,\wp}\\
&\quad + \frac{\mathbb{K}_1(\mathbf{S}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}I_{\varkappa,\wp}\bigg),\\
\mathbf{W}_h(t_{g+1}) &= \mathbf{W}_h(0) + \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_2(\mathbf{W}_h(t_g),t_g) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{g}\bigg(\frac{\mathbb{K}_2(\mathbf{W}_h(t_\varkappa),t_\varkappa)}{\hbar}I_{\varkappa-1,\wp}\\
&\quad + \frac{\mathbb{K}_2(\mathbf{W}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}I_{\varkappa,\wp}\bigg),\\
\mathbf{C}_h(t_{g+1}) &= \mathbf{C}_h(0) + \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_3(\mathbf{C}_h(t_g),t_g) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{g}\bigg(\frac{\mathbb{K}_3(\mathbf{C}_h(t_\varkappa),t_\varkappa)}{\hbar}I_{\varkappa-1,\wp}\\
&\quad + \frac{\mathbb{K}_3(\mathbf{C}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}I_{\varkappa,\wp}\bigg), \qquad (28)\\
\mathbf{G}_h(t_{g+1}) &= \mathbf{G}_h(0) + \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_4(\mathbf{G}_h(t_g),t_g) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{g}\bigg(\frac{\mathbb{K}_4(\mathbf{G}_h(t_\varkappa),t_\varkappa)}{\hbar}I_{\varkappa-1,\wp}\\
&\quad + \frac{\mathbb{K}_4(\mathbf{G}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}I_{\varkappa,\wp}\bigg),\\
\mathbf{J}_h(t_{g+1}) &= \mathbf{J}_h(0) + \frac{1-\wp}{\mathbb{N}(\wp)}\mathbb{K}_5(\mathbf{J}_h(t_g),t_g) + \frac{\wp}{\mathbb{N}(\wp)\Gamma(\wp)}\sum_{\varkappa=0}^{g}\bigg(\frac{\mathbb{K}_5(\mathbf{J}_h(t_\varkappa),t_\varkappa)}{\hbar}I_{\varkappa-1,\wp}\\
&\quad + \frac{\mathbb{K}_5(\mathbf{J}_h(t_{\varkappa-1}),t_{\varkappa-1})}{\hbar}I_{\varkappa,\wp}\bigg),
\end{aligned}$$

where:

$$I_{\varkappa-1,\wp} = \int_{t_\varkappa}^{t_{\varkappa+1}}(t-t_{\varkappa-1})(t_{g+1}-t)^{\wp-1}dt,$$

and:

$$I_{\varkappa,\wp} = \int_{t_\varkappa}^{t_{\varkappa+1}} (t - t_\varkappa)(t_{g+1} - t)^{\wp-1} dt.$$

Now, we simplify the integrals $I_{\varkappa-1,\wp}$ and $I_{\varkappa,\wp}$ as follows:

$$\begin{aligned} I_{-1+\varkappa,\wp} &= -\frac{1}{\wp}\Big[(-t_{-1+\varkappa} + t_{\varkappa+1})(t_{g+1} - t_{\varkappa+1})^{\wp} - (-t_{-1+\varkappa} + t_\varkappa)(-t_\varkappa + t_{g+1})^{\wp}\Big] \\ &\quad -\frac{1}{\wp(-1+\wp)}\Big[(-t_{1+\varkappa} + t_{g+1})^{\wp+1} - (-t_\varkappa + t_{g+1})^{1+\wp}\Big], \\ I_{\varkappa,\wp} &= -\frac{1}{\wp}\Big[(t_{1+\varkappa} - t_\varkappa)(t_{1+g} - t_{1+\varkappa})^{\wp}\Big] - \frac{1}{\wp(-1+\wp)}\Big[(-t_{1+\varkappa} + t_{1+g})^{1+\wp} - (-t_\varkappa + t_{1+g})^{1+\wp}\Big]. \end{aligned}$$

By setting $t_\varkappa = i\hbar$, one can easily deduce:

$$I_{\varkappa-1,\wp} = -\frac{\hbar^{\wp+1}}{\wp(\wp+1)}\Big[(-\varkappa+g+1)^{\wp}(-\varkappa+g+2+\wp) \tag{29}$$

$$-(-\varkappa+g)^{\wp}(-\varkappa+2\wp+g+2)\Big], \tag{30}$$

and:

$$I_{\varkappa,\wp} = \frac{\hbar^{\wp+1}}{\wp(\wp+1)}\Big[(g+1-\varkappa)^{\wp+1} - (g-\varkappa)^{\wp}(\wp+1+g-\varkappa)\Big]. \tag{31}$$

Plugging Equations (29) and (31) into (28), one can get:

$$\begin{aligned} \mathbf{S}_h(t_{g+1}) &= \mathbf{S}_h(t_0) + \frac{(1-\wp)}{\mathbb{N}(\wp)}[\mathbb{K}_1(\mathbf{S}_h(t_g), t_g)] + \frac{\wp}{\mathbb{N}(\wp)}\sum_{\varkappa=0}^{g}\Bigg(\frac{\mathbb{K}_1(\mathbf{S}_h(t_g), t_g)}{\wp(\wp+2)} \\ &\quad \times\hbar^{\wp}\Big[(-\varkappa+g+1)^{\wp}(g-\varkappa+\wp+2) - (g-\varkappa)^{\wp}(-\varkappa+g+2+2\wp)\Big] \\ &\quad -\frac{\mathbb{K}_1(\mathbf{S}_h(t_{-1+g}), t_{-1+g})}{\wp(\wp+2)}\hbar^{\wp} && (32) \\ &\quad \times\Big[(-\varkappa+1+g)^{1+\wp} - (-\varkappa+g)^{\wp}(g-\varkappa+1+\wp)\Big]\Bigg), && (33) \end{aligned}$$

$$\begin{aligned} \mathbf{W}_h(t_{g+1}) &= \mathbf{W}_h(t_0) + \frac{(1-\wp)}{\mathbb{N}(\wp)}[\mathbb{K}_2(\mathbf{W}_h(t_g), t_g)] + \frac{\wp}{\mathbb{N}(\wp)}\sum_{\varkappa=0}^{g}\Bigg(\frac{\mathbb{K}_2(\mathbf{W}_h(t_g), t_g)}{\wp(\wp+2)} \\ &\quad \times\hbar^{\wp}\Big[(-\varkappa+g+1)^{\wp}(g-\varkappa+\wp+2) - (g-\varkappa)^{\wp}(-\varkappa+g+2+2\wp)\Big] \\ &\quad -\frac{\mathbb{K}_2(\mathbf{W}_h(t_{g-1}), t_{-1+g})}{\wp(\wp+2)} && (34) \\ &\quad \times\hbar^{\wp}\Big[(-\varkappa+1+g)^{1+\wp} - (-\varkappa+g)^{\wp}(g-\varkappa+1+\wp)\Big]\Bigg), && (35) \end{aligned}$$

$$\begin{aligned} \mathbf{C}_h(t_{g+1}) &= \mathbf{C}_h(t_0) + \frac{(1-\wp)}{\mathbb{N}(\wp)}[\mathbb{K}_3(\mathbf{C}_h(t_g), t_g)] + \frac{\wp}{\mathbb{N}(\wp)}\sum_{\varkappa=0}^{g}\Bigg(\frac{\mathbb{K}_3(\mathbf{C}_h(t_g), t_g)}{\wp(\wp+2)} \\ &\quad \times\hbar^{\wp}\Big[(-\varkappa+g+1)^{\wp}(g-\varkappa+\wp+2) - (g-\varkappa)^{\wp}(-\varkappa+g+2+2\wp)\Big] \\ &\quad -\frac{\mathbb{K}_3(\mathbf{C}_h(t_{g-1}), t_{-1+g})}{\wp(\wp+2)} && (36) \\ &\quad \hbar^{\wp}\Big[(-\varkappa+1+g)^{\wp+1} - (-\varkappa+g)^{\wp}(g-\varkappa+1+\wp)\Big]\Bigg), && (37) \end{aligned}$$

$$
\begin{aligned}
\mathbf{G}_h(t_{g+1}) \quad = \quad & \mathbf{G}_h(t_0) + \frac{(1-\wp)}{\mathbb{N}(\wp)}\left[\mathbb{K}_4(\mathbf{G}_h(t_g), t_g)\right] + \frac{\wp}{\mathbb{N}(\wp)} \sum_{\varkappa=0}^{g} \left(\frac{\mathbb{K}_1(\mathbf{G}_h(t_g), t_g)}{\wp(\wp+2)} \right. \\
& \times \hbar^{\wp}\left[(-\varkappa+g+1)^{\wp}(g-\varkappa+\wp+2) - (g-\varkappa)^{\wp}(-\varkappa+g+2+2\wp)\right] \\
& - \frac{\mathbb{K}_4(\mathbf{G}_h(t_{g-1}), t_{-1+g})}{\wp(\wp+2)} \qquad (38) \\
& \left. \hbar^{\wp}\left[(-\varkappa+1+g)^{\wp+1} - (-\varkappa+g)^{\wp}(g-\varkappa+1+\wp)\right]\right), \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{J}_h(t_{g+1}) \quad = \quad & \mathbf{J}_h(t_0) + \frac{(1-\wp)}{\mathbb{N}(\wp)}\left[\mathbb{K}_5(\mathbf{J}_h(t_g), t_g)\right] + \frac{\wp}{\mathbb{N}(\wp)} \sum_{\varkappa=0}^{g} \left(\frac{\mathbb{K}_5(\mathbf{J}_h(t_g), t_g)}{\wp(\wp+2)} \right. \\
& \times \hbar^{\wp}\left[(-\varkappa+g+1)^{\wp}(g-\varkappa+\wp+2) - (g-\varkappa)^{\wp}(-\varkappa+g+2+2\wp)\right] \\
& - \frac{\mathbb{K}_5(\mathbf{J}_h(t_{g-1}), t_{-1+g})}{\wp(\wp+2)} \qquad (40) \\
& \left. \hbar^{\wp}\left[(-\varkappa+1+g)^{\wp+1} - (-\varkappa+g)^{\wp}(g-\varkappa+1+\wp)\right]\right). \qquad (41)
\end{aligned}
$$

5. Numerical Simulations and Discussion

In this section, we establish the approximate solution of our considered Model (2) using various parameters given in Table 2 for verification of the proposed scheme. The initial values for all cases of the given system are $\mathbf{S}_h(0) = 1000, \mathbf{W}_h(0) = 20, \mathbf{C}_h(0) = 20$, $\mathbf{G}_h(0) = 10$ and $\mathbf{J}_h(0) = 10$. We have taken four different sets of parameter, one without control and the remaining three simulated by applying some control strategies for all of the compartments in problem (2) and at a different fractional order of $\wp$. Figure 1a–d represents the the dynamics of susceptible humans (free of crimes) $\mathbf{S}_h(t)$ at a different fractional order of $\wp$ before and after control strategies. Figure 1a is the plot before control parameters in which the susceptible people are decreasing and transferring to the serial killers. While Figure 1b–d are the plots after we applied the control strategies by changing the values of the most affected parameter, as given in Table 2. In these three figures, the susceptible class is controlled and increasing. Figure 1e is the combined graph. Figure 2a–d show the dynamics of weaponization in humans $\mathbf{W}_h(t)$ at different fractional-orders of $\wp$ before and after control strategies. Figure 2a is the plot before the control parameters in which the weaponized people are increasing and, thus, the number of serial killers also increases. While Figure 2a–d are the graphs after applying the control strategies by changing the values of most affected parameters as given in Table 2. The weapons class is controlled, reduced, and approaching zero in these three figures. Figure 2e is the combined graph. In Figure 3a–d, the dynamics of serial killer humans $\mathbf{C}_h(t)$ have been shown at different fractional-orders of $\wp$ before and after control strategies. Figure 1a is the plot before the control parameters in which the serial killers are increasing and converging. While Figure 3b–d are the simulations after applying control strategies by changing the values of most affected parameters, as given in Table 2. In these three figures, the serial killers' class is controlled, decreasing, and tends to zero. Figure 3e is the combined graph. In Figure 4a–d, the dynamics of gang member humans $\mathbf{G}_h(t)$ have been shown at different fractional-orders of $\wp$ before and after control strategies. Figure 4a is the plot before the control parameters in which the gang members are increasing and diverging. While Figure 4b–d are the simulations after applying the control strategies by changing the values of the most affected parameters, as given in Table 2. In these three figures, the gang member class is controlled, decreased, and tends to zero. Figure 4e is the combined graph. In the first four subfigures (a–d) of Figure 5, the behavior of serial killers detained in jail $\mathbf{J}_h(t)$ have been shown at different fractional-orders of $\wp$ before and after control strategies. Figure 5a is the plot before control parameters in which the jailed members are extremely increasing and diverging. The Figure 5b–d are the representations of jailed members after applying the control strategies by changing the values of the most influencing parameters, as given in Table 2. In these three figures, the gang member class is controlled, decreased, and tends to zero. Figure 5e is the combined graph.

Table 2. Values of the parameters used in Model (2) with different control strategies.

Parameter	Value	Control Strategy	Control Strategy-I	Control Strategy-II
b	0.29	0.1	0.02	0.01
μ_1	0.001	0.001	0.001	0.001
μ_2	0.005	0.005	0.005	0.005
μ_3	0.003	0.007	0.008	0.009
μ_4	0.05833	0.58	0.59	0.78
$\beta 1$	0.056	0.006	0.001	0.0001
$\beta 2$	0.0093	0.0093	0.0093	0.0093
$\beta 3$	0.067	0.08	0.089	0.09
$\beta 4$	0.0027	0.0027	0.0027	0.0027
Λ	0.09586	0.09586	0.09586	0.09586
π	0.00039	0.00039	0.00039	0.00039

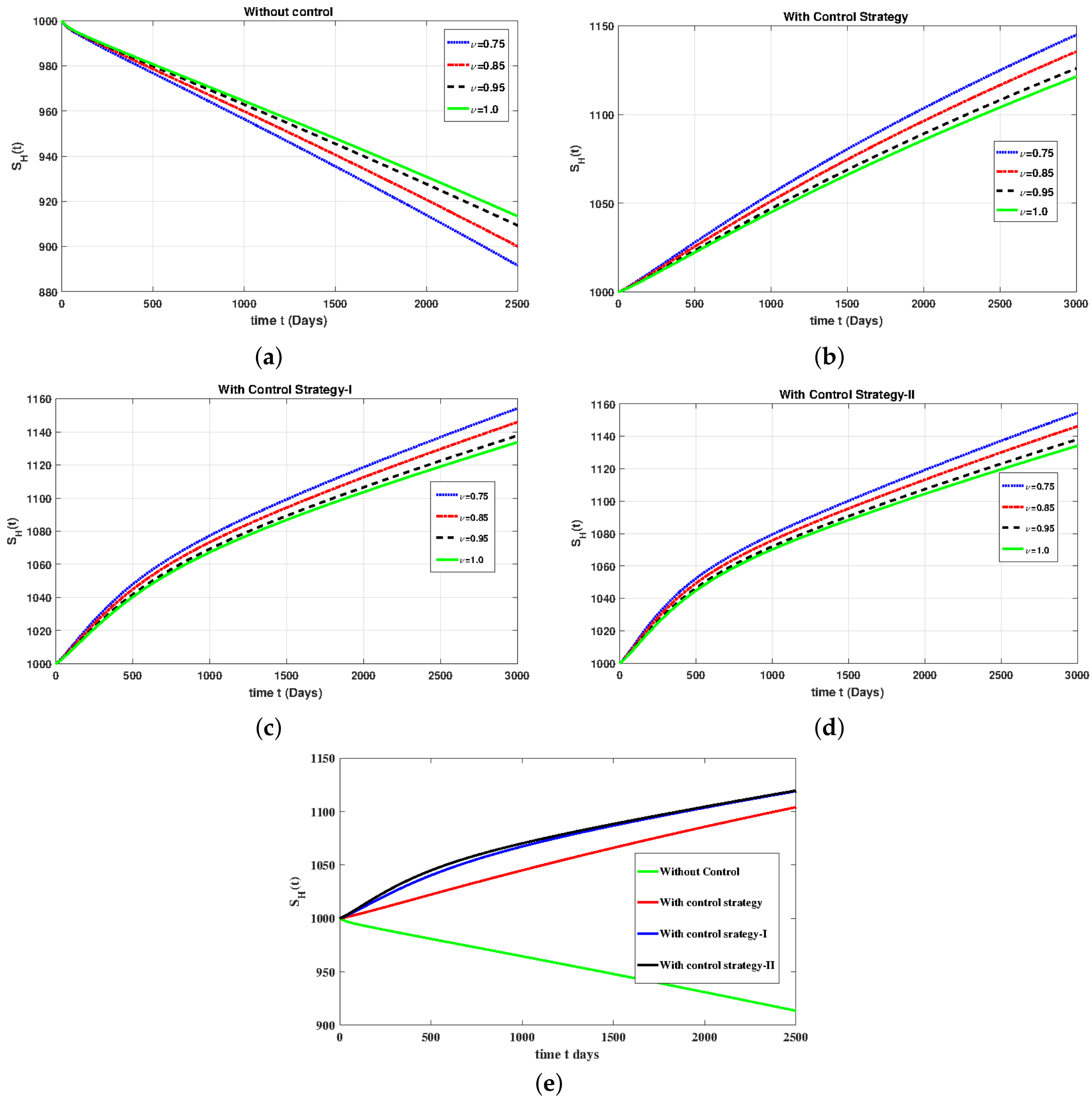


Figure 1. (**a**–**e**) Plots of $\mathbf{S}_h(t)$ having four different sets, before and after control strategies in the problem under analysis (2) at various arbitrary orders.

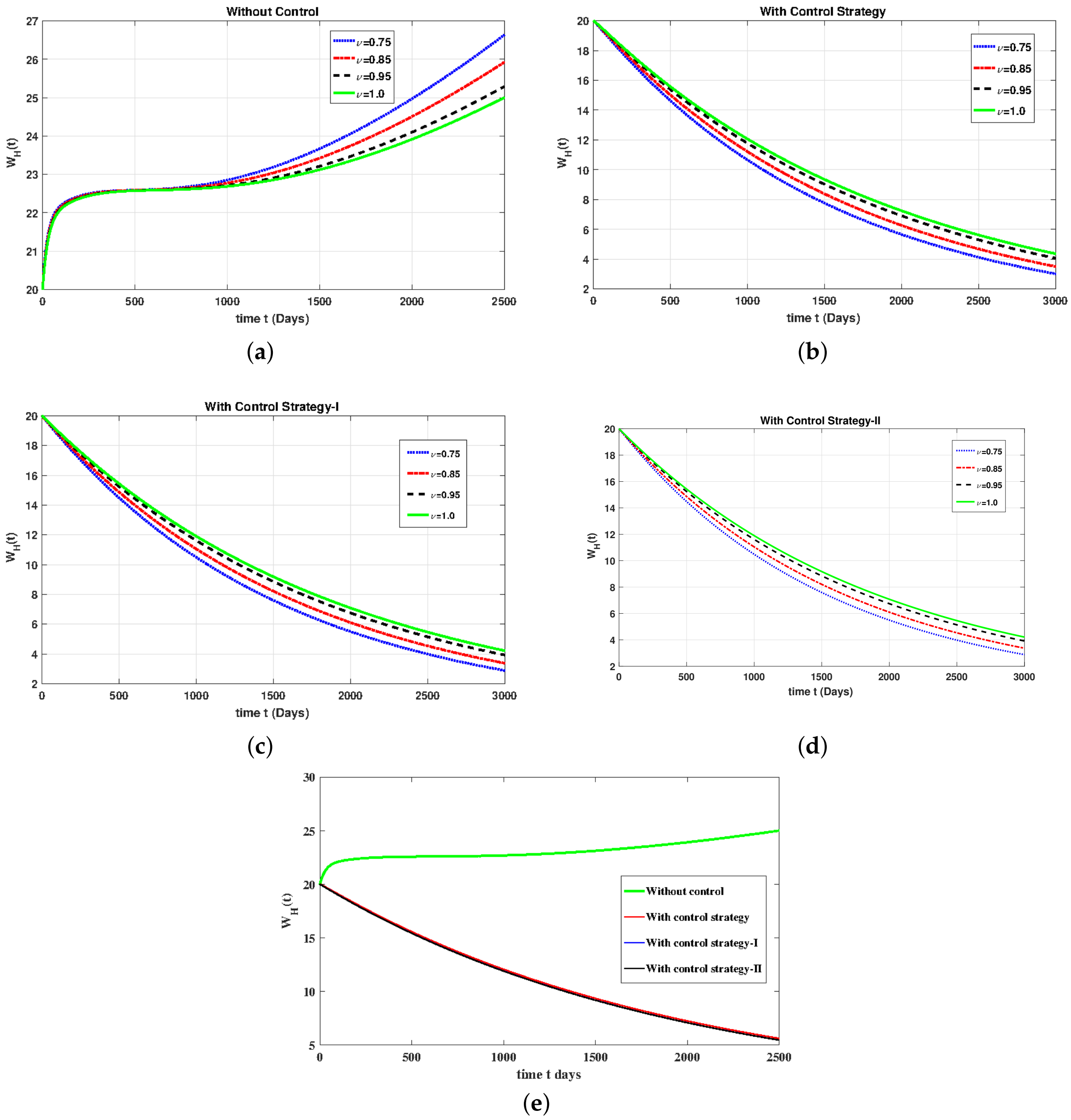


Figure 2. (**a**–**e**) Plots of $\mathbf{W}_h(t)$ having four different sets, before and after control strategies in the problem under analysis (2) at various arbitrary orders.

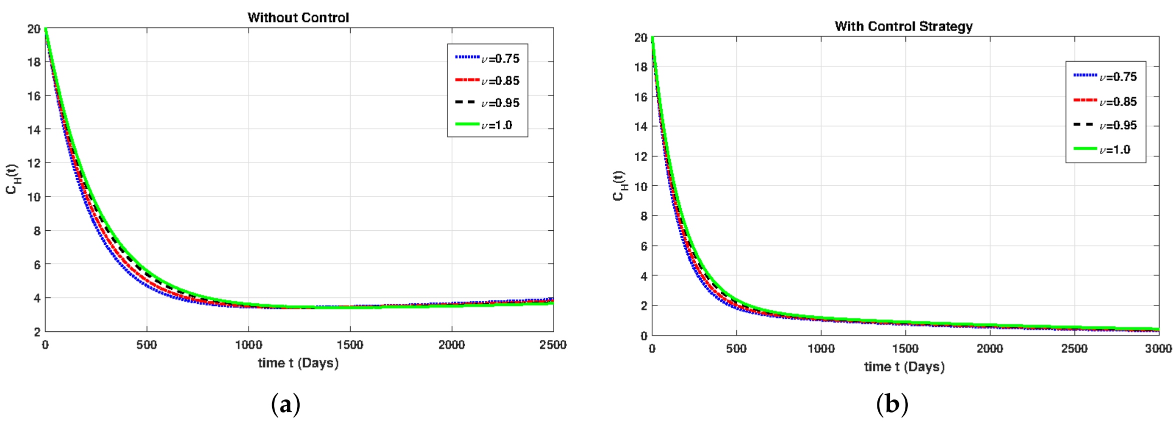


Figure 3. *Cont.*

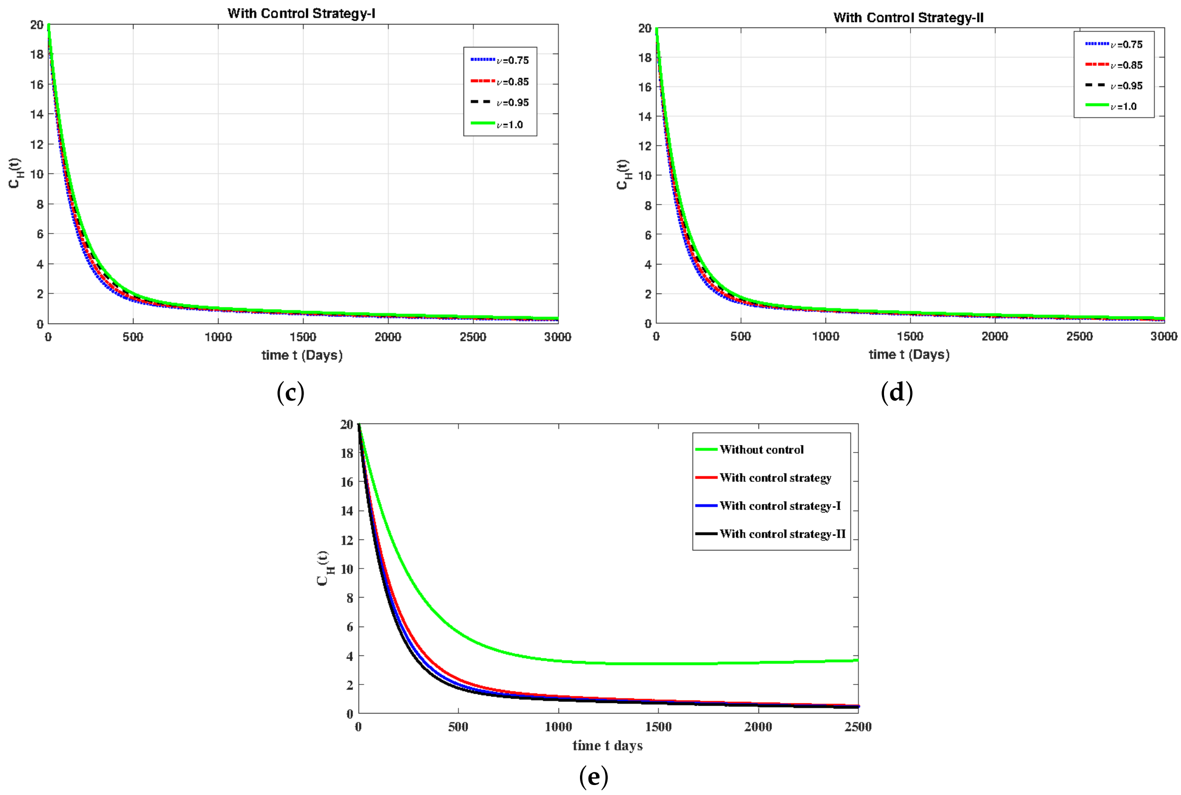


Figure 3. (**a**–**e**) Plots of serial killers $\mathbf{C}_h(t)$ having four different sets, before and after control strategies in the problem under analysis (2) at various arbitrary orders.

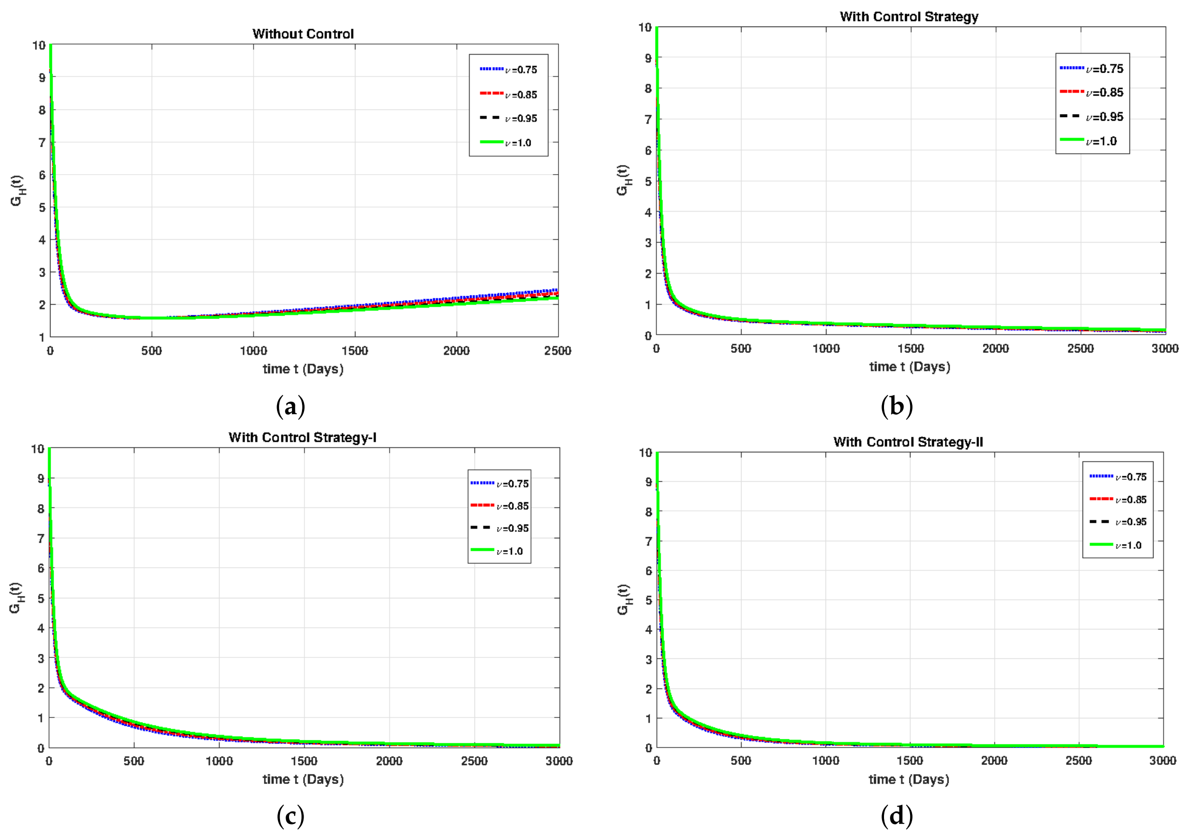
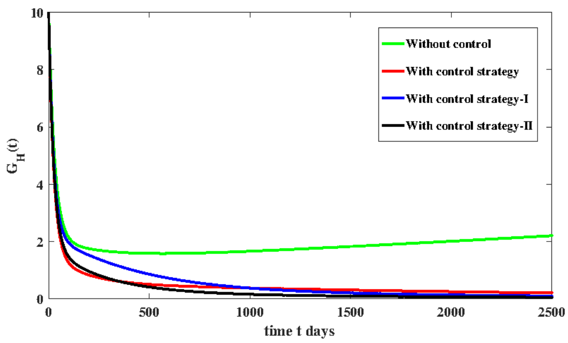
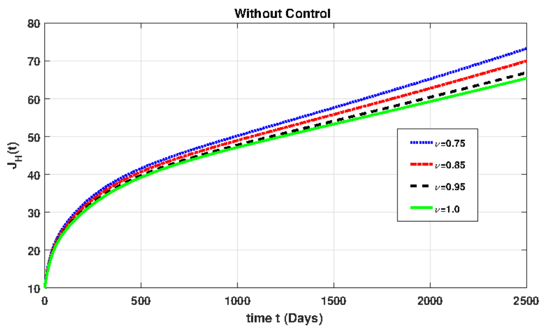


Figure 4. *Cont.*

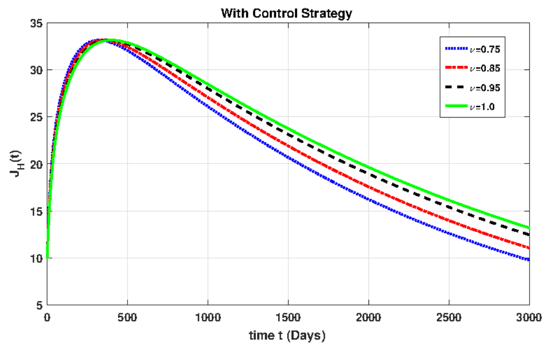


(**e**)

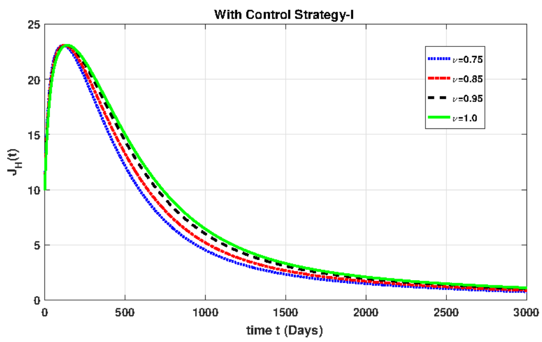
Figure 4. (**a**–**e**) Plots of gang members $\mathbf{G}_h(t)$ having four different sets, before and after control strategies in the problem under analysis (2) at various arbitrary orders.



(**a**)



(**b**)



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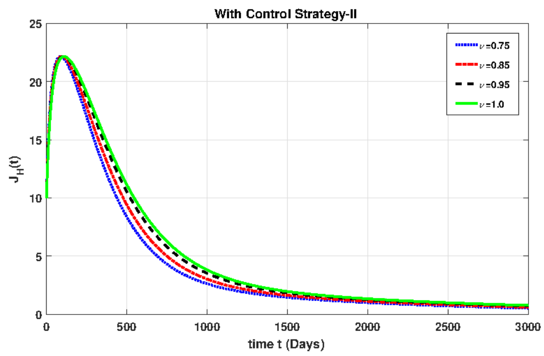
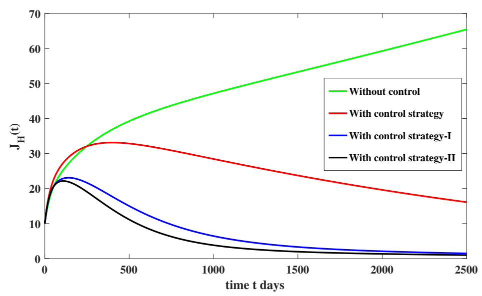


(**d**)



(**e**)

Figure 5. (**a**–**e**) Plots of $\mathbf{J}_h(t)$ having four different sets, before and after control strategies in the problem under analysis (2) at various arbitrary orders.

6. Conclusions

In the current paper, we have studied a fractional-order mathematical model of serial killing under the ABC operator. We have proved the existence of the solution to the problem under examination by using the fixed point theorem. We used the Adams–Basforth method to attain an approximate solution to the given model. We showed the stability of the given model through the nonlinear concepts of Ulam–Hyers. We performed numerical simulations for two different cases to support our analytical findings. We discussed the given model for control with and without control strategies. All components of the presented model have attained stability and convergence. The stability of the decay process adjusts quickly to small fractional orders, whereas the stability of the growth process adjusts quickly to higher orders. The given model provided global information due to fractional analysis of an extra degree of freedom. Thus, the fractional-order Model (2) is superior to the integer-order Model (1). In the future, more global and generalized operators might be used to investigate the given model.

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