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Symmetric space σ-model dynamics: Internal metric formalism

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Abstract

For the symmetric space sigma model in the internal metric formalism we explicitly construct the Lagrangian in terms of the axions and the dilatons of the solvable Lie algebra gauge and then we exactly derive the axion–dilaton field equations.

1. Introduction

It is well known that the sigma model Lagrangian of the Riemannian globally symmetric space G/K can be formulated by using a definition of an internal metric. The construction can be referred in [1–6]. The choice of the solvable Lie algebra gauge [7] for parameterizing the coset representatives brings further simplicity in the construction. Under a specified trace condition the field equations of the symmetric space sigma model are derived and further studied in [3,4] and [1,2] respectively. However since the Lagrangian is not explicitly constructed in terms of the coset scalars the formulation of [3,4] is based on the Lagrange multiplier methods and the field equations are written in terms of the field strengths of the axions which are treated as independent fields. It is also mentioned in [3,4] that if one can express the Lagrangian in terms of the scalar fields explicitly one can directly vary it to obtain the exact field equations of the dilatons and the axions. On the other hand in [1,2] the Cartan-forms in terms of which the symmetric space sigma model Lagrangian can be expressed are calculated exactly. This promises an explicit formulation of the symmetric space sigma model Lagrangian in terms of the coset scalar fields and the derivation of the field equations for a generic trace convention.

In this Letter we go in the above mentioned direction to obtain the most general form of the field equations of the sigma model which is based on the Riemannian globally symmetric space G/K. By using the exact form of the Cartan-form we express the SSSM Lagrangian explicitly in terms of the coset scalars in a generic trace convention then by varying it directly we obtain the field equations of the theory. We will assume the solvable Lie algebra gauge to parameterize the coset manifold G/K and we will classify the scalar fields as axions and the dilatons referring to the non-perturbative effective string theory and the supergravity literature where the symmetric space sigma model plays a central role governing the scalar sector which reveals the global symmetry and the U-duality structure of the supergravity and string theories respectively [8,9].

In section two leaving some of the details to the references we will present a concise formulation of the Lagrangian for the axion–dilaton parametrization. Without choosing a specific trace convention which generalizes the formalism of [1–4] we will construct the Lagrangian explicitly in terms of the scalar fields for a generic trace convention. Then in section three we will vary the symmetric space sigma model Lagrangian to derive the general field equations of the axion and the dilaton scalar fields.

2. Lagrangian in the axion–dilaton parameterization

The construction of the symmetric space sigma model is based on a set of G-valued maps $\{\nu(x)\}$ which are onto C^{∞}-maps from the D-dimensional spacetime to the coset space G/K. Thus they parameterize the coset manifold G/K. Here G is in general a non-compact real form of any other semi-simple Lie group and K is a maximal compact subgroup of G. The

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coset manifold G/K has a unique analytical structure induced by the quotient topology of G. The scalar manifold G/K is a Riemannian globally symmetric space for all the G-invariant Riemannian structures on G/K [10]. The solvable Lie algebra gauge is a parametrization of the coset manifold G/K which is due to the Iwasawa decomposition

$$\begin{aligned} g &= \mathbf{k} \oplus \mathbf{s} \\ &= \mathbf{k} \oplus \mathbf{h}_p \oplus \mathbf{n}, \end{aligned} \tag{2.1}$$

of the Lie algebra g of G. In (2.1) $\mathbf{k}$ is the Lie algebra of K and $\mathbf{s} = \mathbf{h}_p \oplus \mathbf{n}$ is a solvable Lie subalgebra of g. The Abelian subalgebra $\mathbf{h}_p$ is generated by r non-compact Cartan generators $\{H_i\}$. Also the nilpotent Lie subalgebra $\mathbf{n}$ is generated by a subset $\{E_m\}$ of the positive root generators of g where $m \in \Delta^+_{\mathrm{nc}}$. The roots in Δ^+_{nc} are the non-compact roots with respect to the Cartan involution associated with the Iwasawa decomposition (2.1) [1–3,10]. The map

$$\mathrm{Exp}: \mathbf{s} \to G/K, \tag{2.2}$$

from the $\mathbb{R}^{\dim \mathbf{s}}$-manifold $\mathbf{s}$ into G/K is a local diffeomorphism [10]. Therefore

$$\nu(x) = e^{\frac{1}{2}\phi^i(x) H_i} e^{\chi^m(x) E_m}, \tag{2.3}$$

is a legitimate parametrization of the coset manifold G/K which is called the solvable Lie algebra gauge [7]. The scalar fields $\{\phi^i\}$ are called the dilatons and $\{\chi^m\}$ are called the axions. In the internal metric formalism [1–6] of the symmetric space sigma model the Lagrangian which is invariant under the right rigid action of G and the left local action of K is constructed as

$$\mathcal{L} = \frac{1}{4} \operatorname{tr}\big({*d\mathcal{M}^{-1}} \wedge d\mathcal{M}\big), \tag{2.4}$$

where the internal metric $\mathcal{M}$ is defined as

$$\mathcal{M} = \nu^{\#} \nu. \tag{2.5}$$

The generalized transpose # over the Lie group G is such that $(\exp(g))^{\#} = \exp(g^{\#})$. It is defined by using the Cartan involution θ over g that is associated with (2.1) as $g^{\#} = -\theta(g)$ [10]. If the subgroup of G generated by the compact generators is an orthogonal group then in the fundamental representation of g we have $g^{\#} = g^T$. Also it is always possible to find a matrix representation of g in which # coincides with the matrix transpose operator [3]. In spite of the fact that the generalized transpose # shares the usual properties of the matrix transpose in our formulation we will assume a matrix representation in which $g^{\#} = g^T$. From the definition of the coset parametrization in (2.3) we have the identities

$$\nu^{-1} d\nu = -d\nu^{-1} \nu, \qquad d\nu\, \nu^{-1} = -\nu\, d\nu^{-1}. \tag{2.6}$$

Also

$$\operatorname{tr}(d\nu_1 \wedge *d\nu_2) = (-1)^{(D-1)} \operatorname{tr}(*d\nu_2 \wedge d\nu_1), \tag{2.7}$$

for two matrix-valued functions ν_1 and ν_2. Now if we define the Cartan–Maurer form $\mathcal{G}$ as

$$\mathcal{G} = d\nu\, \nu^{-1}, \tag{2.8}$$

in the light of the above mentioned identities, the properties of the coset representatives and the generalized transpose we can express the Lagrangian (2.4) in terms of the Cartan–Maurer form $\mathcal{G}$ as

$$\mathcal{L} = -\frac{1}{2} \operatorname{tr}\big({*\mathcal{G}} \wedge \mathcal{G}^{\#} + {*\mathcal{G}} \wedge \mathcal{G}\big). \tag{2.9}$$

The Cartan–Maurer form $\mathcal{G}$ is explicitly calculated in terms of the axions and the dilatons in [1,2]

$$\mathcal{G} = \frac{1}{2} d\phi^i\, H_i + \vec{\mathbf{E}}' \boldsymbol{\Omega} \vec{\mathbf{d}\boldsymbol{\chi}}. \tag{2.10}$$

The row vector $\vec{\mathbf{E}}'$ is

$$(\vec{\mathbf{E}}')_\alpha = e^{\frac{1}{2}\alpha_i \phi^i} E_\alpha. \tag{2.11}$$

Also $\vec{\mathbf{d}\boldsymbol{\chi}}$ is a column vector of the field strengths of the axions $\{d\chi^i\}$. In (2.10) $\boldsymbol{\Omega}$ is a $\dim \mathbf{n} \times \dim \mathbf{n}$ matrix

$$\boldsymbol{\Omega} = \big(e^{\omega} - I\big)\omega^{-1}. \tag{2.12}$$

The $\dim \mathbf{n} \times \dim \mathbf{n}$ matrix ω is also defined as

$$\omega^\gamma_\beta = \chi^\alpha K^\gamma_{\alpha\beta}. \tag{2.13}$$

The structure constants $K^\gamma_{\alpha\beta}$ and the root vector components α_i are defined as

$$[E_\alpha, E_\beta] = K^\gamma_{\alpha\beta} E_\gamma, \qquad [H_i, E_\alpha] = \alpha_i E_\alpha. \tag{2.14}$$

Since now we have the exact form of the Cartan–Maurer form $\mathcal{G}$ we can express the Lagrangian (2.4) explicitly in terms of the axions and the dilatons. Inserting (2.10) in (2.9) we obtain

$$\begin{aligned} \mathcal{L} = &-\frac{1}{8} \mathcal{A}_{ij} * d\phi^i \wedge d\phi^j - \frac{1}{4} \mathcal{B}_{i\alpha} * d\phi^i \wedge e^{\frac{1}{2}\alpha_i \phi^i} U^\alpha \\ &- \frac{1}{2} \mathcal{C}_{\alpha\beta} e^{\frac{1}{2}\alpha_i \phi^i} * U^\alpha \wedge e^{\frac{1}{2}\beta_i \phi^i} U^\beta, \end{aligned} \tag{2.15}$$

in which we slightly change the notation introduced above and use

$$U^\alpha = \boldsymbol{\Omega}^\alpha_\beta\, d\chi^\beta. \tag{2.16}$$

For the sake of generality in (2.15) we have not specified any trace convention and we have defined the generic trace coefficients as

$$\begin{aligned} \mathcal{A}_{ij} &= \operatorname{tr}\big(H_i H_j^{\#}\big) + \operatorname{tr}(H_i H_j), \\ \mathcal{B}_{i\alpha} &= \operatorname{tr}\big(H_i E_\alpha^{\#}\big) + \operatorname{tr}\big(E_\alpha H_i^{\#}\big) + \operatorname{tr}(H_i E_\alpha) + \operatorname{tr}(E_\alpha H_i), \\ \mathcal{C}_{\alpha\beta} &= \operatorname{tr}\big(E_\alpha E_\beta^{\#}\big) + \operatorname{tr}(E_\alpha E_\beta). \end{aligned} \tag{2.17}$$

By using the properties of the generalized transpose # $\mathcal{B}_{i\alpha}$ can further be expressed as

$$\mathcal{B}_{i\alpha} = 2\big(\operatorname{tr}\big(E_\alpha H_i^{\#}\big) + \operatorname{tr}(E_\alpha H_i)\big). \tag{2.18}$$

3. Field equations for the axions and the dilatons

Now that we have obtained the Lagrangian (2.15) explicitly in terms of the axions and the dilatons we can derive the field equations. We should first observe that

$$\omega=\omega(\chi_m),\qquad \mathbf{\Omega}=\mathbf{\Omega}(\chi_m). \tag{3.1}$$

Thus we see that while the variation of (2.15) with respect to the dilatons $\{\phi^i\}$ is a straightforward task we should examine the variation of $\mathbf{\Omega}$ with respect to the axions $\{\chi^m\}$ from a closer look. When we vary the Lagrangian (2.15) with respect to the dilaton ϕ^k we obtain the dilatonic field equations as

$$\begin{aligned}&(-1)^{(D-1)}d\left(\frac{1}{2}(\mathcal{A}_{ik}+\mathcal{A}_{ki})*d\phi^i+\mathcal{B}_{k\alpha}e^{\frac{1}{2}\alpha_i\phi^i}\mathbf{\Omega}^\alpha_\beta*d\chi^\beta\right)\\&\quad=\frac{1}{2}\mathcal{B}_{i\alpha}\alpha_k*d\phi^i\wedge e^{\frac{1}{2}\alpha_i\phi^i}\mathbf{\Omega}^\alpha_\beta\,d\chi^\beta\\&\qquad+\mathcal{C}_{\alpha\beta}(\alpha_k+\beta_k)e^{\frac{1}{2}\alpha_i\phi^i}\mathbf{\Omega}^\alpha_\tau*d\chi^\tau\wedge e^{\frac{1}{2}\beta_i\phi^i}\mathbf{\Omega}^\beta_\gamma\,d\chi^\gamma.\end{aligned} \tag{3.2}$$

Before writing down the axion field equations we will mention about the variation of $\mathbf{\Omega}$. Firstly from (2.13) we have

$$\omega'\equiv\frac{\partial\omega}{\partial\chi^m}=K_m, \tag{3.3}$$

where the components of the matrix K^m are defined as

$$(K_m)^\gamma_\beta=K^\gamma_{m\beta}. \tag{3.4}$$

Before going further we should define the adjoint representation of g. The set of endomorphisms namely the linear maps on g form a vector space with the addition and the scalar product induced from g. They also form a Lie algebra denoted as $gl(g)$ under the product $[\alpha,\beta]=\alpha\cdot\beta-\beta\cdot\alpha$. The non-singular (invertible) endomorphisms of g form an analytical Lie group which we will refer as $GL(g)$. Naturally $gl(g)$ is isomorphic to the Lie algebra of $GL(g)$. Now if we assign the map

$$ad_X=[X,\ ],\quad \forall X\in g, \tag{3.5}$$

such that

$$ad_X(Y)=[X,Y],\quad \forall Y\in g, \tag{3.6}$$

then ad_X is an endomorphism. The map

$$ad_g(g)\equiv ad_X:X\to ad_X, \tag{3.7}$$

is an algebra homomorphism from g into $gl(g)$ and it is called the adjoint representation of the Lie algebra g. The image of $ad_g(g)$ in $gl(g)$ is a subalgebra and we will denote it as $ad(g)$. Now after introducing the elements of the adjoint representation we can write down the partial derivative of e^ω as [11,12]

$$\begin{aligned}\frac{\partial e^\omega}{\partial\chi^m}&=e^\omega\left(\frac{I-e^{-ad_\omega}}{ad_\omega}\right)(\omega')\\&=e^\omega\left(\omega'-\frac{1}{2!}[\omega,\omega']+\frac{1}{3!}\big[\omega,[\omega,\omega']\big]-\cdots\right).\end{aligned} \tag{3.8}$$

We should observe that the commutation series in (3.8) will terminate after a finite number of terms since from their definitions in (2.13) and (3.3) both ω and ω' lie in the adjoint representation of $\mathbf{n}$ which is a nilpotent Lie algebra so is its image $ad(\mathbf{n})$ which is composed of nilpotent endomorphisms [10]. We may see this fact as follows; if we define the ideals

$$\varphi^{p+1}ad(\mathbf{n})=\big[ad(\mathbf{n}),\varphi^p ad(\mathbf{n})\big], \tag{3.9}$$

where $\varphi^0 ad(\mathbf{n})=ad(\mathbf{n})$ then the series

$$\varphi^0 ad(\mathbf{n})\supset\varphi^1 ad(\mathbf{n})\supset\varphi^2 ad(\mathbf{n})\supset\cdots, \tag{3.10}$$

is called the central descending series and we observe that the growing terms of the series (3.8) belong to the growing ideals of (3.10). Due to the nilpotency of $ad(\mathbf{n})$ (3.10) terminates with $\varphi^m ad(\mathbf{n})=\{0\}$ for some $m\geqslant\dim(ad(\mathbf{n}))$ [10,13,14]. This justifies the termination of (3.8) after a finite number of terms. The expansion of e^ω which is $e^\omega=I+\omega+1/2!\omega^2+\cdots$ also terminates after a finite number of terms since the matrix ω as an element of $ad(\mathbf{n})$ is the representative of a nilpotent endomorphism and for any nilpotent endomorphism N $N^k=0$ for some $k\in\mathbb{Z}$. This fact also brings termination following a finite number of terms in the expansion of $\mathbf{\Omega}$ in (2.12). If we vary $\mathbf{\Omega}$ with respect to the axion χ^m we find that

$$\mathcal{D}_m\equiv\frac{\partial\mathbf{\Omega}}{\partial\chi^m}=e^\omega\left(\frac{I-e^{-ad_\omega}}{ad_\omega}\right)(\omega')\omega^{-1}-\mathbf{\Omega}\omega'\omega^{-1}, \tag{3.11}$$

where we have also used

$$\frac{\partial\omega^{-1}}{\partial\chi^m}=-\omega^{-1}\omega'\omega^{-1}. \tag{3.12}$$

Now we are ready to vary the Lagrangian (2.15) with respect to the axion χ^m. Performing the variation while keeping in mind the definitions we have introduced we obtain the axionic field equations

$$\begin{aligned}&(-1)^{(D-1)}d\left(\frac{1}{2}\mathcal{B}_{i\alpha}e^{\frac{1}{2}\alpha_i\phi^i}\mathbf{\Omega}^\alpha_m*d\phi^i\right.\\&\qquad\left.+\,\mathcal{C}_{\alpha\beta}e^{\frac{1}{2}\alpha_i\phi^i}e^{\frac{1}{2}\beta_i\phi^i}\big(\mathbf{\Omega}^\alpha_\gamma\mathbf{\Omega}^\beta_m+\mathbf{\Omega}^\alpha_m\mathbf{\Omega}^\beta_\gamma\big)*d\chi^\gamma\right)\\&\quad=\frac{1}{2}\mathcal{B}_{i\alpha}\mathcal{D}^\alpha_{m\beta}e^{\frac{1}{2}\alpha_i\phi^i}*d\phi^i\wedge d\chi^\beta\\&\qquad+\mathcal{C}_{\alpha\beta}e^{\frac{1}{2}\alpha_i\phi^i}e^{\frac{1}{2}\beta_i\phi^i}\big(\mathcal{D}^\alpha_{m\tau}\mathbf{\Omega}^\beta_\gamma+\mathbf{\Omega}^\alpha_\tau\mathcal{D}^\beta_{m\gamma}\big)*d\chi^\tau\wedge d\chi^\gamma.\end{aligned} \tag{3.13}$$

4. Conclusion

By using the exact form of the Cartan-form in the symmetric space sigma model Lagrangian we have expressed the Lagrangian explicitly in terms of the dilatons and the axions which parameterize the coset manifold of the SSSM in the solvable Lie algebra gauge. In this formulation we have kept the coefficients of a generic trace convention. Then we have directly varied this basic form of the Lagrangian to obtain the dilatonic and the axionic field equations.

Our formulation generalizes the one in [3,4] which is based on a special trace convention. In [3,4] the Lagrangian is not derived exactly, however as we have mentioned before a dualisation method is used to find the first-order field equations for the undetermined field strengths of the axions which take part in the Cartan–Maurer form. Since the Cartan–Maurer form is derived in [1,2] by using the results of [1,2] in this work we exactly construct the Lagrangian and obtain the field equations directly for the coset scalars namely the dilatons and the axions. The formulation presented in this work is purely in algebraic terms. Our derivation expresses both the Lagrangian and the field equations in terms of the unspecified structure constants of a generic global symmetry group G without assigning a representation. Thus the results are powerful in applying to any specific symmetric space sigma model example. As we mentioned in the introduction due to the special role of the SSSM in the low energy effective string theory the construction presented here also serves as a direct and an exact method of calculation in the non-perturbative string dynamics.

We are also working on a similar formulation for the vielbein formalism of the symmetric space sigma model whose construction differs from the one presented here. Different coset parametrizations can further be studied. Finally starting from the field equations obtained here one can work on the first-order formulation of the theory.

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