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# Fractional partial random differential equations with infinite delay

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ABSTRACT

The present paper deals with some existence results for the Darboux problem of partial fractional random differential equations with infinite delay. The arguments are based on a random fixed point theorem with stochastic domain combined with the measure of noncompactness.

## Introduction

The fractional calculus is concerned with noninteger order extensions of derivatives and integrals. Differential and integral equations of fractional order have a wide range of applications, see e.g. [1–7] for more information. In recent years, there has been substantial progress in ordinary and partial fractional differential and integral equations; see papers of Abbas et al. [8–17], Kilbas et al. [18], Ahmad and Nieto [19], Salim et al. [20,21], Stanek [22], Vityuk and Golushkov [23], and the sources within.

The essence of a dynamic system in natural sciences or engineering is determined by the precision of the knowledge we have about the system's characteristics. A deterministic dynamical system emerges when information about a dynamic system is exact. However, most of the data obtainable for the modeling and assessment of dynamic system characteristics is incorrect, imprecise, or unclear. In other terms, finding the parameters of a dynamical system is fraught with uncertainty. When we have statistical understanding of the parameters of a dynamic system, that is, when the knowledge is probabilistic, the most popular strategy in mathematical modeling of such systems is to employ random differential equations or stochastic differential equations. As natural extensions of deterministic differential equations, random differential equations appear in numerous applications and have been studied by several mathematicians; see [24–26] and references therein.

Prompted by the aforementioned papers, in this paper, we consider the following problem:

$$({}^cD_0^{\zeta}\mathfrak{p})(\vartheta,\eta,\delta)=\psi(\vartheta,\eta,\mathfrak{p}_{(\vartheta,\eta)},\delta),\ \text{ if } (\vartheta,\eta)\in\Theta:=[0,\theta_1]\times[0,\theta_2],\delta\in\Psi, \tag{1}$$

$$\mathfrak{p}(\vartheta,\eta,\delta)=\varpi(\vartheta,\eta,\delta),\ \text{ if } (\vartheta,\eta)\in\tilde{\Theta}:=(-\infty,\theta_1]\times(-\infty,\theta_2]\backslash(0,\theta_1]\times(0,\theta_2],\delta\in\Psi, \tag{2}$$

$$\mathfrak{p}(\vartheta,0,\delta)=\varpi_1(\vartheta,\delta),\ \vartheta\in[0,\theta_1],\ \mathfrak{p}(0,\eta,\delta)=\varpi_2(\eta,\delta),\ \eta\in[0,\theta_2],\delta\in\Psi, \tag{3}$$

where $\theta_1,\theta_2>0$, ${}^cD_0^{\zeta}$ is the standard Caputo's fractional derivative of order $\zeta=(\zeta_1,\zeta_2)\in(0,1]\times(0,1]$, $(\Psi,\mathcal{A})$ is a measurable space, $(\mathbf{E},\|\cdot\|)$ is a Banach space, $\psi:\Theta\times\mathcal{S}\times\Psi\to\mathbf{E}$ is a given function, $\varpi:\tilde{\Theta}\times\Psi\to\mathbf{E}$ is a given continuous function, $\varpi_1:[0,\theta_1]\times\Psi\to\mathbf{E}$, $\varpi_2:[0,\theta_2]\times\Psi\to\mathbf{E}$ are absolutely continuous functions where $\varpi_1(\vartheta,\delta)=\varpi(\vartheta,0,\delta)$, $\varpi_2(\eta,\delta)=\varpi(0,\eta,\delta)$ for each $\vartheta\in[0,\theta_1]$, $\eta\in[0,\theta_2]$, $\delta\in\Psi$, $\mathbb{R}^-:=\mathbb{R}^-$ and $\mathcal{S}$ is a phase space, which will be defined later. Let $\mathfrak{p}_{(\vartheta,\eta)}$ be the element of $\mathcal{S}$ given by

$$\mathfrak{p}_{(\vartheta,\eta)}(\varrho,\kappa,\delta)=\mathfrak{p}(\vartheta+\varrho,\eta+\kappa,\delta);\ (\varrho,\kappa)\in\mathbb{R}^-\times\mathbb{R}^-.$$

The following is how this manuscript is structured. Section "Preliminaries" is reserved for introduction. Section "Main result" is dedicated to our primary result. Section "An example" provides a relevant illustration.

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## Preliminaries

First, we introduce and explain the notations and concepts used in this study.

Consider the space $AC(\Theta)$ of absolutely continuous functions from $\Theta$ into $\mathbf{E}$.

Denote $L^1(\Theta)$ the space of Bochner-integrable functions $\mathfrak{p} : \Theta \to \mathbf{E}$ with the norm

$$\|\mathfrak{p}\|_{L^1} = \int_0^{\theta_1}\int_0^{\theta_2} \|\mathfrak{p}(\vartheta,\eta)\|_{\mathbf{E}} d\eta d\vartheta.$$

Let $L^\infty(\Theta)$ be the Banach space of functions $\mathfrak{p} : \Theta \to \mathbb{R}$ which are essentially bounded.

Consider the $\sigma$-algebra $\mathfrak{D}_{\mathbf{E}}$ of Borel subsets of $\mathbf{E}$. The map $\bar{\mathfrak{p}} : \Psi \to \mathbf{E}$ is measurable if for any $\Omega \in \mathfrak{D}_{\mathbf{E}}$, we have

$$\bar{\mathfrak{p}}^{-1}(\Omega) = \{\delta \in \Psi : \bar{\mathfrak{p}}(\delta) \in \Omega\} \subset \mathcal{A}.$$

**Definition 1.** A mapping $\mathfrak{S} : \Psi \times \mathbf{E} \to \mathbf{E}$ is jointly measurable if for any $\Omega \in \mathfrak{D}_{\mathbf{E}}$, we have

$$\mathfrak{S}^{-1}(\Omega) = \{(\delta,\bar{\mathfrak{p}}) \in \Psi \times \mathbf{E} : \mathfrak{S}(\delta,\bar{\mathfrak{p}}) \in \Omega\} \subset \mathcal{A} \times \mathfrak{D}_{\mathbf{E}},$$

where $\mathcal{A} \times \mathfrak{D}_{\mathbf{E}}$ is the direct product of the $\sigma$-algebras $\mathcal{A}$ and $\mathfrak{D}_{\mathbf{E}}$ those defined in $\Psi$ and $\mathbf{E}$ respectively.

**Lemma 1.** *Let $\mathfrak{S} : \Psi \times \mathbf{E} \to \mathbf{E}$ be a mapping such that $\mathfrak{S}(\cdot,\bar{\mathfrak{p}})$ is measurable for all $\bar{\mathfrak{p}} \in \mathbf{E}$, and $\mathfrak{S}(\delta,.)$ is continuous for all $\delta \in \Psi$. Then the map $(\delta,\bar{\mathfrak{p}}) \mapsto \mathfrak{S}(\delta,\bar{\mathfrak{p}})$ is jointly measurable.*

**Definition 2.** A function $\psi : \Theta \times \mathbf{E} \times \Psi \to \mathbf{E}$ is called random Carathéodory if the assumptions that follow are verified:

- The map $(\vartheta,\eta,\delta) \to \psi(\vartheta,\eta,\mathfrak{p},\delta)$ is jointly measurable for all $\mathfrak{p} \in \mathbf{E}$, and
- $\mathfrak{p} \to \psi(\vartheta,\eta,\mathfrak{p},\delta)$ is continuous for almost all $(\vartheta,\eta) \in \Theta$ and $\delta \in \Psi$.

The map $\mathfrak{S} : \Psi \times \mathbf{E} \to \mathbf{E}$ is a random operator if $\mathfrak{S}(\delta,\mathfrak{p})$ is measurable in $\delta$ for all $\mathfrak{p} \in \mathbf{E}$ and it is given as $\mathfrak{S}(\delta)\mathfrak{p} = \mathfrak{S}(\delta,\mathfrak{p})$. We also can say that $\mathfrak{S}(\delta)$ is a random operator on $\mathbf{E}$. A random operator $\mathfrak{S}(\delta)$ on $\mathbf{E}$ is called continuous if $\mathfrak{S}(\delta,\mathfrak{p})$ is continuous in $\mathfrak{p}$ for all $\delta \in \Psi$. (See [27] for more details).

**Definition 3** (*[28]*)**.** Let $\mathcal{P}(\mathfrak{W})$ be the family of all nonempty subsets of $\mathfrak{W}$ and $\mathfrak{F}$ be a mapping from $\Psi$ into $\mathcal{P}(\mathfrak{W})$. A mapping $\mathfrak{S} : \{(\delta,\eta) : \delta \in \Psi,\ \eta \in \mathfrak{F}(\delta)\} \to \mathfrak{W}$ is a random operator with stochastic domain $\mathfrak{F}$ if $\mathfrak{F}$ is measurable (i.e., for all closed $\Omega \subset \mathfrak{W}$, $\{\delta \in \Psi, \mathfrak{F}(\delta) \cap \Omega \neq \emptyset\}$ is measurable) and for all open $\tilde{\Omega} \subset \mathfrak{W}$ and all $\eta \in \mathfrak{W}$, $\{\delta \in \Psi : \eta \in \mathfrak{F}(\delta), \mathfrak{S}(\delta,\eta) \in \tilde{\Omega}\}$ is measurable. $\mathfrak{S}$ is continuous if every $\mathfrak{S}(\delta)$ is continuous. A mapping $\eta : \Psi \to \mathfrak{W}$ is a random fixed point of $\mathfrak{S}$ if for $P$−almost all $\delta \in \Psi$, $\eta(\delta) \in \mathfrak{F}(\delta)$ and $\mathfrak{S}(\delta)\eta(\delta) = \eta(\delta)$ and for all open $\tilde{\Omega} \subset \mathfrak{W}$, $\{\delta \in \Psi : \eta(\delta) \in \tilde{\Omega}\}$ is measurable.

Let $\mathcal{M}_{\mathbf{E}*}$ denote the class of all bounded subsets of a metric space $\mathbf{E}*$.

**Definition 4** (*[29]*)**.** Let $\mathbf{E}*$ be a complete metric space. A map $\mu : \mathcal{M}_{\mathbf{E}*} \to [0,\infty)$ is called a measure of noncompactness on $\mathbf{E}*$ if it verifies the following for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{M}_{\mathbf{E}*}$.

(MNC.1) $\mu(\Omega) = 0$ if and only if $\Omega$ is precompact (Regularity),
(MNC.2) $\mu(\Omega) = \mu(\overline{\Omega})$ (Invariance under closure),
(MNC.3) $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$ (Semi-additivity).

**Example 1.** In every metric space $\mathbf{E}*$, the map $\varpi : \mathcal{M}_{\mathbf{E}*} \to [0,\infty)$ with $\varpi(\Omega) = 0$ if $\Omega$ is relatively compact and $\varpi(\Omega) = 1$ otherwise is a measure of noncompactness ([30], Example1,… p. 19).

Let $\varepsilon = (0,0)$, $\zeta_1, \zeta_2 > 0$ and $\zeta = (\zeta_1,\zeta_2)$. For $\psi \in L^1(\Theta)$, the left-sided mixed Riemann–Liouville integral of order $\zeta$ is given by:

$$(I_\varepsilon^\zeta \psi)(\vartheta,\eta) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^\vartheta\int_0^\eta (\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}\psi(\varrho,\kappa)d\kappa d\varrho.$$

In particular,

$$(I_\varepsilon^0 \mathfrak{p})(\vartheta,\eta) = \mathfrak{p}(\vartheta,\eta),$$

$$(I_\varepsilon^\omega \mathfrak{p})(\vartheta,\eta) = \int_0^\vartheta\int_0^\eta \mathfrak{p}(\varrho,\kappa)d\kappa d\varrho;\ \text{for a.a } (\vartheta,\eta) \in \Theta,$$

where $\omega = (1,1)$ and $1-\zeta$ means $(1-\zeta_1, 1-\zeta_2) \in [0,1) \times [0,1)$. Denote by $D_{\vartheta\eta}^2 := \frac{\partial^2}{\partial\vartheta\partial\eta}$, the mixed second order partial derivative.

**Definition 5** (*[23]*)**.** Let $\zeta \in (0,1] \times (0,1]$ and $\mathfrak{p} \in AC(\Theta)$. The Caputo fractional-order derivative of order $\zeta$ of $\mathfrak{p}$ is given by:

$${}^c D_\varepsilon^\zeta \mathfrak{p}(\vartheta,\eta) = (I_\varepsilon^{1-\zeta} D_{\vartheta\eta}^2 \mathfrak{p})(\vartheta,\eta).$$

The case $\omega = (1,1)$ is included and we have

$$({}^c D_\varepsilon^\omega \mathfrak{p})(\vartheta,\eta) = (D_{\vartheta\eta}^2 \mathfrak{p})(\vartheta,\eta);\ \text{for a.a } (\vartheta,\eta) \in \Theta.$$

**Lemma 2** (*[31]*)**.** *If $\mathfrak{W}$ is a bounded subset of Banach space $\mathbf{E}*$, then for each $\alpha > 0$, there is a sequence $\{\eta_\beta\}_{\beta=1}^\infty \subset \mathfrak{W}$ such that*

$$\mu(\mathfrak{W}) \le 2\mu(\{\eta_\beta\}_{\beta=1}^\infty) + \alpha,$$

*where $\mu$ is the Kuratowskii measure of noncompactness on the space $\mathbf{E}*$.*

**Lemma 3** (*[32]*)**.** *If $\{\mathfrak{p}_\beta\}_{\beta=1}^\infty \subset L^1(\Theta)$, then $\mu(\{\mathfrak{p}_\beta\}_{\beta=1}^\infty)$ is measurable and for each $(\vartheta,\eta) \in \Theta$,*

$$\mu\left(\left\{\int_0^\vartheta\int_0^\eta \mathfrak{p}_\beta(\varrho,\kappa)d\kappa d\varrho\right\}_{\beta=1}^\infty\right) \le 2\int_0^\vartheta\int_0^\eta \mu(\{\mathfrak{p}_\beta(\varrho,\kappa)\}_{\beta=1}^\infty)d\kappa d\varrho,$$

*where $\mu$ is the Kuratowskii measure of noncompactness on the space $\mathbf{E}*$.*

**Lemma 4** (*[33]*)**.** *Consider the continuous operator $\mathfrak{S} : \Lambda \to \Lambda$ where $\mathfrak{S}(\Lambda)$ is bounded and $\Lambda$ is a convex and closed subset of a real Banach space. If there exists a constant $\beta \in [0,1)$ such that for each bounded subset $\Omega \subset \Lambda$,*

$$\mu(\mathfrak{S}(\Omega)) \le \beta\mu(\Omega),$$

*then $\mathfrak{S}$ has a fixed point in $\Lambda$.*

### *The phase space $\mathcal{S}$*

The phase space $\mathcal{S}$ is fundamental in the analysis of functional differential equations. A semi-normed space meeting acceptable axioms is an appropriate option, as presented by Hale and Kato (see [34]). For other examples, check the book [35], and its sources.

For any $(\vartheta,\eta) \in \Theta$ denote $\gamma_{(\vartheta,\eta)} := [0,\vartheta]\times\{0\}\cup\{0\}\times[0,\eta]$, furthermore in case $\vartheta = \theta_1$, $\eta = \theta_2$, we denote $\gamma$. Let $(\mathcal{S}, \|(\cdot,\cdot)\|_{\mathcal{S}})$ be a seminormed linear space of functions from $\mathbb{R}^- \times \mathbb{R}^-$ to $\mathbb{R}^n$, and verifying:

$(A_1)$ If $\mathfrak{q} : (-\infty,\theta_1]\times(-\infty,\theta_2] \to \mathbb{R}^n$ continuous on $\Theta$ and $\mathfrak{q}_{(\vartheta,\eta)} \in \mathcal{S}$, for all $(\vartheta,\eta) \in \gamma$, then there are constants $\chi_1, \chi_2, \chi_3 > 0$ such that for any $(\vartheta,\eta) \in \Theta$ the assumptions that follow are met:
 (i) $\mathfrak{q}_{(\vartheta,\eta)}$ is in $\mathcal{S}$;
 (ii) $\|\mathfrak{q}(\vartheta,\eta)\| \le \chi_1 \|\mathfrak{q}_{(\vartheta,\eta)}\|_{\mathcal{S}}$,
 (iii) $\|\mathfrak{q}_{(\vartheta,\eta)}\|_{\mathcal{S}} \le \chi_2 \sup_{(\varrho,\kappa)\in[0,\vartheta]\times[0,\eta]} \|\mathfrak{q}(\varrho,\kappa)\| + \chi_3 \sup_{(\varrho,\kappa)\in\gamma_{(\vartheta,\eta)}} \|\mathfrak{q}_{(\varrho,\kappa)}\|_{\mathcal{S}}$,
$(A_2)$ For the function $\mathfrak{q}(\cdot,\cdot)$ in $(A_1)$, $\mathfrak{q}_{(\vartheta,\eta)}$ is a $\mathcal{S}$-valued continuous function on $\Theta$.
$(A_3)$ The space $\mathcal{S}$ is complete.

Now we will look at some phase space examples [36].

**Example 2.** Let $\mathcal{S}$ be the set of all functions $\varpi : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^n$ that are continuous on $[-\theta_1, 0] \times [-\theta_2, 0]$, $\theta_1, \theta_2 \geq 0$, with the seminorm

$$\|\varpi\|_{\mathcal{S}} = \sup_{(\varrho,\kappa)\in[-\theta_1,0]\times[-\theta_2,0]} \|\varpi(\varrho,\kappa)\|.$$

Thus we obtain $\chi_1 = \chi_2 = \chi_3 = 1$. The quotient space $\hat{\mathcal{S}} = \mathcal{S}/\|\cdot\|_{\mathcal{S}}$ is isometric to the space $C([-\theta_1, 0] \times [-\theta_2, 0], \mathbb{R}^n)$ of all continuous functions from $[-\theta_1, 0] \times [-\theta_2, 0]$ into $\mathbb{R}^n$ with the supremum norm.

**Example 3.** Let $C_\varsigma$ be the set of the continuous functions $\varpi : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^n$ where $\lim_{\|(\varrho,\kappa)\|\to\infty} e^{\varsigma(\varrho+\kappa)}\varpi(\varrho,\kappa)$ exists, with the norm

$$\|\varpi\|_{C_\varsigma} = \sup_{(\varrho,\kappa)\in\mathbb{R}^-\times\mathbb{R}^-} e^{\varsigma(\varrho+\kappa)}\|\varpi(\varrho,\kappa)\|.$$

Then we have $\chi_1 = 1$ and $\chi_2 = \chi_3 = \max\{e^{-(\theta_1+\theta_2)}, 1\}$.

**Example 4.** Let $\theta_1, \theta_2, \varsigma \geq 0$ and

$$\|\varpi\|_{CL_\varsigma} = \sup_{(\varrho,\kappa)\in[-\theta_1,0]\times[-\theta_2,0]} \|\varpi(\varrho,\kappa)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\varsigma(\varrho+\kappa)}\|\varpi(\varrho,\kappa)\|d\kappa d\varrho.$$

be the seminorm for the space $CL_\varsigma$ of all functions $\varpi : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^n$ which are continuous on $[-\theta_1, 0] \times [-\theta_2, 0]$ measurable on $(-\infty, -\mu] \times \mathbb{R}^- \cup \mathbb{R}^- \times (-\infty, -\theta_2]$, and such that $\|\varpi\|_{CL_\varsigma} < \infty$. Then

$$\chi_1 = 1,\ \chi_2 = \int_{-\theta_1}^{0}\int_{-\theta_2}^{0} e^{\varsigma(\varrho+\kappa)}d\kappa d\varrho,\ \chi_3 = 2.$$

## Main result

Let us start by giving the following result.

**Lemma 5** ([9,12]). *Let $\xi \in L^1(\Theta)$. The linear problem*

$$\begin{cases} {}^cD_\varepsilon^\zeta \mathfrak{p}(\vartheta,\eta) = \xi(\vartheta,\eta); & \text{for a.a. } (\vartheta,\eta) \in \Theta := [0,a]\times[0,\theta_2],\\ \mathfrak{p}(\vartheta,0) = \varpi_1(\vartheta); & \vartheta \in [0,\theta_1],\\ \mathfrak{p}(0,\eta) = \varpi_2(\eta); & \eta \in [0,\theta_2],\\ \varpi_1(0) = \varpi_2(0). \end{cases}$$

*has the following unique solution:*

$$\mathfrak{p}(\vartheta,\eta) = \varkappa(\vartheta,\eta) + I_\varepsilon^\zeta \xi(\vartheta,\eta); \quad \text{for a.a. } (\vartheta,\eta) \in \Theta,$$

*where*

$$\varkappa(\vartheta,\eta) = \varpi_1(\vartheta) + \varpi_2(\eta) - \varpi_1(0).$$

Suppose that $\psi$ is random Carathéodory on $\Theta\times\mathcal{S}\times\Psi$. The following Lemma 6 is derived from the preceding Lemma 5. Let the space

$$\Upsilon = \{\mathfrak{p} : (-\infty,\theta_1]\times(-\infty,\theta_2] \to \mathbf{E} :\ \mathfrak{p}_{(\vartheta,\eta)} \in \mathcal{S} \text{ for } (\vartheta,\eta) \in \gamma \text{ and } \mathfrak{p}|_\Theta \text{ is continuous}\}.$$

**Lemma 6.** *Let $0 < \zeta_1, \zeta_2 \leq 1$, $\varkappa(\vartheta,\eta,\delta) = \varpi_1(\vartheta,\delta) + \varpi_2(\eta,\delta) - \varpi_1(0,\delta)$. A function $\mathfrak{p} \in \Psi \times \Upsilon$ is a solution of (1)–(3) if $\mathfrak{p}$ verifies (2) for $(\vartheta,\eta) \in \tilde{\Theta}, \delta \in \Psi$ and $\mathfrak{p}$ is a solution of the equation*

$$\mathfrak{p}(\vartheta,\eta,\delta) = \varkappa(\vartheta,\eta,\delta) + \int_0^\vartheta\int_0^\eta \frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\vartheta,\eta,\mathfrak{p}_{(\varrho,\kappa)},\delta)d\kappa d\varrho$$

*for $(\vartheta,\eta) \in \Theta, \delta \in \Psi$.*

**The hypotheses**:

$(H_1)$ The functions $\delta \mapsto \varpi_1(\vartheta,0,\delta)$ *and* $\delta \mapsto \varpi_2(0,\eta,\delta)$ are measurable and bounded for $(\vartheta,\eta) \in \Theta$.
$(H_2)$ The function $\varpi$ is measurable for $(\vartheta,\eta) \in \tilde{\Theta}$.
$(H_3)$ The function $\psi$ is random Carathéodory on $\Theta\times\mathcal{S}\times\Psi$.
$(H_4)$ There exist functions $\sigma_1, \sigma_2 : \Theta\times\Psi \to [0,\infty)$ with

$$\sigma_j(\cdot,\delta) \in L^\infty(\Theta,[0,\infty));\ j = 1,2,$$

such that for each $\delta \in \Psi$,

$$\|\psi(\vartheta,\eta,\mathfrak{p},\delta)\|_{\mathbf{E}} \leq \sigma_1(\vartheta,\eta,\delta) + \sigma_2(\vartheta,\eta,\delta)\|\mathfrak{p}\|_{\mathcal{S}},$$

for all $\mathfrak{p} \in \mathcal{S}$ and a.e. $(\vartheta,\eta) \in \Theta$.
$(H_5)$ For any bounded $\Omega \subset \mathbf{E}$

$$\mu(\psi(\vartheta,\eta,\Omega,\delta)) \leq \sigma_2(\vartheta,\eta,\delta)\mu(\Omega),\ for\ a.e.\ (\vartheta,\eta) \in \Theta,$$

where

$$\sigma_i^*(\delta) = \sup ess_{(\vartheta,\eta)\in\Theta}\sigma_i(\vartheta,\eta,\delta);\ i = 1,2.$$

**Theorem 1.** *Suppose that $(H_1)-(H_5)$ are met. If*

$$\mathfrak{L} := \frac{4\sigma_2^*(\delta)\theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)} < 1,$$

*then (1)–(3) has a random solution.*

**Proof.** Define the operator $\mathfrak{T} : \Psi\times\Upsilon \to \Upsilon$ by

$$(\mathfrak{T}(\delta)\mathfrak{p})(\vartheta,\eta) = \begin{cases} \varpi(\vartheta,\eta,\delta), & (\vartheta,\eta) \in \tilde{\Theta}, \delta \in \Psi\\ \varkappa(\vartheta,\eta,\delta) + \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^\vartheta\int_0^\eta(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1} & \\ \times\psi(\varrho,\kappa,\mathfrak{p}_{(\varrho,\kappa)},\delta)d\kappa d\varrho, & (\vartheta,\eta) \in \Theta, \delta \in \Psi. \end{cases} \tag{4}$$

Since the functions $\varpi_1, \varpi_2$ and $\psi$ are absolutely continuous, then $\mathfrak{T}(\delta)$ defines a mapping $\mathfrak{T} : \Psi\times\Upsilon \to \Upsilon$. Hence $\mathfrak{p}$ is a solution for the problem (1)–(3) if and only if $\mathfrak{p} = (\mathfrak{T}(\delta))\mathfrak{p}$.

Let $\bar{\mathfrak{p}}(\cdot,\cdot,\cdot) : (-\infty,\theta_1]\times(-\infty,\theta_2]\times\Psi \to \mathbf{E}$ be a function defined by,

$$\bar{\mathfrak{p}}(\vartheta,\eta,\delta) = \begin{cases} \varpi(\vartheta,\eta,\delta), & (\vartheta,\eta) \in \tilde{\Theta}', \delta \in \Psi,\\ \varkappa(\vartheta,\eta,\delta), & (\vartheta,\eta) \in \Theta, \delta \in \Psi. \end{cases}$$

Then $\bar{\mathfrak{p}}_{(\vartheta,\eta)} = \varpi$ for all $(\vartheta,\eta) \in \gamma$. For each continuous function $I$ defined on $\Theta$ with $I(\vartheta,\eta,\delta) = 0$ for each $(\vartheta,\eta) \in \gamma$ we denote by $\overline{I}$ the function defined by

$$\overline{I}(\vartheta,\eta,\delta) = \begin{cases} 0, & (\vartheta,\eta) \in \tilde{\Theta}', \delta \in \Psi,\\ I(\vartheta,\eta,\delta) & (\vartheta,\eta) \in \Theta, \delta \in \Psi. \end{cases}$$

If $\mathfrak{p}(\cdot,\cdot,\cdot)$ verifies the equation:

$$\mathfrak{p}(\vartheta,\eta,\delta) = \varkappa(\vartheta,\eta,\delta) + \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^\vartheta\int_0^\eta(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1} \times \psi(\varrho,\kappa,\mathfrak{p}_{(\varrho,\kappa)},\delta)d\kappa d\varrho,$$

we can decompose $\mathfrak{p}(\cdot,\cdot,\cdot)$ as $\mathfrak{p}(\vartheta,\eta,\delta) = \overline{I}(\vartheta,\eta,\delta) + \bar{\mathfrak{p}}(\vartheta,\eta,\delta)$; $(\vartheta,\eta) \in \Theta$, which implies $\mathfrak{p}_{(\vartheta,\eta)} = \overline{I}_{(\vartheta,\eta)} + \bar{\mathfrak{p}}_{(\vartheta,\eta)}$, for every $(\vartheta,\eta) \in \Theta$, and the function $I(\cdot,\cdot,\cdot)$ satisfies

$$I(\vartheta,\eta,\delta) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^\vartheta\int_0^\eta(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1} \times \psi(\varrho,\kappa,\overline{I}_{(\varrho,\kappa)} + \bar{\mathfrak{p}}_{(\varrho,\kappa)},\delta)d\kappa d\varrho.$$

Set

$$\Omega_0 = \{I \in C(\Theta,\mathbf{E}) :\ I(\vartheta,\eta) = 0 \ \text{ for } (\vartheta,\eta) \in \gamma\},$$

and let $\|\cdot\|_{(\theta_1,\theta_2)}$ be the norm in $\Omega_0$ given by

$$\|I\|_{(\theta_1,\theta_2)} = \sup_{(\vartheta,\eta)\in\gamma}\|I_{(\vartheta,\eta)}\|_{\mathcal{S}} + \sup_{(\vartheta,\eta)\in\Theta}\|I(\vartheta,\eta)\| = \sup_{(\vartheta,\eta)\in\Theta}\|I(\vartheta,\eta)\|,\ I \in \Omega_0.$$

$\Omega_0$ is a Banach space with norm $\|\cdot\|_{(\theta_1,\theta_2)}$. Let $\mathfrak{H} : \Psi\times\ \Omega_0 \to \Omega_0$ be defined by:

$$(\mathfrak{H}(\delta)I)(\vartheta,\eta) = \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^\vartheta\int_0^\eta(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1} \times \psi(\varrho,\kappa,\overline{I}_{(\varrho,\kappa)} + \bar{\mathfrak{p}}_{(\varrho,\kappa)},\delta)d\kappa d\varrho, \tag{5}$$

for each $(\vartheta,\eta) \in \Theta$. Then $\mathfrak{T}$ has a fixed point is equivalent to $\mathfrak{H}$ has a fixed point. Now, we will demonstrate that $\mathfrak{H}$ verifies all the requirements of Lemma 4.

**Claim 1.** *$\mathfrak{H}(\delta)$ is a random operator with stochastic domain on $\Omega_0$.*

*Since $\psi(\vartheta,\eta,\mathfrak{p},\delta)$ is random Carathéodory, the map $\delta\to\psi(\vartheta,\eta,\mathfrak{p},\delta)$ is measurable in view of Definition 1. Also, the product $(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}\psi(\varrho,\kappa,\mathfrak{p}_{(\varrho,\kappa)},\delta)$ is measurable. Then*

$$\delta\mapsto\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\vartheta}\int_0^{\eta}(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}\psi(\varrho,\kappa,\overline{I}_{(\varrho,\kappa)}+\bar{\mathfrak{p}}_{(\varrho,\kappa)},\delta)d\kappa d\varrho,$$

*is measurable. Consequently, $\mathfrak{H}$ is a random operator on $\Psi\times\Omega_0$ into $\Omega_0$.*

*Let $\mathfrak{X}:\Psi\to\mathcal{P}(\Omega_0)$ be given by*

$$\mathfrak{X}(\delta)=\{I\in\Omega_0:\|I\|_{(\theta_1,\theta_2)}\le\rho(\delta)\},$$

*where*

$$\rho(\delta)\ge\frac{((\chi_2\|\varpi(0,0)\|+\chi_3\|\varpi\|)\sigma_2^*(\delta)+\sigma_1^*(\delta))\frac{\theta_1{}^{\zeta_1}\theta_2{}^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}}{1-\chi_2\sigma_2^*(\delta)\frac{\theta_1{}^{\zeta_1}\theta_2{}^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}}.$$

*$\mathfrak{X}(\delta)$ is a bounded, closed, convex and solid for all $\delta\in\Psi$. Then $\mathfrak{X}$ is measurable by Lemma 17 of [28]. Let $\delta\in\Psi$ be fixed, then by $(H_4)$, for any $\mathfrak{p}\in\delta(\delta)$, we obtain*

$$\begin{aligned}
&\|(\mathfrak{H}(\delta)I)(\vartheta,\eta)\|\\
&\le\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\|\psi(\varrho,\kappa,\overline{I}_{(\varrho,\kappa)}+\bar{\mathfrak{p}}_{(\varrho,\kappa)},\delta)\|d\kappa d\varrho\\
&\le\frac{\sigma_1^*(\delta)}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\vartheta}\int_0^{\eta}(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}d\kappa d\varrho\\
&\quad+\frac{\sigma_2^*(\delta)\rho^*(\delta)}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\vartheta}\int_0^{\eta}(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}d\kappa d\varrho\\
&\le\frac{(\sigma_1^*(\delta)+\sigma_2^*(\delta)\rho^*(\delta))\theta_1{}^{\zeta_1}\theta_2{}^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}\\
&\le\rho(\delta),
\end{aligned}$$

*where*

$$\begin{aligned}
\|\overline{I}_{(\varrho,\kappa)}+\bar{\mathfrak{p}}_{(\varrho,\kappa)}\|_S&\le\|\overline{I}_{(\varrho,\kappa)}\|_S+\|\bar{\mathfrak{p}}_{(\varrho,\kappa)}\|_S\\
&\le\chi_2\rho(\delta)+\chi_2\|\varpi(0,0)\|+\chi_3\|\varpi\|_S\\
&:=\rho^*(\delta).
\end{aligned}$$

*Thus, $\mathfrak{H}$ is a random operator with stochastic domain $\mathfrak{X}$ and $\mathfrak{H}(\delta):\mathfrak{X}(\delta)\to\mathfrak{X}(\delta)$. Moreover, $\mathfrak{H}(\delta)$ maps bounded sets into bounded sets in $\Omega_0$.*

**Claim 2.** *$\mathfrak{H}(\delta)$ is continuous.*

*Let $\{I_n\}$ be a sequence such that $I_n\to\mathfrak{p}$ in $\Omega_0$. Hence, for each $(\vartheta,\eta)\in\Theta$ and $\delta\in\Psi$, we get*

$$\begin{aligned}
&\|(\mathfrak{H}(\delta)I_n)(\vartheta,\eta)-(\mathfrak{H}(\delta)I)(\vartheta,\eta)\|_{\mathbf{E}}\\
&\le\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^{\vartheta}\int_0^{\eta}(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}\\
&\times\|\psi(\varrho,\kappa,\overline{I}_{n(\varrho,\kappa)}+\bar{\mathfrak{p}}_{n(\varrho,\kappa)},\delta)-\psi(\varrho,\kappa,\overline{I}_{(\varrho,\kappa)}+\bar{\mathfrak{p}}_{(\varrho,\kappa)},\delta)\|_F d\kappa d\varrho.
\end{aligned}$$

*Thus*

$$\|\mathfrak{H}(\delta)I_n-\mathfrak{H}(\delta)I\|_\infty\to0\text{ as }n\to\infty.$$

*As a result, we can deduce that $\mathfrak{H}(\delta):\mathfrak{X}(\delta)\to\mathfrak{X}(\delta)$ is a continuous random operator with stochastic domain $\mathfrak{X}$, and $\mathfrak{H}(\delta)(\mathfrak{X}(\delta))$ is bounded.*

**Claim 3.** *For each bounded subset $\Omega$ of $\mathfrak{X}(\delta)$ we obtain*

$$\mu_C(\mathfrak{H}(\delta)\Omega)\le\mathfrak{L}\mu_C(\Omega),$$

*where $\mu_C$ is a measure of noncompactness defined on $C(\Theta,\mathbf{E})$ by*

$$\mu_C(\Omega)=\sup_{(\vartheta,\eta)\in\Theta}\mu(\Omega(\vartheta,\eta)).$$

Let $\delta\in\Psi$ be fixed. From Lemmas 2 and 3, for any $\Omega\subset\mathfrak{X}$ and any $\alpha>0$, there exists a sequence $\{I_n\}_{n=0}^{\infty}\subset\Omega$, such that for all $(\vartheta,\eta)\in\Theta$, we have

$$\begin{aligned}
&\mu(\mathfrak{H}(\delta)\Omega)(\vartheta,\eta)\\
&=\mu\left(\left\{\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\varrho,\kappa,\overline{I}_{(\varrho,\kappa)}+\bar{\mathfrak{p}}_{(\varrho,\kappa)},\delta)d\kappa d\varrho;\ I\in\Omega\right\}\right)\\
&\le2\mu\left(\left\{\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\varrho,\kappa,\overline{I}_{n(\varrho,\kappa)}+\bar{\mathfrak{p}}_{n(\varrho,\kappa)},\delta)d\kappa d\varrho\right\}_{n=1}^{\infty}\right)+\alpha\\
&\le4\int_0^{\vartheta}\int_0^{\eta}\mu\left(\left\{\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\varrho,\kappa,\overline{I}_{n(\varrho,\kappa)}+\bar{\mathfrak{p}}_{n(\varrho,\kappa)},\delta)\right\}_{n=1}^{\infty}\right)d\kappa d\varrho+\alpha\\
&\le4\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\mu\left(\{\psi(\varrho,\kappa,\overline{I}_{n(\varrho,\kappa)}+\bar{\mathfrak{p}}_{n(\varrho,\kappa)},\delta)\}_{n=1}^{\infty}\right)d\kappa d\varrho+\alpha\\
&\le4\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\sigma_2(\varrho,\kappa,\delta)\mu\left(\{\overline{I}_{n(\varrho,\kappa)}+\bar{\mathfrak{p}}_{n(\varrho,\kappa)}\}_{n=1}^{\infty}\right)d\kappa d\varrho+\alpha\\
&\le\left(4\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\sigma_2(\varrho,\kappa,\delta)d\varrho d\kappa\right)\mu\left(\{I_{n(\varrho,\kappa)}\}_{n=1}^{\infty}\right)+\alpha\\
&\le\left(4\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\sigma_2(\varrho,\kappa,\delta)d\varrho d\kappa\right)\mu_C\left(\{I_n\}_{n=1}^{\infty}\right)+\alpha\\
&\le\left(4\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\sigma_2(\varrho,\kappa,\delta)d\varrho d\kappa\right)\mu_C\left(\{I_n\}_{n=1}^{\infty}\right)+\alpha\\
&\le\left(4\int_0^{\vartheta}\int_0^{\eta}\frac{(\vartheta-\varrho)^{\zeta_1-1}(\eta-\kappa)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\sigma_2(\varrho,\kappa,\delta)d\kappa d\varrho\right)\mu_C(\Omega)+\alpha\\
&\le\frac{4\sigma_2^*(\delta)\theta_1{}^{\zeta_1}\theta_2{}^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}\mu_C(\Omega)+\alpha\\
&=\mathfrak{L}\mu_C(\Omega)+\alpha.
\end{aligned}$$

Since $\alpha>0$, we get

$$\mu(\mathfrak{H}(\delta)\Omega)(\vartheta,\eta)\le\mathfrak{L}\mu_C(\Omega).$$

Then

$$\mu_C(\mathfrak{H}(\delta)\Omega)\le\mathfrak{L}\mu_C(\Omega).$$

Lemma 4 implies that for each $\delta\in\Psi$, $\mathfrak{H}$ has at least one fixed point in $\mathfrak{X}$. Since $\bigcap_{\delta\in\Psi}int\mathfrak{X}(\delta)\ne\emptyset$ and a measurable selector of $int\mathfrak{X}$ exists, By Lemma 4, $\mathfrak{T}$ has a stochastic fixed point, hence the existence of at least one random solution of (1)–(3).

## An example

Let $\mathbf{E}=\mathbb{R}$, $\Psi=(-\infty,0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. Consider the following problem:

$$\begin{aligned}
&({}^cD_\varepsilon^\zeta\mathfrak{p})(\vartheta,\eta,\delta)\\
&=\frac{ce^{\vartheta+\eta-\varsigma(\vartheta+\eta)}\|\mathfrak{p}_{(\vartheta,\eta)}\|}{(e^{\vartheta+\eta}+e^{-\vartheta-\eta})(1+\delta^2+\|\mathfrak{p}_{(\vartheta,\eta)}\|)};\\
&\text{a.a. }(\vartheta,\eta)\in\Theta=[0,1]\times[0,1],\ \delta\in\Psi,
\end{aligned}\tag{6}$$

$$\mathfrak{p}(\vartheta,\eta,\delta)=\vartheta\sin\delta+\eta^2\cos\delta,\ (\vartheta,\eta)\in(-\infty,1]\times(-\infty,1]\backslash(0,1]\times(0,1],\ \delta\in\Psi,\tag{7}$$

$$\mathfrak{p}(\vartheta,0,\delta)=\vartheta\sin\delta;\ \vartheta\in[0,1],\ \mathfrak{p}(0,\eta,\delta)=\eta^2\cos\delta;\ \eta\in[0,1],\ \delta\in\Psi,\tag{8}$$

where $\nu=\frac{8}{\Gamma(\zeta_1+1)\Gamma(\zeta_2+1)}$ and $\varsigma>0$.

Let

$$\mathcal{S}_\varsigma=\{\mathfrak{p}\in C(\mathbb{R}^-\times\mathbb{R}^-,\mathbb{R}):\lim_{\|(\varepsilon,\lambda)\|\to\infty}e^{\varsigma(\varepsilon+\lambda)}\mathfrak{p}(\varepsilon,\lambda)\text{ exists in }\mathbb{R}\},$$

with the norm

$$\|\mathfrak{p}\|_\varsigma=\sup_{(\varepsilon,\lambda)\in\mathbb{R}^-\times\mathbb{R}^-}e^{\varsigma(\varepsilon+\lambda)}|\mathfrak{p}(\varepsilon,\lambda)|.$$

Let

$$\mathbf{E}:=[0,1]\times\{0\}\cup\{0\}\times[0,1],$$

and $\mathfrak{p}:(-\infty,1]\times(-\infty,1]\to\mathbb{R}$ where $\mathfrak{p}_{(\vartheta,\eta)}\in\mathcal{S}_\varsigma$ for $(\vartheta,\eta)\in\mathbf{E}$, thus

$$\lim_{\|(\varepsilon,\lambda)\|\to\infty}e^{\varsigma(\varepsilon+\lambda)}\mathfrak{p}_{(\vartheta,\eta)}(\varepsilon,\lambda)=\lim_{\|(\varepsilon,\lambda)\|\to\infty}e^{\varsigma(\varepsilon-\vartheta+\lambda-\eta)}\mathfrak{p}(\varepsilon,\lambda)$$

$$=e^{-\varsigma(\vartheta+\eta)}\lim_{\|(\varepsilon,\lambda)\|\to\infty}e^{\varsigma(\varepsilon+\lambda)}\mathfrak{p}(\varepsilon,\lambda)<\infty.$$

Thus, $\mathfrak{p}_{(\vartheta,\eta)} \in \mathcal{S}_\varsigma$. We demonstrate that

$$\|\mathfrak{p}_{(\vartheta,\eta)}\|_\varsigma = \chi_2 \sup\{|\mathfrak{p}(\varrho,\kappa)| : (\varrho,\kappa) \in [0,\vartheta]\times[0,\eta]\} + \chi_3 \sup\{\|\mathfrak{p}_{(\varrho,\kappa)}\|_\varsigma : (\varrho,\kappa) \in E_{(\vartheta,\eta)}\},$$

where $\chi_2 = \chi_3 = 1$ and $\chi_1 = 1$.

If $\vartheta+\varepsilon \le 0,\ \eta+\lambda \le 0$ we obtain

$$\|\mathfrak{p}_{(\vartheta,\eta)}\|_\varsigma = \sup\{|\mathfrak{p}(\varrho,\kappa)| : (\varrho,\kappa) \in \mathbb{R}^-\times\mathbb{R}^-\},$$

and if $\vartheta+\varepsilon \ge 0,\ \eta+\lambda \ge 0$ then we get

$$\|\mathfrak{p}_{(\vartheta,\eta)}\|_\varsigma = \sup\{|\mathfrak{p}(\varrho,\kappa)| : (\varrho,\kappa) \in [0,\vartheta]\times[0,\eta]\}.$$

Then for all $(\vartheta+\varepsilon,\eta+\lambda) \in [0,1]\times[0,1]$, we have

$$\|\mathfrak{p}_{(\vartheta,\eta)}\|_\varsigma = \sup\{|\mathfrak{p}(\varrho,\kappa)| : (\varrho,\kappa) \in \mathbb{R}^-\times\mathbb{R}^-\} + \sup\{|\mathfrak{p}(\varrho,\kappa)| : (\varrho,\kappa) \in [0,\vartheta]\times[0,\eta]\}.$$

Then

$$\|\mathfrak{p}_{(\vartheta,\eta)}\|_\varsigma = \sup\{\|\mathfrak{p}_{(\varrho,\kappa)}\|_\varsigma : (\varrho,\kappa) \in \mathbf{E}\} + \sup\{|\mathfrak{p}(\varrho,\kappa)| : (\varrho,\kappa) \in [0,\vartheta]\times[0,\eta]\}.$$

$(\mathcal{S}_\varsigma, \|\cdot\|_\varsigma)$ is a Banach space. We conclude that $\mathcal{S}_\varsigma$ is a phase space. Set

$$\psi(\vartheta,\eta,\mathfrak{p}_{(\vartheta,\eta)}) = \frac{c e^{\vartheta+\eta-\varsigma(\vartheta+\eta)}\|\mathfrak{p}_{(\vartheta,\eta)}\|}{(e^{\vartheta+\eta}+e^{-\vartheta-\eta})(1+\|\mathfrak{p}_{(\vartheta,\eta)}\|)},\ (\vartheta,\eta) \in [0,1]\times[0,1].$$

The functions $\delta \mapsto \varpi_1(\vartheta,0,\delta) = \vartheta\sin\delta$, $\delta \mapsto \varpi_2(0,\eta,\delta) = \eta^2\cos\delta$ and $\delta \mapsto \varpi(\vartheta,\eta,\delta) = \vartheta\sin\delta + \eta^2\cos\delta$ are measurable and bounded with

$$|\varpi_1(\vartheta,0,\delta)| \le 1,\ |\varpi_2(0,\eta,\delta)| \le 1,$$

Thus, $(H_1)$ is verified.

Obviously, $(\vartheta,\eta,\delta) \mapsto \psi(\vartheta,\eta,\mathfrak{p},\delta)$ is jointly continuous for all $\mathfrak{p} \in \mathcal{S}_\varsigma$, thus jointly measurable for all $\mathfrak{p} \in \mathcal{S}_\varsigma$. $\mathfrak{p} \mapsto \psi(\vartheta,\eta,\mathfrak{p},\delta)$ is continuous for all $(\vartheta,\eta) \in \Theta$ and $\delta \in \Psi$. So the function $\psi$ is Carathéodory on $[0,1]\times[0,1]\times\mathcal{S}_\varsigma\times\Psi$.

For each $\mathfrak{p} \in \mathcal{S}_\varsigma$, $(\vartheta,\eta) \in [0,1]\times[0,1]$ and $\delta \in \Psi$, we have

$$|\psi(\vartheta,\eta,\mathfrak{p}_{(\vartheta,\eta)})| \le 1 + \frac{1}{\nu}\|\mathfrak{p}\|_{\mathcal{S}}.$$

Thus $(H_4)$ is verified with

$$\sigma_1(\vartheta,\eta,\delta) = \sigma_1^* = 1,\ \sigma_2(\vartheta,\eta,\delta) = \sigma_2^* = \frac{1}{\nu}.$$

Also, $(H_5)$ is met.

We will prove that $\mathfrak{L} < 1$ with $\theta_1 = \theta_2 = 1$. For each $(\zeta_1,\zeta_2) \in (0,1]\times(0,1]$ we get

$$\begin{aligned}\mathfrak{L} &= \frac{4\sigma_2^*\theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}\\ &= \frac{4}{\nu\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}\\ &< \frac{1}{2}\\ &< 1.\end{aligned}$$

Consequently, Theorem 1 implies that the problem (6)-(8) has a random solution defined on $(-\infty,1]\times(-\infty,1]$.

## Conclusion

In this paper, we demonstrated certain existence results for the Darboux problem of partial fractional random differential equations with infinite delay using a random fixed point theorem with stochastic domain paired with the measure of noncompactness. Finally, we have provided a clear example to highlight the applicability of our main result. As a result, we expect that our work will pave the way for us to pursue new applications and broader problems, such as generalizing the problem using newly defined fractional derivatives.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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