Research Article

Ved Prakash Dubey, Devendra Kumar*, Hashim M. Alshehri, Jagdev Singh, and Dumitru Baleanu

Generalized invexity and duality in multiobjective variational problems involving non-singular fractional derivative

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Abstract: In this article, we extend the generalized invexity and duality results for multiobjective variational problems with fractional derivative pertaining to an exponential kernel by using the concept of weak minima. Multiobjective variational problems find their applications in economic planning, flight control design, industrial process control, control of space structures, control of production and inventory, advertising investment, impulsive control problems, mechanics, and several other engineering and scientific problems. The proposed work considers the newly derived Caputo–Fabrizio (CF) fractional derivative operator. It is actually a convolution of the exponential function and the first-order derivative. The significant characteristic of this fractional derivative operator is that it provides a non-singular exponential kernel, which describes the dynamics of a system in a better way. Moreover, the proposed work also presents various weak, strong, and converse duality theorems under the diverse generalized invexity conditions in view of the CF fractional derivative operator.

Keywords: Multiobjective variational problem, weak minima, Caputo–Fabrizio fractional derivative, generalized invexity, duality

*** Corresponding author: Devendra Kumar,** Department of Mathematics, University of Rajasthan, Jaipur-302004, Rajasthan, India; Department of Mathematics, Faculty of Sciences, Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, King Abdulaziz University, Jeddah, 21589, Saudi Arabia, e-mail: devendra.maths@gmail.com, tel: +91-9460905223
Ved Prakash Dubey: Faculty of Mathematical and Statistical Sciences, Shri Ramswaroop Memorial University, Lucknow-Deva Road, Uttar Pradesh-225003, India, e-mail: vpdubey19@gmail.com
Hashim M. Alshehri: Department of Mathematics, Faculty of Sciences, Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, King Abdulaziz University, Jeddah, 21589, Saudi Arabia, e-mail: hmalshehri@kau.edu.sa
Jagdev Singh: Department of Mathematics, Faculty of Sciences, Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, King Abdulaziz University, Jeddah, 21589, Saudi Arabia; Department of Mathematics, JECRC University, Jaipur-303905, Rajasthan, India, e-mail: jagdevsinghrathore@gmail.com
Dumitru Baleanu: Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Eskisehir Yolu 29. Km, Yukarıyurtcu Mahallesi Mimar Sinan Caddesi No: 4 06790, Etimesgut, Turkey; Institute of Space Sciences, Magurele-Bucharest, Romania; Lebanese American University, Beirut, Lebanon, e-mail: dumitru@cankaya.edu.tr

1 Introduction

The present scenario indicates that the fractional differential equations (FDEs) and fractional variational problems (FVPs) are being used to delineate the physical models and engineering processes in a better way. The clear reason is that the standard mathematical models of integer-order derivatives incorporating models of nonlinear nature do not perform efficiently in many instances according to desired results. Recently, the field of fractional calculus has portrayed a significant part in various areas of knowledge such as chemistry [1], biology [2,3], mechanics [4,5], and finance [6]. The application area of fractional modelling and fractional operators encompasses anomalous diffusion [7], physics [8], heat conduction [9], geophysics [10], epidemiology [11], fractals and fractional derivative [12], computational fractional derivative equations [13], fractional predator–prey system [14], and porous media [15]. The models related to these fields utilize fractional derivative operators frequently.

There are various types of fractional derivative operators in the literature of fractional calculus founded by so many famous mathematicians. But the most popular definitions of them are Riemann–Liouville (RL) fractional derivative and Riesz fractional derivative described in the studies of Samko *et al.* [16] and Podlubny [17], Caputo fractional derivative in refs. [18,19], Weyl fractional derivative [20], Hadamard fractional derivative

[16,17], Jumarie's fractional derivative propounded in works of Jumarie [21,22], Atangana–Baleanu derivative proposed in ref. [23], and Liouville–Caputo derivative described in ref. [24]. The interesting fact is that these definitions of fractional derivative operators have their own significance and their uses vary according to the structure and behaviour of particular models along with initial conditions. A wide literature is available on different perceptions of fractional derivatives. But the most celebrated fractional calculi are the Caputo fractional derivative and the RL derivative. The Caputo fractional derivative handles initial value problems efficiently in comparison to the RL derivative. The newly introduced Caputo–Fabrizio (CF) fractional derivative operator propounded by Caputo and Fabrizio in ref. [25] is actually a convolution of an exponential function and the first-order derivative. In this definition, the derivative of a constant is equal to zero like the usual Liouville–Caputo definition but it also provides the non-singular kernel which was not a characteristic of the Liouville–Caputo fractional derivative. The main purpose of the CF definition was to introduce a new fractional derivative with an exponential kernel to describe even better the dynamics of systems with memory effect.

Recently, some authors presented a new analysis on fractional modelling of real-world problems and application of fractional order Lagrangian approach towards study of problems arising in physical sciences and engineering. Some recent works related to these fields are necessary to be cited here. Jajarmi *et al.* [26] suggested a general fractional formulation for immunogenic tumour dynamics. Baleanu *et al.* [27] presented a new study on the general fractional model of COVID-19 with isolation and quarantine effects. Erturk *et al.* [28] utilized a new fractional-order Lagrangian to describe the dynamics of a beam on nanowire. Jajarmi *et al.* [29] implemented a new fractional Lagrangian approach to study the case of capacitor microphone. Dubey *et al.* [30] solved the fractional model of Phytoplankton–Toxic Phytoplankton–Zooplankton system with convergence analysis. Moreover, a fractional model of atmospheric dynamics of carbon dioxide gas [31] and a fractional-order hepatitis E virus model [32] were also recently investigated with efficient computational methods.

Multiobjective variational problems proficiently handle the problems of science, engineering, logistics, and economics where optimal decisions have to be decided between two or more clashing objectives. To derive the optimality conditions it is necessary to study the behaviour of functions and their derivatives at that point. In the theory of mathematical optimization, the duality principle indicates two perspectives of optimization problems: the primal problem and the dual problem. If the primal is a minimization problem, then the dual is a maximization problem, and if the primal is a maximization problem, then the dual is a minimization problem. The concept of duality considers a problem with less number of variables and constraints and so it is much advantageous regarding computational procedure. Duality results play a major role in construction of numerical algorithms for solving some specific types of optimization problems. The duality theory is applied mainly in economics, management, physics, *etc.* On the other hand, calculus of variations significantly deals with the solution of several problems arising in theory of variations, optimization of orbits, dynamics of rigid bodies, *etc.* It is closely related to optimization of functional and is expressed in terms of definite integrals pertaining to functions and their derivatives. In the past few years, a number of contributions have been made towards the duality results for multiobjective variational problems. For the first time, Hanson [33] established and developed the linkage between classical calculus of variation and mathematical programming. After that, Mond and Hanson [34] derived optimality and duality results for scalar valued variational problems in view of convexity assumptions. Chandra *et al.* [35] studied optimality and duality for a class of non-differentiable variational problems. In this sequence, Bector and Husain [36] investigated duality for multiobjective variational problems. Nahak and Nanda [37] and Chen [38] constructed duality results for multiobjective variational problems with invexity. Some years later, Bhatia and Mehra [39] extended further the results of Mond *et al.* [40] and explored the optimality conditions and duality results for multiobjective variational problems with generalized B-invexity.

The concept of invexity is of great significance in variational problems and mathematical programming. Hanson [41] introduced the notion of invexity to mathematical programming. Mishra and Mukherjee [42] presented duality results for multiobjective FVPs. Furthermore, Mond and Husain [43] also investigated sufficient optimality criteria and duality for variational problems with generalized invexity. It is clearly observed that the duality results derived for variational problems presented in refs. [40,42,43] that hold for convex functions are also well-fitted for the wide range of invex functions. Weir and Mond [44] considered the concept of weak minima to derive the duality results for multiobjective programming problems. Different scalar duality results have also been extended for multiobjective programming problems by Weir and Mond [44]. Mukherjee and Mishra [45] have considered the concept of weak minima in the continuous case and have delivered a complete generalization of the results of Weir and Mond [44] to multiobjective variational problems. Moreover, they also relaxed the generalized convexity conditions to generalized invexity conditions.

Recently, Kumar [46] extended the invexity for continuous functions to invexity of order m. They further generalized the invexity of order m to ρ-pseudoinvexity type-I of order m, ρ-pseudoinvexity type-II of order m, as well as ρ-quasi-invexity type-I and type-II of order m. In 2016, Kumar *et al.* [47] also analysed the multiobjective FVP under F-Kuhn–Tucker (KT) pseudoinvexity conditions. Hachimi and Aghezzaf [48] established the mixed duality results and the sufficient optimality conditions concerning multiobjective variational problems under generalized (F,α,ρ,d)-type I functions which assimilate the several concepts of generalized type-I functions successfully. Later on, Mishra *et al.* [49] extended the generalized type-I invexity and duality for non-differentiable multiobjective variational problems. In 2014, Wolfe-type and Mond–Weir-type duality results were formulated for multiobjective variational control models under (ϕ,ρ)-invexity conditions by Antczak [50]. More recently, Upadhyay *et al.* [51] presented optimality conditions and duality for multiobjective semi-infinite programming problems on Hadamard manifolds utilizing generalized geodesic convexity. Moreover, Upadhyay *et al.* [52] also investigated Minty's variational principle for non-smooth multiobjective optimization problems on Hadamard manifolds. Guo *et al.* [53] showed applications of symmetric gH-derivative to dual interval-valued optimization problems in a very efficient way. Furthermore, optimality conditions and duality for a class of generalized convex interval-valued optimization problems are recently investigated in works of Guo *et al.* [54].

The main purpose of this study is to derive the weak and strong duality results for multiobjective variational problems pertaining to a CF fractional derivative operator with exponential kernel. The CF fractional derivative possesses the non-singular kernel and so is better than the Caputo and RL fractional derivative operators. The proposed work presents the derivation of duality as well as strict converse duality theorems for variational problems with CF fractional derivative by employing some propositions and theorems of fractional calculus. In this article, we propounded first the optimality conditions for the variational problem. Furthermore, we present Theorem 1 which proves that a minimizer of the variational problem is a solution of the fractional Euler–Lagrange equation containing the CF fractional derivative. Now, we derived the formula for integration by parts for the CF fractional derivative in Proposition 1. The extended invexity definitions in view of CF fractional derivative operator along with Proposition 1 and Theorem 1 have been the key motivation behind the study of the variational problems with fractional calculus approach. Theorem 2 proves the fact that if a function is convex, the solution of the fractional Euler–Lagrange equation containing the CF fractional derivative will be a minimizer of the variational problem. Theorems 3 and 4 present the results for primal variational problems having a weak minimum. Furthermore, Theorems 5–10 are concerned with weak and strong duality results depending on the CF fractional derivative. Finally, Theorems 11 and 12 provide the strict converse duality results in view of the CF fractional derivative.

In the present work, the concept of weak minima has been considered and the generalizations of weak, strong, and strict converse duality results of Mukherjee and Mishra [45] have been extended to multiobjective variational problems pertaining to the CF fractional derivative operator. The remaining part of the article is organized as follows: In Section 2, we present the elemental definitions, formulae, and theorems regarding invexity and fractional derivative operators. Section 3 derives weak and strong duality results. Section 4 presents a strict converse duality result. Finally, Section 5 records the epilogue for the proposed work.

2 Basic definitions, theorems, and symbols

We follow these definitions and symbols in the present article.

Definition 1. [55]: The left and the right RL fractional derivatives of order α are defined by

$$ {}_aD_\xi^\alpha y(\xi) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_a^\xi (\xi-\tau)^{-\alpha}y(\tau)\mathrm{d}\tau, $$

$$ {}_\xi D_b^\alpha y(\xi) = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_\xi^b (\tau-\xi)^{-\alpha}y(\tau)\mathrm{d}\tau, \quad \alpha \in (0,\ 1). $$

Definition 2. [56]: The Caputo fractional derivative of $y(\xi) : [a, b] \to \mathfrak{R}$ of order $\alpha \in (0, 1)$ is stated as:

$$ {}^CD_{a+}^\alpha y\ (\xi) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_a^\xi \frac{1}{(\xi-\tau)^\alpha}[y(\tau)-y(a)]\mathrm{d}\tau. $$

If $y \in C^1$, then

$$ {}^{C}D_{a+}^{\alpha}y\ (\xi) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{\xi}\frac{1}{(\xi-\tau)^{\alpha}}y'(\tau)\mathrm{d}\tau. $$

As $\alpha \to 1$, ${}^{C}D_{a+}^{\alpha}y\ (\xi)$ approaches to $y'(\xi)$.

Definition 3. [25]: The new CF fractional derivative operator is described as follows:

$$ {}^{\mathrm{CF}}D_{a+}^{\alpha}y\ (\xi) = \frac{K(\alpha)}{(1-\alpha)}\int_{a}^{\xi}\exp\left(-\frac{\alpha(\xi-\tau)}{1-\alpha}\right)y'(\tau)\mathrm{d}\tau, \quad \alpha \in (0,\ 1), $$

where $K(\alpha)$ signifies the normalization function with the property $K(0) = K(1) = 1$. Clearly, ${}^{\mathrm{CF}}D_{a+}^{\alpha}y\ (\xi) = 0$ if $y(\xi)$ is a constant function, *i.e.* the CF derivative of a constant function vanishes to zero same as the Caputo derivative but kernel of CF derivative does not have singularity for $\xi = \tau$ like the Caputo fractional derivative. It is remarkable that the CF fractional derivative has an exponential kernel.

Remark 1. Here we consider the value of $K(\alpha)$ as $(1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}$.

Remark 2. As $\alpha \to 1$, ${}^{\mathrm{CF}}D_{a+}^{\alpha}y\ (\xi)$ approaches to $y'(\xi)$ and as $\alpha \to 0$, ${}^{\mathrm{CF}}D_{a+}^{\alpha}y\ (\xi)$ approaches to $y(\xi) - y(a)$.

Definition 4. Abdeljawad and Baleanu [57] have defined the right CF fractional derivative as

$$ {}^{\mathrm{CF}}D_{b-}^{\alpha}y\ (\xi) = -\frac{K(\alpha)}{(1-\alpha)}\int_{\xi}^{b}\exp\left(-\frac{\alpha(\tau-\xi)}{1-\alpha}\right)y'(\tau)\mathrm{d}\tau, \quad \alpha \in (0,\ 1). $$

Definition 5. The first-order Sobolev space defined in the interval (a, b) is stated as $H^1(a,\ b) = \{x \in L^2(a,\ b)\ |x' \in L^2(a,\ b)\}$, where x' denotes the weak derivative of x.

Definition 6. [25]: Let $y \in H^1(a, b)$, $b > a$, $0 < \alpha < 1$, then the CF fractional derivative is stated as in Definition (3), where $K(\alpha)$ specifies the normalization function with characteristic $K(0) = K(1) = 1$. If the function $y \notin H^1(a, b)$, then the derivative is formulated as follows:

$$ {}^{\mathrm{CF}}D_{a+}^{\alpha}y\ (\xi) = \frac{\alpha\ K(\alpha)}{(1-\alpha)}\int_{a}^{\xi}\exp\left(-\frac{\alpha(\xi-\tau)}{1-\alpha}\right)[y\ (\xi) - y\ (\tau)]\mathrm{d}\tau. $$

Here, the CF fractional derivative has an exponential kernel.

Definition 7. [57]: Let x be a function in such a way that $x \in H^1(a, b)$ $a < b$. The left Riemann fractional derivative of order α in the CF sense is given by

$$ {}^{\mathrm{CFR}}D_{a+}^{\alpha}y(\xi) = \frac{K(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_{a}^{\xi}\exp\left[-\frac{\alpha}{1-\alpha}(\xi-\tau)\right]y(\tau)\mathrm{d}\tau, $$

where $a \le \xi$, $\alpha(0 < \alpha < 1)$ is a real number and $K(\alpha)$ is a normalization function depending on α with $K(0) = K(1) = 1$.

Similarly, the right Riemann fractional derivative of order α in the CF sense can be written as follows:

$$ {}^{\mathrm{CFR}}D_{b-}^{\alpha}y(\xi) = -\frac{K(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_{\xi}^{b}\exp\left[-\frac{\alpha}{1-\alpha}(\tau-\xi)\right]y(\tau)\mathrm{d}\tau, $$

where $\xi \le b$.

Remark 3. When $\alpha \to 0$, $\lim_{\alpha\to 0}{}^{\mathrm{CFR}}D_{a+}^{\alpha}y(\xi) = \frac{\mathrm{d}}{\mathrm{d}\xi}\int_{a}^{\xi}y(\tau)\mathrm{d}\tau = y(\xi)$.

Proposition 1. *Let $\alpha \in (0,\ 1)$ and $y, z : [a, b] \to \Re$ be two continuous functions of class $C^1[a, b]$. Then the following formula for integration by parts holds:*

$$ \int_{a}^{b}y\ (\xi){}^{\mathrm{CF}}D_{a+}^{\alpha}z\ (\xi)\mathrm{d}\xi = [z(\xi)\ I_{b-}^{1-\alpha}y\ (\xi)]_{\xi=a}^{\xi=b} + \int_{a}^{b}z(\xi){}^{\mathrm{CFR}}D_{b-}^{\alpha}y\ (\xi)\ \mathrm{d}\xi. $$

Proof. We define the left and right auxiliary fractional integrals as

$$ I_{a+}^{1-\alpha}z(\xi) = \frac{K(\alpha)}{(1-\alpha)}\int_{a}^{\xi}\exp\left(-\frac{\alpha}{1-\alpha}(\xi-\tau)\right)z(\tau)\mathrm{d}\tau, \tag{1} $$

$$ I_{b-}^{1-\alpha}z(\xi) = \frac{K(\alpha)}{(1-\alpha)}\int_{\xi}^{b}\exp\left(-\frac{\alpha}{1-\alpha}(\tau-\xi)\right)z(\tau)\mathrm{d}\tau. \tag{2} $$

Now in view of Definition (3) and Eq. (1), it is concluded that

$$ {}^{\mathrm{CF}}D_{a+}^{\alpha}z\ (\xi) = I_{a+}^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}\xi}z\ (\xi). \tag{3} $$

In the next step, we evaluate the integral $\int_{a}^{b}y\ (\xi)\ {}^{\mathrm{CF}}D_{a+}^{\alpha}z(\xi)\mathrm{d}\xi$ as follows.

Using Eq. (3) along with further utilization of Theorem 1 of ref. [57] and integration by parts for classical derivatives, we obtain

$$\int_a^b y(\xi)^{\mathrm{CF}}D_{a+}^{\alpha}z\,(\xi)\,\mathrm{d}\xi = \int_a^b y(\xi)\left\{I_{a+}^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}\xi}z(\xi)\right\}\mathrm{d}\xi$$
$$= \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}\xi}z(\xi)\right)I_{b-}^{1-\alpha}y(\xi)\mathrm{d}\xi$$
$$= \int_a^b I_{b-}^{1-\alpha}y(\xi)\left(\frac{\mathrm{d}}{\mathrm{d}\xi}z(\xi)\right)\mathrm{d}\xi$$
$$= [z(\xi)I_{b-}^{1-\alpha}y(\xi)]_{\xi=a}^{\xi=b} - \int_a^b z(\xi) \times\left[\frac{K(\alpha)}{(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_{\xi}^{b}\exp\left(-\frac{\alpha}{1-\alpha}(\xi-\tau)\right)y(\tau)\mathrm{d}\tau\right]\mathrm{d}\xi.$$

Now in view of Definition 7, we obtain

$$\int_a^b y(\xi)^{\mathrm{CF}}D_{a+}^{\alpha}z(\xi)\mathrm{d}\xi = [z(\xi)I_{b-}^{1-\alpha}y\,(\xi)]_{\xi=a}^{\xi=b} + \int_a^b z(\xi)^{\mathrm{CFR}}D_{b-}^{\alpha}y(\xi)\mathrm{d}\xi. \qquad \square$$

Definition 8. (Optimality conditions for variational problems):

The following variational problem with the CF fractional derivative is considered here for given $y \in C^1(a,b)$,

$$\min V(y) = \int_a^b Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi, \tag{4}$$

with $y(a) = y_a$ and $y(b) = y_b$, where $y(a),\ y(b) \in \Re$. The assumptions are as follows:

1. $Q : [a,b] \times \Re^2 \to \Re$ is continuously differentiable w.r.t. the second and third arguments.
2. Given any x, the map $\xi \to {}^{\mathrm{CFR}}D_{b-}^{\alpha}(\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y\,(\xi))) = 0$ is continuous.

Here, we denote $\partial_i g(y_1, y_2, \dots y_n) = \frac{\partial g}{\partial y_i}(y_1, y_2, \dots y_n)$ for a function $g : T \subseteq \Re^n \to \Re$.

Theorem 1. *Let y be a minimizer of the variational $V(y)$ defined on $E = \{y \in C^1(a,b) : y(a) = y_a, y(b) = y_b\}$, where $y_a, y_b \in \Re$ are fixed. Then y is a solution of the following fractional Euler–Lagrange equation*:

$$\partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y\,(\xi)) + {}^{\mathrm{CFR}}D_{b-}^{\alpha}(\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y\,(\xi))) = 0, \forall \xi \in [a,b]\,. \tag{5}$$

Proof. Let y be a solution for the functional $V(y)$. Assume $y + \delta\omega$ be a variation of y with $|\delta| \ll 1$, and $\omega : [a,\ b] \to R$ be a function of class $C^1[a,b]$ in such a way that the conditions $\omega\,(a) = \omega\,(b) = 0$ hold. Let $\vartheta(\delta) = V(y + \delta\omega)$. Since y satisfies Eq. (4) as a solution, the first variation of V must vanish, and hence $\vartheta'(0) = 0$. Now, computing $\vartheta'(\delta)|_{\delta=0}$, equating to zero, and further employing Proposition 1, we have

$$\int_a^b \partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\omega(\xi)\mathrm{d}\xi + \int_a^b \partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y\,(\xi))^{\mathrm{CF}}D_{a+}^{\alpha}\omega(\xi)\mathrm{d}\xi$$
$$= \int_a^b \partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\omega(\xi)\mathrm{d}\xi + \int_a^b \omega(\xi)^{\mathrm{CFR}}D_{b-}^{\alpha}(\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)))\mathrm{d}\xi + [\omega(\xi)I_{b-}^{1-\alpha}(\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)))]_{\xi=a}^{\xi=b}$$
$$= \int_a^b [\partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + {}^{\mathrm{CFR}}D_{b-}^{\alpha}(\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y\,(\xi)))]\omega(\xi)\mathrm{d}\xi, \quad (\because \omega(a) = \omega(b) = 0) = 0.$$

Now utilizing the boundary conditions $\omega\,(a) = \omega\,(b) = 0$ along with the assumption of arbitrariness of ω, we obtain the desired equation as:

$$\partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + {}^{\mathrm{CFR}}D_{b-}^{\alpha}(\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))) = 0\ \forall \xi \in [a,b]. \qquad \square$$

Remark 4. Eq. (5) is called the Euler–Lagrange equation associated with the variational $V(y)$ and the solutions for this equation are termed as extremals.

Remark 5. It is notable that Eq. (5) provides necessary criterion only. Now to obtain sufficient criterion, the concept of convex function is necessary to recall.

Definition 9. A function $Q(\xi,\ \wp,\ \aleph)$ is said to be convex in $T \subseteq R_3$ if Q possesses continuous derivatives in respect of the second and third arguments and also satisfies the following inequality:

$$Q(\xi, \wp + \wp_1, \aleph + \aleph_1) - Q(\xi, \wp, \aleph) \geq \partial_2 Q(\xi, \wp, \aleph)\wp_1 + \partial_3 Q(\xi, \wp, \aleph)\aleph_1,$$
$$\forall(\xi, \wp, \aleph), (\xi, \wp + \wp_1, \aleph + \aleph_1) \in T.$$

Theorem 2. *If the function Q as described in Eq.* (4) *is convex in* $[a, b] \times \Re^2$, *then each solution of the fractional Euler–Lagrange* Eq. (5) *minimizes V in E.*

Proof. Let y be a solution for the fractional Euler–Lagrange Eq. (5). Assume $y + \delta\omega$ to be a variation of y with $|\delta| \ll 1$, and $\omega : [a, b] \to \Re$ is a function that belongs to $C^1[a, b]$ such that the boundary conditions $\omega(a) = \omega(b) = 0$ hold. Now, we compute $V(y + \delta\omega) - V(y)$ in view of Definition 9 and Proposition 1 as follows:

$$\begin{aligned}
&V(y + \delta\omega) - V(y)\\
&= \int_a^b [Q(\xi, y(\xi) + \delta\omega(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi) + \delta\, {}^{\mathrm{CF}}D_{a+}^{\alpha}\omega(\xi))\\
&\quad - Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))]\mathrm{d}\xi\\
&\geq \int_a^b [\partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\delta\omega(\xi)\\
&\quad + \partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\delta\, {}^{\mathrm{CF}}D_{a+}^{\alpha}\omega(\xi)]\mathrm{d}\xi\\
&= \int_a^b \partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\delta\omega(\xi)\mathrm{d}\xi\\
&\quad + \int_a^b \partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\delta\, {}^{\mathrm{CF}}D_{a+}^{\alpha}\omega(\xi)\mathrm{d}\xi\\
&= \int_a^b \partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\delta\omega(\xi)\mathrm{d}\xi \qquad (6)\\
&\quad + \delta[\omega(\xi)I_{b-}^{1-\alpha}\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))]_{\xi=a}^{\xi=b}\\
&\quad + \int_a^b \delta\omega(\xi)\, {}^{\mathrm{CFR}}D_{b-}^{\alpha}\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi\\
&= \int_a^b \partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\delta\omega(\xi)\mathrm{d}\xi\\
&\quad + \int_a^b \delta\omega(\xi)\, {}^{\mathrm{CFR}}D_{b-}^{\alpha}\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi\\
&(\because \omega(a) = \omega(b) = 0),\\
&= \int_a^b [\partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\\
&\quad + {}^{\mathrm{CFR}}D_{b-}^{\alpha}\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))]\delta\omega(\xi)\mathrm{d}\xi.
\end{aligned}$$

Since y is a solution of the fractional Euler–Lagrange Eq. (5), we have

$$\partial_2 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + {}^{\mathrm{CFR}}D_{b-}^{\alpha}\partial_3 Q(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) = 0. \qquad (7)$$

Hence in view of Eq. (7), Eq. (6) provides the inequality as follows:

$$V(y + \delta\omega) - V(y) \geq 0. \qquad (8)$$

Consequently, $V(y + \delta\omega) \geq V(y)$, which implies that y is a local minimizer of V. □

Definition 10. Invexity definitions

Let $\Omega = [a, b]$ be a real interval. Let $g : \Omega \times \Re^n \times \Re^n \to \Re$ be a continuously differentiable function. Consider $g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))$, where $y : \Omega \to \Re^n$ is a function of class $C^1[a, b]$ and ${}^{\mathrm{CF}}D_{a+}^{\alpha}y$ represents the CF derivative of order $0 < \alpha < 1$ of a function y. We denote the partial derivatives of g by

$$g_\xi = \frac{\partial g}{\partial \xi}, g_y = \left[\frac{\partial g}{\partial y^1}, \frac{\partial g}{\partial y^2}, \frac{\partial g}{\partial y^3}, \ldots, \frac{\partial g}{\partial y^n}\right],$$

$$g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}y} = \left[\frac{\partial g}{\partial({}^{\mathrm{CF}}D_{a+}^{\alpha}y^1)}, \frac{\partial g}{\partial({}^{\mathrm{CF}}D_{a+}^{\alpha}y^2)}, \frac{\partial g}{\partial({}^{\mathrm{CF}}D_{a+}^{\alpha}y^3)}, \ldots, \frac{\partial g}{\partial({}^{\mathrm{CF}}D_{a+}^{\alpha}y^n)}\right].$$

Let Y be the space of piecewise smooth functions $y : \Omega \to \Re^n$ along with the norm $\|y\| = \|y\|_\infty + \|Dy\|_\infty$, where the differential operator D is described as follows:

$$v = Dy \Leftrightarrow y(\xi) = y_0 + \int_a^\xi v(s)\mathrm{d}s,$$

where y_0 signifies the boundary value.

Let $G : Y \to \Re$ defined by $G(y) = \int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi$ be Fréchet differentiable. For notational convenience $g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))$ will be written as $g(\xi, y, {}^{\mathrm{CF}}D_{a+}^{\alpha}y)$. Here, it is assumed that $g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))$ is convex in $\Re^3$ if $\partial_2 g$ and $\partial_3 g$ exist and are continuous, and the condition

$$g(\xi, y + y_1, {}^{\mathrm{CF}}D_{a+}^{\alpha}y + {}^{\mathrm{CF}}D_{a+}^{\alpha}y_2) - g(\xi, y, {}^{\mathrm{CF}}D_{a+}^{\alpha}y) \geq g_y(\xi, y, {}^{\mathrm{CF}}D_{a+}^{\alpha}y)y_1 + g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}y}(\xi, y, {}^{\mathrm{CF}}D_{a+}^{\alpha}y)\, {}^{\mathrm{CF}}D_{a+}^{\alpha}y_2,$$

holds for every $(\xi, y, {}^{\mathrm{CF}}D_{a+}^{\alpha}y)$, $(\xi, y + y_1, {}^{\mathrm{CF}}D_{a+}^{\alpha}y + {}^{\mathrm{CF}}D_{a+}^{\alpha}y_2) \in \Re^3$. Here, $\partial_2 g$ and $\partial_3 g$ denote the partial derivatives of g with respect to y and ${}^{\mathrm{CF}}D_{a+}^{\alpha}y$, respectively.

Let $\bar{y}$ be a solution of the variational functional $G(\bar{y}) = \int_a^b g(\xi, \bar{y}(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))\mathrm{d}\xi\ \forall \bar{y} \in Y$ and $\xi \in [a, b]$. Let $y = \bar{y} + \eta\,(\xi, y, \bar{y})$ and $\eta \in C^1[a, b]$ with $\eta\,(\xi, y, \bar{y})|_{\xi=a} = \eta\,(\xi, y, \bar{y})|_{\xi=b}$. Clearly also $\eta(\xi, y, y) = 0$.

Now utilizing the linearity property of the CF derivative operator and further the convexity assumption of $g(\xi, \bar{y}\,(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))$, we have

$$\begin{aligned}
G(y) - G(\bar{y}) &= G(\bar{y} + \eta) - G(\bar{y})\\
&= \int_a^b [g(\xi, \bar{y}(\xi) + \eta(\xi, y, \bar{y}), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi) + {}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))\\
&\quad - g(\xi, \bar{y}(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))]\mathrm{d}\xi\\
&\geq \int_a^b [g_{\bar{y}}(\xi, \bar{y}(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))\eta\,(\xi, y, \bar{y})\\
&\quad + g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y})]\mathrm{d}\xi\\
&= \int_a^b [\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}\,(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))\\
&\quad + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))]\mathrm{d}\xi.
\end{aligned}$$

Clearly,

$$\begin{aligned}
G(y) - G(\bar{y}) \geq \int_a^b &\{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\\
&+ ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi,
\end{aligned}$$

$\forall y, \bar{y} \in Y$.

Clearly as $\alpha \to 1$, the above obtained inequality reduces to

$$\begin{aligned}
&\int_a^b g\left(\xi, y(\xi), \frac{\mathrm{d}}{\mathrm{d}\xi}y(\xi)\right)\mathrm{d}\xi - \int_a^b g\left(\xi, \bar{y}(\xi), \frac{\mathrm{d}}{\mathrm{d}\xi}\bar{y}(\xi)\right)\mathrm{d}\xi\\
&\quad\geq \int_a^b \left\{\eta\,(\xi, y, \bar{y})g_{\bar{y}}\left(\xi, \bar{y}, \frac{\mathrm{d}}{\mathrm{d}\xi}\bar{y}\right)\right.\\
&\qquad\left. + \left(\frac{\mathrm{d}}{\mathrm{d}\xi}\eta(\xi, y, \bar{y})\right)g_{\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{y}}\left(\xi, \bar{y}, \frac{\mathrm{d}}{\mathrm{d}\xi}\bar{y}\right)\right\}\mathrm{d}\xi,
\end{aligned}$$

which is the definition of invexity in the continuous case extended by Mond *et al.* [40]. It is notable that if the function g is independent of ξ, the above given definition of invexity transforms to the inequality $g(y) - g(\bar{y}) \geq \eta\,(y, \bar{y})g_{\bar{y}}(\bar{y})$, which is the fundamental interpretation of invexity prescribed by Hanson [41].

Example 1. The proposed inequality which is derived earlier is given as follows:

$$\begin{aligned}
&G(y) - G(\bar{y})\\
&\geq \int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\\
&\quad + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi.
\end{aligned}\tag{9}$$

Let $\bar{y} = \xi$, $y = 2\xi$, $\eta(\xi, y, \bar{y}) = y - \bar{y}$, $g(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) = \xi + \bar{y} + {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}$, and $g(\xi, y, {}^{\mathrm{CF}}D_{a+}^{\alpha}y) = \xi + y + {}^{\mathrm{CF}}D_{a+}^{\alpha}y$.

Then

$$\eta(\xi, y, \bar{y}) = y - \bar{y} = \xi,\ {}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}) = {}^{\mathrm{CF}}D_{a+}^{\alpha}\xi,\tag{10}$$

$$g(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) = \xi + \bar{y} + {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y} = 2\xi + {}^{\mathrm{CF}}D_{a+}^{\alpha}\xi.\tag{11}$$

Now utilizing the formula of CF derivative, we obtain

$$\begin{aligned}
{}^{\mathrm{CF}}D_{a+}^{\alpha}\xi &= \frac{K(\alpha)}{(1-\alpha)}\int_a^{\xi}\exp\left(-\frac{\alpha}{1-\alpha}(\xi-\tau)\right)\mathrm{d}\tau\\
&= \frac{K(\alpha)}{\alpha}\left[1 - \exp\left(-\frac{\alpha}{1-\alpha}(\xi - a)\right)\right],
\end{aligned}\tag{12}$$

where $K(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}$ signifies the normalization function with the property $K(0) = K(1) = 1$.

Thus,

$$\begin{aligned}
&{}^{\mathrm{CF}}D_{a+}^{\alpha}\eta\,(\xi, y, \bar{y})\\
&\quad= \frac{K(\alpha)}{\alpha}\left[1 - \exp\left(-\frac{\alpha}{1-\alpha}(\xi - a)\right)\right],
\end{aligned}\tag{13}$$

and

$$\begin{aligned}
&g(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\\
&\quad= 2\xi + \frac{K(\alpha)}{\alpha}\left[1 - \exp\left(-\frac{\alpha}{1-\alpha}(\xi - a)\right)\right].
\end{aligned}\tag{14}$$

Now

$$\begin{aligned}
&g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) = 2 + \frac{K(\alpha)}{(1-\alpha)}\exp\left(-\frac{\alpha}{1-\alpha}(\xi - a)\right),\\
&g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) = 1.
\end{aligned}\tag{15}$$

Now, we evaluate the term $G(y) - G(\bar{y})$ of the aforementioned proposed inequality (9) in view of Eq. (14) as follows:

$$G(y) - G(\bar{y}) = \int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi - \int_a^b g(\xi, \bar{y}(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}(\xi))\mathrm{d}\xi = \frac{1}{2}(b^2 - a^2) + \frac{K(\alpha)}{\alpha}(b-a) + \frac{K(\alpha)(1-\alpha)}{\alpha^2}\exp\left(-\frac{\alpha(b-a)}{1-\alpha}\right) - \frac{K(\alpha)(1-\alpha)}{\alpha^2}. \tag{16}$$

In this sequence, we also evaluate the integral term of the proposed inequality (9) in view of Eqs. (10) and (15) as follows:

$$\int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi = (b^2 - a^2) - b\frac{K(\alpha)}{\alpha}\exp\left(-\frac{\alpha(b-a)}{1-\alpha}\right) + 2\frac{K(\alpha)(1-\alpha)}{\alpha^2}\exp\left(-\frac{\alpha(b-a)}{1-\alpha}\right) - 2\frac{K(\alpha)(1-\alpha)}{\alpha^2} + b\frac{K(\alpha)}{\alpha}. \tag{17}$$

Case I: For $a = 0,\ b = 1,\ \alpha = 0.5,\ K(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}$, the proposed inequality is satisfied.

Case II: For $a = 0,\ b = 1,\ \alpha = 0.2,\ K(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}$, the proposed inequality is also satisfied.

Consequently, it is concluded that the proposed inequality (9) with CF fractional derivative holds well for the function $g(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) = 2\xi + \left\{\frac{1-\alpha}{\alpha} + \frac{1}{\Gamma(\alpha)}\right\}\left[1 - \exp\left(-\frac{\alpha}{1-\alpha}\xi\right)\right]$, where $0 < \alpha < 1$.

Now, we extend the definitions of invex, pseudoinvex (PIX), strictly pseudoinvex (SPIX), and quasi-invex (QIX) as described in ref. [58] with the CF fractional derivative of order $0 < \alpha < 1$ in the following way:

Definition 11. Invex

The functional G is stated as invex with respect to η if there exists a differentiable vector function $\eta(\xi, y, \bar{y}) \in C^1[a, b]$ with $\eta(\xi, y, y) = 0$ such that $\forall\ y, \bar{y} \in Y$,

$$G(y) - G(\bar{y}) \geq \int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi.$$

Definition 12. PIX

The functional G is stated as PIX w.r.t. η if $\exists$ a differentiable vector function $\eta(\xi, y, \bar{y}) \in C^1[a, b]$ with $\eta(\xi, y, y) = 0$ such that $\forall y, \bar{y} \in Y$,

$$\int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi \geq 0 \Rightarrow G(y) \geq G(\bar{y}),$$

or equivalently,

$$G(y) < G(\bar{y}) \Rightarrow \int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi < 0.$$

Definition 13. SPIX

The functional G is stated as SPIX w.r.t. η if $\exists$ a differentiable vector function $\eta(\xi, y, \bar{y}) \in C^1[a, b]$ with $\eta(\xi, y, y) = 0$ such that $\forall y, \bar{y} \in Y$,

$$\int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi \geq 0 \Rightarrow G(y) > G(\bar{y}),$$

or equivalently,

$$G(y) \leq G(\bar{y}) \Rightarrow \int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi < 0.$$

Definition 14. QIX

The functional G is stated as QIX w.r.t. η if $\exists$ a differentiable vector function $\eta(\xi, y, \bar{y}) \in C^1[a, b]$ with $\eta(\xi, y, y) = 0$ such that $\forall y, \bar{y} \in Y$,

$$\int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi > 0 \Rightarrow G(y) > G(\bar{y}),$$

or equivalently,

$$G(y) \leq G(\bar{y}) \Rightarrow \int_a^b \{\eta(\xi, y, \bar{y})g_{\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, \bar{y}))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y}}(\xi, \bar{y}, {}^{\mathrm{CF}}D_{a+}^{\alpha}\bar{y})\}\mathrm{d}\xi \leq 0.$$

In the aforementioned definitions, ${}^{\mathrm{CF}}D^{\alpha}_{a+}\eta\ (\xi,\ y, \bar{y})$ is the vector whose ith component is $(\mathrm{d}^{\alpha}/\mathrm{d}\xi^{\alpha})\ \eta^{i}(\xi,\ y, \bar{y})$. Let $g(\xi, y, {}^{\mathrm{CF}}D^{\alpha}_{a+}y\ (\xi))$ be a real scalar function and $h(\xi, y, {}^{\mathrm{CF}}D^{\alpha}_{a+}y\ (\xi))$ be an m-dimensional function with continuous derivatives up to the second order with respect to each of its arguments. Here, y is an n-dimensional function of ξ and ${}^{\mathrm{CF}}D^{\alpha}_{a+}y\ (\xi)$ denotes the CF fractional derivative of order α with respect to ξ where $0 < \alpha < 1$.

Now we deal with the multiobjective variational primal problem, as discussed in the work of Mukherjee and Mishra [45], with the CF fractional derivative operator in the following way:

(P) Minimize

$$\int_a^b g(\xi, y\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi))\mathrm{d}\xi,$$

subject to $y\ (a) = y_0,\ y\ (b) = y_1$,

$$h(\xi, y(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y\ (\xi)) \leq 0,$$

where $g : [a, b] \times \mathfrak{R}^n \times \mathfrak{R}^n \rightarrow \mathfrak{R}^p$ and $h : [a, b] \times \mathfrak{R}^n \times \mathfrak{R}^n \rightarrow \mathfrak{R}^m$, $0 < \alpha < 1$.

For the primal problem (P), a point y_0 is referred to as a weak minimum if there exists no other feasible point y for which the following inequality will hold

$$\int_a^b g(\xi, y_0\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))\mathrm{d}\xi > \int_a^b g(\xi, y\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi))\mathrm{d}\xi. \tag{18}$$

Now, we frame the continuous versions of Theorems 2.1 and 2.2, as described in the work of Weir and Mond [44], involving fractional derivative operators with exponential kernel in the following way:

Theorem 3. *Let $y = y_0$ be a weak minimum for the primal problem (P). Then $\exists\ \lambda \in \mathfrak{R}^p$, $z \in \mathfrak{R}^m$ such that*

$$\begin{aligned}\lambda^T g_y(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) &+ z(\xi)^T h_y(\xi, y_0\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) \\ = -\ {}^{\mathrm{CFR}}D^{\alpha}_{b-}[\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)}(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) &+ z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)}(\xi, y_0\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))]\ ,\end{aligned} \tag{19}$$

$$z(\xi)^T h(\xi, y_0\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) = 0, \tag{20}$$

$$(\lambda, y) \geq 0. \tag{21}$$

Proof. The proof is easily established through Theorem 1. □

Theorem 4. *Let the primal problem (P) have a weak minimum at a point y_0, which satisfies the KT constraint qualification. Then $\exists\ \lambda \in \mathfrak{R}^p$, $z \in \mathfrak{R}^m$ such that*

$$\begin{aligned}\lambda^T g_y(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) &+ z(\xi)^T h_y(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) \\ = -\ {}^{\mathrm{CFR}}D^{\alpha}_{b-}[\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)}(\xi, y_0\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) &+ z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)}(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))],\end{aligned} \tag{22}$$

$$z(\xi)^T h\ (\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) = 0, \tag{23}$$

$$z(\xi) \geq 0, \tag{24}$$

$$\lambda(\xi) \geq 0,\ \lambda^T e = 1, \tag{25}$$

where $e = (1,\ \ldots..\ , 1) \in \mathfrak{R}^p$.

Proof. The proof is easily established through Theorem 1. □

3 Duality

In relation to the primal problem (P), the dual problem (D) as discussed in ref. [45] is considered with fractional derivative operators pertaining to exponential kernel in the following way:

(D) Maximize

$$\int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e\}\mathrm{d}\xi,$$

subject to

$$y\ (a) = y_0,\ \ y\ (b) = y_1, \tag{26}$$

$$\begin{aligned}g_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v\ (\xi)) &+ z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e \\ = -\ {}^{\mathrm{CFR}}D^{\alpha}_{b-}[g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) &+ z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e],\end{aligned} \tag{27}$$

$$z \geq 0. \tag{28}$$

$$\lambda \in \Lambda,$$

where

$$\Lambda = \{\lambda \in \mathfrak{R}^p : \lambda \geq 0,\ \lambda^T e = 1\}\ . \tag{29}$$

In upcoming steps, we discuss the duality theorems, as discussed in the work of Mukherjee and Mishra [45], with the CF fractional derivative operator of order $0 < \alpha < 1$ as follows:

Theorem 5. (Weak duality): *If, for all feasible* (y, v, z, λ),

(*a*) $\int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}$ dξ *is PIX or*

(*b*) $\int_a^b \{\lambda^T g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}$ dξ *is PIX, then*

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi \nless \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi.$$

Proof. (a) Let y be feasible for (P) and (v, z, λ) feasible for (D).

From Eq. (27), we have

$$\int_a^b \eta(\xi, y, v)\{g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi = -\int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}d\xi. \tag{30}$$

Suppose contrary to the result given in statement of Theorem 5, *i.e.*

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi < \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi$$
$$\Rightarrow \int_a^b \{g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))e\}\mathrm{d}\xi < \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi. \tag{31}$$

Now, use of pseudoinvexity of $\int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi$ along with the above obtained inequality (31) provides

$$\int_a^b \{\eta(\xi, y, v)[g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e] + {}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v)[g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e]\}\mathrm{d}\xi < 0,$$

or

$$\int_a^b \eta(\xi, y, v)[g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e]\mathrm{d}\xi + \int_a^b [g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e]{}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v)\mathrm{d}\xi < 0. \tag{32}$$

Now using Proposition 1 in the above obtained inequality (32), we have

$$\int_a^b \eta(\xi, y, v)[g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e]\mathrm{d}\xi + [\eta(\xi, y, v)I_{b-}^{1-\alpha}\{g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}]_{\xi=a}^{\xi=b} + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi < 0.$$

$$\because \eta(\xi, y, y) = 0,$$

we obtain

$$\int_a^b \eta(\xi, y, v)[g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e]\mathrm{d}\xi + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi < 0. \tag{33}$$

Now in view of Eq. (30) and inequality (33), we have

$$\int_a^b \eta\ (\xi, y, v)[g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e]\mathrm{d}\xi + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v)e\}\mathrm{d}\xi = 0 < 0,$$

which is a contradiction.

Hence,

$$\int_a^b \{g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))e\}\mathrm{d}\xi \nless \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi.$$

Thus, the supposition (31), which is contrary to the result given in the statement of Theorem 5, is wrong and consequently

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi \nless \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi.$$

□

(**b**) Let y be feasible for (P) and (v, z, λ) feasible for (D).

Multiplying Eq. (30) by λ^T and using $\lambda^T e = 1$, we obtain

$$\int_a^b \eta(\xi, y, v)\{\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi = -\int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi. \tag{34}$$

Suppose contrary to the result given in the statement of Theorem 5, *i.e.*

$$\int_a^b \{g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))e\}\mathrm{d}\xi < \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e\}\mathrm{d}\xi. \tag{35}$$

Multiplying the aforementioned obtained inequality (35) by λ^T and using $\lambda^T e = 1$, we obtain

$$\int_a^b \{\lambda^T g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) + z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\}\mathrm{d}\xi < \int_a^b \{\lambda^T g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi. \tag{36}$$

Now, the pseudoinvexity of $\int_a^b\{\lambda^T g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi$ along with the above obtained inequality (36) provides

$$\int_a^b \{\eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))] + {}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v)[\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))]\}\mathrm{d}\xi < 0,$$

or

$$\begin{aligned}&\int_a^b \eta\ (\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))]\mathrm{d}\xi\\&\quad + \int_a^b [\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))]^{\mathrm{CF}}D^{\alpha}_{a+}\\&\quad \eta(\xi, y, v)\mathrm{d}\xi < 0.\end{aligned} \tag{37}$$

Now by using Proposition 1 in the above obtained inequality (37), we have

$$\begin{aligned}&\int_a^b \eta(\xi, y, v)[\lambda^T g_v(\xi, v\ (\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))]\mathrm{d}\xi\\&\quad + [\eta(\xi, y, v)I^{1-\alpha}_{b-}\{\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\qquad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\}]^{\xi=b}_{\xi=a}\\&\quad + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D^{\alpha}_{b-}\{\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\}\mathrm{d}\xi < 0.\end{aligned} \tag{38}$$

$\because\ \eta\ (\xi, y, y) = 0$, the above obtained inequality (38) reduces to

$$\begin{aligned}&\int_a^b \eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))]\mathrm{d}\xi\\&\quad + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D^{\alpha}_{b-}\{\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\}\mathrm{d}\xi < 0.\end{aligned} \tag{39}$$

Now in view of Eq. (34) and inequality (39), we have

$$\begin{aligned}&\int_a^b \eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))]\mathrm{d}\xi\\&\quad + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D^{\alpha}_{b-}\{\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D^{\alpha}_{a+}v)\}\mathrm{d}\xi = 0 < 0,\end{aligned}$$

which is a contradiction. Thus, the supposition (35) is false. Consequently,

$$\begin{aligned}&\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi))\mathrm{d}\xi\\&\quad \nless \int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\\&\quad + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e\}\mathrm{d}\xi.\end{aligned}$$

□

Theorem 6. (**Strong duality**): *Let the primal (P) have a weak minimum at y_0, which satisfies the KT constraint qualification. Then there exists (z, λ) in such a way that (y_0, z, λ) is feasible for (D) and the objective values of (P) and (D) are equal. If also,*

(*a*) $\int_a^b \{g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e\}\mathrm{d}\xi$ *is PIX or*

(*b*) $\int_a^b \{\lambda^T g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))\}\mathrm{d}\xi$ *is PIX,*

then (y_0, z, λ) is a weak maximum for (D).

Proof. Since the primal (P) has a weak minimum at y_0, which satisfies the KT constraint qualification, by Theorem 4, $\exists\ z \geq 0,\ \lambda \geq 0,\ \lambda^T e = 1$ such that

$$\begin{aligned}&\lambda^T g_y(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) + z(\xi)^T h_y(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))\\&\quad = -{}^{\mathrm{CFR}}D^{\alpha}_{b-}[\lambda^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)}(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))\\&\qquad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)}(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))],\end{aligned}$$

and $z(\xi)^T h\ (\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi)) = 0$. Hence, (y_0, z, λ) is feasible for dual (D) and the objective values of primal (P) and dual (D) are equal.

If (y_0, z, λ) is not a weak maximum for (D), then a feasible (v^*, z^*, λ^*) for dual (D) occurs so that

$$\begin{aligned}&\int_a^b \{g(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))\\&\quad + z^*(\xi)^T h(\xi, y_0(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y_0(\xi))e\}\mathrm{d}\xi\\&\quad < \int_a^b \{g(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi))\\&\quad + z^*(\xi)^T h(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi))e\}\mathrm{d}\xi.\end{aligned} \tag{40}$$

(**a**) The pseudoinvexity of $\int_a^b \{g(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)) + z^*(\xi)^T h(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi))e\}\mathrm{d}\xi$ together with the above obtained inequality (40) gives

$$\int_a^b \{\eta(\xi, y_0, v)[g_v(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_v(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi))e] + {}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y_0, v)[g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)e]\}\mathrm{d}\xi < 0. \tag{41}$$

Since $\lambda^* \geq 0$ thus multiplying inequality (41) by λ^* and using $\lambda^*(\xi)^T e = 1$, we obtain

$$\int_a^b \{\eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi))] + {}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)]\}\mathrm{d}\xi < 0. \tag{42}$$

Now by using Proposition 1 in inequality (42), we have

$$\int_a^b \eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_v^*(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi))]\mathrm{d}\xi + [\eta(\xi, y_0, v^*) I_{b-}^{1-\alpha}\{\lambda^*(\xi)^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)} (\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)\}]_{\xi=a}^{\xi=b} + \int_a^b \eta(\xi, y_0, v^*)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^{*T} g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)\}\mathrm{d}\xi < 0. \tag{43}$$

$\because$ $\eta(\xi, y, y) = 0$, inequality (43) reduces to

$$\int_a^b \eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_v^*(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v^*)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^{*T} g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)\}\mathrm{d}\xi < 0. \tag{44}$$

Since (v^*, z^*, λ^*) is feasible for (D) thus from Eq. (34), we have

$$\int_a^b \eta(\xi, y_0, v^*)\{\lambda^{*T} g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi))\}\mathrm{d}\xi = -\int_a^b \eta(\xi, y_0, v^*)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^{*T} g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)\}\mathrm{d}\xi. \tag{45}$$

Using Eq. (45) in inequality (44), we obtain

$$\int_a^b \eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)) + z^*(\xi)^T h_v^*(\xi, v^*(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v^*)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^{*T} g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D_{a+}^{\alpha}v^*)\}\mathrm{d}\xi = 0 < 0.$$

This is clearly a contradiction. Hence, the supposition of the existence of feasibility of (v^*, z^*, λ^*) for (D) is false and consequently (y_0, z, λ) is a weak maximum for (D). □

(b) The proof of part (b) is very similar to that of part (a).

We now consider the following dual problem (D1) in relation to the primal problem (P).

(D1) Maximize

$$\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi \tag{46}$$

subject to $y(a) = y_0, \quad y(b) = y_1,$

$$\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))e = -{}^{\mathrm{CFR}}D_{b-}^{\alpha}[\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))], \tag{47}$$

$$z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) \geq 0, \tag{48}$$

$$z \geq 0. \tag{49}$$

$\lambda \in \Lambda$, where

$$\Lambda = \{\lambda \in \Re^p : \lambda \geq 0,\ \lambda^T e = 1\}. \tag{50}$$

The next step is to establish the weak and strong duality theorems, as discussed in the work of Mukherjee and Mishra [45], with the CF fractional derivative operators. □

Theorem 7. (Weak duality): *If for all feasible* (y, v, z, λ),

(a) $\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is PIX and* $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is QIX or*

(*b*) $\int_a^b \lambda^T g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is PIX and* $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is QIX or*

(*c*) $\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is QIX and* $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is SPIX or*

(*d*) $\int_a^b \lambda^T g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\,\mathrm{d}\xi$ *is QIX and* $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ *is SPIX.*

Then

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi \nless \int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi.$$

Proof. (a) Let y be feasible for primal (P) and (v, z, λ) feasible for dual (D1).

Suppose contrary to the result, *i.e.*

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi < \int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi. \tag{51}$$

Thus, in view of the pseudoinvexity of $\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$, we have

$$\int_a^b \{\eta(\xi, y, v)g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v))g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi < 0.$$

$$\because\ \lambda \geq 0,$$

we have

$$\int_a^b \{\eta(\xi, y, v)\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v))\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi < 0. \tag{52}$$

From the constraint of the primal problem (P), we have

$$h(\xi, y(\xi), {}^{CF}D_{a+}^{\alpha}y(\xi)) \leq 0.$$

$\because z(\xi) \geq 0$ therefore

$$z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) \leq 0. \tag{53}$$

Now from the constraint of the dual problem (D1), we have

$$z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) \geq 0. \tag{54}$$

Combining both inequalities (53) and (54), we obtain

$$z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) \leq 0 \leq z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)),$$

or

$$z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi)) \leq z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)),$$

or

$$\int_a^b z(\xi)^T h(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}y(\xi))\mathrm{d}\xi \leq \int_a^b z(\xi)^T h\,(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi. \tag{55}$$

Now, inequality (55) together with the quasi-invexity of $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ implies that

$$\int_a^b \{\eta(\xi, y, v)z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v))z(t)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha}v)\}\mathrm{d}\xi \leq 0. \tag{56}$$

Now adding inequalities (52) and (56), we obtain

$$\begin{aligned}&\int_a^b \{\eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))]\}\mathrm{d}\xi \\ &+ \int_a^b \{({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v))[\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha}v)]\}\mathrm{d}\xi < 0.\end{aligned} \tag{57}$$

Now by using Proposition 1 in inequality (57), we have

$$\begin{aligned}&\int_a^b \{\eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))]\}\mathrm{d}\xi \\ &+ [\eta(\xi, y, v)I_{b-}^{1-\alpha}\{\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}]_{\xi=a}^{\xi=b} \\ &+ \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v\,(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi < 0.\end{aligned} \tag{58}$$

Using Eq. (47) and $\eta(\xi, y, y) = 0$, inequality (58) reduces to

$$\int_a^b \{\eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))]\}\mathrm{d}\xi + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha}v)\}\mathrm{d}\xi = 0 < 0,$$

which is a contradiction. Thus,

$$\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi \nless \int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi.$$ □

(**b**) The proof of part (b) is very similar to the proof of part (a).

(**c**) Let y be feasible for (P) and (v, z, λ) feasible for (D1).

Suppose contrary to the result, *i.e.*

$$\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi < \int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi. \quad (59)$$

Then in view of the quasi-invexity of $\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$, we have

$$\int_a^b \{\eta(\xi, y, v) g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v)) g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi \le 0.$$

$$\because \lambda \ge 0$$

, we have

$$\int_a^b \{\eta(\xi, y, v)\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v))\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)} (\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\}\mathrm{d}\xi \le 0. \quad (60)$$

$$\because \int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi \le \int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi. \quad (61)$$

Thus, the strict pseudoinvexity of $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi$ along with inequality (61) provides

$$\int_a^b \{\eta(\xi, y, v) z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, v, v)) z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha}v)\}\mathrm{d}\xi < 0. \quad (62)$$

Now adding inequalities (60) and (62), we obtain

$$\int_a^b \{\eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))]\}\mathrm{d}\xi + \int_a^b \{({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y, v))[\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha}v)]\}\mathrm{d}\xi < 0. \quad (63)$$

Now using Proposition 1 in inequality (63) along with $\eta(\xi, y, y) = 0$, and from Eq. (47), we obtain

$$\int_a^b \{\eta(\xi, y, v)[\lambda^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))]\}\mathrm{d}\xi + \int_a^b \eta(\xi, y, v)^{\mathrm{CFR}}D_{b-}^{\alpha}\{\lambda^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)) + z(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi)}(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha}v)\}\mathrm{d}\xi = 0 < 0,$$

which is a contradiction. Consequently,

$$\int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi \nless \int_a^b g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha}v(\xi))\mathrm{d}\xi.$$ □

(e) The proof of part (d) is very similar to the proof of part (c). □

Theorem 8. (Strong duality): *Let the primal (P) have a weak minimum at the point y_0, which satisfies the KT constraint qualification. Then $\exists\,(z,\lambda)$ so that (y_0,z,λ) is feasible for (D1) and the objective values of primal (P) and dual (D1) are equal. If also,*

(a) $\int_a^b g(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is PIX and $\int_a^b z(t)^T h(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is QIX or

(b) $\int_a^b \lambda^T g(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is PIX and $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is QIX or

(c) $\int_a^b g(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is QIX and $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is SPIX or

(d) $\int_a^b \lambda^T g(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is QIX and $\int_a^b z(\xi)^T h(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi))d\xi$ is SPIX,

then (y_0,z,λ) is a weak maximum for (D1).

Proof. Since the primal (P) have a weak minimum at y_0 for which the KT constraint qualification is satisfied, then by Theorem 4, there exists $z \geq 0$, $\lambda \geq 0$, $\lambda^T e = 1$ such that

$$\lambda^T g_y(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) + z(\xi)^T h_y(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) = -{}^{CFR}D^{\alpha}_{b-}[\lambda^T g_{{}^{CF}D^{\alpha}_{a+}y(\xi)}(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) + z(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}y(\xi)}(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi))],$$

and $z(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) = 0$. Therefore, (y_0,z,λ) is feasible for (D1) and the objective values of primal (P) and dual (D1) are equal.

If (y_0,z,λ) is not a weak maximum for (D1), then a feasible solution (v^*, z^*, λ^*) for dual (D1) occurs such that

$$\int_a^b g(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi))d\xi < \int_a^b g(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))d\xi. \tag{64}$$

(**a**) The pseudoinvexity of $\int_a^b g(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))d\xi$ together with inequality (64) gives

$$\int_a^b \{\eta(\xi, y_0, v^*) g_v(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)) + ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*)) g_{{}^{CF}D^{\alpha}_{a+}v(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)\}d\xi < 0. \tag{65}$$

Since $\lambda^* \geq 0$ thus multiplying the above obtained inequality (65) by λ^*, and using $\lambda^*(\xi)^T e = 1$, we obtain

$$\int_a^b \{\eta(\xi, y_0, v^*)\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)) + ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*))\lambda^*(\xi)^T g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\} d\xi < 0. \tag{66}$$

From the constraint of the primal problem (P), we have

$$h(\xi, y(\xi), {}^{CF}D^{\alpha}_{a+}y(\xi)) \leq 0.$$

Since y_0 is feasible for primal problem (P) thus it satisfies the aforementioned constraint and so $h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) \leq 0$.

$$\because z^*(t) \geq 0$$

thus

$$z^*(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) \leq 0. \tag{67}$$

Now from the constraint of the dual problem (D1), we have

$$z(\xi)^T h(\xi, v(\xi), {}^{CF}D^{\alpha}_{a+}v(\xi)) \geq 0.$$

Since (v^*, z^*, λ^*) is feasible for (D1) thus

$$z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)) \geq 0. \tag{68}$$

Combining both inequalities (67) and (68), we obtain

$$z^*(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) \leq 0 \leq z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)),$$

or

$$z^*(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) \leq z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)),$$

or

$$\int_a^b z^*(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi))d\xi \leq \int_a^b z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))d\xi. \tag{69}$$

The quasi-invexity of $\int_a^b z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))d\xi$ together with the above obtained inequality (69) provides

$$\int_a^b \{\eta(\xi, y_0, v^*) z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)) + ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*)) z^*(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\} d\xi \leq 0. \tag{70}$$

Now adding inequalities (66) and (70), we obtain

$$\begin{aligned}\int_a^b &\{\eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))]\\ &+({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*))[\lambda^{*T} g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)\\ &+ z^*(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)]\}\mathrm{d}\xi < 0.\end{aligned} \tag{71}$$

Now using Proposition 1 in the above obtained inequality (71) and $\eta(\xi, y_0, y_0) = 0$, we obtain

$$\begin{aligned}\int_a^b &\eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))]\mathrm{d}\xi\\ &+ \int_a^b (\eta(\xi, y_0, v^*))^{CFR}D^{\alpha}_{b-}\,[\lambda^{*T} g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)\\ &+ z^*(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)]\mathrm{d}\xi < 0.\end{aligned} \tag{72}$$

Since (v^*, z^*, λ^*) is feasible for (D1) thus from Eq. (47), we have

$$\begin{aligned}\int_a^b &\eta(\xi, y_0, v^*)\{\lambda^{*T} g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\}\mathrm{d}\xi\\ &= -\int_a^b \eta(\xi, y_0, v^*)^{CFR}D^{\alpha}_{b-}\{\lambda^{*T} g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*(\xi),\\ &{}^{CF}D^{\alpha}_{a+}v^*(\xi)) + z^*(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)\}\mathrm{d}\xi.\end{aligned} \tag{73}$$

Using Eq. (73) in inequality (72), we obtain

$$\begin{aligned}\int_a^b &\eta(\xi, y_0, v^*)\{\lambda^{*T} g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\}\mathrm{d}\xi\\ &+ \int_a^b (\eta(\xi, y_0, v^*))^{CFR}D^{\alpha}_{b-}[\lambda^{*T} g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ z^{*T} h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)]\mathrm{d}\xi = 0 < 0,\end{aligned}$$

which is a discrepancy and consequently contradicts the feasibility of (v^*, z^*, λ^*). □

(**b**) The proof of part (b) is very similar to the proof of part (a).

(**c**) The quasi-invexity of $\int_a^b g(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\mathrm{d}\xi$ together with inequality (64) gives

$$\begin{aligned}\int_a^b &\{\eta(\xi, y_0, v^*)g_v(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*))g_{{}^{CF}D^{\alpha}_{a+}v(\xi)}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\}\mathrm{d}\xi \le 0.\end{aligned} \tag{74}$$

Since $\lambda^* \ge 0$ thus multiplying inequality (74) by λ^* and using $\lambda^*(\xi)^T e = 1$, we obtain

$$\begin{aligned}\int_a^b &\{\eta(\xi, y_0, v^*)\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*))\lambda^*(\xi)^T g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*(\xi),\\ &{}^{CF}D^{\alpha}_{a+}v^*(\xi))\}\mathrm{d}\xi \le 0.\end{aligned} \tag{75}$$

Since

$$\int_a^b z^*(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi))\mathrm{d}\xi \le \int_a^b z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\mathrm{d}\xi. \tag{76}$$

$(\because \int_a^b z^*(\xi)^T h(\xi, y_0(\xi), {}^{CF}D^{\alpha}_{a+}y_0(\xi)) \le 0$ and $\int_a^b z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi)) \ge 0)$.

Thus, in view of strictly pseudoinvexity of $\int_a^b z^*(\xi)^T h(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\mathrm{d}\xi$, we have

$$\begin{aligned}\int_a^b &\{\eta(\xi, y_0, v^*)z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*))z^*(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*(\xi),\\ &{}^{CF}D^{\alpha}_{a+}v^*(\xi))\}\mathrm{d}\xi < 0.\end{aligned} \tag{77}$$

Now combining inequalities (75) and (77), we obtain

$$\begin{aligned}\int_a^b &\{\eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))\\ &+ z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{CF}D^{\alpha}_{a+}v^*(\xi))]\\ &+ ({}^{CF}D^{\alpha}_{a+}\eta(\xi, y_0, v^*))[\lambda^{*T} g_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)\\ &+ z^*(\xi)^T h_{{}^{CF}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{CF}D^{\alpha}_{a+}v^*)]\}\mathrm{d}\xi < 0.\end{aligned} \tag{78}$$

Now using Proposition 1 in inequality (78) and utilizing $\eta(\xi, y_0, y_0) = 0$, we obtain

$$\int_a^b \eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)) + z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi))]\mathrm{d}\xi + \int_a^b (\eta(\xi, y_0, v^*))^{\mathrm{CFR}}D^{\alpha}_{b-}[\lambda^{*T} g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*) + z^*(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*)]\mathrm{d}\xi < 0. \tag{79}$$

Now in view of Eq. (73), inequality (79) can be expressed as

$$\int_a^b \eta(\xi, y_0, v^*)[\lambda^*(\xi)^T g_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)) + z^*(\xi)^T h_{v^*}(\xi, v^*(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi))]\mathrm{d}\xi + \int_a^b (\eta(\xi, y_0, v^*))^{\mathrm{CFR}}D^{\alpha}_{b-}[\lambda^{*T} g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*) + z^{*T} h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v^*(\xi)}(\xi, v^*, {}^{\mathrm{CF}}D^{\alpha}_{a+}v^*)]\mathrm{d}\xi = 0 < 0,$$

which clearly contradicts the feasibility of (v^*, z^*, λ^*). □

(**d**) The proof of part (d) is very much identical to the proof of part (c).

Now we state the continuous form of a general primal (CP) and dual (CD) for the multiobjective variational optimization problem, as discussed in the work of Mukherjee and Mishra [45], involving fractional derivative operator with exponential kernel. Consider the problem given as follows:

(**CP**) Minimize

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi))\mathrm{d}\xi,$$

subject to $y(a) = y_0, y(b) = y_1$

$$h(\xi, y(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)) \leq 0,$$

$$\vartheta(\xi, y(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}y(\xi)) = 0,$$

where $g: I \times \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^k$, $h: I \times \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^m$, and $\vartheta: I \times \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^l$ are all differentiable.

Let $K = \{1, 2, \ldots, k\}$, $U = \{1, 2, \ldots, u\}$, $I_\beta \subseteq K$, $\beta = 0, 1, \ldots, \kappa$ with $I_\beta \cap I_\gamma = \varphi$, $\beta \neq \gamma$, and $\cup_{\beta=0}^{\kappa} I_\beta = K$ and $J_\beta \subseteq U$, $\beta = 0, 1, \ldots, \kappa$ with $J_\beta \cap J_\gamma = \varphi$, $\beta \neq \gamma$, and $\cup_{\beta=0}^{\kappa} J_\beta = U$. It is notable that any particular I_β or J_β may be empty. Hence, if K has κ_1 disjoint subsets and U has κ_2 disjoint subsets, $\kappa = \max[\kappa_1, \kappa_2]$. So, if $\kappa_1 > \kappa_2$, then J_β, $\beta > \kappa_2$ is empty.

In connection to (CP), we investigate the problem:

(**CD**) Maximize

$$\int_a^b \left\{ g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{i \in I_0} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e + \sum_{j \in J_0} w_j(\xi)^T \vartheta_j(\xi, v, {}^{\mathrm{CF}}D^{\alpha}_{a+}v)e \right\}\mathrm{d}\xi,$$

subject to

$$y(a) = y_0, y(b) = y_1$$

$$\begin{aligned}
&\lambda(\xi)^T g_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + z(\xi)^T h_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \\
&\quad + w(\xi)^T \vartheta_v(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \\
&\quad = -{}^{\mathrm{CFR}}D^{\alpha}_{b-}[\lambda(\xi)^T g_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \\
&\qquad + z(\xi)^T h_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \\
&\qquad + w(\xi)^T \vartheta_{{}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)}(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))],
\end{aligned}$$

$$\sum_{i \in I_\beta} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{j \in J_\beta} w_j(\xi)^T \vartheta_j(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \geq 0,$$

$\beta = 0, 1, \ldots, \kappa$, $y \geq 0$, $\lambda \in \Lambda$, where $\Lambda = \{\lambda \in \mathfrak{R}^p : \lambda \geq 0, \lambda^T e = 1\}$.

The weak and strong duality theorems, as described in ref. [45], are articulated here with the CF fractional derivative operators without proof because their proof may be delivered in a very identical manner to that of Theorems (5)–(8). □

Theorem 9. (**Weak duality**): *If, for all feasible* (y, v, z, w, λ),

(*a*) $\int_a^b \{ g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{i \in I_0} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi))e + \sum_{j \in J_0} w_j(\xi)^T \vartheta_j(\xi, v, {}^{\mathrm{CF}}D^{\alpha}_{a+}v)e \}\mathrm{d}\xi$

is PIX and $\int_a^b \{ \sum_{i \in I_\alpha} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{j \in J_\alpha} w_j(\xi)^T \vartheta_j(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \}\mathrm{d}\xi$,

$\beta = 1, 2, \ldots, \kappa$, *is QIX; or*

(*b*) $\int_a^b \{ \lambda(t)^T g(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{i \in I_0} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{j \in J_0} w_j(\xi)^T \vartheta_j(\xi, v, {}^{\mathrm{CF}}D^{\alpha}_{a+}v) \}\mathrm{d}\xi$

is PIX and $\int_a^b \{ \sum_{i \in I_\alpha} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) + \sum_{j \in J_\alpha} w_j(\xi)^T \vartheta_j(\xi, v(\xi), {}^{\mathrm{CF}}D^{\alpha}_{a+}v(\xi)) \}\mathrm{d}\xi$,

$\beta = 1, 2, \ldots, \kappa$, *is QIX; or* (*c*) $I_0 \neq K$ and $J_0 \neq U$,

$$\int_a^b \left\{ g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) + \sum_{i \in I_0} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi))e + \sum_{j \in J_0} w_j(\xi)^T \vartheta_j(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha} v)e \right\} \mathrm{d}\xi,$$

is QIX and $\int_a^b \left\{ \sum_{i \in I_\alpha} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) + \sum_{j \in J_\alpha} w_j(\xi)^T \vartheta_j(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) \right\} \mathrm{d}\xi$, $\alpha = 1, 2, \ldots, v$, is SPIX; or

(*d*) $I_0 \neq K$ and $J_0 \neq U$,

$$\int_a^b \left\{ \lambda(t)^T g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) + \sum_{i \in I_0} z_i(\xi)^T h_i(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha} v) + \sum_{j \in J_0} w_j(\xi)^T \vartheta_j(\xi, v, {}^{\mathrm{CF}}D_{a+}^{\alpha} v) \right\} \mathrm{d}\xi,$$

is QIX and $\int_a^b \left\{ \sum_{i \in I_\alpha} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) + \sum_{j \in J_\alpha} w_j(\xi)^T \vartheta_j(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) \right\} \mathrm{d}\xi$, $\alpha = 1, 2, \ldots, v$, is SPIX,

then

$$\int_a^b g(\xi, y(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y(\xi)) \mathrm{d}\xi \nless \int_a^b \left\{ g(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) + \sum_{i \in I_0} z_i(\xi)^T h_i(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi))e + \sum_{j \in J_0} w_j(\xi)^T \vartheta_j(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi))e \right\}.$$

Theorem 10. **(Strong duality):** *Let (CP) have a weak minimum at y_0, which satisfies the KT constraint qualification. Then $\exists$ (z, w, λ), so that (y_0, z, w, λ) is feasible for dual (CD) and the objective values of primal (CP) and dual (CD) are equal. If one of the presumptions (a), (b), (c), or (d) of Theorem 9 is fulfilled, then (y_0, z, w, λ) is a weak maximum for (CD).*

Now, we establish the strict converse duality theorem as stated in ref. [45] with CF fractional derivative operator in the forthcoming section.

4 Strict converse duality

Theorem 11. *Let y_0 be a weak minimum for (P) and (v_0, z_0, λ_0) be a weak maximum for (D1) such that*

$$\int_a^b \lambda_0{}^T(\xi) g(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi \leq \int_a^b \lambda_0{}^T g(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) \mathrm{d}\xi.$$

Assume that

(*a*) $\int_a^b \lambda_0{}^T(\xi) g(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi$ *is SPIX at v_0 and* $\int_a^b z_0(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) \mathrm{d}\xi$ *is QIX at v_0; or*

(*b*) $\int_a^b \lambda_0{}^T(\xi) g(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi$ *is QIX at v_0 and* $\int_a^b z_0(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) \mathrm{d}\xi$ *is SPIX at v_0;*

then $y_0 = v_0$, i.e. v_0 is a weak minimum for (P).

Proof. It is assumed that $y_0 \neq v_0$. Since y_0 and (v_0, z_0, λ_0) are feasible for (P) and (D1), respectively. Thus, $h(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \leq 0$ and $z_0(\xi)^T h(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) \geq 0$.

Now, the aforementioned inequalities imply that $\int_a^b z_0(\xi)^T h(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi \leq 0$, and $\int_a^b z_0(\xi)^T h(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) \mathrm{d}\xi \geq 0$. Finally, they can be written in the combined form as

$$\int_a^b z_0(\xi)^T h(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi \leq \int_a^b z_0(\xi)^T h(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) \mathrm{d}\xi. \tag{80}$$

(a) Now, inequality (80) in view of quasi-invexity of $\int_a^b z_0(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v(\xi)) \mathrm{d}\xi$ at v_0 provides the following inequality:

$$\int_a^b \{ \eta(\xi, y_0, v_0) z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) + ({}^{\mathrm{CF}}D_{a+}^{\alpha} \eta(\xi, y_0, v_0)) z_0(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) \} \mathrm{d}\xi \leq 0. \tag{81}$$

Since it is given that $\int_a^b \lambda_0{}^T(\xi) g(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi \leq \int_a^b \lambda_0{}^T g(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) \mathrm{d}\xi$, thus in view of strict pseudoinvexity of $\int_a^b \lambda_0(\xi)^T g(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) \mathrm{d}\xi$, we have

$$\int_a^b \{\eta(\xi, y_0, v_0)\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + ({}^{\mathrm{CF}}D^\alpha_{a+}\eta(\xi, y_0, v_0))\lambda_0(\xi)^T g_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))\}\mathrm{d}\xi < 0. \tag{82}$$

Now combining inequalities (81) and (82), we obtain

$$\int_a^b \{\eta(\xi, y_0, v_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))] + ({}^{\mathrm{CF}}D^\alpha_{a+}\eta(\xi, y_0, v_0))[\lambda_0(\xi)^T g_{{}^{CF}D^\alpha_{a+}v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0) + z_0(\xi)^T h_{{}^{CF}D^\alpha_{a+}v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0)]\}\mathrm{d}\xi \le 0. \tag{83}$$

Now using Proposition 1 in inequality (83) and $\eta(\xi, y_0, y_0) = 0$, we obtain

$$\int_a^b \{\eta(\xi, y_0, y_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v_0)^{\mathrm{CFR}}D^\alpha_{b-}[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))]\mathrm{d}\xi \le 0. \tag{84}$$

Since (v_0, z_0, λ_0) is feasible for (D1) thus from Eq. (47), we have

$$\int_a^b \eta(\xi, y_0, v_0)\{\lambda_0{}^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))\}\mathrm{d}\xi = -\int_a^b \eta(\xi, y_0, v_0)^{\mathrm{CFR}}D^\alpha_{b-}\{\lambda_0{}^T g_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0) + z_0(\xi)^T h_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0)\}\mathrm{d}\xi. \tag{85}$$

In view of Eq. (85), inequality (84) can be written as:

$$\int_a^b \eta(\xi, y_0, v_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v_0)^{\mathrm{CFR}}D^\alpha_{b-}[\lambda_0(\xi)^T g_{v_0}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0) + z_0(\xi)^T h_{v_0}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0)]\mathrm{d}\xi = 0 \le 0,$$

which contradicts the assumption $y_0 \ne v_0$. Consequently, $y_0 = v_0$ and since y_0 is a weak minimum for (P), thus v_0 is a weak minimum for (P). □

(b) Since $\int_a^b z_0(\xi)^T h(\xi, y_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}y_0(\xi))\mathrm{d}\xi \le \int_a^b z_0(\xi)^T h(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))\mathrm{d}\xi$ thus in view of the strictly pseudoinvexity of $\int_a^b z_0(\xi)^T h(\xi, v(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v(\xi))\mathrm{d}\xi$ at v_0, we have

$$\int_a^b \{\eta(\xi, y_0, v_0)z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + ({}^{\mathrm{CF}}D^\alpha_{a+}\eta(\xi, y_0, v_0))z_0(\xi)^T h_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))\}\mathrm{d}\xi < 0. \tag{86}$$

Since it is given that $\int_a^b \lambda_0{}^T(\xi)g(\xi, y_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}y_0(\xi))\mathrm{d}\xi \le \int_a^b \lambda_0{}^T g(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))\mathrm{d}\xi$, thus in view of the quasi-invexity of $\int_a^b \lambda_0(\xi)^T g(\xi, y_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}y_0(\xi))\mathrm{d}\xi$, we have

$$\int_a^b \{\eta(\xi, y_0, v_0)\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + ({}^{\mathrm{CF}}D^\alpha_{a+}\eta(\xi, y_0, v_0))\lambda_0(\xi)^T g_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))\}\mathrm{d}\xi \le 0. \tag{87}$$

Now combining inequalities (86) and (87), we obtain

$$\int_a^b \{\eta(\xi, y_0, v_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))] + ({}^{\mathrm{CF}}D^\alpha_{a+}\eta(\xi, y_0, v_0))[\lambda_0(\xi)^T g_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0) + z_0(\xi)^T h_{{}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D^\alpha_{a+}v_0)]\}\mathrm{d}\xi \le 0. \tag{88}$$

Now using Proposition 1 in inequality (88) along with $\eta(\xi, y_0, y_0) = 0$, we obtain

$$\int_a^b \eta(\xi, y_0, v_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v_0)^{\mathrm{CFR}}D^\alpha_{b-}[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D^\alpha_{a+}v_0(\xi))]\mathrm{d}\xi \le 0. \tag{89}$$

In view of Eq. (85), inequality (89) can be written as

$$\int_a^b \eta(\xi, y_0, v_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))]d\xi + \int_a^b \eta(\xi, y_0, v_0)^{CFR}D_{b-}^{\alpha}[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0, {}^{CF}D_{a+}^{\alpha}v_0)]d\xi = 0 \leq 0,$$

which contradicts the assumption $y_0 \neq v_0$. Consequently, $y_0 = v_0$ and v_0 is a weak minimum for (P). □

In the next step, we state the theorem as stated in ref. [45] with the CF fractional derivative operator in the following way:

Theorem 12. *Let y_0 be a weak minimum for (CP) and $(v_0, z_0, w_0, \lambda_0)$ be a weak maximum for (CD) such that*

$$\int_a^b \lambda_0(\xi)^T g(\xi, y_0(\xi), {}^{CF}D_{a+}^{\alpha}y_0(\xi))d\xi \leq \int_a^b \left\{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{i\in I_0} z_{0_i}(\xi)^T \times h_i(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{j\in J_0} w_{0_j}(\xi)^T \times \vartheta_j(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\right\}d\xi.$$

If

(a) $\int_a^b\left\{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{i\in I_0} z_{0_i}(\xi)^T \times h_i(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{j\in J_0} w_{0_j}(\xi)^T \times \vartheta_j(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\right\}d\xi$ *is SPIX at v_0 and each* $\int_a^b\left\{\sum_{i\in I_\alpha} z_{0_i}(\xi)^T \times h_i(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{j\in J_\alpha} w_{0_j}(\xi)^T \times \vartheta_j(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\right\}d\xi,$ *$\beta = 1, 2, 3,\ldots \kappa$, is QIX at v_0; or*

(b) $\int_a^b\left\{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{i\in I_0} z_{0_i}(\xi)^T, \times h_i(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{j\in J_0} w_{0_j}(\xi)^T \times \vartheta_j(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\right\}d\xi$ *is QIX at v_0 and each* $\int_a^b\left\{\sum_{i\in I_\alpha} z_{0_i}(\xi)^T \times h_i(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + \sum_{j\in J_\alpha} w_{0_j}(\xi)^T \times \vartheta_j(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\right\}d\xi,$ *$\beta = 1, 2, 3, \ldots \kappa$, is SPIX at v_0 then $y_0 = v_0$, i.e. v_0 is a weak minimum for (P).*

Corollary. *Suppose y_0 is a weak minimum for (P) and (v_0, z_0, λ_0) is a weak maximum for (D) so that*

$$\int_a^b \lambda_0(\xi)^T g(\xi, y_0(\xi), {}^{CF}D_{a+}^{\alpha}y_0(\xi))d\xi \leq \int_a^b \{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\}d\xi.$$

If $\int_a^b \{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\}d\xi$ *is SPIX at v_0, then $y_0 = v_0$, i.e. v_0 is a weak minimum for (P).*

Proof. It is assumed that $y_0 \neq v_0$. Since y_0 is feasible for (P) thus $h(\xi, y_0(\xi), {}^{CF}D_{a+}^{\alpha}y_0(\xi)) \leq 0$, and (v_0, z_0, λ_0) is feasible for (D) thus from Eq. (27), we have

$$\int_a^b \eta(\xi, y_0, v_0)\{g_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))e\}d\xi = -\int_a^b \eta(\xi, y_0, v_0)^{CFR}D_{b-}^{\alpha}\{g_{{}^{CF}D_{a+}^{\alpha}v_0(\xi)}(\xi, v_0, {}^{CF}D_{a+}^{\alpha}v_0) + z_0(\xi)^T h_{{}^{CF}D_{a+}^{\alpha}v_0(\xi)}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))e\}d\xi. \quad (90)$$

Multiplying Eq. (90) by $\lambda_0(\xi)^T$ and further using $\lambda_0(\xi)^T e = 1$, we obtain

$$\int_a^b \eta(\xi, y_0, v_0)\{\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\}d\xi = -\int_a^b \eta(\xi, y_0, v_0)^{CFR}D_{b-}^{\alpha}\{\lambda_0(\xi)^T g_{{}^{CF}D_{a+}^{\alpha}v_0(\xi)}(\xi, v_0, {}^{CF}D_{a+}^{\alpha}v_0) + z_0(\xi)^T h_{{}^{CF}D_{a+}^{\alpha}v_0(\xi)}(\xi, v_0, {}^{CF}D_{a+}^{\alpha}v_0)\}d\xi. \quad (91)$$

Since

$$\int_a^b \lambda_0(\xi)^T g(\xi, y_0(\xi), {}^{CF}D_{a+}^{\alpha}y_0(\xi))d\xi \leq \int_a^b \{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi)) + z_0(\xi)^T h(\xi, v_0(\xi), {}^{CF}D_{a+}^{\alpha}v_0(\xi))\}d\xi, \quad (92)$$

and also $h(\xi, y_0(\xi), {}^{CF}D_{a+}^{\alpha}y_0(\xi)) \leq 0$, which implies $\int_a^b z_0(\xi)^T h(\xi, y_0(\xi), {}^{CF}D_{a+}^{\alpha}y_0(\xi))d\xi \leq 0$, thus inequality (92) can be written as

$$\int_a^b \{\lambda_0(\xi)^T g(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi)) + z_0(\xi)^T h(\xi, y_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} y_0(\xi))\}\mathrm{d}\xi \le \int_a^b \{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) + z_0(\xi)^T h(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi))\}\mathrm{d}\xi,$$

and since $\int_a^b \{\lambda_0(\xi)^T g(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) + z_0(\xi)^T h(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi))\}\mathrm{d}\xi$ is SPIX at v_0, so we have

$$\int_a^b \{\eta(\xi, y_0, v_0)\,[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi))]\mathrm{d}\xi + ({}^{\mathrm{CF}}D_{a+}^{\alpha}\eta(\xi, y_0, v_0))[\lambda_0(\xi)^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0) + z_0(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0)]\}\mathrm{d}\xi < 0. \tag{93}$$

Now using Proposition 1 in inequality (93) and $\eta(\xi, y_0, y_0) = 0$, we obtain

$$\int_a^b \eta(\xi, y_0, v_0)\,[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v_0)^{CFR}D_{b-}^{\alpha}\,[\lambda_0(\xi)^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0) + z_0(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0)]\mathrm{d}\xi < 0. \tag{94}$$

Now using Eq. (91) in inequality (94), we have

$$\int_a^b \eta(\xi, y_0, v_0)[\lambda_0(\xi)^T g_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)) + z_0(\xi)^T h_{v_0}(\xi, v_0(\xi), {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi))]\mathrm{d}\xi + \int_a^b \eta(\xi, y_0, v_0)^{\mathrm{CFR}}D_{b-}^{\alpha}[\lambda_0{}^T g_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0) + z_0(\xi)^T h_{{}^{\mathrm{CF}}D_{a+}^{\alpha} v_0(\xi)}(\xi, v_0, {}^{\mathrm{CF}}D_{a+}^{\alpha} v_0)]\mathrm{d}\xi = 0 < 0, \tag{95}$$

which is a contradiction. Thus, the assumption $y_0 \neq v_0$ is false. Consequently, $y_0 = v_0$ and v_0 is a weak minimum for (P). □

5 Conclusions

The proposed work extends and derives the generalized invexity and duality results for multiobjective variational problems with the framing of a non-singular fractional derivative pertaining to the exponential kernel by utilizing the concept of weak minima. This work considers the CF fractional derivative operator possessing a non-singular exponential kernel. Moreover, several duality results of weak, strong, and converse categories have also been derived for various types of generalized invexity conditions in view of the CF fractional derivative operator. Some basic theorems and formulas for integration by parts for the fractional derivative with an exponential kernel have played a significant role in proving the weak, strong, and converse duality theorems. This article also presents the derivation of strict converse duality theorems for multiobjective variational problems with the CF fractional derivative by employing some propositions and theorems of fractional calculus. The variational problems with CF fractional derivative may be helpful in analysing the optimization problems and physical processes. Problems related to production planning, oil refinery scheduling, portfolio selection, management sciences, and economics can be modelled successfully in the form of multiobjective variational problems. The results derived in this article are important for the growth of generalized invexity and duality results for a class of multiobjective variational problems involving a non-singular fractional derivative. However, it is difficult to illustrate any practical application on the basis of derived results. But there is an ample scope to explore the optimality and duality results for multiobjective variational problems within the scope of fractional calculus. This work can be further extended to study the multiobjective variational problems and non-differentiable multiobjective variational problems involving other kinds of fractional derivatives. As a future scope of the work, FVPs can also be studied with the theorems and propositions applied in this article.

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