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Research article

Mellin transform for fractional integrals with general analytic kernel

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Abstract: Many different operators of fractional calculus have been proposed, which can be organized in some general classes of operators. According to this study, the class of fractional integrals and derivatives can be classified into two main categories, that is, with and without general analytical kernel (introduced in 2019). In this article, we define the Mellin transform for fractional differential operator with general analytic kernel in both Riemann-Liouville and Caputo derivatives of order $\varsigma \geq 0$ and ϱ be a fixed parameter. We will also establish relation between Mellin transform with Laplace and Fourier transforms.

Keywords: Mellin transform; fractional integrals; Caputo fractional derivative; Laplace and Fourier transforms
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1. Introduction

Fractional calculus is a generalization of the ordinary calculus and it has been used prosperously in sundry fields of engineering and science [1, 2]. It was introduced in 1695 when L'Hospital raised the question in a letter written to Leibnitz [3]. Leibnitz's prophetic answer to this profound question enclosed the innovative idea for all generations of experts and also continues to refresh the minds of that time researchers. Then after the Liouville's work, in 1847, Riemann [4] derived the formula which can be linked with the fractional integral formula of Liouville. From 1900 till now, a huge number of results and the cluster of booklets on fractional calculus appeared in the literature. Detailed discussions of the history of fractional calculus may be found in [5–13].

There are variants of fractional derivatives in the current literature. Riemann-Liouville's (R-L) definition of fractional differentiation and integration is the most popular model. The Caputo derivative (also known as Caputo-Fabrizio derivative) is another commonly used derivative [8,14,15]. Numerical methods were also used to solve the fractional parabolic differential problems [16] and subdiffusion equations involving Caputo fractional derivatives to get the convergence of results in more effective way [17].

An important subtopic of fractional calculus is the class of fractional derivatives and integrals with analytic kernel proposed in 2019 by Fernandez et al. [18]. Each of the above mentioned class of fractional calculus covers major portions of the subject and capture diverse behaviors in fractional systems. Combining both ideas gives a more general class of fractional integrals and derivatives with analytic kernels with respect to functions. Taylor series and asymptotic approximations of the integral whose integrands depends on the fabrication from two hypergeometric functions is most commonly solved by the method of Mellin transform and by using this approach gives the exceptional results along with the evaluation of electromagnetic propagation in a turbulent medium, optical multi-image encryption and image compression [19–21]. The Mellin transform is used to solve different biological models, (see [22–24]).

General properties of the Mellin transform are typically treated in detail in books on integral transforms, see [25–31]. In 1959, Francis [32] discussed the applications of complex Mellin transform to networks with time-variant parameters. In 1995, Flajolet et al. [33] used Mellin transform for the asymptotic evaluation of harmonic sums. In 2016, Kilicman and Omran [21] studied some effects of Mellin transform on fractional integral and differential operators and discussed their properties. The Mellin transform is mostly used to solve the axisymmetric problems and to simplify a number of derivations. Till now, the Mellin integral transform has been sporadically employed within the fractional calculus guides.

The arrangement of this article is as follows: In Section 2, some basic definitions and features of the fractional models with general analytic kernel are defined. In Section 3, we define some basic concepts of Mellin transform and the notion of Mellin transform for both R-L and Caputo models are discussed. In Section 4, the concept of Mellin transform is extended for the R-L and Caputo type fractional derivatives in the presence of general analytic kernel. In Section 5, the relation of Mellin transforms with the previously defined Laplace and Fourier transforms, applied on the fractional differential equations having the general analytic kernel is discussed. Using the method of Fourier and Laplace transforms, we analyzed and solved some simple ordinary differential equations in the new general framework. In Section 6, the conclusion of the article is given.

2. Preliminaries

In this section, some basic definitions regarding R-L and Caputo fractional derivative and integral operator are stated. Also the notion of general analytic kernel and some related properties are given.

Definition 2.1. *[34] A function $h(t), t > 0$ is said to be in the space $C_b, (b \in \mathbb{R})$ if $h(t) = x^q h_1(t)$ for some $q > b$, where $h_1(t)$ is continuous in $[0, \infty)$ and it is said to be in the space C_b^m if and only if $h^m \in C_b, m \in \mathbb{N}$. For more details see [35–37].*

Definition 2.2. *[1] The R-L fractional integral of order $\varsigma > 0$ of a function $h(t)$ is defined as*

$$^{RL}\mathcal{I}_{c+}^{\varsigma}h(t) = \frac{1}{\Gamma(\varsigma)}\int_c^t (t-\varpi)^{\varsigma-1}h(\varpi)d\varpi, \qquad 0<\varpi<t, \qquad \varsigma>0.$$

Definition 2.3. *[1] The R-L fractional derivative of order ς of a function $h(t)$ is defined as*

$$^{RL}\mathcal{D}^{\varsigma}h(t) = \frac{d^m}{dt^m}\left(^{RL}\mathcal{I}^{m-\varsigma}h(t)\right), \qquad m-1<\varsigma<m.$$

Definition 2.4. *[38] Let $0 \le m-1 < \varsigma < m$ and the function $h(t)$ has $(m+1)$ continuous bounded derivative in $[0,T]$ for every $T>0$. Then Caputo derivative of a function is defined as*

$$\mathcal{D}_{0+}^{\varsigma}h(t) = \frac{1}{\Gamma(m-\varsigma)}\int_0^t (t-\varpi)^{m-\varsigma-1}h^m(\varpi)d\varpi, \qquad m-1<\varsigma<m.$$

Definition 2.5 (Semigroup property)**.** *[2] It is also important to note that if $\varsigma, \varrho, \eta, \xi$ be any complex parameters, then $\mathcal{I}^{\varsigma}\mathcal{I}^{\varrho}h(t) = \mathcal{I}^{\varsigma+\varrho}h(t)$.*

Thus, we have composition property of differential operators

$$\mathcal{D}^{\eta}(\mathcal{D}^{\xi}h(t)) = \mathcal{D}^{\eta+\xi}h(t).$$

Remark 1. *[39] By convention, we have*

$$^{RL}\mathcal{D}_c^{-\varsigma}h(t) = {}^{RL}\mathcal{I}_c^{\varsigma}h(t),$$

so that the fractional operators $^{RL}\mathcal{D}_c^{\varsigma}h(t)$ and $^{RL}\mathcal{I}_c^{\varsigma}h(t)$ are well-defined for all $\varsigma \in \mathbb{C}$.

Fractional model with general analytic kernel is two parameter fractional model defined on the analytic disc. Due to their analytic behavior, this kernel is also known as "singular kernel". The following definitions and theorems are taken from [18]:

Definition 2.6. *Let $[c,d] \in \mathbb{R}$, ς and ϱ be complex parameters with positive real parts and positive real number R satisfying $R > (d-c)^{Re(\varrho)}$. Let $\mathcal{A}$ be an analytic complex function defined on the disc $D(0,R)$ by locally uniformly convergent power series*

$$\mathcal{A}(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where the coefficients c_k may depend on complex parameters.

Definition 2.7. *For any analytic function as in Definition 2.6, modified analytic function $\mathcal{A}_\Gamma$ is defined as*

$$\mathcal{A}_\Gamma(x) = \sum_{k=0}^{\infty} c_k\Gamma(\varrho k+\varsigma)x^k, \tag{2.1}$$

where the radius of convergence of the series (2.1) is given by

$$\lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}(\varrho k+\varrho+\varsigma)^{-\varrho}\right|.$$

Note: *The convergence of the series $\mathcal{A}$ depends upon the convergence of series $\mathcal{A}_\Gamma$ but the converse is not necessarily true.*

Definition 2.8. *The fractional integral operator with general analytic kernel, acting on the function* $h:[c,d]\to\mathbb{R}$ *as*

$$^{\mathcal{A}}\mathcal{I}_{c+}^{\varsigma,\varrho}h(t)=\int_c^t(t-\varpi)^{\varsigma-1}\mathcal{A}\left((t-\varpi)^{\varrho}\right)h(\varpi)d\varpi.$$

Theorem 2.9. *With all the representations as in Definition 2.6, for any function* $h\in L^1[c,d]$, *the following locally uniformly convergent power series for* $^{\mathcal{A}}\mathcal{I}_{c+}^{\varsigma,\varrho}h$ *is defined as*

$$^{\mathcal{A}}\mathcal{I}_{c+}^{\varsigma,\varrho}h(t)=\sum_{k=0}^{\infty}c_k\Gamma(\varrho k+\varsigma)^{RL}\mathcal{I}_{c+}^{\varrho k+\varsigma}h(t), \tag{2.2}$$

as a function on $[c,d]$. *Similarly, fractional integral can also be written in the form of modified analytic function* (2.1) *as*

$$^{\mathcal{A}}\mathcal{I}_{c+}^{\varsigma,\varrho}h(t)=\mathcal{A}_{\Gamma}(^{RL}\mathcal{I}_{c+}^{\varrho})\,^{RL}\mathcal{I}_{c+}^{\varsigma}h(t).$$

Definition 2.10. *For R-L and Caputo type the fractional derivative for general analytic kernel is defined as*

$$^{\mathcal{A}}_{RL}\mathcal{D}_{c+}^{\varsigma,\varrho}h(t)=\frac{d^m}{dt^m}\left(^{\mathcal{A}}\mathcal{I}_{c+}^{\varsigma',\varrho'}h(t)\right), \tag{2.3}$$

$$^{\mathcal{A}}_{C}\mathcal{D}_{c+}^{\varsigma,\varrho}h(t)=\,^{\mathcal{A}}\mathcal{I}_{c+}^{\varsigma',\varrho'}\left(\frac{d^m}{dt^m}h(t)\right), \tag{2.4}$$

where $m\in\mathbb{N}$, $m+\varsigma'=\varsigma$ *and* $\varrho'=\varrho$.

In many areas of applied mathematics, optics [20], quantum mechanics [40] and signal processing [21, 41], integral transforms of fractional derivatives provide well-proven and valuable methods for solving integral and differential problems. Since, in 1980, Namias' introduced the Fourier transform for fractional operators [42], many other mathematicians and applied physicists have turned their goals not only to Fourier transform of fractional derivatives but also towards many other transforms, such as the Mellin transform, Laplace transform and the Hilbert transform of fractional differential problems [43, 44]. Laplace and Fourier transforms have a major role in solving fractional differential and integral equations so Fernandez et al. [18] additionally defined these transforms for the fractional integral with general analytic kernel utilizing convolution property.

Theorem 2.11. *Let* $c=0, d>0$, $\varsigma,\varrho,\mathcal{A}$ *be as in Definition 2.6 and* $h\in L^2[c,d]$ *with Laplace transform* $\bar{h}$. *Then Laplace transform of (2.2) is as follows*

$$\overline{^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(s)}=s^{-\varsigma}\mathcal{A}_{\Gamma}(s^{-\varrho})\bar{h}(s),$$

where $\mathcal{A}_{\Gamma}$ *is defined in Definition 2.7.*

Theorem 2.12. *Let* $c=-\infty, d\in\mathbb{R}$, $\varsigma,\varrho,\mathcal{A}$ *be as in Definition 2.6 and* $h\in L^2[c,d]$ *with Fourier transform* $\tilde{h}$. *Fourier transform for fractional integral (2.2) is defined as*

$$^{\mathcal{A}}\widetilde{\mathcal{I}_{+}^{\varsigma,\varrho}h}(k)=k^{-\varsigma}e^{i\varsigma\pi/2}\mathcal{A}_{\Gamma}(k^{-\varrho}e^{i\varrho\pi/2})\tilde{h}(k),$$

where A_{Γ} *is defined in Definition 2.7.*

3. Mellin transform

According to Flajolet et al. [33], Mellin gave his denomination to the Mellin transform $\mathcal{M}[h(t); u]$ that associates to a function $h(t)$ defined over the positive reals. It is approximately cognate to the Fourier and Laplace transform. From now onwards, we recall some definitions and some basic properties of the Mellin transform.

Definition 3.1. *Let $h(t)$ be the function defined on the interval $(0, \infty)$, the Mellin transform of the function $h(t)$, denoted by $H(u)$, is as follows*

$$H(u) = \mathcal{M}\{h(t); u\} = \int_0^\infty h(t)t^{u-1}dt, \tag{3.1}$$

where s is complex.

The function $h(t)$ can be restored using inverse Mellin formula

$$h(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} H(u)t^{-u}du. \tag{3.2}$$

Some important properties of Mellin transform are:

(1) $\mathcal{M}(t^{-\varsigma}h(t)) = H(u - \varsigma)$,
(2) $\mathcal{M}(h(t^{\mu})) = \frac{1}{\mu}H(\frac{u}{\mu})$,
(3) $\mathcal{M}\left\{t^{\lambda}\int_0^\infty \varpi^{\mu}h(t\varpi)g(\varpi)d\varpi\right\} = H(u + \lambda)G(1 - u - \lambda - \mu)$.

To define the Mellin transform of R-L and Caputo derivatives, let $m - 1 < \varsigma < m$ and h be the function defined in Definition 4. The Mellin transform is defined as follows [1]:

$$\mathcal{M}\left\{{}^{RL}\mathcal{D}^{\varsigma}h(t)\right\} = \mathcal{M}\left\{{}^{C}\mathcal{D}_a^{\varsigma}h(t)\right\} = \frac{(1 - u + \varsigma)}{(1 - u)}H(u - \varsigma). \tag{3.3}$$

The Mellin transform of R-L fractional integral of order $\varsigma > 0$ of a function $h \in C_{\mu}$, where $\mu \geq -1$ is defined as

$$\mathcal{M}\left\{{}^{RL}\mathcal{D}^{-\varsigma}h(t)\right\} = \frac{(1 - u - \varsigma)}{(1 - u)}H(u + \varsigma). \tag{3.4}$$

4. Results and discussion

In this section, we define the Mellin transform of R-L fractional integral operator with general analytic kernel, and Laplace, Fourier and Mellin transforms of the Caputo fractional integrals in the presence of general analytic kernel, indeed, Laplace and Fourier transforms of fractional integral with general analytic kernel are already defined in [18]. It is straight forward to find formulae for the transformed function using Theorem 2.

4.1. Mellin transform of R-L fractional integral with general analytic kernel

Theorem 4.1. *Let $c = 0, d > 0$ and $\varsigma, \varrho, \mathcal{A}$ as in Definition 2.6, and let $h \in L^2[c, d]$ with the Mellin transform $\widehat{h}$. The function ${}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(t)$ has the Mellin transform given by the following formula:*

$$\widehat{{}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}}h(u) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\varsigma + n\varrho)\Gamma(1 - u - \varsigma - n\varrho)}{\Gamma(1 - u)}H(u + \varsigma + n\varrho). \tag{4.1}$$

Proof. Consider series formula from Theorem 2

$$ {}^{\mathcal{A}}\mathcal{I}^{\varsigma,\varrho}_{0+}h(t) = \sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\ {}^{RL}\mathcal{I}^{\varsigma+n\varrho}_{0+}h(t). $$

Taking Melllin transform on both sides

$$ \begin{aligned} \mathcal{M}\left({}^{\mathcal{A}}\mathcal{I}^{\varsigma,\varrho}_{0+}h(t)\right) &= \mathcal{M}\left(\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\ {}^{RL}\mathcal{I}^{\varsigma+n\varrho}_{0+}h(t)\right), \\ \widehat{{}^{\mathcal{A}}\mathcal{I}^{\varsigma,\varrho}_{0+}}h(u) &= \sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\mathcal{M}\left({}^{RL}\mathcal{I}^{\varsigma+n\varrho}_{0+}h(t)\right), \\ &= \sum_{n=0}^{\infty} a_n \mathcal{M}\left(\int_0^t (t-\varpi)^{\varsigma+n\varrho-1}h(\varpi)d\varpi\right). \end{aligned} \tag{4.2} $$

Consider

$$ \begin{aligned} \mathcal{M}\left(\int_0^t (t-\varpi)^{\varsigma+n\varrho-1}h(\varpi)d\varpi\right) &= \mathcal{M}\left(\int_0^1 t(t-t\xi)^{\varsigma+n\varrho-1}h(t\xi)d\xi\right), \\ &= \mathcal{M}\left(\int_0^1 t.t^{\varsigma+n\varrho-1}(1-\xi)^{\varsigma+n\varrho-1}h(t\xi)d\xi\right), \\ &= \mathcal{M}\left(\int_0^1 t^{\varsigma+n\varrho}(1-\xi)^{\varsigma+n\varrho-1}h(t\xi)d\xi\right), \\ &= \mathcal{M}\left(\int_0^\infty t^{\varsigma+n\varrho}g(\xi)h(t\xi)d\xi\right), \end{aligned} \tag{4.3} $$

where

$$ g(t) = \begin{cases} (1-t)^{\varsigma+n\varrho-1} & 0 \le t < 1; \\ 0 & t \ge 1. \end{cases} $$

The Mellin transform of the function $g(t)$ is simply the Euler beta function.

$$ \mathcal{M}\{g(t)\} = \frac{\Gamma(\varsigma+n\varrho)\Gamma(u)}{\Gamma(\varsigma+n\varrho+u)}. $$

By using the convolution theorem of Mellin transform given by

$$ \mathcal{M}\left\{t^{\lambda}\int_0^\infty g(\xi)h(t\xi)d\xi\right\} = H(u+\lambda)G(1-u-\lambda), \tag{4.4} $$

so from (4.3), we have

$$ \begin{aligned} \mathcal{M}\left\{t^{\varsigma+n\varrho}\int_0^\infty g(\xi)h(t\xi)d\xi\right\} &= \frac{\Gamma(\varsigma+n\varrho)\Gamma(1-u-\varsigma-n\varrho)}{\Gamma(\varsigma+n\varrho+1-u-\varsigma-n\varrho)}H(u+\varsigma+n\varrho), \\ &= \frac{\Gamma(\varsigma+n\varrho)\Gamma(1-u-\varsigma-n\varrho)}{\Gamma(1-u)}H(u+\varsigma+n\varrho). \end{aligned} \tag{4.5} $$

By using (4.5) in (4.2), we have

$$\begin{aligned}\mathcal{M}\left({}^{A}\mathcal{I}_{0+}^{\varsigma,\varrho}h(t)\right) &= \sum_{n=0}^{\infty} a_n \mathcal{M}\left(\int_0^t (t-\varpi)^{\varsigma+n\varrho-1} h(\varpi)d\varpi\right),\\ &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(\varsigma+n\varrho)\Gamma(1-u-\varsigma-n\varrho)}{\Gamma(1-u)} H(u+\varsigma+n\varrho).\end{aligned}$$

□

4.2. Laplace transform of Caputo derivative with general analytic kernel

Theorem 4.2. *Let $c = 0, d > 0$ and $\varsigma, \varrho, \mathcal{A}$ as in Definition 2.6 and let $h \in L^2[c,d]$ with the Laplace transform $\overline{h}$. The function ${}^{\mathcal{A}}_{C}\mathcal{D}_{0+}^{\varsigma,\varrho}h(t)$ has a Laplace transform given by the following formula:*

$$\overline{{}^{\mathcal{A}}_{C}D_{0^+}^{\varsigma,\varrho}}h(s) = s^{\varsigma-2\xi}A_\Gamma(s^{-\varrho})\overline{h}(s). \tag{4.6}$$

Proof. The Caputo fractional derivative with general analytic kernel is given in (2.4) and defined as

$${}^{\mathcal{A}}_{C}\mathcal{D}_{0+}^{\varsigma,\varrho}h(t) = {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma',\varrho'}\left(\frac{d^m}{dt^m}h(t)\right), \tag{4.7}$$

where

$$m + \varsigma' = \varsigma, \qquad \varrho' = \varrho. \tag{4.8}$$

By using (4.8) in (4.28), we get

$${}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma',\varrho'}\left(\frac{d^m}{dt^m}h(t)\right) = {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma-m,\varrho}\left(h^{(m)}(t)\right).$$

Consider

$$\begin{aligned}{}^{\mathcal{A}}\mathcal{I}_{0^+}^{\varsigma-m,\varrho}h^{(m)}(t) &= \sum_{n=0}^{\infty} a_n\Gamma(\varsigma-m+n\varrho)\,{}^{RL}\mathcal{I}_{0^+}^{\varsigma-m+n\varrho}h^{(m)}(t),\\ &= \sum_{n=0}^{\infty}\frac{a_n\Gamma(\varsigma-m+n\varrho)}{\Gamma(\varsigma-m+n\varrho)}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi,\\ &= \sum_{n=0}^{\infty} a_n \int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi.\end{aligned}$$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\left({}^{\mathcal{A}}\mathcal{I}_{0^+}^{\varsigma-m,\varrho}h^{(m)}(t)\right) = \sum_{n=0}^{\infty} a_n\mathcal{L}\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi\right). \tag{4.9}$$

Consider

$$\mathcal{L}\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi\right) = \int_0^\infty e^{-st}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi dt. \tag{4.10}$$

Let $t-\varpi=u,\ dt=du$, then

$$\int_0^\infty e^{-st}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi$$
$$=\int_0^\infty e^{-s(u+\varpi)}\int_0^t u^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi du.$$

Using Fubini's theorem, we have

$$\int_0^\infty e^{-s(u+\varpi)}\int_0^t u^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi$$
$$=\int_0^\infty\int_0^\infty e^{-s(u+\varpi)}u^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi du,$$
$$=\int_0^\infty e^{-su}u^{\varsigma-m+n\varrho-1}du\int_0^\infty e^{-s\varpi}h^{(m)}(\varpi)d\varpi, \tag{4.11}$$

where the integral (4.11) is equals to

$$\int_0^\infty e^{-su}u^{\varsigma-m+n\varrho-1}du=\frac{\Gamma(\varsigma-m+n\varrho)}{s^{\varsigma-m+n\varrho}}, \tag{4.12}$$

and

$$\int_0^\infty e^{-s\varpi}h^{(m)}(\varpi)d\varpi = s^mH(s)-\sum_{k=0}^{m-1}s^kh^{(m-k-1)}(0). \tag{4.13}$$

We assume that the function $h(t)$ and all it's derivatives gives zero at $t=0$, so (4.10) becomes

$$\mathcal{L}\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi\right)=\frac{\Gamma(\varsigma-m+n\varrho)}{s^{\varsigma-m+n\varrho}}\left(s^mH(s)-\sum_{k=0}^{m-1}s^k\quad\times h^{(m-k-1)}(0)\right),$$
$$=\Gamma(\varsigma-m+n\varrho)s^{-\varsigma+m-n\varrho}s^mH(s),$$
$$=\Gamma(\varsigma-m+n\varrho)s^{-\varsigma+2m-n\varrho}H(s). \tag{4.14}$$

By using (4.14) in (4.9), we get

$$\mathcal{L}\left({}^{\mathcal{A}}\mathcal{I}_{0^+}^{\varsigma-m,\varrho}h^{(m)}(t)\right)=\sum_{n=0}^\infty a_n\Gamma(\varsigma-m+n\varrho)s^{-\varsigma+2m-n\varrho}H(s). \tag{4.15}$$

Let us substitute $m=\varsigma-\xi$, then (4.15) becomes

$$\overline{{}^{\mathcal{A}}_{C}D_{0^+}^{\varsigma,\varrho}}h(s)=\sum_{n=0}^\infty a_n\Gamma(\xi+n\varrho)s^{-\xi+\varsigma-\xi-n\varrho}H(s),$$
$$=s^{\varsigma-2\xi}\mathcal{A}_\Gamma(s^{-\varrho})\mathcal{L}\{h(t);s\}.$$

Hence the result. □

4.3. Fourier transform of Caputo fractional derivative with general analytic kernel

Theorem 4.3. *Let $c = 0, d > 0$ and $\varsigma, \varrho, \mathcal{A}$ as in Definition 2.6, and let $h \in L^2[c, d]$ with the Fourier transform $\widetilde{h}$. The function ${}^{\mathcal{A}}_{C}\mathcal{D}^{\varsigma,\varrho}_{0+}h(t)$ has a Fourier transform given by the following formula:*

$$\widetilde{{}^{\mathcal{A}}_{C}D^{\varsigma,\varrho}_{0^+}}h(\omega) = (e^{-i\pi/2}\omega)^{\varsigma-2\xi}\mathcal{A}_{\Gamma}(e^{i\varrho\pi/2}\omega^{-\varrho})\widetilde{h}(\omega). \tag{4.16}$$

Proof. The Caputo fractional derivative with general analytic kernel is given in (2.4) and defined as

$${}^{\mathcal{A}}_{C}\mathcal{D}^{\varsigma,\varrho}_{0+}h(t) = {}^{\mathcal{A}}\mathcal{I}^{\varsigma',\varrho'}_{0+}\left(\frac{d^m}{dt^m}h(t)\right), \tag{4.17}$$

where

$$m + \varsigma' = \varsigma, \qquad \varrho' = \varrho. \tag{4.18}$$

By using (4.18) in (4.28), we get

$${}^{\mathcal{A}}\mathcal{I}^{\varsigma',\varrho'}_{0+}\left(\frac{d^m}{dt^m}h(t)\right) = {}^{\mathcal{A}}\mathcal{I}^{\varsigma-m,\varrho}_{0+}\left(h^{(m)}(t)\right).$$

Consider

$$\begin{aligned}
{}^{\mathcal{A}}\mathcal{I}^{\varsigma-m,\varrho}_{0^+}h^{(m)}(t) &= \sum_{n=0}^{\infty} a_n\Gamma(\varsigma - m + n\varrho)^{RL}\mathcal{I}^{\varsigma_m+n\varrho}_{0^+}h^{(m)}(t),\\
&= \sum_{n=0}^{\infty} a_n\Gamma(\varsigma - m + n\varrho)\frac{1}{\Gamma(\varsigma - m + n\varrho)}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1} \times h^{(m)}(\varpi)d\varpi,\\
&= \sum_{n=0}^{\infty} a_n \int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi.
\end{aligned}$$

Taking Fourier transform on both sides, we get

$$F\left({}^{\mathcal{A}}\mathcal{I}^{\varsigma-m,\varrho}_{0^+}h^{(m)}(t)\right) = \sum_{n=0}^{\infty} a_n F\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi\right). \tag{4.19}$$

Consider

$$F\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi\right) = \int_{-\infty}^{\infty} e^{-ikt}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi dt. \tag{4.20}$$

Let $t - \varpi = u,\ dt = du$, then

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-ikt}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi dt\\
&\qquad = \int_{-\infty}^{\infty} e^{-ik(u+\varpi)}\int_{-\infty}^{t} u^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi du.
\end{aligned}$$

Using Fubini's theorem, we have

$$\int_{-\infty}^{\infty} e^{-ik(u+\varpi)} \int_0^t u^{\varsigma-m+n\varrho-1} h^{(m)}(\varpi) d\varpi du$$
$$= \int_{-\infty}^{\infty} \int_0^{\infty} e^{-ik(u+\varpi)} u^{\varsigma-m+n\varrho-1} h^{(m)}(\varpi) d\varpi du,$$
$$= \int_0^{\infty} e^{-iku} u^{\varsigma-m+n\varrho-1} du \int_0^{\infty} e^{-ik\varpi} h^{(m)}(\varpi) d\varpi, \tag{4.21}$$

where the integral (4.21) is equals to

$$\int_{-\infty}^{\infty} e^{-iku} u^{\varsigma-m+n\varrho-1} du = \frac{\Gamma(\varsigma - m + n\varrho)}{(-i\omega)^{\varsigma-m+n\varrho}}, \tag{4.22}$$

and

$$\int_0^{\infty} e^{-ik\varpi} h^{(m)}(\varpi) d\varpi = (-i\omega)^m H(\omega). \tag{4.23}$$

So, (4.21) becomes

$$\int_{-\infty}^{\infty} e^{-ik(u+\varpi)} \int_0^t u^{\varsigma-m+n\varrho-1} h^{(m)}(\varpi) d\varpi$$
$$= \left(\frac{\Gamma(\varsigma - m + n\varrho)}{(-i\omega)^{\varsigma-m+n\varrho}} \right) ((-i\omega)^m H(\omega)),$$
$$= \Gamma(\varsigma - m + n\varrho)(-i\omega)^{-\varsigma+m-n\varrho}(-i\omega)^m H(\omega),$$
$$= \Gamma(\varsigma - m + n\varrho)(-i\omega)^{-\varsigma+2m-n\varrho} H(\omega). \tag{4.24}$$

By using (4.24) in (4.30), we get

$$F\left({}^{\mathcal{A}}\mathcal{I}_{0^+}^{\varsigma-m,\varrho} h^{(m)}(t) \right) = \sum_{n=0}^{\infty} a_n \Gamma(\varsigma - m + n\varrho)(-i\omega)^{-\varsigma+2m-n\varrho} H(\omega). \tag{4.25}$$

Let us substitute $\varsigma - m = \eta,\ m = \varsigma - \eta$, then (4.25) becomes

$$\begin{aligned} \widetilde{{}^{\mathcal{A}}_{C}D_{0^+}^{\varsigma,\varrho}} h(\omega) &= \sum_{n=0}^{\infty} a_n \Gamma(\eta + n\varrho)(-i\omega)^{-\eta+\varsigma-\eta-n\varrho} H(\omega), \\ &= (-i\omega)^{\varsigma-2\eta} \mathcal{A}_\Gamma(s^{-\varrho}) F\{h(t); \omega\}, \\ &= (e^{-i\pi/2}\omega)^{\varsigma-2\xi} \mathcal{A}_\Gamma(e^{i\varrho\pi/2}\omega^{-\varrho}) F\{h(t); \omega\}; \qquad -i = e^{-i\pi/2}. \end{aligned} \tag{4.26}$$

□

4.4. *Mellin transform of Caputo fractional derivative with general analytic kernel*

Theorem 4.4. *Let $c = 0, d > 0$ and $\varsigma, \varrho, \mathcal{A}$ as in Definition 2.6, and let $h \in L^2[c, d]$ with the Mellin transform $\widehat{h}$. The function ${}^{\mathcal{A}}_{C}\mathcal{D}_{0+}^{\varsigma,\varrho} h(t)$ has a Mellin transform given by the following formula:*

$$\widehat{{}^{\mathcal{A}}_{C}\mathcal{D}_{0+}^{\varsigma,\varrho}} h(u) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(\lambda + n\varrho)\Gamma(1 - u + \varsigma - 2\lambda - n\varrho)}{\Gamma(1 - u)} H(u - \varsigma + n\varrho + 2\lambda). \tag{4.27}$$

Proof. The Caputo fractional derivative with general analytic kernel is given in (2.4) and defined as

$$ {}_{C}^{\mathcal{A}}\mathcal{D}_{0+}^{\varsigma,\varrho}h(t) = {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma',\varrho'}\left(\frac{d^m}{dt^m}h(t)\right), \tag{4.28} $$

$$ m+\varsigma' = \varsigma, \qquad \varrho' = \varrho. \tag{4.29} $$

By using (4.29) in equation (4.28), we get

$$ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma',\varrho'}\left(\frac{d^m}{dt^m}h(t)\right) = {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma-m,\varrho}\left(h^{(m)}(t)\right). $$

Consider

$$ \begin{aligned} {}^{\mathcal{A}}\mathcal{I}_{0^+}^{\varsigma-m,\varrho}h^{(m)}(t) &= \sum_{n=0}^{\infty} a_n\Gamma(\varsigma-m+n\varrho)^{RL}\mathcal{I}_{0^+}^{\varsigma-m+n\varrho}h^{(m)}(t), \\ &= \sum_{n=0}^{\infty} a_n\Gamma(\varsigma-m+n\varrho)\frac{1}{\Gamma(\varsigma-m+n\varrho)}\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}\times h^{(m)}(\varpi)d\varpi, \\ &= \sum_{n=0}^{\infty} a_n\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi. \end{aligned} $$

Taking Mellin transform on both sides, we get

$$ \mathcal{M}\left({}^{\mathcal{A}}\mathcal{I}_{0^+}^{\varsigma-m,\varrho}h^{(m)}(t)\right) = \sum_{n=0}^{\infty} a_n\mathcal{M}\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varpi-1}h^{(m)}(\varpi)d\varpi\right). \tag{4.30} $$

Consider

$$ \begin{aligned} &\mathcal{M}\left(\int_0^t (t-\varpi)^{\varsigma-m+n\varrho-1}h^{(m)}(\varpi)d\varpi\right) \\ &\qquad = \mathcal{M}\left(\int_0^1 t(t-t\xi)^{\varsigma-m+n\varrho-1}h^{(m)}(t\xi)d\xi\right), \\ &\qquad = \mathcal{M}\left(\int_0^1 t.t^{\varsigma-m+n\varrho-1}(1-\xi)^{\varsigma-m+n\varrho-1}h^{(m)}(t\xi)d\xi\right), \\ &\qquad = \mathcal{M}\left(\int_0^1 t^{\varsigma-m+n\varrho}(1-\xi)^{\varsigma-m+n\varrho-1}h^{(m)}(t\xi)d\xi\right), \\ &\qquad = \mathcal{M}\left(\int_0^\infty t^{\varsigma-m+n\varrho}g(\xi)h^{(m)}(t\xi)d\xi\right), \end{aligned} $$

where

$$ g(t) = \begin{cases} (1-t)^{\varsigma-m+n\varrho-1}, & 0\le t<1, \\ 0, & t\ge 1. \end{cases} $$

The Mellin transform of the function $g(t)$ is simply the beta function.

$$ \mathcal{M}\{g(t)\} = \frac{\Gamma(\varsigma-m+n\varrho)\Gamma(u)}{\Gamma(\varsigma-m+n\varrho+u)}, $$

and the Mellin transform of *mth* derivative of function $h(t)$ is

$$\mathcal{M}\{h^{(m)}t; u\} = \left\{\frac{\Gamma(1-u+m)}{\Gamma(1-u)}H(u-m)\right\}.$$

By using convolution theorem of Mellin transform

$$\mathcal{M}\left\{t^{\lambda}\int_0^{\infty} g(\xi)h(t\xi)d\xi\right\} = H(u+\lambda)G(1-u-\lambda). \tag{4.31}$$

So, we have

$$\begin{aligned}\mathcal{M}\left\{t^{\varsigma-m+n\varrho}\int_0^{\infty} g(\xi)h^{(m)}(t\xi)d\xi\right\} &= \frac{\Gamma(\varsigma-m+n\varrho)\Gamma(1-u-\varsigma+m-n\varrho+m)}{\Gamma(\varsigma-m+n\varrho+1-u-\varsigma+m-n\varrho)} \\ &\quad \times H(u+\varsigma-m+n\varrho-m), \qquad (4.32) \\ &= \frac{\Gamma(\varsigma-m+n\varrho)\Gamma(1-u-\varsigma+2m-n\varrho)}{\Gamma(1-u)} \\ &\quad \times H(u+\varsigma-2m+n\varrho). \qquad (4.33)\end{aligned}$$

By using (4.33) in (4.3), we have

$$\sum_{n=0}^{\infty} a_n\mathcal{M}\left(\int_0^t (t-\varpi)^{\varsigma+n\varrho-1}h(\varpi)d\varpi\right) = \sum_{n=0}^{\infty} a_n\frac{\Gamma(\varsigma-m+n\varrho)\Gamma\left(\begin{array}{c}1-u-\varsigma\\+2m-n\varrho\end{array}\right)}{\Gamma(1-u)} \times H(u+\varsigma+n\varrho-2m). \tag{4.34}$$

By substituting $\varsigma - m = \lambda,\ m = \varsigma - \lambda$, then

$$\widehat{{}^{\mathcal{A}}_{C}\mathcal{D}^{\varsigma,\varrho}_{0+}}h(u) = \sum_{n=0}^{\infty} a_n\frac{\Gamma(\lambda+n\varrho)\Gamma(1-u+\varsigma-2\lambda-n\varrho)}{\Gamma(1-u)} H(u-\varsigma+n\varrho+2\lambda). \tag{4.35}$$

Hence the result. □

Mellin transform is mostly used in applied sciences such as physics, engineering and computer sciences because of it's invariance to scale. In reality, this property of scale invariance is analogous to the shift invariance property of the Fourier transform. This transform is being considered very useful for the problems in which the inputs and the outputs are functions of time.

Here, we consider an axisymmetric initial value fractional differential equation. In the following example, we have $c = 0, d > 0$ and ς, ϱ and r are complex parameters with $Re(\varsigma) > 0, Re(\varrho) > 0$ and $r > 0$. The expression $\frac{\partial^{\varsigma,\varrho}}{\partial t^{\varsigma,\varrho}} = {}^{\mathcal{A}}\mathcal{D}^{\varsigma,\varrho}_{0^+}$ represents the generalized fractional differential operator with general analytic kernel. We apply "*Combined Laplace and Mellin Transform Method*" to get the required solution.

Example.

Consider the following initial value fractional differential equation

$$\frac{\partial^{\varsigma,\varrho} u(x,t)}{\partial t^{\varsigma,\varrho}} = x^{2r+n\varrho}\frac{\partial^{2r,\varrho} u(x,t)}{\partial x^{2r,\varrho}} + x^{r+n\varrho}\frac{\partial^{r,\varrho} u(x,t)}{\partial x^{r,\varrho}}, \tag{4.36}$$

with initial condition $u(0, x) = h(x)$.

Applying Laplace transform with respect to t on both sides of (4.36) and using Theorem 2.9, we have

$$\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\left\{ s^{\varsigma+n\varrho}\overline{u}(x,s) - \sum_{k=0}^{n-1} s^{\varsigma+n\varrho-k-1} u(x,0)\right\} = x^{2r+n\varrho}\ {}^{\mathcal{A}}\mathcal{D}_{0^+}^{2r,\varrho}\overline{u}(r,s) + x^{r+n\varrho}\ {}^{\mathcal{A}}\mathcal{D}_{0^+}^{r,\varrho}\overline{u}(r,s),$$

which implies

$$\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\left\{ s^{\varsigma+n\varrho}\overline{u}(x,s) - s^{\varsigma+n\varrho-1} h(x)\right\} = x^{2r+n\varrho}\ {}^{\mathcal{A}}\mathcal{D}_{0^+}^{2r,\varrho}\overline{u}(r,s) + x^{r+n\varrho}\ {}^{\mathcal{A}}\mathcal{D}_{0^+}^{r,\varrho}\overline{u}(r,s).$$

Now applying Mellin transform on both sides

$$\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\left\{ s^{\varsigma+n\varrho}\tilde{\overline{u}}(p,s) - s^{\varsigma+n\varrho-1} \tilde{h}(p)\right\} = \sum_{n=0}^{\infty} a_n \Gamma(2r+n\varrho)\frac{\Gamma(1-s)}{\Gamma(1-s-2r-n\varrho)}\tilde{\overline{u}}(p,s) + \sum_{n=0}^{\infty} a_n \frac{\Gamma(r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-r-n\varrho)}\tilde{\overline{u}}(p,s),$$

$$\left\{ \begin{array}{c} \sum_{n=0}^{\infty} a_n \Gamma(\varsigma+n\varrho) s^{\varsigma+n\varrho}\tilde{\overline{u}}(p,s) \\ - \sum_{n=0}^{\infty} a_n \left(\frac{\Gamma(2r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-2r-n\varrho)} + \frac{\Gamma(r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-r-n\varrho)}\right) \end{array} \right\}\tilde{\overline{u}}(p,s) = \sum_{n=0}^{\infty} a_n \Gamma(\varsigma+n\varrho) s^{\varsigma+n\varrho-1}\tilde{h}(p), \tag{4.37}$$

which implies

$$\tilde{\overline{u}}(p,s) = \frac{\sum_{n=0}^{\infty} a_n \Gamma(\varsigma+n\varrho) s^{\varsigma+n\varrho-1}\tilde{h}(p)}{\sum_{n=0}^{\infty} a_n \Gamma(\varsigma+n\varrho) s^{\varsigma+n\varrho} - \sum_{n=0}^{\infty} a_n \left(\frac{\Gamma(2r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-2r-n\varrho)} + \frac{\Gamma(r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-r-n\varrho)}\right)}, = \frac{\sum_{n=0}^{\infty} a_n \Gamma(\varsigma+n\varrho) s^{\varsigma+n\varrho-1}\tilde{h}(p)}{\sum_{n=0}^{\infty} a_n \Gamma(\varsigma+n\varrho) s^{\varsigma+n\varrho} - \aleph^2},$$

where

$$\aleph^2 = \sum_{n=0}^{\infty} a_n \left(\frac{\Gamma(2r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-2r-n\varrho)} + \frac{\Gamma(r+n\varrho)\Gamma(1-s)}{\Gamma(1-s-r-n\varrho)}\right).$$

Now,

$$\begin{aligned}
\widetilde{\widetilde{u}}(p,s) &= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varsigma+n\varrho-1}\tilde{h}(p)}{a_0\Gamma(\varsigma)s^{\varsigma}+a_1\Gamma(\varsigma+\varrho)s^{\varsigma+\varrho}+\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varsigma+n\varrho}-\aleph^2},\\
&= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{n\varrho-\varrho-1}\tilde{h}(p)}{a_0\Gamma(\varsigma)s^{-\varrho}+a_1\Gamma(\varsigma+\varrho)+\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}-\aleph^2 s^{-\varsigma-\varrho}},\\
&= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{a_0\Gamma(\varsigma)\left[s^{-\varrho}+\frac{a_1\Gamma(\varsigma+\varrho)}{a_0\Gamma(\varsigma)}+\frac{\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}}{a_0\Gamma(\varsigma)}-\frac{\aleph^2 s^{-\varsigma-\varrho}}{a_0\Gamma(\varsigma)}\right]},\\
&= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{a_0\Gamma(\varsigma)\left(s^{-\varrho}+(a_1/a_0\Gamma(\varsigma))\Gamma(\varsigma+\varrho)\right)}\\
&\quad\left(1+\frac{\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}}{a_0\Gamma(\varsigma)}-\frac{\aleph^2 s^{-\varsigma-\varrho}}{a_0\Gamma(\varsigma)}\right)^{-1} s^{-\varrho}+(a_1/a_0\Gamma(\varsigma))\Gamma(\varsigma+\varrho).
\end{aligned}$$

Using Taylor's Theorem, we have

$$\begin{aligned}
\widetilde{\widetilde{u}}(p,s) &= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{a_0\Gamma(\varsigma)\left(s^{-\varrho}+(a_1/a_0\Gamma(\varsigma))\Gamma(\varsigma+\varrho)\right)}\sum_{k=0}^{\infty}(-1)^k\\
&\qquad\times\left(\frac{\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}-\aleph^2 s^{-\varsigma-\varrho}}{a_0\Gamma(\varsigma)\left(s^{-\varrho}+(a_1/a_0\Gamma(\varsigma))\Gamma(\varsigma+\varrho)\right)}\right)^k,\\
&= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{a_0\Gamma(\varrho)\left(s^{-\varrho}+A\right)}\sum_{k=0}^{\infty}(-1)^k\\
&\qquad\times\left(\frac{\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}-\aleph^2 s^{-\varsigma-\varrho}}{a_0\Gamma(\varsigma)\left(s^{-\varrho}+A\right)}\right)^k,\\
&= \frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{(a_0\Gamma(\varsigma))^{k+1}\left(s^{-\varrho}+A\right)^{k+1}}\sum_{k=0}^{\infty}(-1)^k\\
&\qquad\times\left(\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}-\aleph^2 s^{-\varsigma-\varrho}\right)^k,\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{(a_0\Gamma(\varsigma))^{k+1}}\frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{(s^{-\varrho}+A)^{k+1}}\\
&\qquad\times\left(\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}-\aleph^2 s^{-\varsigma-\varrho}\right)^k,
\end{aligned}$$

where $A=(a_1/a_0\Gamma(\varsigma))\Gamma(\varsigma+\varrho)$. Now, using binomial expansion

$$\begin{aligned}
\widetilde{\widetilde{u}}(p,s) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(a_0\Gamma(\varsigma))^{k+1}}\frac{\sum_{n=0}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)-1}\tilde{h}(p)}{(s^{-\varrho}+A)^{k+1}}\sum_{m=0}^{k}\binom{k}{m}\\
&\quad\times\left[\sum_{n=2}^{\infty} a_n\Gamma(\varsigma+n\varrho)s^{\varrho(n-1)}\right]^m\left[\aleph^2 s^{-\varsigma-\varrho}\right]^{k-m},
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(a_0\Gamma(\varsigma))^{k+1}} \sum_{m=0}^{k} (\aleph^2)^{k-m} \binom{k}{m} \left[\frac{\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho) s^{\varrho(n-1)-1} \tilde{h}(p)}{(s^{-\varrho} + A)^{k+1}} \right]$$
$$\times \left[\sum_{n=2}^{\infty} a_n \Gamma(\varsigma + n\varrho) s^{\varrho(n-1)} \right]^m [s^{-\varsigma-\varrho}]^{k-m}.$$

Now using multinomial expansion and Laplace inverse, we get

$$\widehat{u}(p,t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(a_0\Gamma(\varsigma))^{k+1}} \sum_{m=0}^{k} (\aleph^2)^{k-m} \binom{k}{m} \binom{k}{n_1, n_2, n_3, ..., n_\infty}$$
$$\times \left[\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho) \right] \left[\sum_{n_2+n_3+n_3+...+n_\infty} \Pi_{i=2}^{\infty} [a_i \Gamma(\varsigma + i\varrho)]^{n_i} \right]$$
$$\times t^{\mu} E_{-\varrho,\eta}^{(m)}(At^{-\varrho}) \overline{h}(p),$$

where

$$\mu = -m\varrho + \left((\varsigma + \varrho)(m-k) - n\varrho - \sum_{i=2}^{\infty} (i-1) n_i + 1 \right) - 1$$

and

$$\eta = (\varsigma + \varrho)(m-k) - n\varrho - \sum_{i=2}^{\infty} (i-1) n_i + 1.$$

Taking Mellin inverse, we get

$$u(x,t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(a_0\Gamma(\varsigma))^{k+1}} \sum_{m=0}^{k} (\aleph^2)^{k-m} \binom{k}{m} \binom{k}{n_1, n_2, n_3, ..., n_\infty}$$
$$\times \left[\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho) \right] \left[\sum_{n_2+n_3+n_3+...+n_\infty} \Pi_{i=2}^{\infty} [a_i \Gamma(\varsigma + i\varrho)]^{n_i} \right]$$
$$\times \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} t^{\mu} E_{-\varrho,\eta}^{(m)}(At^{-\varrho}) h(p) x^{-p} dp.$$

By using [45] (Eq (8.2.6) on page 340), we get

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} t^{\mu} E_{-\varrho,\eta}^{(m)}(At^{-\varrho}) h(p) x^{-p} dp \quad = \quad t^{\mu} E_{-\varrho,\eta}^{(m)}(At^{-\varrho}) \overline{h}(x),$$

implies

$$u(x,t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(a_0\Gamma(\varsigma))^{k+1}} \sum_{m=0}^{k} (p^2)^{k-m} \binom{k}{m} \binom{k}{n_1, n_2, n_3, ..., n_\infty}$$
$$\times \left[\sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho) \right] \left[\sum_{n_2+n_3+n_3+....+n_\infty} \Pi_{i=2}^{\infty} [a_i \Gamma(\varsigma + i\varrho)]^{n_i} \right]$$
$$\times t^{\mu} E_{-\varrho,\eta}^{(m)}(At^{-\varrho}) \overline{h}(x)$$

which is the solution of (4.36).

5. Relation between Mellin transform with Laplace and Fourier transforms of fractional differential operators with general analytic kernel

In this section, some relationships between Mellin transform with Laplace and Fourier transform of fractional differential operator with general analytic kernel are presented, which can be useful from application aspect. These relations has great importance in different fields of applied mathematics and signal processing [21, 41].

Theorem 5.1. *Let $c = 0, d > 0$ and $\varsigma, \varrho, \mathcal{A}$ as in Definition 2.6 and let $h \in L^2[c, d]$ with the Mellin transform $\widehat{h}$, then the following relation holds:*

(1) $\mathcal{M}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(t)\right\} = \mathcal{L}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(e^{-t})\right\}$.

(2) $\mathcal{M}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(t)\right\} = F\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(e^{-k})e^{-at}\right\}$.

Proof. (1) Consider series formula from Theorem 2

$$ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(t) = \sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\, {}^{RL}\mathcal{I}_{0+}^{\varsigma+n\varrho} h(t). $$

Replacing t by e^{-t},

$$ \begin{aligned} {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(e^{-t}) &= \sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho)\, {}^{RL}\mathcal{I}_{0+}^{\varsigma+n\varrho} h(e^{-t}), \\ &= \sum_{n=0}^{\infty} a_n \int_0^{e^{-t}} (e^{-t} - \varpi)^{\varsigma+n\varrho-1} h(\varpi) d\varpi. \end{aligned} $$

Applying Laplace transform on both sides

$$ \begin{aligned} \mathcal{L}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(e^{-t})\right\} &= \mathcal{L}\left\{ \sum_{n=0}^{\infty} a_n \int_0^{e^{-t}} (e^{-t} - \varpi)^{\varsigma+n\varrho-1} h(\varpi) d\varpi \right\}, \\ &= \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-st} \int_0^{e^{-t}} (e^{-t} - \varpi)^{\varsigma+n\varrho-1} h(\varpi) d\varpi dt. \end{aligned} $$

Let setting $e^{-t} = y$, we have

$$ \begin{aligned} \mathcal{L}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho} h(e^{-t})\right\} &= -\sum_{n=0}^{\infty} a_n \int_{\infty}^{0} y^s \int_0^{y} (y - \varpi)^{\varsigma+n\varrho-1} h(\varpi) d\varpi \frac{dy}{y}, \\ &= \sum_{n=0}^{\infty} a_n \int_0^{\infty} y^{s-1} \int_0^{y} (y - \varpi)^{\varsigma+n\varrho-1} h(\varpi) d\varpi dy, \\ &= \sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho) \mathcal{M}\left\{ \frac{1}{\Gamma(\varsigma + n\varrho)} \int_0^{y} (y - \varpi)^{\varsigma+n\varrho-1} h(y) dy \right\}, \\ &= \sum_{n=0}^{\infty} a_n \Gamma(\varsigma + n\varrho) \frac{\Gamma(1 - s - \varsigma - n\varrho)}{\Gamma(1 - s)} H(s + \varsigma + n\varrho), \end{aligned} $$

$$=\mathcal{M}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(t)\right\}.$$

Therefore, we get the required result.
(2) We have

$${}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(t) = \sum_{n=0}^{\infty} a_n\Gamma(\varsigma + n\varrho)\, {}^{RL}\mathcal{I}_{0+}^{\varsigma+n\varrho}h(t).$$

Also, we have

$$\begin{aligned}
\mathcal{M}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(t)\right\} =&\mathcal{L}\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(e^{-t})\right\},\\
=&\sum_{n=0}^{\infty} a_n\Gamma(\varsigma + n\varrho)\int_{-\infty}^{\infty} e^{-st}\int_0^t (t-\varpi)^{\varsigma+n\varrho-1}h(e^{-t})dtdt,\\
=&\sum_{n=0}^{\infty} a_n\Gamma(\varsigma + n\varrho)\int_{-\infty}^{\infty} e^{-(a+2\pi i\varrho)t}\int_0^t (t-\varpi)^{\varsigma+n\varrho-1}h(e^{-t})dtdt,\\
=&\sum_{n=0}^{\infty} a_n\Gamma(\varsigma + n\varrho)\int_0^{\infty} e^{-at}e^{2\pi i\varrho t}\int_0^t (t-\varpi)^{\varsigma+n\varrho-1}h(e^{-t})dtdt,\\
=&\sum_{n=0}^{\infty} a_n\Gamma(\varsigma + n\varrho)(-ik)^{-\varsigma-n\varrho}\tilde{h}(e^{-k})e^{-at},\\
=&(-ik)^{-\varsigma}\sum_{n=0}^{\infty} a_n\Gamma(\varsigma + n\varrho)(-ik)^{-n\varrho}\tilde{h}(e^{-k})e^{-at},\\
=&F\left\{ {}^{\mathcal{A}}\mathcal{I}_{0+}^{\varsigma,\varrho}h(e^{-k})e^{-at}\right\}.
\end{aligned}$$

Hence, the result follows. □

6. Conclusions

In this work, we defined the Mellin transform for fractional differential equation with general analytic kernel using the method of Laplace and Fourier transform of fractional differential operators. Also, we have established the relationship between Mellin transform with Laplace and Fourier transform of fractional operators which can play a significant role in various fields of applied mathematics.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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