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Research Article

Multipoint BVP for the Langevin Equation under φ-Hilfer Fractional Operator

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In this research paper, we consider a class of boundary value problems for a nonlinear Langevin equation involving two generalized Hilfer fractional derivatives supplemented with nonlocal integral and infinite-point boundary conditions. At first, we derive the equivalent solution to the proposed problem at hand by relying on the results and properties of the generalized fractional calculus. Next, we investigate and develop sufficient conditions for the existence and uniqueness of solutions by means of semigroups of operator approach and the Krasnoselskii fixed point theorems as well as Banach contraction principle. Moreover, by means of Gronwall's inequality lemma and mathematical techniques, we analyze Ulam-Hyers and Ulam-Hyers-Rassias stability results. Eventually, we construct an illustrative example in order to show the applicability of key results.

1. Introduction

In recent decades, the subject of fractional calculus (FC) becomes a very significant tool to characterize memory phenomena in many branches of engineering and sciences. Some properties of solutions like the existence and uniqueness of solutions for fractional boundary value problems (FBVPs) have been widely investigated [1–5], and a broad rundown of references is given in that regard. The importance of FBVPs comes from their applicability in several fields like science and engineering. Langevin [6] devised an equation to describe the progression of physical processes in fluctuating settings in 1908, which is known as the Langevin equation. Mainardi and Pironi in 1990 [7] presented the fractional Langevin equation (FLE). Yukunthorn et al. [8] via Banach's, Krasnoselskii's, and Leray-Schauder's nonlinear alternative and Leray-Schauder studied the existence and uniqueness of solution for a sequential nonlinear fractional Caputo-Langevin equation. Baghani [9] discussed the solvability of initial value problems for nonlinear Langevin equation involving two fractional orders. Fazli and Nieto [10] by means of coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces investigated the existence and uniqueness of solutions for the nonlinear Langevin equation involving two fractional orders with antiperiodic boundary conditions. Salem et al. [11] studied the existence and uniqueness of solution for fractional integrodifferential Langevin equation involving two fractional orders with three-point multiterm fractional integral boundary conditions. For further works on characteristics of solutions, such as existence, uniqueness, and stability results for fractional Langevin equations (FLE), see [12–15]. We also include some recent works on qualitative analysis of similar situations with the generalized fractional operators (see [16–20]). Samet et al. [21, 22] introduced a new concept of fixed point theorems for mappings in complete metric spaces. System stability is one of the most essential qualitative characteristics of solutions to FDEs. However, there are few results of Ulam-Hyers (UH) and generalized Ulam-Hyers (GUH) stability of solutions of FDEs in the literature. On the other hand, Guo et al. [23] studied the existence and Hyers-Ulam stability of the virtually periodic solution to the fractional differential equation with impulse and fractional Brownian motion under nonlocal conditions via the semigroups of operator approach and the Mönch fixed point method. Li et al. [24] by using Krasnoselskii's fixed point method investigated the existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delay. Shu and Shi [25] studied the mild solution of impulsive fractional evolution equations under Caputo fractional derivative.

Recently, Almalahi and Panchal [26] by means of some fixed point theorems studied the qualitative properties of solution for the following problem:

$$\begin{cases} \left({}^{H}\mathbf{D}_{a^+}^{\mathfrak{q},\delta;\varphi} - \beta\right)\mathfrak{u}(\varkappa) = f(\varkappa, \mathfrak{u}(\varkappa)), & \sigma \in \mathcal{J} \coloneqq (a, b], \\ \mathfrak{u}(a) = 0, \mathfrak{u}(b) = \sum_{i=1}^{m} \kappa_i \mathbf{I}_{a^+}^{\zeta,\phi}\mathfrak{u}(\tau_i), & \tau_i \in (a, b], \end{cases} \tag{1}$$

where ${}^{H}\mathbf{D}_{a^+}^{\mathfrak{q},\delta;\varphi}$ is the φ-Hilfer FD of order $\mathfrak{q} \in (1, 2)$ with type $\delta \in [0, 1]$, $\gamma = \mathfrak{q} + 2\delta - \mathfrak{q}\delta$, $\beta < 0$, $m \in \mathbb{N}$ and $\mathbf{I}_{a^+}^{\zeta,\varphi}$ is the φ-RL fractional integral of order $\zeta > 0, \beta, \kappa_i \in \mathbb{R}$, $a < \tau_1 < \tau_2 < \cdots \cdots < b$, and $\varphi \in \mathcal{C}^1(\mathcal{J})$ is an increasing function with $\varphi'(\varkappa) \neq 0$, for all $\varkappa \in \mathcal{J}$.

The recent works regarding of the FLE can be found in [27–29]. Li et al. [27] via some fixed point techniques and degree theory obtained some existence results of Caputo-type Langevin FBVPs with infinite-point boundary conditions

$$\begin{cases} {}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_1}\left({}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_2} - \beta\right)\mathfrak{u}(\varkappa) = f(\varkappa, \mathfrak{u}(\varkappa)), & \varkappa \in (0, 1], \\ \mathfrak{u}(0) = 0, {}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_2}\mathfrak{u}(0) = 0, {}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_2}\mathfrak{u}(1) = \sum_{i=1}^{m} \kappa_i^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_2}\mathfrak{u}(\tau_i), \end{cases} \tag{2}$$

where ${}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_1}, {}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_2}$ are the Caputo FDs of order $\mathfrak{q}_1 \in (0, 1], \mathfrak{q}_2 \in (1, 2]$.

Seemab et al. in [29] discussed the existence and uniqueness of solution for the following problem:

$$\begin{cases} {}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_1;\varphi}\left({}^{c}\mathbf{D}_{0^+}^{\mathfrak{q}_2;\varphi} - \beta\right)\mathfrak{u}(\varkappa) = f\left(\varkappa, \mathfrak{u}(\varkappa), {}^{c}\mathbf{D}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa)\right), & \varkappa \in (0, b], \\ \mathfrak{u}(0) = 0, \mathfrak{u}(\eta) = 0, \mathfrak{u}(b) = \mu\mathbf{I}_{0^+}^{\sigma,\varphi}\mathfrak{u}(\varkappa), & \mu > 0, \end{cases} \tag{3}$$

where ${}^{c}\mathbf{D}_{a^+}^{m,\varphi}$ denotes the φ-Caputo FD of order to $m \in \{\mathfrak{q}_1, \mathfrak{q}_2, \theta\}$, $\mathfrak{q}_1 \in (1, 2]$, $\mathfrak{q}_2, \theta \in (0, 1]$.

Nuchpong et al. [28] by using Banach, Leray–Schauder, and Krasnoselskii fixed point theorems discussed the existence and uniqueness results of φ-Hilfer-type Langevin FBVP nonlocal integral boundary conditions

$$\begin{cases} {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi} - \beta\right)\mathfrak{u}(\varkappa) = f(\varkappa, \mathfrak{u}(\varkappa)), & \sigma \in \mathcal{J} \coloneqq (a, b], \\ \mathfrak{u}(a) = 0, \mathfrak{u}(b) = \sum_{i=1}^{m} \kappa_i I_{a^+}^{\zeta,\phi}\mathfrak{u}(\tau_i), & \tau_i \in (a, b]. \end{cases} \tag{4}$$

Motivated by the aforementioned discussions, in this research paper, we study the existence and uniqueness as well as Ulam-Hyers stability results for Langevin-type generalized FDEs with infinite-point boundary conditions of the form

$$\begin{cases} {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi} - \beta\right)\mathfrak{u}(\varkappa) = f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right), & \varkappa \in \mathcal{J} \coloneqq (0, b], \\ \mathfrak{u}(0) = 0, {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(0) = 0, \mathfrak{u}(b) = \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathfrak{u}(\lambda_i), \end{cases} \tag{5}$$

where ${}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}, {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}$ are the φ-Hilfer FDs of order $\mathfrak{q}_1 \in (0, 1]$, $\mathfrak{q}_2 \in (1, 2]$, $2 < \mathfrak{q}_1 + \mathfrak{q}_2 \leq 3$ with type $\delta_1, \delta_2 \in [0, 1]$ and $\mathbf{I}_{0^+}^{k,\varphi}$ is a φ-RL fractional integral of order $k \in \{\theta, \sigma_i\}$ such that $\theta, \sigma_i > 0, \beta \in \mathbb{R}, \eta_i \in (0, 1), 0 < \lambda_1 < \lambda_2 < \cdots < 1$, and $\varphi \in \mathcal{C}^1(\mathcal{J})$ is an increasing function with $\varphi'(\varkappa) \neq 0$, for all $\varkappa \in \mathcal{J}$, and $f : (0, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a given function.

Observe that the results that will be obtained according to the problem (5) cover results of [27–29] as follows:

(i) Li et al. in [27] (for $a = 0, \delta_1 = \delta_2 = 1$, and $\varphi(\varkappa) = \varkappa$)

(ii) Nuchpong et al. [28] results (for $\theta = 0, i = 1, 2, .., n$, and $0 < \mathfrak{q}_1, \mathfrak{q}_2 \leq 1$)

(iii) Seemab et al., in [29] (for $a = 0, \delta_1 = \delta_2 = 1$, and $\theta = 0, i = 1, \eta = \mu$)

We anticipate that the results presented in this paper will be groundbreaking and contribute to the existence of knowledge on the Langevin equation. The results acquired in this study are applicable to a wide range by choosing suitable values of function φ and may be used to a variety of other challenges.

The arrangement of this paper is as per the following. In Section 2, we will give some definitions and lemmas to demonstrate our primary outcomes. In Section 3, Krasnoselskii and Banach fixed point techniques are used to acquire the existence and uniqueness of solutions of the suggested problem (5). In Section 4, we analyze the stability results in Ulam-Hyers sense. In the last section, we introduce a numerical model to represent the fundamental results.

2. Auxiliary Results

In this section, to analyze our main results, we present here some important definitions and auxiliary lemmas. Assume that $\mathcal{J} \coloneqq [0, b]$ and $\mathcal{C}(\mathcal{J}) \coloneqq \mathcal{C}(\mathcal{J}, \mathbb{R})$ denote the Banach space of all continuous functions defined from $\mathcal{J}$ into $\mathbb{R}$

with the norm $\|\mathfrak{u}\| = \sup\{|\mathfrak{u}(\varkappa)|: \varkappa \in \mathcal{J}\}$. Let $\varphi \in \mathscr{C}^1(\mathcal{J})$ be an increasing function with $\varphi'(\varkappa) \neq 0$, for all $\varkappa \in \mathcal{J}$.

Definition 1 (see [4]). Let $\mathfrak{q} > 0$ and $\mathfrak{g} : [0,\infty) \longrightarrow \mathbb{R}$. Then, the following representation

$$\mathbf{I}_{0^+}^{\mathfrak{q},\varphi}\mathfrak{g}(\varkappa) = \frac{1}{\Gamma(\mathfrak{q})}\int_0^{\varkappa}\varphi'(s)(\varphi(\varkappa)-\varphi(s))^{\mathfrak{q}-1}\mathfrak{g}(s)ds \tag{6}$$

is called φ-RL fractional integral of $\mathfrak{g}$ of order $\mathfrak{q}$.

Definition 2 (see [31]). Let $\mathfrak{q} \in (n-1, n]$, $n \in \mathbb{N}$, and the functions $\mathfrak{g}, \varphi \in \mathscr{C}^n(\mathcal{J})$. Then, the φ-Hilfer FD of $\mathfrak{g}$ with order $\mathfrak{q}$ and type $0 \leq \delta \leq 1$ is defined by

$$\begin{aligned}{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},\delta,\varphi}\mathfrak{g}(\varkappa) &= \mathbf{I}_{0^+}^{\delta(n-\mathfrak{q});\varphi}\mathfrak{g}_{\varphi}^{[n]}\mathbf{I}_{0^+}^{(1-\delta)(n-\mathfrak{q}),\varphi}\mathfrak{g}(\varkappa)\\ &= \mathbf{I}_{0^+}^{\delta(n-\mathfrak{q});\varphi}\mathfrak{g}_{\varphi}^{[n]}\,\mathbf{I}_{0^+}^{n-\gamma,\varphi}\mathfrak{g}(\varkappa)\\ &= \mathbf{I}_{0^+}^{\delta(n-\mathfrak{q});\varphi}\mathbf{D}_{a^+}^{\gamma;\varphi}\mathfrak{g}(\varkappa), \quad \gamma = \mathfrak{q}+\delta(n-\mathfrak{q}),\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathbf{D}_{0^+}^{\gamma;\varphi}\mathfrak{g}(\varkappa) &= \mathfrak{g}_{\varphi}^{[n]}\mathbf{I}_{0^+}^{(1-\delta)(n-\mathfrak{q});\varphi}\mathfrak{g}(\varkappa),\\ \mathfrak{g}_{\varphi}^{[n]} &= \left(\frac{1}{\varphi'(\varkappa)}\frac{d}{d\varkappa}\right)^n.\end{aligned} \tag{8}$$

Remark 3. In Definition 2, type δ and function φ allow ${}^{H}\boldsymbol{D}_{0^+}^{\mathfrak{q},\delta,\varphi}$ to interpolate continuously between the Riemann-Liouville FD and the Caputo FD. More precisely, we have the following:

(i) φ-Hilfer FD corresponds to the Riemann-Liouville FD for $(\delta = 0, \varphi(\varkappa) = \varkappa)$, i.e.,

$${}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},0,\varkappa}\mathfrak{g}(\varkappa) = \mathfrak{g}_{\varkappa}^{[n]}\mathbf{I}_{0^+}^{n-\mathfrak{q},\varkappa}\mathfrak{g}(\varkappa) = \mathbf{D}^n\,\mathbf{I}_{0^+}^{n-\mathfrak{q}}\mathfrak{g}(\varkappa), \quad \varkappa > 0 \tag{9}$$

(ii) φ-Hilfer FD corresponds to the Caputo FD for $(\delta = 1, \varphi(\varkappa) = \varkappa)$, i.e.,

$${}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},1,\varkappa}\mathfrak{g}(\varkappa) = \mathbf{I}_{0^+}^{n-\mathfrak{q},\varkappa}\,\mathfrak{g}_{\varkappa}^{[n]}\,\mathfrak{g}(\varkappa) = \mathbf{I}_{0^+}^{n-\mathfrak{q}}\,\mathbf{D}^n\,\mathfrak{g}(\varkappa), \quad \varkappa > 0 \tag{10}$$

(iii) φ-Hilfer FD corresponds to the φ-Riemann-Liouville FD for $\delta = 0$, i.e.,

$$\begin{aligned}{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},0,\varphi}\mathfrak{g}(\varkappa) &= \mathfrak{g}_{\varphi}^{[n]}\mathbf{I}_{0^+}^{n-\mathfrak{q},\varphi}\mathfrak{g}(\varkappa) = \left(\frac{1}{\varphi'(\varkappa)}\frac{d}{d\varkappa}\right)^n\mathbf{I}_{0^+}^{n-\mathfrak{q},\varphi}\mathfrak{g}(\varkappa)\\ &= \mathbf{D}_{\varphi}^n\,\mathbf{I}_{0^+}^{n-\mathfrak{q},\varphi}\mathfrak{g}(\varkappa), \quad \varkappa > 0\end{aligned} \tag{11}$$

(iv) φ-Hilfer FD corresponds to the φ-Caputo FD for $\delta = 1$, i.e.,

${}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},1}\,,\varphi\mathfrak{g}(\varkappa) = \mathbf{I}_{0^+}^{n-\mathfrak{q},\varphi}\mathfrak{g}_{\varphi}^{[n]}\,\mathfrak{g}(\varkappa) = \mathbf{I}_{0^+}^{n-\mathfrak{q},\varphi}((1/\varphi'(\varkappa))(d/d\varkappa))^n$ $\mathfrak{g}(\varkappa) = \mathbf{I}_{0^+}^{n-\mathfrak{q},\varphi}\mathbf{D}_{\varphi}^n\mathfrak{g}(\varkappa), \ \varkappa > 0$

Theorem 4 (see [4]). *Let $\mathfrak{g} \in \mathscr{C}(\mathcal{J})$ be a function. Then, $\mathbf{I}_{0^+}^{\mathfrak{q};\varphi}$ $\mathfrak{g}(0) = \lim_{\varkappa \longrightarrow 0^+}\mathbf{I}_{0^+}^{\mathfrak{q};\varphi}\mathfrak{g}(\varkappa) = 0$.*

Lemma 5 (see [4, 30]). *Let $\mathfrak{q}, \delta > 0$ and $\eta > 0$. Then,*

$$\begin{aligned}\mathbf{I}_{0^+}^{\mathfrak{q},\varphi}\mathbf{I}_{0^+}^{\delta,\varphi}\mathfrak{g}(\varkappa) &= \mathbf{I}_{0^+}^{\mathfrak{q}+\delta,\varphi}\mathfrak{g}(\varkappa),\\ \mathbf{I}_{0^+}^{\mathfrak{q},\varphi}(\varphi(\varkappa)-\varphi(0))^{\eta-1} &= \frac{\Gamma(\eta)}{\Gamma(\mathfrak{q}+\eta)}(\varphi(\varkappa)-\varphi(0))^{\mathfrak{q}+\eta-1},\end{aligned} \tag{12}$$

$${}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},\delta,\varphi}(\varphi(\varkappa)-\varphi(0))^{\gamma-1} = 0, \gamma = \mathfrak{q}+\delta(n-\mathfrak{q}). \tag{13}$$

Lemma 6 (see [30]). *If $\mathfrak{g} \in \mathscr{C}^n(\mathcal{J})$, $\mathfrak{q} \in (n-1, n)$, and $0 \leq \delta \leq 1$, then*

$$\begin{aligned}\mathbf{I}_{0^+}^{\mathfrak{q};\varphi H}\mathbf{D}_{0^+}^{\mathfrak{q},\delta,\varphi}\mathfrak{g}(\varkappa) &= \mathfrak{g}(\varkappa) - \sum_{k=1}^{n}\frac{(\varphi(\varkappa)-\varphi(0))^{\gamma-k}}{\Gamma(\gamma-k+1)}\mathfrak{g}_{\varphi}^{[n-k]}\mathbf{I}_{a^+}^{(1-\delta)(n-\mathfrak{q});\varphi}\mathfrak{g}(0),\\ {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q},\delta,\varphi}\mathbf{I}_{0^+}^{\mathfrak{q};\varphi}\mathfrak{g}(\varkappa) &= \mathfrak{g}(\varkappa).\end{aligned} \tag{14}$$

Theorem 7 (see [31]) (Banach theorem). *Let $\mathcal{O}$ be a closed subset of Banach space $\mathscr{H}$, and the operator $\mathcal{Q} : \mathcal{O} \longrightarrow \mathcal{O}$ be a strict contraction that means $\|\mathcal{Q}(\mathfrak{u}) - \mathcal{Q}(v)\| \leq \mathscr{L}\|\mathfrak{u} - v\|$ for some $0 < \mathscr{L} < 1$, $\mathfrak{u}, v \in \mathcal{O}$. Then, $\mathcal{Q}$ has a fixed point in $\mathcal{O}$.*

Theorem 8 (see [32]) (Krasnoselskii's theorem). *Let $\mathcal{O}$ be a nonempty, closed, convex, and bounded subset from Banach space $\mathscr{H}$. If there exist two operators $\mathcal{Q}, \mathcal{Q}^\star$ such that (i) for all $\mathfrak{u}, v \in \mathscr{H}$, imply $\mathcal{Q}\mathfrak{u} + \mathcal{Q}^\star v \in \mathscr{H}$, (ii) $\mathcal{Q}$ is compact and continuous, and (iii) $\mathcal{Q}^\star$ is a contraction mapping, then there exists a function $z \in \mathcal{O}$ such that $z = \mathcal{Q}z + \mathcal{Q}^\star z$.*

3. Equivalent Integral Equations for Problem (5)

In order to convert the problem (5) into a fixed point problem, we will present the following lemma with linear function $\mathbf{h}(\varkappa)$.

Lemma 9. *For $j=1,2$, let $\gamma_j = \mathbf{q}_j + j\delta_j - \mathbf{q}_j\delta_j$, $\mathbf{q}_1 \in (0,1]$, $\mathbf{q}_2 \in (1,2]$, $\delta_j \in [0,1]$, $\boldsymbol{h} \in \mathscr{C}(\mathscr{J})$. Then, the function $\mathfrak{u} \in \mathscr{C}(\mathscr{J})$ is a solution of the linear problem*

$$\begin{cases} {}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa) - \beta\mathfrak{u}(\varkappa)\right) = \mathbf{h}(\varkappa), \quad \varkappa \in (0,b], \\ \mathfrak{u}(0) = 0, {}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(0) = 0, \mathfrak{u}(b) = \sum_{i=1}^{\infty}\eta_i \mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathfrak{u}(\lambda_i), \end{cases} \tag{15}$$

if and only if

$$\begin{aligned} \mathfrak{u}(\varkappa) = {} & \frac{(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)} \\ & \cdot \left[\sum_{i=1}^{\infty}\eta_i\left(\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i,\varphi}\beta\mathfrak{u}(\lambda_i) + \mathbf{I}_{0^+}^{\mathbf{q}_1+\mathbf{q}_2+\sigma_i,\varphi}\mathbf{h}(\lambda_i)\right) - \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(b) - \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(b)\right] \\ & + \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(\varkappa), \end{aligned} \tag{16}$$

where

$$\Theta = \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Gamma(\gamma_2)} - \sum_{i=1}^{\infty}\eta_i\frac{(\varphi(\lambda_i)-\varphi(0))^{\gamma_2+\sigma_i-1}}{\Gamma(\gamma_2+\sigma_i)} \neq 0. \tag{17}$$

Proof. Let $\mathfrak{u} \in \mathscr{C}(\mathscr{J})$ be a solution of the problem (15). Take $\mathbf{I}_{0^+}^{\mathbf{q}_1,\varphi}$ taking into consideration that $\mathbf{q}_1 \in (0,1]$ on both sides of the following equation:

$${}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa) - \beta\mathfrak{u}(\varkappa)\right) = \mathbf{h}(\varkappa). \tag{18}$$

Using Lemma 6, we have

$${}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa) = \beta\mathfrak{u}(\varkappa) + \frac{c_0}{\Gamma(\gamma_1)}(\varphi(\varkappa)-\varphi(0))^{\gamma_1-1} + \mathbf{I}_{0^+}^{\mathbf{q}_1,\varphi}\mathbf{h}(\varkappa), \tag{19}$$

where c_0 is an arbitrary constant. By conditions $\mathfrak{u}(0)=0$ and ${}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(0)=0$, we obtain $c_0=0$ and hence, (19) reduces to

$${}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa) = \beta\mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathbf{q}_1,\varphi}\mathbf{h}(\varkappa). \tag{20}$$

Taking again $\mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}$ taking into consideration that $\mathbf{q}_2 \in (1,2]$ on both sides of (20) and using Lemmas 6 and 5, we have

$$\begin{aligned} \mathfrak{u}(\varkappa) = {} & \frac{c_1}{\Gamma(\gamma_2)}(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1} + \frac{c_2}{\Gamma(\gamma_2-1)}(\varphi(\varkappa)-\varphi(0))^{\gamma_2-2} \\ & + \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(\varkappa). \end{aligned} \tag{21}$$

According to the condition $\mathfrak{u}(0)=0$, we obtain $c_2=0$ and hence, (21) reduces to

$$\mathfrak{u}(\varkappa) = \frac{c_1}{\Gamma(\gamma_2)}(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1} + \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(\varkappa). \tag{22}$$

By replacing $\varkappa$ with b in equation (22), we get

$$\mathfrak{u}(b) = \frac{c_1}{\Gamma(\gamma_2)}(\varphi(b)-\varphi(0))^{\gamma_2-1} + \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(b) + \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(b). \tag{23}$$

Replacing again $\varkappa$ with λ_i in equation (22) with multiplication by $\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}$ with the use of semigroup property defined in Lemma 5, we get

$$\begin{aligned} \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathfrak{u}(\lambda_i) = {} & \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\frac{c_1}{\Gamma(\gamma_2)}(\varphi(\lambda_i)-\varphi(0))^{\gamma_2-1} \\ & + \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(\lambda_i) + \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(\lambda_i). \end{aligned} \tag{24}$$

Now, by equations (23) and (24) and second condition $\mathfrak{u}(b) = \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathfrak{u}(\lambda_i)$, we get

$$\begin{aligned} & \frac{c_1}{\Gamma(\gamma_2)}(\varphi(b)-\varphi(0))^{\gamma_2-1} + \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(b) + \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(b) \\ & \quad = \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\frac{c_1}{\Gamma(\gamma_2)}(\varphi(\lambda_i)-\varphi(0))^{\gamma_2-1} + \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i,\varphi}\beta\mathfrak{u}(\lambda_i) \\ & \qquad + \sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i,\varphi}\mathbf{h}(\lambda_i). \end{aligned} \tag{25}$$

Hence,

$$c_1 = \frac{1}{\Theta}\left[\sum_{i=1}^{\infty}\eta_i\left(\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i,\varphi}\beta\mathfrak{u}(\lambda_i) + \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i,\varphi}\mathbf{h}(\lambda_i)\right) - \mathbf{I}_{0^+}^{\mathbf{q}_2,\varphi}\beta\mathfrak{u}(b) - \mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1,\varphi}\mathbf{h}(b)\right]. \tag{26}$$

Putting c_1 into (22), we get (16). Conversely, we assume that the solution $\mathfrak{u}$ satisfies (16). Then, one can get $\mathfrak{u}(0)=0$ and ${}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(0)=0$. Furthermore, applying $\mathbf{I}_{0^+}^{\sigma_i;\varphi}$ on both sides of (16) replacing $\varkappa$ by λ_i and multiplying by $\sum_{i=1}^{\infty}\eta_i$, we get

$$\begin{aligned}\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i;\varphi}\mathfrak{u}(\lambda_i) &= \sum_{i=1}^{\infty}\eta_i\frac{(\varphi(\lambda_i)-\varphi(0))^{\gamma_2+\sigma_i-1}}{\Theta\Gamma(\gamma_2+\sigma_i)} \\ &\cdot\left[\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i;\varphi}\beta\mathfrak{u}(\lambda_i)+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i;\varphi}\mathbf{h}(\lambda_i)-\mathbf{I}_{0^+}^{\mathbf{q}_2;\varphi}\beta\mathfrak{u}(b)-\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1;\varphi}\mathbf{h}(b)\right] \\ &+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i;\varphi}\beta\mathfrak{u}(\lambda_i)+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i;\varphi}\mathbf{h}(\lambda_i)=\frac{1}{\Theta}\left(\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Gamma(\gamma_2)}-\Theta\right) \\ &\cdot\left[\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i;\varphi}\beta\mathfrak{u}(\lambda_i)+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i;\varphi}\mathbf{h}(\lambda_i)-\mathbf{I}_{0^+}^{\mathbf{q}_2;\varphi}\beta\mathfrak{u}(b)-\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1;\varphi}\mathbf{h}(b)\right] \\ &+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i;\varphi}\beta\mathfrak{u}(\lambda_i)+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i;\varphi}\mathbf{h}(\lambda_i)=\mathfrak{u}(b).\end{aligned} \tag{27}$$

Thus, all conditions are satisfied. Next, applying ${}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}$ on both sides of (16), we have

$$\begin{aligned}{}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa)&={}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\frac{(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)} \\ &\cdot\left[\sum_{i=1}^{\infty}\eta_i\left(\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i;\varphi}\beta\mathfrak{u}(\lambda_i)+\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i;\varphi}\mathbf{h}(\lambda_i)\right)-\mathbf{I}_{0^+}^{\mathbf{q}_2;\varphi}\beta\mathfrak{u}(b)-\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1;\varphi}\mathbf{h}(b)\right] \\ &+{}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathbf{I}_{0^+}^{\mathbf{q}_2;\varphi}\beta\mathfrak{u}(\varkappa)+{}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1;\varphi}\mathbf{h}(\varkappa).\end{aligned} \tag{28}$$

Using Lemmas 5 and 6, we get

$${}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa)-\beta\mathfrak{u}(\varkappa)=\mathbf{I}_{0^+}^{\mathbf{q}_1;\varphi}\mathbf{h}(\varkappa). \tag{29}$$

Applying ${}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}$ on both sides of equation (29), we have

$${}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa)-\beta\mathfrak{u}(\varkappa)\right)={}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\mathbf{I}_{0^+}^{\mathbf{q}_1;\varphi}\mathbf{h}(\varkappa). \tag{30}$$

Using Lemma 5, we get

$${}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathbf{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa)-\beta\mathfrak{u}(\varkappa)\right)=\mathbf{h}(\varkappa). \tag{31}$$

The proof is completed. □

Lemma 10. *For $j=1,2$, let $\gamma_j=\mathbf{q}_j+j\delta_j-\mathbf{q}_j\delta_j$, $\mathbf{q}_1\in(0,1]$, $\mathbf{q}_2\in(1,2]$, $\delta_j\in[0,1]$, and $f:J\times\mathbb{R}^3\longrightarrow\mathbb{R}$ be a continuous function. Then, the solution of the problem (5) is given by*

$$\begin{aligned}\mathfrak{u}(\varkappa)=\frac{(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}&\left[\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\sigma_i;\varphi}\beta\mathfrak{u}(\lambda_i)+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1+\sigma_i;\varphi}f\left(\lambda_i,\mathfrak{u}(\lambda_i),\mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\lambda_i),{}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\mathfrak{u}(\lambda_i)\right)-\mathbf{I}_{0^+}^{\mathbf{q}_2;\varphi}\beta\mathfrak{u}(b)-\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1;\varphi}f\left(b,\mathfrak{u}(b),\mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(b),{}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\mathfrak{u}(b)\right)\right] \\ &+\mathbf{I}_{0^+}^{\mathbf{q}_2;\varphi}\beta\mathfrak{u}(\varkappa)+\mathbf{I}_{0^+}^{\mathbf{q}_2+\mathbf{q}_1;\varphi}f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathbf{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right).\end{aligned} \tag{32}$$

In order to simplify our analysis, we will use the following notation:

$$\Pi_A^B=\frac{(\varphi(A)-\varphi(0))^B}{\Gamma(B+1)}, \tag{33}$$

where $A\in\{\lambda_i,b\}$ and $B\in\{\mathbf{q}_2+\sigma_i,\mathbf{q}_2,\mathbf{q}_2+\mathbf{q}_1+\sigma_i,\mathbf{q}_2+\mathbf{q}_1\}$.

4. Existence and Uniqueness Solution

In this section, we will discuss the existence and uniqueness of solutions for φ-Hilfer FDE (5) by applying Theorems 8 and 7. To demonstrate our main results, the following assumptions must be fulfilled.

(H_1) $f:\mathcal{J}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is a continuous and there exists a constant number $\mu_f>0$ such that

$$\begin{aligned}&|f(\varkappa,\mathfrak{u},\nu,z)-f(\sigma,\widehat{\mathfrak{u}},\widehat{\nu},\widehat{z})| \\ &\quad\le\mu_f[|\mathfrak{u}-\widehat{\mathfrak{u}}|+|\nu-\widehat{\nu}|+|z-\widehat{z}|],\varkappa\in\mathcal{J},\mathfrak{u},\widehat{\mathfrak{u}},\nu,\widehat{\nu},z,\widehat{z}\in\end{aligned} \tag{34}$$

(H_2) $f:\mathcal{J}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is continuous and there exists

$\omega_f \in \mathscr{C}(\mathscr{J})$ such that

$$|f(\varkappa, \mathfrak{u}(\varkappa), v(\varkappa), z(\varkappa))| \leq \omega_f(\varkappa), (\varkappa, \mathfrak{u}, v, z) \in \mathscr{J} \times \mathbb{R}^3, \quad (35)$$

with $\sup_{\varkappa \in \mathscr{J}} |\omega_f(\varkappa)| = \omega_f^*$.

Theorem 11. *Assume that (H_1) and (H_2) hold. Then, the problem (5) has at least one solution, provided that $\mathscr{M} + \mathscr{V} < 1$, where*

$$\mathscr{M} = \frac{(\varphi(b) - \varphi(0))^{\gamma_2 - 1}}{\Theta \Gamma(\gamma_2)} \sum_{i=1}^{\infty} \eta_i |\beta| \Pi_{\lambda_i}^{\mathfrak{q}_2 + \sigma_i} + |\beta| \Pi_b^{\mathfrak{q}_2},$$
$$\mathscr{V} = \frac{(\varphi(b) - \varphi(0))^{\gamma_2 - 1}}{\Theta \Gamma(\gamma_2)} \frac{\mu_f}{1 - \mu_f} \left(1 + \frac{1}{\Gamma(\theta + 1)}\right) \cdot \left(\sum_{i=1}^{\infty} \eta_i \Pi_{\lambda_i}^{\mathfrak{q}_2 + \mathfrak{q}_1 + \sigma_i} + \Pi_b^{\mathfrak{q}_2 + \mathfrak{q}_1}\right). \quad (36)$$

Proof. Consider the continuous operator $\mathscr{Q} : \mathscr{C}(\mathscr{J}) \longrightarrow \mathscr{C}(\mathscr{J})$, which is defined by

$$\begin{aligned} \mathscr{Q}(\mathfrak{u}(\varkappa)) = {} & \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta \Gamma(\gamma_2)} \cdot \left[\sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \sigma_i, \varphi} \beta \mathfrak{u}(\lambda_i) + \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1 + \sigma_i, \varphi} f \cdot \left(\lambda_i, \mathfrak{u}(\lambda_i), I_{0^+}^{\theta, \varphi} \mathfrak{u}(\lambda_i), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(\lambda_i)\right) \right. \\ & \left. - \mathbf{I}_{0^+}^{\mathfrak{q}_2, \varphi} \beta \mathfrak{u}(b) - \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1, \varphi} f\left(b, \mathfrak{u}(b), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(b), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(b)\right)\right] \\ & + \mathbf{I}_{0^+}^{\mathfrak{q}_2, \varphi} \beta \mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1, \varphi} f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(\varkappa), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(\varkappa)\right). \end{aligned} \quad (37)$$

Clearly, the fixed points of the operator $\mathscr{Q}$ defined by (37) is a solution of the problem (5). Now, we will prove that the operator $\mathscr{Q}$ has a fixed point by using Theorem 8. Let $\mathbb{B}_r$ be a closed ball defined by

$$\mathbb{B}_r = \{\mathfrak{u} \in \mathscr{C}(\mathscr{J}) : \|\mathfrak{u}\| \leq r\}, \quad (38)$$

with

$$r \geq \frac{\left((\varphi(b) - \varphi(0))^{\gamma_2 - 1} / \Theta \Gamma(\gamma_2) \sum_{i=1}^{\infty} \eta_i \Pi_{\lambda_i}^{\mathfrak{q}_2 + \mathfrak{q}_1 + \sigma_i} + (\varphi(b) - \varphi(0))^{\gamma_2 - 1} / \Theta \Gamma(\gamma_2) \Pi_b^{\mathfrak{q}_2 + \mathfrak{q}_1} + \Pi_b^{\mathfrak{q}_2 + \mathfrak{q}_1}\right) \omega_f^*}{1 - \left((\varphi(b) - \varphi(0))^{\gamma_2 - 1} / \Theta \Gamma(\gamma_2) \sum_{i=1}^{\infty} \eta_i \Pi_{\lambda_i}^{\mathfrak{q}_2 + \sigma_i} + (\varphi(b) - \varphi(0))^{\gamma_2 - 1} / \Theta \Gamma(\gamma_2) \Pi_b^{\mathfrak{q}_2} + \Pi_b^{\mathfrak{q}_2}\right) |\beta|}. \quad (39)$$

Let $\mathscr{Q}^1, \mathscr{Q}^2 \in \mathbb{B}_r$ be two operators such that $\mathscr{Q}^1 + \mathscr{Q}^2 = \mathscr{Q}$, where

$$\mathscr{Q}^1 \mathfrak{u}(\varkappa) = \mathbf{I}_{0^+}^{\mathfrak{q}_2, \varphi} \beta \mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1, \varphi} f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(\varkappa), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(\varkappa)\right),$$
$$\mathscr{Q}^2 \mathfrak{u}(\varkappa) = \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta \Gamma(\gamma_2)} \left[\sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \sigma_i, \varphi} \beta \mathfrak{u}(\lambda_i) + \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1 + \sigma_i, \varphi} f\left(\lambda_i, \mathfrak{u}(\lambda_i), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(\lambda_i), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(\lambda_i)\right) - \mathbf{I}_{0^+}^{\mathfrak{q}_2, \varphi} \beta \mathfrak{u}(b) - \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1, \varphi} f\left(b, \mathfrak{u}(b), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(b), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(b)\right)\right]. \quad (40)$$

In order to achieve conditions of Theorem 8, the proof is divided into the following steps.

Step 1. $\mathscr{Q} \in \mathbb{B}_r$ for all $\mathfrak{u} \in \mathbb{B}_r$. For any $\mathfrak{u} \in \mathbb{B}_r, \varkappa \in \mathscr{J}$, we have

$$\begin{aligned} \|\mathscr{Q}^1 \mathfrak{u}\| &= \sup_{\varkappa \in \mathscr{J}} \left|\mathbf{I}_{0^+}^{\mathfrak{q}_2, \varphi} \beta \mathfrak{u}(\varkappa) + \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1, \varphi} f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(\varkappa), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(\varkappa)\right)\right| \\ &\leq \sup_{\varkappa \in \mathscr{J}} \left\{\mathbf{I}_{0^+}^{\mathfrak{q}_2, \varphi} |\beta| |\mathfrak{u}(\varkappa)| + \mathbf{I}_{0^+}^{\mathfrak{q}_2 + \mathfrak{q}_1, \varphi} \left|f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta, \varphi} \mathfrak{u}(\varkappa), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1, \delta_1; \varphi} \mathfrak{u}(\varkappa)\right)\right|\right\} \leq |\beta| \Pi_b^{\mathfrak{q}_2} \|\mathfrak{u}\| + \Pi_b^{\mathfrak{q}_2 + \mathfrak{q}_1} \omega_f^*, \\ \|\mathscr{Q}^2 \mathfrak{u}\| &\leq \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta \Gamma(\gamma_2)} \left[\sum_{i=1}^{\infty} \eta_i \left(\beta \|\mathfrak{u}\| \Pi_{\lambda_i}^{\mathfrak{q}_2 + \sigma_i} + \omega_f^* \Pi_{\lambda_i}^{\mathfrak{q}_2 + \mathfrak{q}_1 + \sigma_i}\right) + \beta \|\mathfrak{u}\| \Pi_b^{\mathfrak{q}_2} + \omega_f^* \Pi_b^{\mathfrak{q}_2 + \mathfrak{q}_1}\right]. \end{aligned} \quad (41)$$

Due to the fact that $\varphi(\varkappa)$ is an increasing function, we have $(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1} \leq (\varphi(b)-\varphi(0))^{\gamma_2-1}$ and hence,

$$\left\|\mathcal{Q}^2\mathbf{u}\right\| \leq \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\left[\sum_{i=1}^{\infty}\eta_i\left(\beta\|\mathbf{u}\|\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}+\omega_f^*\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\right)+\beta\|\mathbf{u}\|\Pi_b^{\mathfrak{q}_2}+\omega_f^*\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\right]. \tag{42}$$

Thus,

$$\begin{aligned}\|\mathcal{Q}\mathbf{u}\| &\leq \left\|\mathcal{Q}^1\mathbf{u}\right\|+\left\|\mathcal{Q}^2\mathbf{u}\right\| \leq \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i|\beta|\|\mathbf{u}\|\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\omega_f^*\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\\ &+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}|\beta|\|\mathbf{u}\|\Pi_b^{\mathfrak{q}_2}+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\omega_f^*\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}+|\beta|\Pi_b^{\mathfrak{q}_2}\|\mathbf{u}\|+\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\omega_f^*\\ &\leq\left(\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\Pi_b^{\mathfrak{q}_2}+\Pi_b^{\mathfrak{q}_2}\right)|\beta|\|\mathbf{u}\|\\ &+\left(\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}+\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\right)\omega_f^*\leq r.\end{aligned} \tag{43}$$

Thus, $\mathcal{Q}(\mathbb{B}_r)\subset\mathbb{B}_r$.

Step 2. $\mathcal{Q}^1$ is compact and continuous.

Let $\mathbf{u}_n$ be a sequence such that $\mathbf{u}_n \longrightarrow \mathbf{u}$ in $\mathbb{B}_r$. Then,

$$\begin{aligned}\left\|\mathcal{Q}^1\mathbf{u}_n-\mathcal{Q}^1\mathbf{u}\right\| &\leq \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta|\mathbf{u}_n(\varkappa)-\mathbf{u}(\varkappa)|+\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}\left|f\left(\varkappa,\mathbf{u}_n(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathbf{u}_n(\varkappa),{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathbf{u}_n(\varkappa)\right)-f\left(\varkappa,\mathbf{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathbf{u}(\varkappa),{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathbf{u}(\varkappa)\right)\right|\\ &\leq \Pi_b^{\mathfrak{q}_2}\|\mathbf{u}_n-\mathbf{u}\|+\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\|\mathbf{u}_n-\mathbf{u}\|\longrightarrow 0\; as\; \mathbf{u}_n\longrightarrow\mathbf{u}.\end{aligned} \tag{44}$$

That means $\mathcal{Q}^1$ is continuous. Moreover, the operator $\mathcal{Q}^1$ is bounded on $\mathbb{B}_r$ due to Step 1. Thus, $\mathcal{Q}^1$ is uniformly bounded on $\mathbb{B}_r$. Next, we will prove that $\mathcal{Q}^1$ is equicontinuous. Let $\varkappa_1,\varkappa_2\in\mathcal{J}$ such that $\varkappa_1<\varkappa_2$. Then,

$$\begin{aligned}\left|\mathcal{Q}^1\mathbf{u}(\varkappa_2)-\mathcal{Q}^1\mathbf{u}(\varkappa_1)\right| &= \left|\mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\mathbf{u}(\varkappa_2)-\mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\mathbf{u}(\varkappa_1)+\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}f\left(\varkappa_2,\mathbf{u}(\varkappa_2),\mathbf{I}_{0^+}^{\theta,\varphi}\mathbf{u}(\varkappa_2),{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathbf{u}(\varkappa_2)\right)-\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}f\left(\varkappa_1,\mathbf{u}(\varkappa_1),\mathbf{I}_{0^+}^{\theta,\varphi}\mathbf{u}(\varkappa_1),{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathbf{u}(\varkappa_1)\right)\right|\\ &\leq \frac{\beta}{\Gamma(\mathfrak{q}_2)}\int_0^{\varkappa_1}\varphi'(s)\left[(\varphi(\varkappa_2)-\varphi(s))^{\mathfrak{q}_2-1}-(\varphi(\varkappa_1)-\varphi(s))^{\mathfrak{q}_2-1}\right]|\mathbf{u}(s)|ds+\frac{\beta}{\Gamma(\mathfrak{q}_2)}\int_{\varkappa_1}^{\varkappa_2}\varphi'(s)(\varphi(\varkappa_2)-\varphi(s))^{\mathfrak{q}_2-1}|\mathbf{u}(s)|ds+\frac{1}{\Gamma(\mathfrak{q}_2+\mathfrak{q}_1)}\\ &\cdot\int_0^{\varkappa_1}\varphi'(s)\left[(\varphi(\varkappa_2)-\varphi(s))^{\mathfrak{q}_2+\mathfrak{q}_1-1}-(\varphi(\varkappa_1)-\varphi(s))^{\mathfrak{q}_2+\mathfrak{q}_1-1}\right]\left|f\left(s,\mathbf{u}(s),\mathbf{I}_{0^+}^{\theta,\varphi}\mathbf{u}(s),{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathbf{u}(s)\right)\right|ds+\frac{1}{\Gamma(\mathfrak{q}_2+\mathfrak{q}_1)}\\ &\cdot\int_{\varkappa_1}^{\varkappa_2}\varphi'(s)(\varphi(\varkappa_2)-\varphi(s))^{\mathfrak{q}_2+\mathfrak{q}_1-1}\left|f\left(s,\mathbf{u}(s),\mathbf{I}_{0^+}^{\theta,\varphi}\mathbf{u}(s),{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathbf{u}(s)\right)\right|ds\leq\frac{\|\mathbf{u}\|\beta}{\Gamma(\mathfrak{q}_2+1)}\left[(\varphi(\varkappa_2)-\varphi(0))^{\mathfrak{q}_2}-(\varphi(\varkappa_1)-\varphi(0))^{\mathfrak{q}_2}\right]\\ &+\frac{\omega_f^*}{\Gamma(\mathfrak{q}_2+\mathfrak{q}_1+1)}\left[(\varphi(\varkappa_2)-\varphi(0))^{\mathfrak{q}_2+\mathfrak{q}_1}-(\varphi(\varkappa_1)-\varphi(0))^{\mathfrak{q}_2+\mathfrak{q}_1}\right].\end{aligned} \tag{45}$$

For $\varkappa_2 \longrightarrow \varkappa_1$ and continuity of φ, we obtain

$$\left|\mathcal{Q}^1\mathfrak{u}(\varkappa_2) - \mathcal{Q}^1\mathfrak{u}(\varkappa_1)\right| \longrightarrow 0 \text{ as } \varkappa_2 \longrightarrow \varkappa_1. \tag{46}$$

Hence, $\mathcal{Q}^1$ is equicontinuous. As a result of the Arzelá-Ascoli theorem, we deduce that the operator $\mathcal{Q}^1$ is compact in $\mathbb{B}_r$. Thus, $\mathcal{Q}^1$ is completely continuous.

Step 3. $\mathcal{Q}^2$ is contraction mapping. For $\mathfrak{u}, \widehat{\mathfrak{u}} \in \mathbb{B}_r$ and $\varkappa \in \mathcal{J}$, we obtain

$$\begin{aligned}\left\|\mathcal{Q}^2\mathfrak{u} - \mathcal{Q}^2\widehat{\mathfrak{u}}\right\| &\le \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)} \\ &\cdot \Bigg[\sum_{i=1}^{\infty}\eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i;\varphi}\beta|\mathfrak{u}(\lambda_i) - \widehat{\mathfrak{u}}(\lambda_i)| \\ &+ \sum_{i=1}^{\infty}\eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i;\varphi}\Big|f\left(\lambda_i, \mathfrak{u}(\lambda_i), \mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\lambda_i), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\lambda_i)\right) \\ &- f\left(\lambda_i, \widehat{\mathfrak{u}}(\lambda_i), \mathbf{I}_{0^+}^{\theta;\varphi}\widehat{\mathfrak{u}}(\lambda_i), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\lambda_i)\right)\Big| + \mathbf{I}_{0^+}^{\mathfrak{q}_2;\varphi}\beta|\mathfrak{u}(b) \\ &- \widehat{\mathfrak{u}}(b)| + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1;\varphi}\Big|f\left(b, \mathfrak{u}(b), \mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(b), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(b)\right) \\ &- f\left(b, \widehat{\mathfrak{u}}(b), \mathbf{I}_{0^+}^{\theta;\varphi}\widehat{\mathfrak{u}}(b), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(b)\right)\Big|\Bigg] \le \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)} \\ &\cdot \Bigg[\sum_{i=1}^{\infty}\eta_i\beta\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}\|\mathfrak{u}-\widehat{\mathfrak{u}}\| + \sum_{i=1}^{\infty}\eta_i\frac{\mu_f}{1-\mu_f} \\ &\cdot \left(1 + \frac{1}{\Gamma(\theta+1)}\right)\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\|\mathfrak{u}-\widehat{\mathfrak{u}}\| + \beta\Pi_b^{\mathfrak{q}_2}\|\mathfrak{u}-\widehat{\mathfrak{u}}\| \\ &+ \frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\|\mathfrak{u}-\widehat{\mathfrak{u}}\|\Bigg] \le (\mathcal{M}+\mathcal{V})\|\mathfrak{u}-\widehat{\mathfrak{u}}\|.\end{aligned} \tag{47}$$

Thus, we conclude that $\mathcal{Q}^2$ is a contraction.

According to the above steps and Theorem 8, we infer that problem (5) has at least one solution on $\mathcal{J}$. □

Theorem 12. *Assume that (H_1) holds. If*

$$\begin{aligned}Q_1 := \Bigg\{&\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\left[\beta\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i} + \frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\right] \\ &+ \left(1 + \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\right)\left[\beta\Pi_b^{\mathfrak{q}_2} + \frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\right]\Bigg\} \\ &< 1,\end{aligned} \tag{48}$$

then the problem (5) has a unique solution on $\mathcal{J}$.

Proof. We noted that the fixed points of the operator $\mathcal{Q}$ defined in (37) are a solution of problem (5). Define a closed ball set $\mathbb{B}_\varpi$ as

$$\mathbb{B}_\varpi = \{\mathfrak{u} \in \mathcal{C}(\mathcal{J}) : \|\mathfrak{u}\| \le \varpi\}, \tag{49}$$

with $\varpi \ge (Q_2/(1-Q_1)1-Q_1)$, where

$$\begin{aligned}Q_2 = \Bigg(&\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i} \\ &+ \left(1 + \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\Bigg)\mathcal{P},\end{aligned} \tag{50}$$

and $\mathcal{P} = \sup_{s\in\mathcal{J}}|f(s,0,0,0)|$. □

First, we show that $\mathcal{Q}(\mathbb{B}_\varpi) \subset \mathbb{B}_\varpi$. By (33) and condition (H_1), we have

$$\begin{aligned}\mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i;\varphi}\beta|\mathfrak{u}(\lambda_i)| &= \frac{\beta}{\Gamma(\mathfrak{q}_2+\sigma_i)}\int_0^{\lambda_i}\varphi'(s)(\varphi(\lambda_i)-\varphi(s))^{\mathfrak{q}_2+\sigma_i-1}|\mathfrak{u}(s)|ds \\ &\le \frac{\beta(\varphi(\lambda_i)-\varphi(0))^{\mathfrak{q}_2+\sigma_i}}{\Gamma(\mathfrak{q}_2+\sigma_i+1)}\|\mathfrak{u}\| = \beta\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}\|\mathfrak{u}\|, \\ \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i;\varphi}&\left|f\left(\lambda_i, \mathfrak{u}(\lambda_i), I_{0^+}^{\theta;\varphi}\mathfrak{u}(\lambda_i), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\lambda_i)\right)\right| \\ &\le \frac{1}{\Gamma(\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i)}\int_0^{\lambda_i}\varphi'(s)(\varphi(\lambda_i)-\varphi(s))^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i-1} \\ &\times\left(\left|f\left(s, \mathfrak{u}(s), \mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(s), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(s)\right) - f(s,0,0,0)\right| + |f(s,0,0,0)|\right)ds \\ &\le \frac{1}{\Gamma(\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i)}\int_0^{\lambda_i}\varphi'(s)(\varphi(\lambda_i)-\varphi(s))^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i-1}ds \\ &\times\left(\frac{\mu_f}{1-\mu_f}\left[\|\mathfrak{u}\| + \mathbf{I}_{0^+}^{\theta;\varphi}\|\mathfrak{u}\|\right] + \mathcal{P}\right) \le \Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i} \\ &\times\left[\frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\|\mathfrak{u}\| + \mathcal{P}\right].\end{aligned} \tag{51}$$

Thus, for $\mathfrak{u} \in \mathbb{B}_\varpi$, we obtain

$$\begin{aligned}|\mathcal{Q}\mathfrak{u}(\varkappa)| &\le \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\Bigg[\sum_{i=1}^{\infty}\eta_i\Big(\mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i;\varphi}\beta|\mathfrak{u}(\lambda_i)| \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i;\varphi}\left|f\left(\lambda_i, \mathfrak{u}(\lambda_i), \mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\lambda_i), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\lambda_i)\right)\right|\Big) \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2;\varphi}\beta|\mathfrak{u}(b)| + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1;\varphi}\left|f\left(b, \mathfrak{u}(b), \mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(b), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(b)\right)\right|\Bigg] \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2;\varphi}\beta|\mathfrak{u}(\varkappa)| + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1;\varphi}\left|f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\varkappa), {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\right| \\ &\le \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\Bigg[\sum_{i=1}^{\infty}\eta_i\Big(\beta\|\mathfrak{u}\|\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i} \\ &+ \left[\frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\|\mathfrak{u}\| + \mathcal{P}\right]\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\Big) \\ &+ \beta\|\mathfrak{u}\|\Pi_b^{\mathfrak{q}_2} + \left[\frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\|\mathfrak{u}\| + \mathcal{P}\right]\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\Bigg] \\ &+ \beta\|\mathfrak{u}\|\Pi_b^{\mathfrak{q}_2} + \left[\frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\|\mathfrak{u}\| + \mathcal{P}\right]\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1} \\ &\le \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\Big[\beta\varpi\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i} \\ &+ \left[\frac{\mu_f}{1-\mu_f}\left(1 + \frac{1}{\Gamma(\theta+1)}\right)\varpi + \mathcal{P}\right]\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\Big] \\ &+ \left(1 + \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\right)\beta\varpi\Pi_b^{\mathfrak{q}_2} + \left(1 + \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\right)\end{aligned}$$

$$\cdot\left[\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\varpi+\mathscr{P}\right]\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}$$
$$\le\frac{\varpi}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i$$
$$\cdot\left[\beta\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}+\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\right]$$
$$+\frac{\mathscr{P}}{\Theta\Gamma(\gamma_2)}\sum_{i=1}^{\infty}\eta_i\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}+\left(1+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\right)\varpi$$
$$\cdot\left[\beta\Pi_b^{\mathfrak{q}_2}+\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\right]$$
$$+\left(1+\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\right)\mathscr{P}\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\le Q_1\varpi+Q_2\le\varpi. \quad (52)$$

Thus, $\mathscr{Q}(\mathbb{B}_\varpi)\subset\mathbb{B}_\varpi$.

Next, we prove that $\mathscr{Q}$ is contraction map. Indeed, for $\mathfrak{u},\hat{\mathfrak{u}}\in\mathbb{B}_\varpi$ and $\varkappa\in\mathscr{J}$, we obtain

$$\|\mathscr{Q}\mathfrak{u}-\mathscr{Q}\hat{\mathfrak{u}}\|\le\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}$$
$$\cdot\left[\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i;\varphi}\beta|\mathfrak{u}(\lambda_i)-\hat{\mathfrak{u}}(\lambda_i)|\right.$$
$$+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i;\varphi}\left|f\left(\lambda_i,\mathfrak{u}(\lambda_i),\mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\lambda_i),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\lambda_i)\right)\right.$$
$$\left.-f\left(\lambda_i,\hat{\mathfrak{u}}(\lambda_i),\mathbf{I}_{0^+}^{\theta;\varphi}\hat{\mathfrak{u}}(\lambda_i),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\hat{\mathfrak{u}}(\lambda_i)\right)\right|+\mathbf{I}_{0^+}^{\mathfrak{q}_2;\varphi}\beta|\mathfrak{u}(b)$$
$$-\hat{\mathfrak{u}}(b)|+\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1;\varphi}\left|f\left(b,\mathfrak{u}(b),\mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(b),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(b)\right)\right.$$
$$\left.\left.-f\left(b,\hat{\mathfrak{u}}(b),\mathbf{I}_{0^+}^{\theta;\varphi}\hat{\mathfrak{u}}(b),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\hat{\mathfrak{u}}(b)\right)\right|\right]+\mathbf{I}_{0^+}^{\mathfrak{q}_2;\varphi}\beta|\mathfrak{u}(\varkappa)$$
$$-\hat{\mathfrak{u}}(\varkappa)|+\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1;\varphi}\left|f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta;\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\right.$$
$$\left.-f\left(\varkappa,\hat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta;\varphi}\hat{\mathfrak{u}}(\varkappa),,{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\hat{\mathfrak{u}}(\varkappa)\right)\right|$$
$$\le\frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\left[\sum_{i=1}^{\infty}\eta_i\beta\Pi_{\lambda_i}^{\mathfrak{q}_2+\sigma_i}\|\mathfrak{u}-\hat{\mathfrak{u}}\|+\sum_{i=1}^{\infty}\eta_i\frac{\mu_f}{1-\mu_f}\right.$$
$$\cdot\left(1+\frac{1}{\Gamma(\theta+1)}\right)\Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i}\|\mathfrak{u}-\hat{\mathfrak{u}}\|+\beta\Pi_b^{\mathfrak{q}_2}\|\mathfrak{u}-\hat{\mathfrak{u}}\|$$
$$\left.+\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\|\mathfrak{u}-\hat{\mathfrak{u}}\|\right]$$
$$+\Pi_b^{\mathfrak{q}_2}\|\mathfrak{u}-\hat{\mathfrak{u}}\|+\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}\|\mathfrak{u}-\hat{\mathfrak{u}}\|$$
$$\le Q_1\|\mathfrak{u}-\hat{\mathfrak{u}}\|. \quad (53)$$

From (48), $\mathscr{Q}$ is a contraction map. Hence, in view of Theorem 7, we conclude that the problem (5) has a unique solution on $\mathscr{J}$.

5. Stability Analysis

In 1940 [33], the problem of stability of functional equations was created by Ulam's question regarding the stability of group homomorphisms. In the following year, Hyers [34] gave a positive interpretation of the Ulam question within the Banach spaces, and this was the first major advance and a step towards more solutions in this field. Since then, many papers have been published regarding various generalizations of the Ulam problem and Hyers theory. In 1978, Rassias [35] succeeded in extending Hyers' theory of mappings between Banach spaces. Rassias's result attracted many mathematicians around the world who began their investigations of the problems of stability of functional equations.

In this regard, we discuss the stability results in the frame of Ulam-Hyers- (UH-) Rassias (UHR). Let $\epsilon>0$ and a continuous function $\alpha_\phi:\mathscr{J}\longrightarrow\mathbb{R}^+$ such that it satisfies the following inequalities:

$$\left|{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\hat{\mathfrak{u}}(\varkappa)-\beta\hat{\mathfrak{u}}(\varkappa)\right)-f\left(\varkappa,\hat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\hat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\hat{\mathfrak{u}}(\varkappa)\right)\right|\le\epsilon \quad (54)$$

$$\left|{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\hat{\mathfrak{u}}(\varkappa)-\beta\hat{\mathfrak{u}}(\varkappa)\right)-f\left(\varkappa,\hat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\hat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\hat{\mathfrak{u}}(\varkappa)\right)\right|\le\epsilon\alpha_\phi(\varkappa). \quad (55)$$

Lemma 13 (see [36]). *Let $\mathfrak{q}>0$ and $\mathfrak{u},v$ be two nonnegative functions locally integrable on $\mathscr{J}$. Assume that g is nonnegative and nondecreasing, and let $\varphi\in\mathscr{C}^1(\mathscr{J})$ be an increasing function such that $\varphi'(t)\ne0$ for all $t\in J$. If*

$$\mathfrak{u}(\varkappa)\le v(\varkappa)+\mathfrak{g}(t)\int_0^{\varkappa}\varphi'(s)(\varphi(\varkappa)-\varphi(s))^{\mathfrak{q}-1}\mathfrak{u}(s)ds,\quad\varkappa\in\mathscr{J}, \quad (56)$$

then

$$\mathfrak{u}(\varkappa)\le v(\varkappa)+\int_0^t\sum_{n=1}^{\infty}\frac{[\mathfrak{g}(\varkappa)\Gamma(p)]^n}{\Gamma(n\mathfrak{q})}\varphi'(s)(\varphi(\varkappa)-\varphi(s))^{n\mathfrak{q}-1}v(s)ds,\quad\varkappa\in\mathscr{J}. \quad (57)$$

If v is a nondecreasing function on $\mathscr{J}$. then, we have

$$\mathfrak{u}(\varkappa)\le v(\varkappa)E_{\mathfrak{q}}\{\mathfrak{g}(\varkappa)\Gamma(p)(\varphi(\varkappa)-\varphi(0))^{\mathfrak{q}}\},\quad\varkappa\in\mathscr{J}. \quad (58)$$

Definition 14. We say that the problem (5) is UH stable if for every$\hat{\mathfrak{u}}\in\mathscr{C}(\mathscr{J})$that satisfies an inequality (54) and$\mathfrak{u}\in\mathscr{C}(\mathscr{J})$ is a solution of the problem (5), there exists a constant number $0<\mathscr{T}\in\mathbb{R}$ such that

$$|\hat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|\le\mathscr{T}\epsilon,\quad\varkappa\in\mathscr{J},\epsilon>0. \quad (59)$$

Definition 15. We say that the problem (5) is UHR stable with respect to nondecreasing function $\alpha_\phi(\varkappa)$ if for every $\hat{\mathfrak{u}}\in\mathscr{C}(\mathscr{J})$ that satisfies an inequality (55) and $\mathfrak{u}\in\mathscr{C}(\mathscr{J})$ is a solution of the problem (5), there exists $0<\mathscr{N}\in\mathbb{R}$ such that

$$|\hat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|\le\mathscr{N}\epsilon\alpha_\phi(\varkappa),\quad\varkappa\in\mathscr{J},\epsilon>0. \quad (60)$$

Remark 16. A function $\hat{\mathfrak{u}}\in\mathscr{C}(\mathscr{J})$ satisfies an inequality (54) if and only if there exist a functions $z \in \mathscr{C}(\mathscr{J})$ such that

$$\begin{aligned} &|z(\varkappa)| \leq \epsilon, \quad \varkappa \in \mathscr{J}, \\ &{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\widehat{\mathfrak{u}}(\varkappa) - \beta\widehat{\mathfrak{u}}(\varkappa)\right) = f\left(\varkappa, \widehat{\mathfrak{u}}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right) + z(\varkappa). \end{aligned} \tag{61}$$

Lemma 17. *Let $\gamma_j = \mathfrak{q}_j + j\delta_j - \mathfrak{q}_j\delta_j (j = 1, 2)$, $\mathfrak{q}_1 \in (0, 1]$, $\mathfrak{q}_2 \in (1, 2]$, and $0 \leq \delta_j \leq 1$. If a function $\widehat{\mathfrak{u}} \in \mathscr{C}(\mathscr{J})$ satisfies (54), then $\widehat{\mathfrak{u}}$ satisfies*

$$\left|\widehat{\mathfrak{u}}(\varkappa) - \mathscr{A}_{\widehat{\mathfrak{u}}} - \mathbf{I}_{0^+}^{\mathfrak{q}_1+\mathfrak{q}_2,\varphi} f\left(\varkappa, \widehat{\mathfrak{u}}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right| \leq \epsilon\mathfrak{P}, \tag{62}$$

where

$$\begin{aligned} \mathscr{A}_{\widehat{\mathfrak{u}}} :=& \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \left[\sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i,\varphi}\beta\widehat{\mathfrak{u}}(\lambda_i)\right. \\ &+ \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i,\varphi} f\left(\lambda_i, \widehat{\mathfrak{u}}(\lambda_i), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\lambda_i), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\lambda_i)\right) \\ &\left.- \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\widehat{\mathfrak{u}}(b) - \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(b, \widehat{\mathfrak{u}}(b), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(b), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(b)\right)\right] \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\widehat{\mathfrak{u}}(\varkappa), \end{aligned}$$

$$\mathfrak{P} := \frac{(\varphi(b) - \varphi(0))^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \sum_{i=1}^{\infty} \eta_i \Pi_{\lambda_i}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i} + \left(\frac{(\varphi(b) - \varphi(0))^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} + 1\right)\Pi_b^{\mathfrak{q}_2+\mathfrak{q}_1}. \tag{63}$$

Proof. By Remark 16, we have

$$\begin{aligned} \widehat{\mathfrak{u}}(\varkappa) =& \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \left[\sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i,\varphi}\beta\widehat{\mathfrak{u}}(\lambda_i)\right. \\ &+ \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i,\varphi}\left(f\left(\lambda_i, \widehat{\mathfrak{u}}(\lambda_i), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\lambda_i), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\lambda_i)\right)\right. \\ &+ z(\lambda_i)) - \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\widehat{\mathfrak{u}}(b) - \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} \\ &\left.\cdot \left(f\left(b, \widehat{\mathfrak{u}}(b), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(b), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(b)\right) + z(b)\right)\right] \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\widehat{\mathfrak{u}}(\varkappa) + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}\left(f\left(\varkappa, \widehat{\mathfrak{u}}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right) + z(\varkappa)\right). \end{aligned} \tag{64}$$

Then, we get

$$\begin{aligned} &\left|\widehat{\mathfrak{u}}(\varkappa) - \mathscr{A}_{\widehat{\mathfrak{u}}} - \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(\varkappa, \widehat{\mathfrak{u}}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right| \\ &\leq \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \left[\sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i,\varphi}|z(\lambda_i)| + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}|z(b)|\right] \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}|z(\varkappa)|. \end{aligned} \tag{65}$$

It follows that

$$\left|\widehat{\mathfrak{u}}(\varkappa) - \mathscr{A}_{\widehat{\mathfrak{u}}} - \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(\varkappa, \widehat{\mathfrak{u}}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right| \leq \varepsilon\mathfrak{P}. \tag{66}$$

□

Theorem 18. *Let us assume that (H_1) holds. Then,*

$$\begin{aligned} &{}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa) - \beta\mathfrak{u}(\varkappa)\right) \\ &= f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa){}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right) \end{aligned} \tag{67}$$

is UH stable.

Proof. Let $\varepsilon > 0$ and $\widehat{\mathfrak{u}} \in \mathscr{C}(\mathscr{J})$ be a function that satisfies an inequality (54) and let $\mathfrak{u} \in \mathscr{C}(\mathscr{J})$ be a solution of

$$\begin{cases} {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa) - \beta\mathfrak{u}(\varkappa)\right) = f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right), \varkappa \in (0, b], \\ \mathfrak{u}(0) = \mathfrak{u}(0) = 0, {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(0) = {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\widehat{\mathfrak{u}}(0) = 0, \\ \mathfrak{u}(b) = \widehat{\mathfrak{u}}(b) = \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\sigma_i,\varphi}\widehat{\mathfrak{u}}(\lambda i). \end{cases} \tag{68}$$

Then, by using Theorem 12, we have

$$\mathfrak{u}(\varkappa) = \mathscr{A}_{\mathfrak{u}} + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(\varkappa, \mathfrak{u}(\varkappa), \mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right), \tag{69}$$

where

$$\begin{aligned} \mathscr{A}_{\mathfrak{u}} :=& \frac{(\varphi(\varkappa) - \varphi(0))^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \\ &\cdot \left[\sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i,\varphi}\beta\mathfrak{u}(\lambda_i) + \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i,\varphi} f\right. \\ &\cdot \left(\lambda_i, \mathfrak{u}(\lambda_i), \mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\lambda_i), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\lambda_i)\right) \\ &\left.- \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\mathfrak{u}(b) - \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(b, \mathfrak{u}(b), \mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(b), {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(b)\right)\right] \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\mathfrak{u}(\varkappa). \end{aligned} \tag{70}$$

□

Since

$$\begin{cases} \mathfrak{u}(0) = \widehat{\mathfrak{u}}(0) = 0, {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(0) = {}^{H}\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\widehat{\mathfrak{u}}(0) = 0, \\ \mathfrak{u}(b) = \widehat{\mathfrak{u}}(b) = \sum_{i=1}^{\infty} \eta_i \mathbf{I}_{0^+}^{\sigma_i,\varphi}\widehat{\mathfrak{u}}(\lambda_i), \end{cases} \tag{71}$$

we can easily prove that $\mathscr{A}_{\mathfrak{u}} = \mathscr{A}_{\widehat{\mathfrak{u}}}$. Hence, according to (H_1)

and Lemma 17, then for each $\varkappa \in \mathcal{J}$,

$$\begin{aligned} |\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)| &= \Big|\widehat{\mathfrak{u}}(\varkappa)-\mathscr{A}_{\widehat{\mathfrak{u}}}-\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right) \\ &- \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)+\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f \\ &\cdot\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\Big| \le \Big|\widehat{\mathfrak{u}}(\varkappa)-\mathscr{A}_{\widehat{\mathfrak{u}}}-\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f \\ &\cdot\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\Big| + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}\Big| f \\ &\cdot\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right) - f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\Big| \\ &\le \mathfrak{P}\varepsilon + \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}\Big| f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right) \\ &- f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\Big| \le \mathfrak{P}\varepsilon \\ &+ \frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|, \end{aligned} \tag{72}$$

Thus, by Lemma 13, we get

$$\begin{aligned} |\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)| &\le \mathfrak{P}\varepsilon + \int_0^{\varkappa} \\ &\cdot\left(\sum_{n=1}^{\infty}\frac{\left[\mu_f/\left(1-\mu_f\right)(1+(1/\Gamma(\theta+1)))\right]^n}{\Gamma[n(\mathfrak{q}_2+\mathfrak{q}_1)]}\varphi'(s)(\varphi(\varkappa)-\varphi(s))^{n(\mathfrak{q}_2+\mathfrak{q}_1)-1}\right)\mathfrak{P}\varepsilon ds \\ &< \mathfrak{P}\varepsilon E_{\mathfrak{q}_2+\mathfrak{q}_1}\left[\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)(\varphi(b)-\varphi(0))^{\mathfrak{q}_2+\mathfrak{q}_1}\right]. \end{aligned} \tag{73}$$

It follows that

$$|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)| \le \mathcal{T}\varepsilon. \tag{74}$$

where $\mathcal{T} = \mathfrak{P}E_{\mathfrak{q}_2+\mathfrak{q}_1}[\mu_f/(1-\mu_f)(1+(1/\Gamma(\theta+1)))(\varphi(b)-\varphi(0))^{\mathfrak{q}_2+\mathfrak{q}_1}]$. Hence, the problem (5) is UH stable.

Corollary 19. *Under the assumptions of Theorem 18, if there exists a function $\phi_f \in \mathscr{C}(\mathcal{J})$, then the problem (5) is generalized UH stable.*

In the forthcoming theorem, we discuss Ulam-Hyers-Rassias stability. For that, the following hypothesis must be satisfied.

(H_3) There exists an increasing function $\alpha_\phi \in \mathscr{C}(\mathcal{J})$, and there exists $\mathscr{R} > 0$ such that for any $\varkappa \in \mathcal{J}$,

$$\mathcal{I}_{0^+}^{\xi,\varphi}\alpha_\phi(\varkappa) \le \mathscr{R}\alpha_\phi(\varkappa), \tag{75}$$

where $\xi \in \{\mathfrak{q}_1+\mathfrak{q}_2, \mathfrak{q}_2+\mathfrak{q}_1+\sigma_i\}$.

Remark 20. A function $\widehat{\mathfrak{u}} \in \mathscr{C}(\mathcal{J})$ satisfies an inequality (55) if and only if there exist a functions $z \in \mathscr{C}(\mathcal{J})$ such that

$$\begin{gathered} |z(\varkappa)| \le \varepsilon\alpha_\phi(\varkappa), \quad \varkappa \in \mathcal{J}, \\ {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\widehat{\mathfrak{u}}(\varkappa)-\beta\widehat{\mathfrak{u}}(\varkappa)\right) \\ = f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)+z(\varkappa). \end{gathered} \tag{76}$$

Lemma 21. *Let $\gamma_j = \mathfrak{q}_j + j\delta_j - \mathfrak{q}_j\delta_j (j = 1, 2)$, $\mathfrak{q}_1 \in (0, 1]$, $\mathfrak{q}_2 \in (1, 2]$, and $0 \le \delta_j \le 1$. If a function $\widehat{\mathfrak{u}} \in \mathscr{C}(\mathcal{J})$ satisfies (55), then $\widehat{\mathfrak{u}}$ satisfies*

$$\begin{aligned} &\left|\widehat{\mathfrak{u}}(\varkappa)-\mathscr{A}_{\widehat{\mathfrak{u}}}-\mathbf{I}_{0^+}^{\mathfrak{q}_1+\mathfrak{q}_2,\varphi} f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right| \\ &\quad\le \varepsilon\mathfrak{P}_1\mathscr{R}\alpha_\phi(\varkappa), \end{aligned} \tag{77}$$

where

$$\begin{aligned} \mathscr{A}_{\widehat{\mathfrak{u}}} &:= \frac{(\varphi(\varkappa)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)} \\ &\cdot\Bigg[\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathfrak{q}_2+\sigma_i,\varphi}\beta\widehat{\mathfrak{u}}(\lambda_i)+\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i,\varphi} f \\ &\cdot\left(\lambda_i,\widehat{\mathfrak{u}}(\lambda_i),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\lambda_i),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\lambda_i)\right)-\mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\widehat{\mathfrak{u}}(b) \\ &-\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(b,\widehat{\mathfrak{u}}(b),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(b),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(b)\right)\Bigg]+\mathbf{I}_{0^+}^{\mathfrak{q}_2,\varphi}\beta\widehat{\mathfrak{u}}(\varkappa), \\ \mathfrak{P}_1 &:= \frac{(\varphi(b)-\varphi(0))^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\left(\sum_{i=1}^{\infty}\eta_i+1\right)+1. \end{aligned} \tag{78}$$

Proof. Indeed, by Remark 20 and Theorem 12, one can easily prove that

$$\begin{aligned} &\left|\widehat{\mathfrak{u}}(\varkappa)-\mathscr{A}_{\widehat{\mathfrak{u}}}-\mathbf{I}_{0^+}^{\mathfrak{q}_1+\mathfrak{q}_2,\varphi} f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right| \\ &\quad\le \varepsilon\mathfrak{P}_1\mathscr{R}\alpha_\phi(\varkappa). \end{aligned} \tag{79}$$

□

Theorem 22. *Assume that (H_1) and (H_3) hold. Then,*

$$\begin{aligned} &{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa)-\beta\mathfrak{u}(\varkappa)\right) \\ &\quad= f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right), \end{aligned} \tag{80}$$

is UHR and generalized UHR stable.

Proof. By the same technique in Theorem 18, one can prove that

$$\begin{aligned} |\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)| &\le \left|\widehat{\mathfrak{u}}(\varkappa)-\mathscr{A}_{\widehat{\mathfrak{u}}}-\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi} f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right| \\ &+ \mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}\Big| f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right) \\ &- f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\Big| \\ &\le \varepsilon\mathfrak{P}_1\mathscr{R}\alpha_\phi(\varkappa)+\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}\Big| f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right) \\ &- f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\Big| \le \varepsilon\mathfrak{P}_1\mathscr{R}\alpha_\phi(\varkappa) \\ &+ \frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\mathbf{I}_{0^+}^{\mathfrak{q}_2+\mathfrak{q}_1,\varphi}|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|. \end{aligned} \tag{81}$$

Thus, by (H_3) and Lemma 13, we have

$$\begin{aligned}|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)| &\le \varepsilon\mathfrak{P}_1\mathscr{R}\alpha_\phi(\varkappa)+\varepsilon\mathfrak{P}_1\mathscr{R}\int_0^\varkappa\left(\sum_{n=1}^{\infty}\frac{\left[\left(\mu_f/\left(1-\mu_f\right)1-\mu_f\right)(1+(1/\Gamma(\theta+1)))\right]^n}{\Gamma[n(\mathfrak{q}_2+\mathfrak{q}_1)]}\varphi'(s)(\varphi(\varkappa)-\varphi(s))^{n(\mathfrak{q}_2+\mathfrak{q}_1)-1}\right)\alpha_\phi(s)ds\\ &\le \varepsilon\mathfrak{P}_1\mathscr{R}\alpha_\phi(\varkappa)\left[1+\sum_{n=1}^{\infty}\left[\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)\mathscr{R}\right]^n\right].\end{aligned} \tag{82}$$

It follows that

$$|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|\le\mathscr{N}\varepsilon\alpha_\phi(\varkappa). \tag{83}$$

where $\mathscr{N}=\mathfrak{P}_1\mathscr{R}[1+\sum_{n=1}^{\infty}[(\mu_f/(1-\mu_f)1-\mu_f)(1+(1/\Gamma(\theta+1)))\mathscr{R}]^n]$. Hence, the problem (5) is UHR stable as well as generalized UHR stable. □

6. An Example

Consider the Langevin-type fractional integrodifferential equation

$$\begin{cases} {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(\varkappa)-\beta\mathfrak{u}(\varkappa)\right)=f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right), & \varkappa\in\mathscr{J}=[0,1],\\ \mathfrak{u}(0)=0,\ {}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\mathfrak{u}(0)=0,\ \mathfrak{u}(b)=\sum_{i=1}^{\infty}\eta_i\mathbf{I}_{0^+}^{\sigma_i,\varphi}\mathfrak{u}(\lambda i), \end{cases} \tag{84}$$

such that $\mathfrak{q}_1=1/2,\delta_1=1/3,\mathfrak{q}_2=3/2,\delta_2=1/3,\gamma_2=5/3,\theta=1/2,\beta=1/10,b=1,\lambda_i=1/2i,\eta_i=2/8i,\ \sigma_i=2/(2i+1),\ \varphi(\varkappa)=\varkappa$, and

$$\begin{aligned}&f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\mathfrak{u}(\varkappa)\right)\\&\quad=\frac{\varkappa}{50+\varkappa}\left(\mathfrak{u}(\varkappa)+\mathbf{I}_{0^+}^{1/5,\varkappa}\mathfrak{u}(\varkappa)+{}^H\mathbf{D}_{0^+}^{1/2,1/3;\varkappa}\mathfrak{u}(\varkappa)\right).\end{aligned} \tag{85}$$

Clearly, for each $\mathfrak{u},\widehat{\mathfrak{u}}\in\mathscr{C}(\mathscr{J})$, we have

$$\begin{aligned}&\left|f\left(\varkappa,\mathfrak{u}(\varkappa),\mathbf{I}_{0^+}^{1/5,\varkappa}\mathfrak{u}(\varkappa),{}^H\mathbf{D}_{0^+}^{1/2,1/3;\varkappa}\mathfrak{u}(\varkappa)\right)-f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{1/5,\varkappa}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{1/2,1/3;\varkappa}\widehat{\mathfrak{u}}(\varkappa)\right)\right|\\&\quad\le\frac{1}{50}\left[|\mathfrak{u}(\varkappa)-\widehat{\mathfrak{u}}(\varkappa)|+\left|\mathbf{I}_{0^+}^{1/5,\varkappa}\mathfrak{u}(\varkappa)-\mathbf{I}_{0^+}^{1/5,\varkappa}\widehat{\mathfrak{u}}(\varkappa)\right|\right.\\&\qquad\left.+\left|{}^H\mathbf{D}_{0^+}^{1/2,1/3;\varkappa}\mathfrak{u}(\varkappa)-{}^H\mathbf{D}_{0^+}^{1/2,1/3;\varkappa}\widehat{\mathfrak{u}}(\varkappa)\right|\right],\end{aligned} \tag{86}$$

with $\mu_f=1/50$. By the given data, we can get $\Theta\approx0.89\ne0$ and hence, $Q_1\approx0.2148<1$. Thus, all assumptions in Theorem 12 hold. Then, problem (84) has a unique solution. Moreover, we have

$$|f(\varkappa,\mathfrak{u}(\varkappa),v(\varkappa),z(\varkappa))|\le\frac{\varkappa}{50+\varkappa}=\omega_f(\varkappa),\quad\text{for all }(\varkappa,\mathfrak{u},v)\in\mathscr{J}\times\mathbb{R}^2. \tag{87}$$

Thus, all the assumptions in Theorem 11 hold. Then, problem (84) has at least one solution. Furthermore, for $\varepsilon>0$, we find that

$$\left|{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\widehat{\mathfrak{u}}(\varkappa)-\beta\widehat{\mathfrak{u}}(\varkappa)\right)-f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right|\le\varepsilon \tag{88}$$

is satisfied. Then, equation (67) is Ulam-Hyers stable with

$$|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|\le\mathscr{T}\varepsilon, \tag{89}$$

where

$$\mathfrak{P}E_{\mathfrak{q}_2+\mathfrak{q}_1}\left[\frac{\mu_f}{1-\mu_f}\left(1+\frac{1}{\Gamma(\theta+1)}\right)(\varphi(b)-\varphi(0))^{\mathfrak{q}_2+\mathfrak{q}_1}\right]>0, \tag{90}$$

and $\Lambda_f\approx0.042<1$. Finally, we consider $\alpha_\phi(\varkappa)=\varphi(\varkappa)-\varphi(0)$, for $\varkappa\in0,1]$. Then, $\alpha_\phi:[0,1]\longrightarrow\mathbb{R}$ is nondecreasing continuous function. Hence, by Lemma 5, we get

$$\begin{aligned}\mathbf{I}_{0^+}^{\xi,\varphi}\alpha_\phi(\varkappa)&=\mathbf{I}_{0^+}^{\xi,\varphi}[\varphi(\varkappa)-\varphi(0)]=\mathbf{I}_{0^+}^{\xi,\varphi}[\varphi(\varkappa)-\varphi(0)]^{2-1}\\&=\frac{\Gamma(2)}{\Gamma(\xi+2)}[\varphi(\varkappa)-\varphi(0)]^{\xi+1}=\frac{[\varphi(\varkappa)-\varphi(0)]^\xi}{\Gamma(\xi+2)}\alpha_\phi(\varkappa)\\&\le\frac{[\varphi(1)-\varphi(0)]^\xi}{\Gamma(\xi+2)}\alpha_\phi(\varkappa)=\mathscr{R}\alpha_\phi(\varkappa),\quad\text{for all }\varkappa\in\mathscr{J},\end{aligned} \tag{91}$$

where $\mathscr{R}=[\varphi(1)-\varphi(0)]^\xi/\Gamma(\xi+2)>0$ for $\xi\in\{\mathfrak{q}_1+\mathfrak{q}_2,\mathfrak{q}_2+\mathfrak{q}_1+\sigma_i\}$ and $\varphi(\varkappa)=\varkappa$. Therefore, Theorem 22 applicable. Furthermore, for $\varepsilon>0$ and a continuous function $\alpha_\phi:\mathscr{J}\longrightarrow\mathbb{R}^+$, we find that

$$\begin{aligned}&\left|{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\left({}^H\mathbf{D}_{0^+}^{\mathfrak{q}_2,\delta_2;\varphi}\widehat{\mathfrak{u}}(\varkappa)-\beta\widehat{\mathfrak{u}}(\varkappa)\right)-f\left(\varkappa,\widehat{\mathfrak{u}}(\varkappa),\mathbf{I}_{0^+}^{\theta,\varphi}\widehat{\mathfrak{u}}(\varkappa),{}^H\mathbf{D}_{0^+}^{\mathfrak{q}_1,\delta_1;\varphi}\widehat{\mathfrak{u}}(\varkappa)\right)\right|\\&\quad\le\varepsilon\alpha_\phi(\varkappa)\end{aligned} \tag{92}$$

is satisfied. Then, equation (80) is UHR stable with

$$|\widehat{\mathfrak{u}}(\varkappa)-\mathfrak{u}(\varkappa)|\le\mathscr{N}\varepsilon\alpha_\phi(\varkappa), \tag{93}$$

where

$$\mathcal{N} = \mathfrak{P}_1 \mathcal{R} \left[1 + \sum_{n=1}^{\infty} \left[\frac{\mu_f}{1-\mu_f} \left(1 + \frac{1}{\Gamma(\theta+1)} \right) \mathcal{R} \right]^n \right] > 0. \tag{94}$$

7. Conclusion

Because the idea of fractional operators in the context of φ-Hilfer is innovative and important, several academics have explored and established various qualitative features of FDE solutions incorporating such operators. To extend these qualitative features, we developed and investigated sufficient conditions for the existence and uniqueness of solutions, as well as Ulam-Hyers stability results for a nonlinear fractional integrodifferential Langevin equation involving φ-Hilfer FD with respect to an increasing function, for a nonlinear fractional integrodifferential Langevin equation involving φ-Hilfer FD.

Our technique was based on reducing the problem (5) to a fractional integral equation and using certain conventional Banach-type and Krasnoselskii-type fixed point theorems. Furthermore, we investigated the stability data in the Ulam-Hyers sense using mathematical analytic tools. To support the main results, an example was given.

In fact, our outcomes generalize those in [26–29] and cover many results not study yet. Due to the wide recent investigations and applications of the Langevin equation, we believe that acquired results here will be important for future investigations on the theory of fractional calculus.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

This work is conducted during our work at Hajjah University (Yemen).

Conflicts of Interest

All authors declare that they have no conflict of interest.

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