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Research Article

n-Dimensional Fractional Frequency Laplace Transform by the Inverse Difference Operator

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With the study of extensive literature on the Laplace transform with one and two variables and its properties, applications are available, but there is no work on n-dimensional Laplace transform. In this research article, we define n-dimensional fractional frequency Laplace transform with shift values. Several theorems are derived with properties of the Laplace transform. The results are numerically analyzed and discussed through MATLAB.

1. Introduction

The fractional calculus is a branch of mathematics that focuses on arbitrary order integrals and derivatives. In spite of that, this type of calculus is as older as the conventional calculus, and it has attracted the interest of researchers for the last few decades. This is because of the results reported by these researchers as consequences of their attempts to model real-world phenomena using the fractional operators [1–4]. The discrete version of these operators fetched the attention of research studies as well. Many good results were reported when fractional sums and differences were used in studying related problems (see [5–17] and the references therein).

The integral transforms such as Mellin, Laplace, and Fourier were applied to obtain the solution of differential equations. These transforms made effectively possible changes of a signal in the time domain into a frequency s-domain in the field of digital signal processing (DSP) [18]. The delta Laplace transform was first defined in a very general way by Bohner and Peterson [19]. In 2015, Ivic discussed the discrete Laplace transforms in the view of fast decay factor e^{-sx} and obtained the Laplace transform of $P(x)$ as $\int_0^\infty P(x)e^{-sx}\mathrm{d}x = \pi s^{-2}\sum_{n=1}^{\infty} r(n)e^{-(\pi^2/n)}$. In practice, many applications of Laplace transform (LT), $L[f(x)] = \int_0^\infty f(x)e^{-sx}\mathrm{d}x$, and the forward discrete Laplace transform (DLT), $L[f(n)] = \sum_{n=0}^{\infty} f(n)e^{-sn}$, are discussed and mentioned by several authors in [20–23]. For physical applications of Laplace transform, refer [24–27].

In the existing Laplace transform, the shifting value of time domains is one. In 2016, Britto Antony Xavier et al. [28] defined the Laplace transform with shift value ℓ using the generalized difference operator and obtained the outcomes of polynomial and trigonometric functions. In this fractional frequency Laplace transform, the shift values $\nu_j's$, $j = 1, 2, \ldots, n$ lie in the interval $[0, 1]$. In [29], the author introduced the double Laplace transform and applied to solve initial and boundary value problems.

In this research work, we extend the work of Laplace transform into an n-dimensional space in discrete case. We present several properties of the fractional transforms for functions such as polynomial factorial, exponential, and trigonometric functions. Also, we derive the relation between Laplace transform and Riemann zeta functions. Furthermore, we present the inverse Laplace transform to compare the results with the existing classical Laplace transform for the particular value of n.

2. Preliminaries

Here, we present some basic definitions and results which will be used further.

Definition 1. Let $u(\bar{t})$ be the function with n-variables and $\bar{h} \in R^n$ be the shift values. Then, the n-dimensional partial difference operator is defined as

$$\underset{h_i}{\Delta} u(\bar{t}) = \frac{u(t_1, t_2, \ldots, t_i + h_i, \ldots, t_n) - u(t_1, t_2, \ldots, t_n)}{h_i}, \tag{1}$$

where $\bar{t} = (t_1, t_2, \ldots, t_n)$ and $\bar{h} = (h_1, h_2, \ldots, h_n)$.

Definition 2 (see [30]). For $h > 0$ and $\mu \in R$, the rising h-polynomial factorial function is defined as

$$t_h^{[\mu]} = h^{\mu} \frac{\Gamma((t/h) + \mu)}{\Gamma(t/h)}, \tag{2}$$

where $t_h^{[0]} = 1$, Γ is the Euler gamma function, and $(t/h) + \mu$, $(t/h) \notin \{0, -1, -2, -3, \ldots\}$ as the division at a pole yields zero.

Lemma 1 (see [31]). *Let $h > 0$ and $u(t)$ and $w(t)$ be real-valued bounded functions. Then,*

$$\Delta_h^{-1}(u(t)w(t)) = u(t)\Delta_h^{-1} w(t) - \Delta_h^{-1}\left(\Delta_h^{-1} w(t+h)\Delta_h u(t)\right). \tag{3}$$

Theorem 1 (see [31]). *Let $t \in (0, \infty)$, $h > 0$, and $s > 0$; then,*

$$L_{h,\nu}\left(t_h^{(\mu)}\right) = \frac{h^{\mu+1}\mu! e^{s^{1/\nu}h}}{\left(e^{s^{1/\nu}h} - 1\right)^{\mu+1}}. \tag{4}$$

2.1. Notations

(i) $\underset{n(h)}{\Delta} u(\bar{t}) = \underset{h_n}{\Delta}\underset{h_{n-1}}{\Delta} \cdots \underset{h_2}{\Delta}\underset{h_1}{\Delta} u(t_1, t_2, \ldots, t_n)$

(ii) $\Delta_{n(h)}^{-1} u(\bar{t}) = \Delta_{h_n}^{-1}\Delta_{h_{n-1}}^{-1} \cdots \Delta_{h_2}^{-1}\Delta_{h_1}^{-1} u(t_1, t_2, \ldots, t_n)$

(iii) $\mathcal{J}(D_n)$ denotes the set of all subsets of $D_n = \{1, 2, \ldots, n\}$

(iv) $n(D_n - \bar{r})$ denotes the number of digits in $D_n - \bar{r}$

Definition 3 (infinite inverse principle law). For the function $u(\bar{t})$, we define the infinite inverse principle law as follows:

$$\Delta_{n(h)}^{-1}\Big|_{\bar{a}}^{\infty} = \left(\prod_{j=1}^{n} h_j\right)\left(\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} u(a_1 + r_1 h_1, a_2 + r_2 h_2, \ldots, a_n + r_2 h_2)\right), \tag{5}$$

where $\bar{a} = (a_1, a_2, \ldots, a_n)$. In particular, if $\bar{a} = (0, 0, \ldots, 0)$, we obtain

$$\Delta_{n(h)}^{-1}\Big|_{0}^{\infty} = \left(\prod_{j=1}^{n} h_j\right)\left(\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} u(r_1 h_1, r_2 h_2, \ldots, r_2 h_2)\right). \tag{6}$$

Theorem 2. *Let $\bar{t} \in R^n, h_j > 0, j = 1, 2, \ldots, n$; then,*

$$\Delta_{n(h)}^{-1} a^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j} = \frac{\left(\prod_{j=1}^{n} h_j\right) a^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}}{\prod_{j=1}^{n}\left(a^{-s_j^{1/\nu_j} h_j} - 1\right)}. \tag{7}$$

Proof. Taking $u(\bar{t}) = a^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}$ in (1) for the shift value h_1, we obtain

$$\underset{h_1}{\Delta} a^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j} = \frac{a^{-\left(s_1^{1/\nu_1}(t_1 + h_1) - \sum_{j=1}^{n} s_j^{1/\nu_j} t_j\right)} - a^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}}{h_1} = \frac{h_1 a^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}}{\left(a^{-s_1^{1/\nu_1} h_1} - 1\right)}. \tag{8}$$

In (8), applying $\Delta_{h_2}^{-1}$ on both sides, we get

$$\Delta_{h_2}^{-1}\Delta_{h_1}^{-1}a^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j} = \frac{h_1}{\left(a^{-s_1^{1/\nu_1}h_1}-1\right)}\Delta_{h_2}^{-1}a^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j}$$
$$= \frac{h_1}{\left(a^{-s_1^{1/\nu_1}h_1}-1\right)}\frac{h_2}{\left(a^{-s_2^{1/\nu_2}h_2}-1\right)}a^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j},$$
$$\Delta_{h_2}^{-1}\Delta_{h_1}^{-1}a^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j} = \frac{\left(\prod_{j=1}^{2}h_j\right)a^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j}}{\prod_{j=1}^{2}\left(a^{-s_j^{1/\nu_j}h_j}-1\right)}. \tag{9}$$

Proceeding like this for the induction on n, we get the proof of (7).

Corollary 1. *Let $\bar{t}\in R^n, h_j>0$ and $j=1,2,\ldots,n$. Then, we have*

$$\Delta_{n(h)}^{-1}e^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j} = \frac{\left(\prod_{j=1}^{n}h_j\right)e^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j}}{\prod_{j=1}^{n}\left(e^{-s_j^{1/\nu_j}h_j}-1\right)}. \tag{10}$$

Proof. In the proof of Theorem 2, replacing a by e, we get (10).

Corollary 2. *In Theorem 2, applying $a=3$, we get*

$$\Delta_{n(h)}^{-1}3^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j} = \frac{\left(\prod_{j=1}^{n}h_j\right)3^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j}}{\prod_{j=1}^{n}\left(3^{-s_j^{1/\nu_j}h_j}-1\right)}. \tag{11}$$

Example 1. Let $n=2$ in (11); we get the result for the shift values h_1 and h_2 as

$$\Delta_{h_2}^{-1}\Delta_{h_1}^{-1}3^{-\left(s_1^{1/\nu_1}t_1+s_2^{1/\nu_2}t_2\right)} = \frac{h_1h_2 3^{-\left(s_1^{1/\nu_1}t_1+s_2^{1/\nu_2}t_2\right)}}{\left(3^{-s_1^{1/\nu_1}h_1}-1\right)\left(3^{-s_2^{1/\nu_2}h_2}-1\right)}. \tag{12}$$

Summing from 0 to ∞ for t_1 and t_2 on both sides yields

$$\Delta_{h_2}^{-1}\Delta_{h_1}^{-1}3^{-\left(s_1^{1/\nu_1}t_1+s_2^{1/\nu_2}t_2\right)}\Big|_0^{\infty}\Big|_0^{\infty} = \frac{h_1h_2}{\left(3^{-s_1^{1/\nu_1}h_1}-1\right)\left(3^{-s_2^{1/\nu_2}h_2}-1\right)}. \tag{13}$$

For $n=2$, the infinite principle law reads

$$\Delta_{h_2}^{-1}\Delta_{h_1}^{-1}u(t_1,t_2)\Big|_0^{\infty}\Big|_0^{\infty} = h_1h_2\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}u(r_1h_1,r_2h_2). \tag{14}$$

Equating (13) and (14) for the function $u(t_1,t_2)=3^{-\left(s_1^{1/\nu_1}t_1+s_2^{1/\nu_2}t_2\right)}$, we obtain

$$h_1h_2\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}3^{-\left(s_1^{1/\nu_1}r_1h_1+s_2^{1/\nu_2}r_2h_2\right)} = \frac{h_1h_2}{\left(3^{-s_1^{1/\nu_1}h_1}-1\right)\left(3^{-s_2^{1/\nu_2}h_2}-1\right)}, \tag{15}$$

which is verified for the particular values $s_1=2, s_2=3, \nu_1=0.3, \nu_2=0.5, h_1=0.4$, and $h_2=0.7$ by MATLAB coding as follows: ((0.4). ∗ (0.7)). ∗ symsum (symsum (3.∧(− (2.∧(1./0.3). ∗ 0.4. ∗ $r1$ + 3.∧(1./0.5). ∗ 0.7. ∗ $r2$)), $r1$, 0, ∞), $r2$, 0, ∞)= (0.4. ∗ 0.7)/ ((3.∧(−(2.∧(1./ 0.3). ∗ 0.4)) − 1). ∗ (3.∧(−(3.∧(1./0.5). ∗ 0.7))− 1)) = 0.2837.

3. n-Dimensional Fractional Frequency Laplace Transform

Definition 4. For the function $u(\bar{t})$ with n-variables $t_1,t_2,\ldots,t_n$, the n-dimensional fractional frequency Laplace transform is defined as

$$\mathscr{L}_{n(h)}[u(\bar{t})] = U_n(\bar{s}) = \Delta_{n(h)}^{-1}u(\bar{t})e^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j}\Big|_{t_j=0,j=1,2,\ldots,n}^{\infty}$$
$$= \left(\prod_{j=1}^{n}h_j\right)\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}u(r_1h_1,r_2h_2,\ldots,r_nh_n)e^{-\sum_{j=1}^{n}s_j^{1/\nu_j}r_jh_j}. \tag{16}$$

Remark 1

(i) The n-dimensional fractional frequency Laplace transform satisfies the linear property.

(ii) In the aforesaid equation (16), we represent the Laplace transform of the functions in two ways: one in the closed-form solution and another one in the summation form solution. In this paper, we numerically verified and analyzed with MATLAB that both solutions are equal.

Theorem 3. *Let $\bar{t}\in R^n, \bar{h}>0$, ν_j be a fraction, and $s_j>0, j=1,2,\ldots,n$. Then, we have*

$$\mathscr{L}_{n(h)}[1] = \frac{\prod_{j=1}^{n}h_j}{\prod_{j=1}^{n}\left(e^{-s_j^{1/\nu_j}h_j}-1\right)}. \tag{17}$$

Proof. Taking $u(\bar{t})=1$ in (16) yields

$$\mathscr{L}_{n(h)}[1] = \Delta_{n(h)}^{-1} e^{-\sum\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}\Big|_{t_j=0, j=1,2,\ldots,n}^{\infty}$$
$$= \Delta_{h_n}^{-1} e^{-s_n^{1/\nu_n} t_n}\Big|_0^{\infty} \cdots \Delta_{h_2}^{-1} e^{-s_2^{1/\nu_2} t_2}\Big|_0^{\infty} \Delta_{h_1}^{-1} e^{-s_1^{1/\nu_1} t_1}\Big|_0^{\infty}$$
$$= \frac{h_n}{\left(e^{-s_n^{1/\nu_n} h_n} - 1\right)} \cdots \frac{h_2}{\left(e^{-s_2^{1/\nu_2} h_2} - 1\right)} \frac{h_1}{\left(e^{-s_1^{1/\nu_1} h_1} - 1\right)}, \tag{18}$$

(using Corollary 1), which completes the proof.

Theorem 4. *Let $\bar{t} \in R^n, \bar{h} > 0$, ν_j be a fraction, and $s_j > 0$, $j = 1, 2, \ldots, n$; then,*

$$\mathscr{L}_{n(h)}\left[e^{\sum_{j=1}^{n} a_j t_j}\right] = \frac{\prod_{j=1}^{n} h_j}{\prod_{j=1}^{n}\left(e^{-\left(s_j^{1/\nu_j} - a_j\right) h_j} - 1\right)}. \tag{19}$$

Proof. Taking $u(\bar{t}) = e^{\sum_{j=1}^{n} a_j t_j}$ in (16), we have

$$\mathscr{L}_{n(h)}\left[e^{\sum_{j=1}^{n} a_j t_j}\right] = \Delta_{n(h)}^{-1} e^{\sum_{j=1}^{n} a_j t_j} e^{-\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}\Big|_{t_j=0, j=1,2,\ldots,n}^{\infty}$$
$$= \Delta_{h_n}^{-1} e^{-\left(s_n^{1/\nu_n} - a_n\right) t_n}\Big|_0^{\infty} \cdots \Delta_{h_2}^{-1} e^{-\left(s_2^{1/\nu_2} - a_2\right) t_2}\Big|_0^{\infty} \Delta_{h_1}^{-1} e^{-\left(s_1^{1/\nu_1} - a_1\right) t_1}\Big|_0^{\infty} \tag{20}$$
$$= \frac{h_n}{\left(e^{-\left(s_n^{1/\nu_n} - a_n\right) h_n} - 1\right)} \cdots \frac{h_2}{\left(e^{-\left(s_2^{1/\nu_2} - a_2\right) h_2} - 1\right)} \frac{h_1}{\left(e^{-\left(s_1^{1/\nu_1} - a_1\right) h_1} - 1\right)},$$

which gives (19).

Example 2. For $n = 2$, the summation solution of the exponential function given by the infinite inverse principle law and the closed form of the solution given as

$$\mathscr{L}_{n(h)}\left[e^{a_1 t_1 + a_2 t_2}\right] = h_1 h_2 \sum_{r_2=0}^{\infty}\sum_{r_1=0}^{\infty} e^{-\left[\left(s_1^{1/\nu_1} - a_1\right) r_1 h_1 + \left(s_2^{1/\nu_2} - a_2\right) r_2 h_2\right]}$$
$$= \frac{h_2}{\left(e^{-\left(s_2^{1/\nu_2} - a_2\right) h_2} - 1\right)} \frac{h_1}{\left(e^{-\left(s_1^{1/\nu_1} - a_1\right) h_1} - 1\right)}, \tag{21}$$

is numerically verified for the particular values $h_1 = 7, h_2 = 3, a_1 = 5, a_2 = 9, \nu_1 = 0.1, \nu_2 = 0.3, s_1 = 11$, and $s_2 = 13$ by MATLAB coding as follows: 21. ∗ (symsum (symsum (exp (− (11.∧ (1./1.0) − 5). ∗ 7. ∗ r1 + (13.∧ (1./0.3) − 9. ∗ 3. ∗ r2)), r1, 0, 10), r2, 0, 10))= 21./ ((exp (− (11.∧(1./0.1) −5). ∗ 7) − 1). ∗ (exp (− (13.∧ (1./0.3) −9). ∗ 3) − 1)).

The following are the graphical representations of the exponential function in time and frequency domains. Figure 1 is the graphical representation of the input function $e^{5t_1+9t_2}$ in the time domain, and Figure 2 is the graphical representation of the output in the frequency domain for the particular values of $\nu_1 = 0.0001, \nu_2 = 0.0003$. One can easily choose the values of fraction $\nu_i' s$ to get the output in the frequency domain.

Theorem 5. *Let $\bar{t} \in R^n, \bar{h} > 0$, ν_j be a fraction, and $s_j > 0$, $j = 1, 2, \ldots, n$; then,*

$$\mathscr{L}_{n(h)}\left[e^{i\sum_{j=1}^{n} a_j t_j}\right] = \frac{\prod_{j=1}^{n} h_j \sum_{\bar{r} \in \mathscr{J}(D_n)} (-1)^{n(D_n - \bar{r})} e_{-i(a_{\bar{r}} h_{\bar{r}})} e_{s_{D_n - \bar{r}} h_{D_n - \bar{r}}}}{2^n \left[\prod_{j=1}^{n}\left(\cosh s_j^{1/\nu_j} h_j - \cos a_j h_j\right)\right]}, \tag{22}$$

where $e_{-i(a_{\bar{r}} h_{\bar{r}})} = e^{-i(a_1 h_1 + a_2 h_2 + \cdots + a_n h_n)}$ for $\bar{r} = \{1, 2, \ldots, n\}$, $e_{s_{D_n - \bar{r}} h_{D_n - \bar{r}}} = e^{s_1^{1/\nu_1} h_1 + s_2^{1/\nu_2} h_2 + \cdots + s_n^{1/\nu_n} h_n}$ for $D_n - \bar{r} = \{1, 2, \ldots, n\}$, and $e_{-i(a_0 h_0)} = e_{s_0 h_0} = 1$.

Proof. From the previous theorem, we obtain the Laplace transform for the trigonometric function $e^{i\sum_{j=1}^{n} a_j t_j}$ as

$$\mathscr{L}_{n(h)}\left[e^{i\sum_{j=1}^{n} a_j t_j}\right] = \frac{\prod_{j=1}^{n} h_j}{\prod_{j=1}^{n}\left(e^{-\left(s_j^{1/\nu_j} - i a_j\right) h_j} - 1\right)}. \tag{23}$$

The proof then can be continued by making use of the conjugate and the product of each term in n-variables.

Theorem 6. *Let $\bar{t} \in R^n, \bar{h} > 0$, ν_j be a fraction, and $s_j > 0$, $j = 1, 2, \ldots, n$; then,*

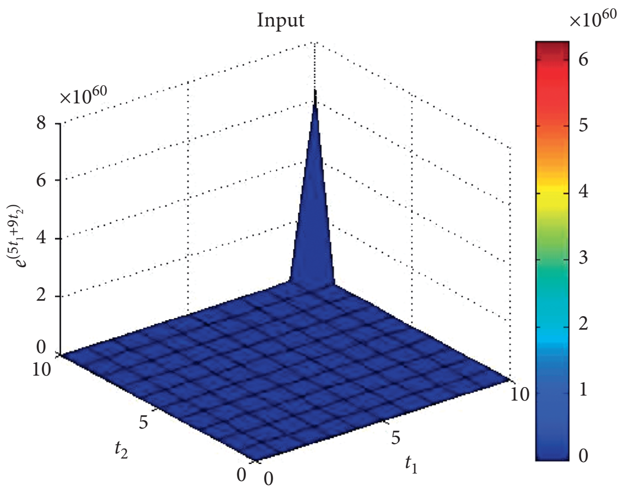

Figure 1: Time signal (function) $e^{5t_1+9t_2}$.

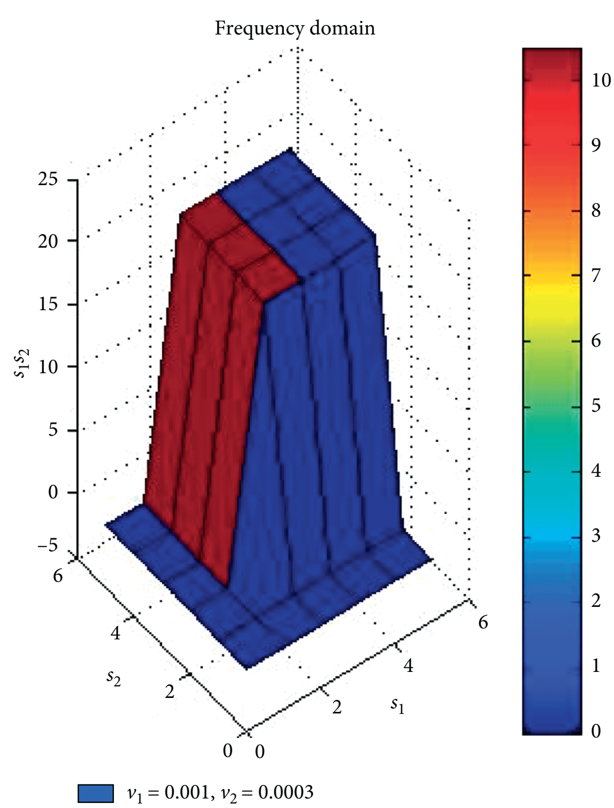

Figure 2: Frequency signal for $v_1 = 0.0001$ and $v_2 = 0.0003$.

$$\mathscr{L}_{2(h)}\left[\sin\left(\sum_{j=1}^{2}a_jt_j\right)\right]=\frac{h_1h_2\left(e^{s_2^{1/\nu_2}h_2}\sin a_1h_1+e^{s_1^{1/\nu_1}h_1}\sin a_2h_2-\sin\left(a_1h_1+a_2h_2\right)\right)}{4\left(\prod_{j=1}^{2}\left(\cosh s_j^{1/\nu_j}h_j-\cos a_jh_j\right)\right)},\tag{24}$$

$$\mathscr{L}_{2(h)}\left[\cos\left(\sum_{j=1}^{2}a_jt_j\right)\right]=\frac{h_1h_2\left(\cos\left(\sum_{j=1}^{2}a_jt_j\right)-e^{s_2^{1/\nu_2}h_2}\cos a_1h_1-e^{s_1^{1/\nu_1}h_1}\cos a_2h_2+e^{\sum_{j=1}^{2}s_j^{1/\nu_j}h_j}\right)}{4\left(\prod_{j=1}^{2}\left(\cosh s_j^{1/\nu_j}h_j-\cos a_jh_j\right)\right)}.\tag{25}$$

Proof. The proof follows by taking $n=2$ in Theorem 5, making the product by its conjugate terms and separating the real and imaginary parts to get the double Laplace transform for the sine and cosine terms.

Example 3. Equation (24) is the closed-form solution of the sine function. Now, for $n=2$, the summation solution of the sine function given by the infinite inverse principle law is

$$\mathscr{L}_{2(h)}\left[\sin\left(\sum_{j=1}^{2}a_jt_j\right)\right]=h_1h_2\sum_{r_2=0}^{\infty}\sum_{r_1=0}^{\infty}\sin\left(a_1r_1h_1+a_2r_2h_2\right)e^{-\left(s_1^{1/\nu_1}r_1h_1+s_2^{1/\nu_2}r_2h_2\right)}.\tag{26}$$

Now, (24) and (26) are numerically verified for the particular values $h_1=2, h_2=3, a_1=1, a_2=4, \nu_1=0.4$, $\nu_2=0.6, s_1=5$, and $s_2=6$ by MATLAB coding as follows: 6. ∗ symsum (symsum (sin (2. ∗ r1 + 12. ∗ r2). ∗ exp (− (5.∧ (1./0.4). ∗ 2. ∗ r1 + 6.∧ (1./0.6). ∗ 3. ∗ r2)), r1, 0, 10), r2, 0, ∞) = (6. ∗ (exp (6.∧ (1./0.6). ∗ 3). ∗ sin (2) − exp ((5.∧ (1./ 0.4). ∗ 2). ∗ sin (12) − sin (14)))/ (4. ∗ (cosh (6.∧ (1./ 0.6). ∗ 3) − cos (12)). ∗ ((cosh (5.∧ (1./0.4)). ∗ 2) − cos (2)))).

The following are the graphical representations of the sine function in time and frequency domains. Figure 3 represents the input time-domain signal (function) for the sine function. Figure 4 represents the output in the frequency domain for the particular values of $\nu_1=0.4$ and $\nu_2=0.6$. Figure 5 represents the output in the frequency domain for the particular values of $\nu_1=0.3$ and $\nu_2=0.5$. Figure 6 represents the output in the frequency domain for the particular values of $\nu_1=0.1$ and $\nu_2=0.7$. Similarly, one can analyze the solution in the frequency domain by choosing diverse values of fraction ν_i's.

Theorem 7. *Let $\bar{t}\in R^n, \bar{h}>0$, ν_j be a fraction, and $s_j,\mu_j>0$, $j=1,2,\ldots,n$; then,*

$$\mathscr{L}_{n(h)}\left[\prod_{j=1}^{n}\left(t_j\right)_{h_j}^{(\mu_j)}\right]=\prod_{j=1}^{n}\frac{h_j^{\mu_j+1}\mu_j!e^{s_j^{1/\nu_j}h_j}}{\left(e^{-s_j^{1/\nu_j}}-1\right)}.\tag{27}$$

Proof. Taking $u(\bar{t})=\prod_{j=1}^{n}(t_j)_{h_j}^{(\mu_j)}$ in (16), we have

$$\begin{aligned}\mathscr{L}_{n(h)}\left[\prod_{j=1}^{n}\left(t_j\right)_{h_j}^{(\mu_j)}\right]&=\Delta_{n(h)}^{-1}\prod_{j=1}^{n}\left(t_j\right)_{h_j}^{(\mu_j)}e^{-\sum_{j=1}^{n}s_j^{1/\nu_j}t_j}\Big|_{t_j=0,j=1,2,\ldots,n}^{\infty}\\&=\Delta_{h_n}^{-1}\left(t_n\right)_{h_n}^{(\mu_n)}e^{-s_n^{1/\nu_n}t_n}\Big|_0^{\infty}\cdots\Delta_{h_2}^{-1}\left(t_2\right)_{h_2}^{(\mu_2)}e^{-s_2^{1/\nu_2}t_2}\Big|_0^{\infty}\Delta_{h_1}^{-1}\left(t_1\right)_{h_1}^{(\mu_1)}e^{-s_1^{1/\nu_1}t_1}\Big|_0^{\infty}\\&=\frac{h_n^{\mu_n+1}\mu_n!e^{s_n^{1/\nu_n}h_n}}{\left(e^{-(s_n^{1/\nu_n}}-1\right)}\cdots\frac{h_2^{\mu_2+1}\mu_2!e^{s_2^{1/\nu_2}h_2}}{\left(e^{-(s_2^{1/\nu_2}}-1\right)}\frac{h_1^{\mu_1+1}\mu_1!e^{s_1^{1/\nu_1}h_1}}{\left(e^{-(s_1^{1/\nu_1}}-1\right)},\end{aligned}\tag{28}$$

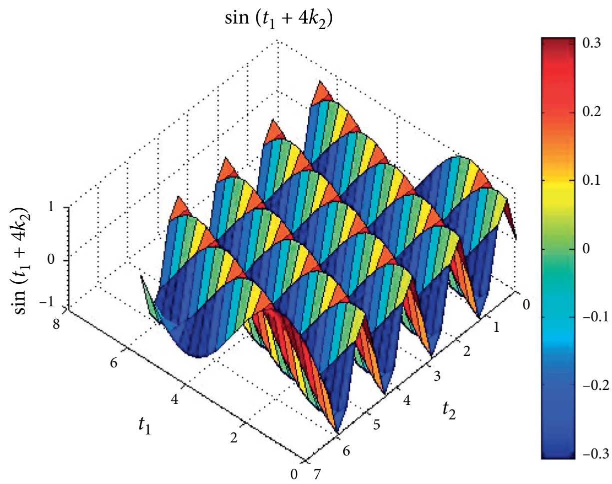

Figure 3: Time signal for $\sin(t_1 + 4t_2)$.

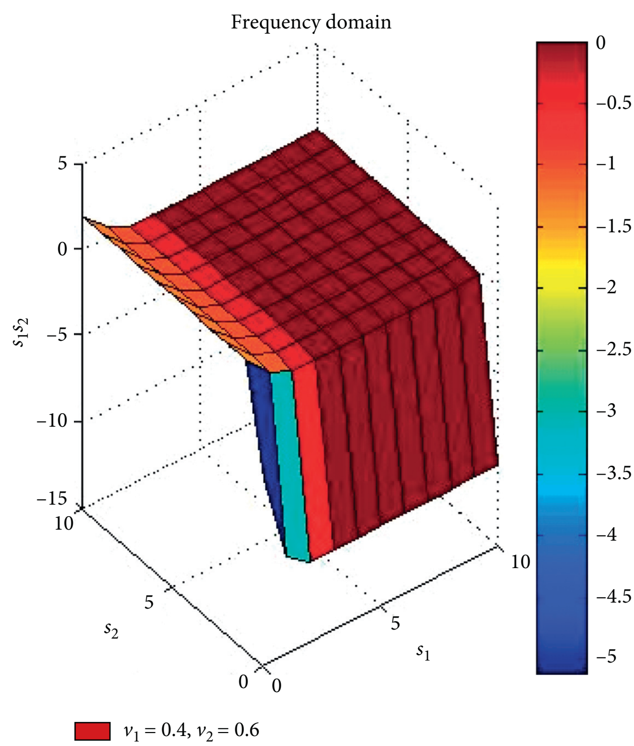

Figure 4: Frequency signal for $\nu_1 = 0.4$ and $\nu_2 = 0.6$.

by Lemma 1, which gives (4).

3.1. n-Kind Riemann Zeta Function in the Discrete Case. In Theorem 7, when $\nu_j = 1$ and $h_j \longrightarrow 0$ for $j = 1, 2, \dots, n.$, we get

$$\mathscr{L}_{n(h)}\left[t_1^{\mu_1-1}t_2^{\mu_2-1}\dots t_n^{\mu_n-1}\right] = \frac{\Gamma(\mu_n)\dots\Gamma(\mu_2)\Gamma(\mu_1)}{s_n^{\mu_n}\dots s_2^{\mu_2}s_1^{\mu_1}}. \tag{29}$$

We know that the Riemann zeta function is defined as

$$\zeta(\mu) = \sum_{s=1}^{\infty}\frac{1}{s^{\mu}}. \tag{30}$$

Equation (29) can be written as

$$\frac{\Gamma(\mu_n)\dots\Gamma(\mu_2)\Gamma(\mu_1)}{s_n^{\mu_n}\dots s_2^{\mu_2}s_1^{\mu_1}} = \Delta_{h_n}^{-1}t_n^{\mu_n-1}e^{-s_n t_n}\Big|_0^{\infty}\dots\Delta_{h_2}^{-1}t_2^{\mu_2-1}e^{-s_2 t_2}\Big|_0^{\infty} \cdot \Delta_{h_1}^{-1}t_1^{\mu_1-1}e^{-s_1 t_1}\Big|_0^{\infty}. \tag{31}$$

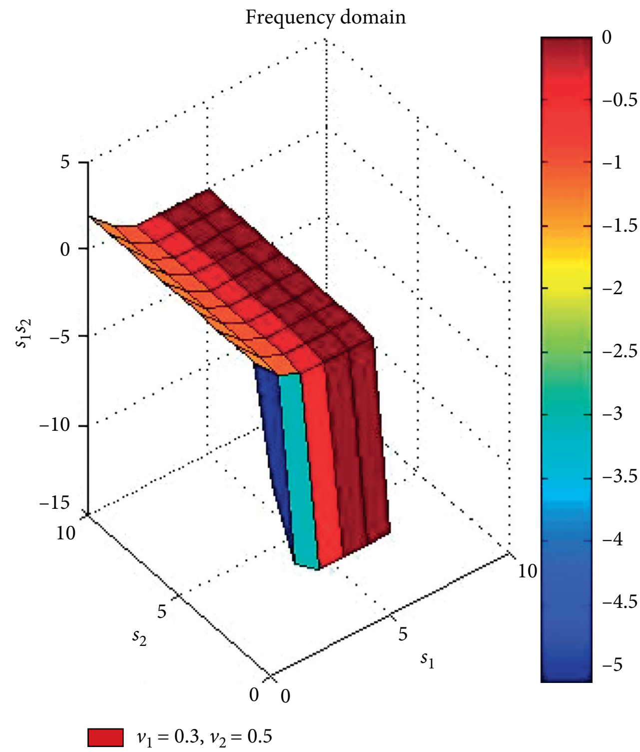

FIGURE 5: Frequency signal for $\nu_1 = 0.3$ and $\nu_2 = 0.5$.

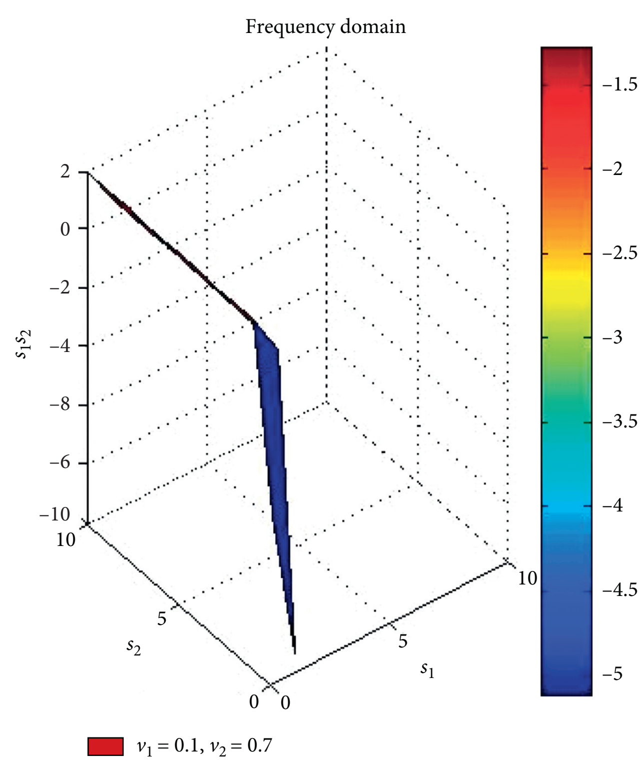

FIGURE 6: Frequency signal for $\nu_1 = 0.1$ and $\nu_2 = 0.7$.

Taking summation on s_j, $j = 1, 2, \ldots, \infty$, on both sides, we get

$$\begin{aligned}
&\prod_{j=1}^{n}\Gamma(\mu_j)\sum_{s_n=1}^{\infty}\frac{1}{s_n^{\mu_n}}\cdots\sum_{s_2=1}^{\infty}\frac{1}{s_2^{\mu_2}}\sum_{s_1=1}^{\infty}\frac{1}{s_1^{\mu_1}} = \Delta_{h_n}^{-1}t_n^{\mu_n-1}\sum_{s_n=1}^{\infty}e^{-s_n t_n}\Big|_0^{\infty}\cdots\Delta_{h_2}^{-1}t_2^{\mu_2-1}\sum_{s_2=1}^{\infty}e^{-s_2 t_2}\Big|_0^{\infty}\Delta_{h_1}^{-1}t_1^{\mu_1-1}\sum_{s_1=1}^{\infty}e^{-s_1 t_1}\Big|_0^{\infty},\\
&\prod_{j=1}^{n}\Gamma(\mu_j)\zeta(\mu_n)\cdots\zeta(\mu_2)\zeta(\mu_1) = \Delta_{h_n}^{-1}t_n^{\mu_n-1}\frac{1}{(e^{t_n}-1)}\Big|_0^{\infty}\cdots\Delta_{h_2}^{-1}t_2^{\mu_2-1}\frac{1}{(e^{t_2}-1)}\Big|_0^{\infty}\Delta_{h_1}^{-1}t_1^{\mu_1-1}\frac{1}{(e^{t_1}-1)}\Big|_0^{\infty},\\
&\zeta(\mu_n)\cdots\zeta(\mu_2)\zeta(\mu_1) = \frac{1}{\Gamma(\mu_n)}\Delta_{h_n}^{-1}\frac{t_n^{\mu_n-1}}{(e^{t_n}-1)}\Big|_0^{\infty}\cdots\frac{1}{\Gamma(\mu_2)}\Delta_{h_2}^{-1}\frac{t_2^{\mu_2-1}}{(e^{t_2}-1)}\Big|_0^{\infty}\frac{1}{\Gamma(\mu_1)}\Delta_{h_1}^{-1}\frac{t_1^{\mu_1-1}}{(e^{t_1}-1)}\Big|_0^{\infty},
\end{aligned}\tag{32}$$

which is the product of n^{th}-kind Riemann zeta function in the discrete case.

3.2. One-Dimensional Laplace Transform on the Fractional Difference Equation. Let $u(t)$ and $v(t)$ be the two functions. The Leibniz rule of noninteger order is $\overset{\nu}{\Delta}[u(t)v(t)] = \sum_{r=0}^{\infty}\binom{\nu}{r}\overset{\nu-r}{\Delta}u(t)\overset{r}{\Delta}v(t+\nu-r)$. Here, we present the product formula on the fractional difference operator as $\Delta_h^{\nu}[u(t)v(t)] = \sum_{r=0}^{\infty}\binom{\nu}{r}\Delta_h^{\nu-r}u(t)\Delta_h^{r}v(t+(\nu-r)h)$.

The following theorem plays an important role in solving the fractional difference equation by one-dimensional Laplace transform.

Theorem 8. *Let $u(t_1)$ be a real-valued function and $s_1, h_1, \nu_1 > 0$. Then, we have*

$$L_{h_1}\left[\Delta_{h_1}^{\nu}u(t)\right] = \frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)^{\nu_1}}{h_1^{\nu_1}}L_{h_1}\left[u(t_1+\nu_1 h_1)\right] - \sum_{r_1=1}^{\infty}\frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)^{\nu_1-r_1}}{h_1^{\nu_1-r_1}}\Delta_{h_1}^{r_1-1}u((\nu_1-r_1)h_1).\tag{33}$$

Proof. Taking $u(t_1) = \underset{h_1}{\Delta}u(t_1)$ in (16), we get $L_{h_1}[\underset{h_1}{\Delta}u(t_1)] = \Delta_{h_1}^{-1}[\underset{h_1}{\Delta}u(t_1)e^{-s_1^{h_1/\nu_1}t_1}]\big|_0^{\infty}$. Now, applying (3) and solving, we get

$$L_{h_1}\left[\underset{h_1}{\Delta}u(t_1)\right] = \frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)}{h_1}L_{h_1}\left[u(t_1+h_1)\right] - u(0).\tag{34}$$

Again taking $u(t_1) = \Delta_{h_1}^{2}u(t_1)$, using (3) and (16), and applying (34) give

$$L_{h_1}\left[\Delta_{h_1}^{2}u(t_1)\right] = \frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)^{2}}{h_1^{2}}L_{h_1}\left[u(t_1+2h_1)\right] - \frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)}{h_1}u(h_1) - \underset{h_1}{\Delta}u(0).\tag{35}$$

Continuing this process for integer n, we arrive at

$$L_{h_1}\left[\Delta_{h_1}^{n}u(t_1)\right] = \frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)^{n}}{h_1^{n}}L_{h_1}\left[u(t_1+nh_1)\right] - \sum_{r_1=1}^{n}\frac{\left(1-e^{-s_1^{1/\nu_1}h_1}\right)^{n-r_1}}{h_1^{n-r_1}}\Delta_{h_1}^{r_1-1}u((n-r_1)h_1).\tag{36}$$

Since the order is a fraction, we consider (36) for fraction ν as mentioned in (33).

4. n-Dimensional Inverse Laplace Transform

The n-dimensional inverse Laplace transform is defined by

$$\mathscr{L}_{n(h)}^{-1}\left[U_n(\bar{s})\right] = u(\bar{t}) = \Delta_{n(h)}^{-1} U_n(\bar{s}) e^{\sum_{j=1}^{n} s_j^{1/\nu_j} t_j}\Big|_{c_j - i\infty}^{c_j + i\infty}, \quad j = 1, 2, \ldots, n. \tag{37}$$

Since we can easily represent the n-dimensional Laplace transform of the functions mentioned, we can present some results listed as follows:

$$\begin{aligned}
&\mathscr{L}_{n(h)}^{-1}\left[\frac{\prod_{j=1}^{n} h_j}{\prod_{j=1}^{n}\left(e^{-s_j^{1/\nu_j} h_j} - 1\right)}\right] = 1,\\
&\mathscr{L}_{n(h)}^{-1}\left[\frac{\prod_{j=1}^{n} h_j}{\prod_{j=1}^{n}\left(e^{-\left(s_j^{1/\nu_j} - a_j\right) h_j} - 1\right)}\right] = e^{\sum_{j=1}^{n} a_j t_j},\\
&\mathscr{L}_{n(h)}^{-1}\left[\frac{\prod_{j=1}^{n} h_j \sum_{\bar{r}\in\mathscr{J}(D_n)} (-1)^{n(D_n - \bar{r})} e_{-i(a_{\bar{r}} h_{\bar{r}})} e_{s_{D_n-\bar{r}} h_{D_n-\bar{r}}}}{2^n\left[\prod_{j=1}^{n}\left(\cosh s_j^{1/\nu_j} h_j - \cos a_j h_j\right)\right]}\right] = e^{i\sum_{j=1}^{n} a_j t_j},\\
&\mathscr{L}_{n(h)}^{-1}\left[\prod_{j=1}^{n} \frac{h_j^{\mu_j+1} \mu_j! e^{s_j^{1/\nu_j} h_j}}{\left(e^{-\left(s_j^{1/\nu_j}\right.} - 1\right)}\right] = \prod_{j=1}^{n} \left(t_j\right)_{h_j}^{(\mu_j)},\\
&\mathscr{L}_{n(h)}^{-1}\left[\frac{\Gamma(\mu_n)\ldots\Gamma(\mu_2)\Gamma(\mu_1)}{s_n^{\mu_n}\ldots s_2^{\mu_2} s_1^{\mu_1}}\right] = t_1^{\mu_1 - 1} t_2^{\mu_2 - 1} \ldots t_n^{\mu_n - 1}.
\end{aligned} \tag{38}$$

5. Results and Discussion

When $n = 2$, $\nu_1 = \nu_2 = 1$, and $h_1, h_2 \longrightarrow 0$, in all the above results, we have

(i) $\mathscr{L}_2[1] = 1/s_1 s_2$.

(ii) $\mathscr{L}_2[e^{a_1 t_1 + a_2 t_2}] = 1/(s_1 - a_1)(s_2 - a_2)$.

(iii) $\mathscr{L}_2[\sin(a_1 t_1 + a_2 t_2)] = a_1 a_2/(s_1^2 + a_1^2)(s_2^2 + a_2^2)$.

(iv) $\mathscr{L}_2[\cos(a_1 t_1 + a_2 t_2)] = s_1 s_2/(s_1^2 + a_1^2)(s_2^2 + a_2^2)$.

Similarly, the following result for the hyperbolic functions can be obtained:

(i) $\mathscr{L}_2[t_1^{\mu_1} t_2^{\mu_2}] = \mu_1! \mu_2! / s_1^{\mu_1+1} s_2^{\mu_2+1}$.

These results match the formulas of the double Laplace transform of functions available in the literature.

6. Conclusion

The fractional frequency is used to derive the n-dimensional Laplace transform with shift values h_j, $j = 1, 2, \ldots, n$, that presents more accuracy outputs of the input functions such as exponential, polynomial factorial, polynomial, and trigonometric functions. Also, the numerical results and the solutions are analyzed graphically by MATLAB. The major application of this research work is also provided by considering the classical Laplace transform according to particular values of n which are $\nu_j = 1$ and $h_j \longrightarrow 0$, $j = 1, 2, \ldots, n$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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