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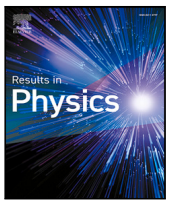

New numerical dynamics of the heroin epidemic model using a fractional derivative with Mittag-Leffler kernel and consequences for control mechanisms

Saima Rashid [a,*], Fahd Jarad [b,c,d,*], Abdulaziz Garba Ahmad [e], Khadijah M. Abualnaja [f]

[a] Department of Mathematics, Government College University, Faisalabad 38000, Pakistan
[b] Department of Mathematics, Cankaya University, Ankara, Turkey
[c] Department of Mathematics, King Abdulaziz University, Jaddeh Saudi Arabia
[d] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
[e] Department of Mathematics, National Mathematical Centre Abuja, Abuja 900211, Nigeria
[f] Department of Mathematics, Faculty of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

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ABSTRACT

Intravenous substance consumption is on the upswing all over the globe, especially in Europe and Asia. It is extremely harmful to society; excessive substance consumption is the leading cause of death. Beyond all prohibited narcotics, heroin is a narcotic that has a substantial negative impact on society and the world at large. In this paper, a heroin epidemic model is developed via an Atangana–Baleanu fractional-order derivative in the Caputo sense describe accurately real world problems, equipped with recovery and persistent immunity. Meanwhile, we have established a globally asymptotically stable equilibrium for both the drug-free and drug-addiction equilibriums. Additionally, we apply a novel scheme that is mingled with the two-step Lagrange polynomial and the basic principle of fractional calculus. The simulation results for various fractional values indicate that as the fractional order decreases from 1, the growth of the epidemic diminishes. The modelling data demonstrates that the suggested containment technique is effective in minimizing the incidence of instances in various categories. Furthermore, modelling the ideal configuration indicated that lowering the fractional-order from 1 necessitates a swift commencement of the implementation of the suggested regulatory technique at the maximum rate and sustaining it throughout a significant proportion of the pandemic time frame.

Introduction

The growing consumption of narcotics as well as similar harmful narcotics is a serious challenge. Heroin addiction has an impact not only on the broader majority's standard of living, but increasingly on the entire scenario of global harmony and financial growth [1–3]. According to statistics from the World Drug Report presented by the United Nations (U.N), 35 million individuals are suffering from severe narcotic misuse abnormalities, with barely one-seventh undergoing rehabilitation [4]. Furthermore, it was revealed that people all over the world abused drugs in 2017, with 3 million people infected with HIV and 6.2 million people infected with the severe hepatitis B virus. The metrics records show that the negative physiological impacts of drug consumption are more significant and pervasive than formerly estimated, and it is critical to regulate the predominance of dangerous substances.

Heroin is manufactured by opioids, often termed "opium", which are formed from the papaver somniferum. Pure heroin is a white granular or white crystallographic particle, (see, Fig. 1). Prolonged heroin intake and transfusion are widely documented to entail behavioural erosion, contemporary perspectives, and a reduction in longevity. There is an increasing proportion of documented heroin addicts, and this figure is expanding [5,6]. Because of its global proliferation, heroin misuse and consumption have placed enormous strains on the worldwide health sector. Because the dissemination of heroin is incurable, there is a growing tendency to examine heroin propagation from the viewpoint of communicable epidemic mechanisms [7,8]. In 2007, White and Comiskey [9] developed an ordinary differential equations (ODEs) framework for heroin contagious illnesses. They examined the consequences, employing $\mathcal{R}_0$ criterion, and discovered that deterrence is

* Corresponding authors.
E-mail addresses: saimarashid@gcuf.edu.pk (S. Rashid), fahd@cankaya.edu.tr (F. Jarad), agarbaahmad@yahoo.com (A.G. Ahmad), Kh.abualnaja@tu.edu.sa (K.M. Abualnaja).

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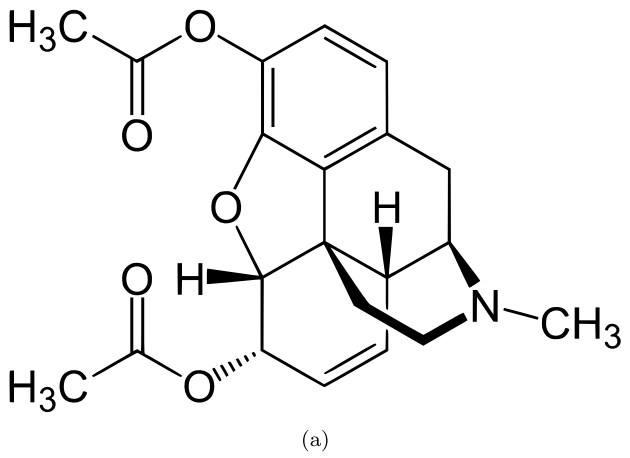

(a)

Fig. 1. Overview of heroin's molecular formula, see (e.g.,[15]).

preferable to therapy. Mulone and Straughan [10] revisited this system in 2009, and the researchers determined the robustness of the systems optimistic equilibria state applying the eigenvalue expression and Poincare–Bendixson hypothesis. Wang et al. [11] employed the bilinear rule occurrence parameter rather than regular incident in 2011, and they often examined the heroin framework transient characteristics. Several more other infectious outbreak methodologies have been introduced and analysed in numerous strategies in an attempt to investigate the determinants of heroin's highly contagious illnesses [12–14].

Many people who were detoxing were hesitant to leave, and many others stayed in contact with people who used narcotics despite having an effective prescription medication, putting them at risk of relapsing substances. As a result, many researchers studied recidivism in the narcotic system as well as its permanence [16–18]. Nonetheless, some heroin addicts have also experienced the negative consequences of illicit substances and discovered the contentment of becoming a regular individual after a fruitful rehabilitation service. Such individuals will be far removed from medications, so people are hopeful that they will resume these healthy practises for the rest of their lives. Furthermore, there are certain individuals who are highly privileged from infancy, reside in a stable setting, and have high self-esteem.

As a result, individuals do not use narcotics from inception to completion. These two classes of individuals are said to be "irreparably immunized" towards medications. Consumers describe a tremendous surge, an abrupt and sublime feeling of euphoria that happens in the nervous system as morphine sulphate is converted into 6-monoacetylmorphine (6-MAM) and heroin, see (e.g., Fig. 2 (a)). Frequent opium consumption alters the visual cortex's physical form and composition, resulting in prolonged abnormalities in neurological and metabolic processes that are difficult to restore. According to research, heroin consumption induces significant degradation of the brain's neural tissue, which might impede judgment capabilities, the power to control behaviour, and reactions to challenging scenarios. Heroin also causes extreme sensitivity and opiate dependence, see (e.g., Fig. 2 (b)).

It has become widely accepted in recent times that spatially and ecological variability have a significant influence on the maintenance and termination of infectious illnesses [19,20]. Because the dispersion pattern of the receptive or affected individual is heterogeneous in the heroin equation system and the concentration can alter at any moment and place, it is generally appropriate to implement the reaction diffusion equations to represent the progression of substance abusers. Furthermore, to the aim of contributing, heroin epidemic disorder systems wherein relative abundance is affected by combining spatial and temporal factors have received far less attention.

A significant innovation has recently been implemented in the applicability domain of fractional calculus (FC), in which novel derivative and integral operators having non-singular and non-local kernels are employed [21–29]. The innovative component proposed uses the generalized Mittag-Leffler function (MLF) including the cornerstone, and the characteristics of this mechanism aggravate the novel formulations to attain several supplemental fascinating characteristics that are identified in serious eventualities, such as mean square deformation interphase and expanding variations. Since Atangana and Baleanu [30] proposed it in 2016, that revolutionary fractional derivative operator has been extensively employed in several domains of science and technology. It was proved that simulation employing the AB-fractional derivative results in a chaotic system for a short period. It has subsequently been discovered that the MLF is a more effective and vital screening mechanism than the power and exponential laws, making the AB-fractional derivative, in the context of Caputo, an efficient arithmetic technique for simulating increasingly intricate critical challenges. Because of their broad implications, such formulations are widely recognized for producing fractional DEs with no contrived anomalies, as in the case of the Riemann–Liouville and Caputo derivatives, due to their inherent non-orientation [31–35]. We have also noticed a surge of curiosity among these operators on the subject of mathematical methods. However, mathematically approximating these derivatives results in a variety of computing issues, see [36,37].

Influenced by the aforesaid explanations, in this research, we establish a heroin framework, including recidivism and persistent immunity, and thereafter investigate their global patterns. To create the fractional derivative framework, which was constructed to assess their numerical results, the newly introduced ABC-fractional derivative and the Toufik–Atangana mathematical formulation [38] are employed. To the extent of the researchers' understanding, no one has analysed the heroin epidemic paradigm by applying the ABC-fractional derivative. Furthermore, the researchers contend that robust regulation evaluation of computational systems from the perspective of ABC-fractional formulations is infrequent in the relevant research. The remainder of this presentation is divided into several parts. In Section "Model configuration and formulation", we extend the model's specification and description. In Section "Global behaviour of the heroin epidemic model", we identify the presence and novelty of the proposed system's equilibrium, as well as its optimism and stability in the context of the AB-fractional operator. Section "Mathematical significance of fractional heroin epidemic model" discusses the drug-free and epidemic equilibria, as well as the associated global consistency analyses. Section "Numerical approaches and simulations" describes the approximate results of the heroin epidemic model employing the numerical method and modelling debated in terms of fractional orders and variation in parameters. In the last section, we speculate on the concluding remarks.

Model configuration and formulation

Let us recall the fundamental concepts of ABC-fractional derivative operators and their related consequences.

Definition 1 (*[30]*)**.** Consider $\mathbf{f} \in \mathbb{C}^1(\mathbf{a},\mathbf{b})$, $\mathbf{b} > \mathbf{a}$, be a mapping and $0 \le \wp \le 1$. Then, AB-fractional derivative in Caputo perspectives is presented as follows:

$$ {}^{ABC}_{\mathbf{a}}\mathbf{D}^{\wp}_{\xi}\mathbf{f}(\xi) = \frac{\mathbf{AB}(\wp)}{1-\wp}\int_{\mathbf{a}}^{\xi}\frac{d\mathbf{f}}{d\mathbf{x}}E_{\wp}\Big(-\frac{\wp}{1-\wp}(\xi-\mathbf{x})^{\wp}\Big)d\mathbf{x}, \tag{1} $$

where $\mathbf{AB}(\wp) = 1-\wp+\wp/\Gamma(\wp)$ represents the normalization function satisfying $\mathbf{AB}(0) = \mathbf{AB}(1) = 1$ and $E_{\wp}(\mathbf{z})$ indicates the MLF defined as

$$ E_{\wp}(\mathbf{z}) = \sum_{\delta=0}^{\infty}\frac{\mathbf{z}^{\delta}}{1+\wp\delta}, \quad \wp,\delta \in \mathbb{C},\ \Re(\wp) > 0. \tag{2} $$

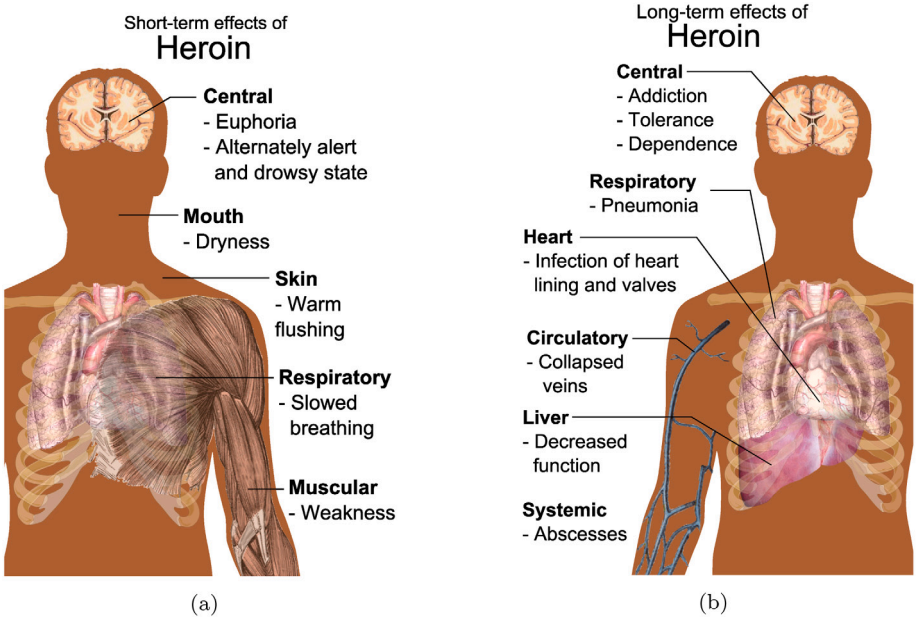

Fig. 2. Short-term and long term impacts of heroin on human body, respectively.

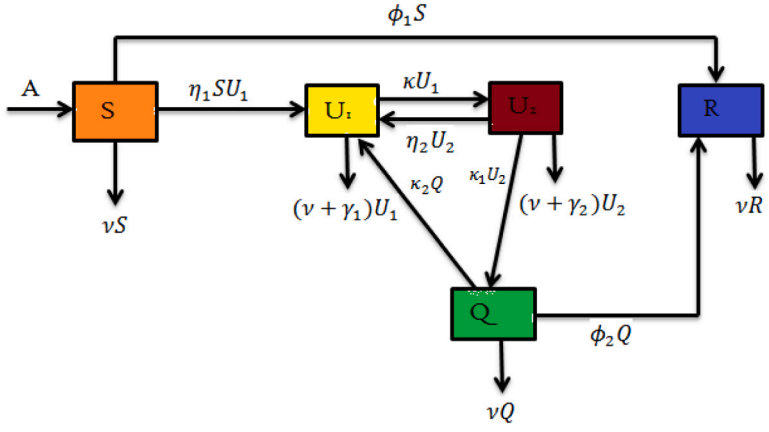

Fig. 3. Diagram illustrating the heroin pandemic idea involving recidivism and persistent immunization.

Definition 2 ([30]). The AB-fractional integral version of the mapping $\mathbf{f} \in \mathbb{C}^1(\mathbf{a}, \mathbf{b})$ is presented as follows:

$$ {}^{AB}_{\mathbf{a}}\mathbf{I}^{\wp}_{\xi}\mathbf{f}(\xi) = \frac{1-\wp}{\mathbf{AB}(\wp)}\mathbf{f}(\xi) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_{\mathbf{a}}^{\xi}\mathbf{f}(\mathbf{x})(\xi-\mathbf{x})^{\wp-1}d\mathbf{x}. \tag{3} $$

Lemma 1 ([39]). *For $\mathbf{f} \in \mathbb{C}^1(\mathbf{a}, \mathbf{b})$, then the AB-fractional derivative and integral operator holds the Newton–Leibniz identity:*

$$ {}^{AB}_{\mathbf{a}}\mathbf{I}^{\wp}_{\xi}\left({}^{ABC}_{\mathbf{a}}\mathbf{D}^{\wp}_{\xi}\mathbf{f}(\xi)\right) = \mathbf{f}(\xi) - \mathbf{f}(\mathbf{a}). \tag{4} $$

Lemma 2 ([30,40]). *For two mappings $\mathbf{f}, \mathbf{g} \in \Delta(\mathbf{a}, \mathbf{b})$, $\mathbf{b} > \mathbf{a}$, then the AB-fractional derivative holds the subsequent variant:*

$$ \left\| {}^{AB}_{\mathbf{a}}\mathbf{D}^{\wp}_{\xi}\mathbf{f}(\xi) - {}^{ABC}_{\mathbf{a}}\mathbf{D}^{\wp}_{\xi}\mathbf{g}(\xi)\right\| = \Delta\left\|\mathbf{f}(\xi) - \mathbf{g}(\xi)\right\|. \tag{5} $$

We shall immediately continue on to the model's construction. The mathematical framework of current research is developed by the flow diagram below (Fig. 3).

The mathematical formalism having numerical approximation employed in this work is represented by the governing equations, which is predicated on the workflow.

$$ \begin{cases} \frac{d\mathbf{S}}{d\xi} = \Theta - \eta_1\mathbf{S}\mathbf{U}_1 - (\nu+\phi_1)\mathbf{S}, \\ \frac{d\mathbf{U}_1}{d\xi} = \eta_1\mathbf{S}\mathbf{U}_1 + \sigma_2\mathbf{Q} + \eta_2\mathbf{U}_2 - (\nu+\gamma_1+\kappa)\mathbf{U}_1, \\ \frac{d\mathbf{U}_2}{d\xi} = \kappa\mathbf{U}_1 - (\nu+\gamma_2+\sigma_1+\eta_2)\mathbf{U}_2, \\ \frac{d\mathbf{Q}}{d\xi} = \sigma_1\mathbf{U}_2 - (\nu+\sigma_2+\phi_2)\mathbf{Q}, \\ \frac{d\mathbf{R}}{d\xi} = \phi_1\mathbf{S} + \phi_2\mathbf{Q} - \nu\mathbf{R}, \end{cases} \tag{6} $$

where $\mathcal{N}(\xi) = \mathbf{S}(\xi) + \mathbf{U}_1(\xi) + \mathbf{U}_2(\xi) + \mathbf{Q}(\xi) + \mathbf{R}(\xi)$ which signifies that $0 \le \mathcal{N}(\xi) \le \frac{\Theta}{\nu} + \mathcal{N}(0)\exp(-\nu\xi)$ with $\mathcal{N}(0)$ in the initial value. Thus, $0 \le \mathcal{N}(\xi) \le \Theta/\nu$, as $\xi \mapsto \infty$.

The overall population is composed of five compartments: $\mathbf{S}$, $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{Q}$, $\mathbf{R}$. Further, $\mathbf{S}$ describes the proportion of people who are vulnerable but have never consumed heroin; treatment; $\mathbf{U}_1$ is the actual population of heroin abusers, while $\mathbf{U}_2$ is the total number of heroin consumers in therapy. $\mathbf{Q}$ is the proportion of individuals who have used drugs in the past and are not using them now, but may do so in the future, and $\mathbf{R}$ denotes the majority of individuals who do not use drugs or who have effectively detoxed and are no longer using narcotics. We consider that heroin addicts are unable to rehabilitate themselves via consciousness and must begin counselling if they wish

Table 1
Table of specified variables and their descriptions.

Parameters	Explanation	Data estimated	References
$\mathbf{S}(\xi)$	Amount of susceptible individuals in time ξ		
$\mathbf{U}_1(\xi)$	Amount of heroin users in time ξ		
$\mathbf{U}_2(\xi)$	Amount of heroin users undergoing treatment in time ξ		
$\mathbf{Q}(\xi)$	Number of individuals who used drugs in time ξ		
$\mathbf{R}(\xi)$	Number of individuals who never used drugs in time ξ		
Θ	Acquisition rate of the population	1	[18].
ν	Natural death rate	0.02	[19].
η_1	Rate of transfer from $\mathbf{S}$ to abuser	Assumed	
η_2	Rate of failure cure	0.0011	[19].
γ_1	The heroin -concerned death rate of $\mathbf{U}_1$	0.01	Estimated
γ_2	The heroin -concerned death rate of being cured	0.005	Estimated
κ	Transition rate from $\mathbf{U}_2$ to $\mathbf{U}_1$	0.0095	[19].
σ_1	The percentage of patients who receive complete remission	Assumed	
σ_2	Insertion rate from $\mathbf{Q}$ to abusers	Assumed	
ϕ_1	The persistent depletion rate from $\mathbf{S}$ to $\mathbf{R}$	Assumed	Estimated
ϕ_2	The persistent depletion rate from $\mathbf{Q}$ to $\mathbf{R}$	0.0001	Estimated

to stop using narcotics. We further expect that not everyone is entirely recovered. People in compartment $\mathbf{U}_2$ will inhabit compartment $\mathbf{Q}$ if the heroin addicts are effectively treated. The individuals who refused the medication will continue to consume medicine even if the intervention is halted or fails. Many of these effective detoxification patients will relapse even though they are unable to fight heroin seduction, while others will never use narcotics and they are aware of the dangers of heroin misuse.

In Theorem 3, we describe the outbreaks viable (positivity and boundedness) research sector and demonstrate that it is positively invariant and bounded.

Theorem 3. *The domain of the system (6) that is epidemiologically viable is determined by*

$$\Lambda := \left\{(\mathbf{S},\mathbf{U}_1,\mathbf{U}_2,\mathbf{Q},\mathbf{R}) \in \mathbf{R}_+^5 : 0 \le \mathbf{S}+\mathbf{U}_1+\mathbf{U}_2+\mathbf{Q}+\mathbf{R} \le \mathcal{N} \le \frac{\Theta}{\nu}\right\}. \quad (7)$$

The existence and uniqueness of system (6) have now been established, and all that requires is to demonstrate that the collection specified in Table 1 and (7) are positively consistent. The demonstration of Theorem 3 will be characterized by the accompanying lemma.

Our next result is the generalized mean-value theorem provided by [40].

Lemma 4 ([40]). *Assume that* $\mathbf{f}(\mathbf{x}) \in \mathbb{C}[\mathbf{a},\mathbf{b}]$ *and suppose* ${}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{f}(\mathbf{x}) \in \mathbb{C}[\mathbf{a},\mathbf{b}]$, $\wp \in (0,1]$. *Then, we have* $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \frac{1}{\Gamma(\wp)}\,{}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{f}(\zeta)(\mathbf{x}-\mathbf{a})^{\wp}$, *when* $\zeta \in [0,\mathbf{x}]$.

Observe that by Lemma 4, if $\mathbf{f}(\mathbf{x}) \in [0,\mathbf{b}]$, ${}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{f}(\mathbf{x}) \in (0,\mathbf{b}]$ and ${}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{f}(\mathbf{x}) \ge 0$, *for all* $\mathbf{x} \in (0,\mathbf{b}]$, $\wp \in (0,1]$, then the mapping $\mathbf{f}(\mathbf{x})$ is increasing, and if ${}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{f}(\mathbf{x}) \le 0$, *for all* $\mathbf{x} \in (0,\mathbf{b}]$, then the mapping $\mathbf{f}(\mathbf{x})$ is decreasing for all $\mathbf{x} \in [0,\mathbf{b}]$.

Employing Lemma 4, we can prove that the collection is positively invariant, we have

$$\begin{cases} {}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{S}\big|_{\mathbf{S}=0} = \Theta \ge 0, \\ {}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{U}_1\big|_{\mathbf{U}_1=0} = \eta_1\mathbf{S}\mathbf{U}_1 \ge 0, \\ {}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{U}_2\big|_{\mathbf{U}_2=0} = \kappa\mathbf{U}_1 \ge 0, \\ {}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{Q}\big|_{\mathbf{Q}=0} = \sigma_1\mathbf{U}_2 \ge 0, \\ {}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathbf{R}\big|_{\mathbf{R}=0} = \phi_1\mathbf{S}+\phi_2\mathbf{Q} \ge 0. \end{cases} \quad (8)$$

Because all solutions of (6) is positive and persists in $\mathbf{R}_+^5$ as a result of (8), the collection described in (7) is positively invariant for system (6).

Furthermore, to illustrate the boundedness of the results of the fractional framework (1), we proceed by accumulating all of the system equations, that provides

$${}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathcal{N}(\xi) = \Theta - \nu\mathcal{N},$$

by means of Laplace transform, we have

$$\mathcal{L}\left({}_0^{ABC}\mathbf{D}_\xi^{\wp}\mathcal{N}(\xi) + \nu\mathcal{N}(\xi)\right) \le \mathcal{L}(\Theta),$$

$$\mathcal{L}(\mathcal{N})\left[(1-\sigma)\mathbf{S}^{\wp} - \frac{\sigma\wp}{1-\wp}\right] - \mathbf{S}^{\wp-1}\mathcal{N}(0) \le \frac{1-\wp}{\mathbf{AB}(\wp)}\left[\mathbf{S}^{\wp} + \frac{\wp}{1-\wp}\right]\frac{\Theta}{\mathbf{S}},$$

which can be expressed as

$$\mathcal{L}(\mathcal{N}) \le \left(1 - \frac{\sigma\wp}{(1-\sigma)(1-\wp)\mathbf{S}^{\wp}}\right)^{-1} \times \left\{\frac{1-\wp}{(1-\sigma)\mathbf{AB}(\wp)}\left[\frac{1-\wp+\wp\mathbf{S}^{-\wp}}{1-\wp}\right]\frac{\Theta}{\mathbf{S}} + \frac{\mathcal{N}(0)}{(1-\sigma)\mathbf{S}}\right\},$$

where $\sigma = -\frac{\nu(1-\wp)}{\mathbf{AB}(\wp)}$.

Employing the inverse Laplace transform, we have

$$\mathcal{N}(\xi) = \frac{\Theta}{\nu} - \frac{\Theta}{\nu(1-\sigma)}\frac{d}{d\xi}\int_0^{\xi}\bar{E}_{\wp}\left(\frac{\sigma\wp}{(1-\sigma)(1-\wp)}(\xi-\mathbf{x})^{\wp}d\mathbf{x}\right) + \frac{1}{1-\sigma}\bar{E}_{\wp}\left(\frac{\sigma\wp}{(1-\sigma)(1-\wp)}\xi^{\wp}\right)\mathcal{N}(0),$$

where the MLF is denoted by $\bar{E}_{\wp_1,\wp_2}$. Considering the assumption that the MLF exhibits asymptotic characteristics

$$\bar{E}_{\wp_1,\wp_2}(\mathbf{z}) \approx \sum_{q=1}^{\theta}\mathbf{z}^{-q}/\Gamma(\wp_2-\wp_1 q) + \mathcal{O}(|\frac{1}{\mathbf{z}^{1+\theta}}|),$$

$$|\mathbf{z}| \mapsto \infty, \quad \frac{\wp_1\pi}{2} < |arg(\mathbf{z})| \le \pi.$$

It is not complicated to understand that $\mathcal{N}(\xi) \mapsto \Theta/\nu$ as ξ tends to ∞. As a result, (7) shows that the system (6) is biologically viable in domain.

Global behaviour of the heroin epidemic model

Drug free equilibrium and configuration of reproductive number

The drug-free equilibria of scheme (6) is simple to obtain:

$$\mathcal{E}_0 = \left(\frac{\Theta}{\nu+\phi_1}, 0, 0, 0, \frac{\phi_1\Theta}{\nu(\nu+\phi_1)}\right). \quad (9)$$

Adopting the Driessche and Watmough [41] and applying the terminology described therein, the matrices $\mathcal{F}$ and $\mathcal{V}$ for the new addicted individuals and the others affected are, respectively, described by

$$\mathcal{F} = \begin{pmatrix} \eta_1\mathbf{S}\mathbf{U}_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} \mathcal{W}_1\mathbf{U}_1 - \kappa_2\mathbf{Q} - \eta_2\mathbf{U}_2 \\ \mathcal{W}_2\mathbf{U}_2 - \kappa\mathbf{U}_1 \\ \mathcal{W}_3\mathbf{Q} - \sigma_1\mathbf{U}_2 \\ \mathcal{W}_4\mathbf{S} + \eta_1\mathbf{S}\mathbf{U}_1 - \Theta \\ \nu\mathbf{R} - \phi_1\mathbf{S} - \phi_2\mathbf{Q} \end{pmatrix}.$$

At the heroin-free state $\mathcal{E}_0$ for $\mathcal{F}$ and $\mathcal{V}$, the Jacobian matrices are presented as:

$$\mathcal{F}(\mathcal{E}_0)=\begin{pmatrix}\mathcal{F}_{3\times3} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\mathcal{V}(\mathcal{E}_0)=\begin{pmatrix}\mathcal{V}_{3\times3} & 0 & 0 & 0 & 0\\ \mathcal{W}_5 & 0 & 0 & \mathcal{W}_4 & 0\\ 0 & 0 & -\phi_2 & -\phi_1 & \nu\end{pmatrix},$$

where

$$\mathcal{F}_{3\times3}=\begin{pmatrix}\mathcal{W}_5 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\mathcal{V}_{3\times3}=\begin{pmatrix}\mathcal{W}_1 & -\eta_2 & -\sigma_2\\ -\kappa & \mathcal{W}_2 & 0\\ 0 & -\sigma_1 & \mathcal{W}_3\end{pmatrix}.$$

Thus, the spectral radius is

$$\mathcal{R}_0=\Phi(\mathcal{F}\mathcal{V}^{-1})=\frac{\eta_1\Theta\mathcal{W}_2\mathcal{W}_1}{\mathcal{W}_1\mathcal{W}_2\mathcal{W}_3\mathcal{W}_4-\kappa\eta_2\mathcal{W}_3-\kappa\sigma_1\sigma_2}, \tag{10}$$

with the representation of parameters, $\mathcal{W}_1=\nu+\gamma_1+\kappa,\ \mathcal{W}_2=\nu+\gamma_2+\sigma_1+\eta_2,\ \mathcal{W}_3=\nu+\sigma_2+\phi_2,\ \mathcal{W}_4=\nu+\phi_1, \mathcal{W}_5=\frac{\eta_1\Theta}{\mathcal{W}_4}$.

Analysing drug-free equilibrium and global stability

To investigate the global stability, assume that the Lyapunov function:

$$Y_1=\mathbf{S}-\mathbf{S}_0-\mathbf{S}_0\ln\frac{\mathbf{S}}{\mathbf{S}_0}+\mathbf{U}_1+\mathbf{U}_2+\mathbf{Q}.$$

Its Lyapunov derivative, as well as the results to the model (6):

$$\begin{aligned}\dot{Y}_1&=\dot{\mathbf{S}}-\dot{\mathbf{S}}\frac{\mathbf{S}_0}{\mathbf{S}}+\dot{\mathbf{U}}_1+\dot{\mathbf{U}}_2+\dot{\mathbf{Q}}\\&=(\nu+\phi_1)\Big(2\mathbf{S}_0-\mathbf{S}-\frac{\mathbf{S}_0}{\mathbf{S}}\mathbf{S}_0\Big)-(\nu+\phi_2)\mathbf{Q}-(\nu+\gamma_2)\mathbf{U}_2-\eta_1\mathbf{S}_0\mathbf{U}_1\\&\le-(\nu+\phi)\Big(\frac{\mathbf{S}_0^2+\mathbf{S}^2-2\mathbf{S}\mathbf{S}_0}{\mathbf{S}}\Big)-(\nu+\gamma_1)\mathcal{R}_0\mathbf{U}_1-(\nu+\phi_2)\mathbf{Q}-(\nu+\gamma_2)\mathbf{U}_2.\end{aligned}$$

Using the fact that $\frac{\mathbf{S}_0^2+\mathbf{S}^2-2\mathbf{S}\mathbf{S}_0}{\mathbf{S}\mathbf{S}_0}\ge0$ if $\mathcal{R}_0\le1$, then $\dot{Y}_1\le0$. It is clear that $\dot{Y}_1\le0$ if and only if $\mathbf{S}=\mathbf{S}_0,\ \ \mathbf{U}_1=\mathbf{U}_2=\mathbf{Q}=0$. Setting $\mathbf{S}=\mathbf{S}_0$ and $\mathbf{U}_1=\mathbf{U}_2=\mathbf{Q}=0$ in (6), we have $\mathcal{R}\mapsto\phi_1\Theta/\nu(\nu+\phi)$ when ξ tends to ∞. By employing LaSalle's Invariance Principle [42], all results to model (6), having initial conditions, tends to $\mathcal{E}_0$ as $\xi\mapsto\infty$. Hence, $\mathcal{E}_0$ is globally asymptotically stable in γ whenever $\mathcal{R}_0<1$.

Drug persistence state and global stability

Utilizing (6), yields

$$\mathcal{E}^*=\begin{cases}\mathbf{S}^*=\frac{\Theta}{\eta_1\mathbf{U}_1^*+\mathcal{W}_4},\\ \mathbf{U}_1^*=\frac{\mathcal{W}_4(\mathcal{R}_0-1)}{\eta_1},\\ \mathbf{U}_2^*=\frac{\kappa\mathbf{U}_1^*}{\mathcal{W}_2},\\ \mathbf{Q}^*=\frac{\kappa\sigma_1\mathbf{U}_1^*}{\mathcal{W}_2\mathcal{W}_3},\\ \mathbf{R}^*=\frac{\Theta\phi_1}{\nu(\eta_1+\mathcal{W}_1)}+\frac{\phi_2\kappa\sigma_1\mathbf{U}_1^*}{\nu\mathcal{W}_2\mathcal{W}_3}.\end{cases} \tag{11}$$

Theorem 5. *For $\mathcal{R}_0>1$, then the unique drug-addictive equilibrium $\mathcal{E}^*$ of the model (6) is global asymptotically stable.*

Proof. Let us introduce the Lyapunov function candidate by

$$\begin{aligned}Y_1=&\widetilde{A}\Big(\mathbf{Q}-\mathbf{Q}^*-\mathbf{Q}^*\big(\ln\mathbf{Q}-\ln\mathbf{Q}^*\big)\Big)+\widetilde{B}\Big(\mathbf{U}_1-\mathbf{U}_1^*-\mathbf{U}_1^*\big(\ln\mathbf{U}_1-\ln\mathbf{U}_1^*\big)\Big)\\&+\widetilde{C}\Big(\mathbf{U}_2-\mathbf{U}_2^*-\mathbf{U}_2^*\big(\ln\mathbf{U}_2-\ln\mathbf{U}_2^*\big)\Big)+\Big(\mathbf{S}-\mathbf{S}^*-\mathbf{S}^*\big(\ln\mathbf{S}-\ln\mathbf{S}^*\big)\Big),\end{aligned} \tag{12}$$

where $\widetilde{A},\widetilde{B}$ and $\widetilde{C}$ are arbitrary constants to be determined. Utilizing the s Lyapunov derivative of Y_1 is

$$\begin{aligned}\dot{Y}_1&=\widetilde{A}\dot{Q}_1\Big(\frac{\mathbf{Q}-\mathbf{Q}^*}{\mathbf{Q}}\Big)+\widetilde{B}\dot{U}_1\Big(\frac{\mathbf{U}_1-\mathbf{U}_1^*}{\mathbf{U}_1}\Big)+\widetilde{C}\dot{U}_2\Big(\frac{\mathbf{U}_2-\mathbf{U}_2^*}{\mathbf{U}_2}\Big)+\dot{\mathbf{S}}_1\Big(\frac{\mathbf{S}-\mathbf{S}^*}{\mathbf{S}}\Big)\\&=\widetilde{A}\Big(\frac{\mathbf{Q}-\mathbf{Q}^*}{\mathbf{Q}}\Big)\Big(\frac{\sigma_1\mathbf{U}_2\mathbf{Q}^*-\sigma_1\mathbf{Q}\mathbf{U}_2^*}{\mathbf{Q}^*}\Big)+\widetilde{B}\frac{1}{\mathbf{U}_1^*}\Big(\frac{\mathbf{U}_1-\mathbf{U}_1^*}{\mathbf{U}_1}\Big)\\&\quad\times\Big\{\mathbf{U}_1^*(\eta_1\mathbf{S}\mathbf{U}_1+\sigma_2\mathbf{Q}+\eta_2\mathbf{U}_2)-\mathbf{U}_1(\eta_1\mathbf{S}^*\mathbf{U}_1^*+\sigma_2\mathbf{Q}^*+\eta_2\mathbf{U}_2^*)\Big\}\\&\quad+\widetilde{C}\frac{1}{\mathbf{U}_2^*}\Big(\frac{\mathbf{U}_2-\mathbf{U}_2^*}{\mathbf{U}_2}\Big)\Big\{\kappa\mathbf{U}_1\mathbf{U}_2^*-\kappa\mathbf{U}_1^*\mathbf{U}_2\Big\}+\Big(\frac{\mathbf{S}-\mathbf{S}^*}{\mathbf{S}}\Big)\Big\{\mathbf{S}^*(\eta_1\mathbf{U}_1^*+\mathcal{W}_1)\\&\quad-\mathbf{S}(\eta_1\mathbf{U}_1-\mathcal{W}_1)\Big\}.\end{aligned} \tag{13}$$

Taking $\zeta_1=\mathbf{S}/\mathbf{S}^*,\ \zeta_2=\mathbf{U}_1/\mathbf{U}_1^*,\ \zeta_3=\mathbf{U}_2/\mathbf{U}_2^*$ and $\zeta_4=\mathbf{Q}/\mathbf{Q}^*$. Then, we have

$$\begin{aligned}\dot{Y}_1&=\zeta_1\zeta_2\big(\widetilde{B}\eta_1\mathbf{S}^*\mathbf{U}_1^*-\eta_1\mathbf{S}^*\mathbf{U}_1^*\big)+\zeta_2\ \big(\eta_1\mathbf{S}^*\mathbf{U}_1^*-\widetilde{B}\eta_1\mathbf{S}^*\mathbf{U}_1^*\\&\quad-\widetilde{B}\sigma_2\mathbf{Q}^*-\widetilde{B}\eta_2\mathbf{U}_2^*+\widetilde{C}\sigma_1\mathbf{U}_2^*\big)\\&\quad+\zeta_4\big(\widetilde{B}\sigma_2\mathbf{Q}^*-\widetilde{A}\sigma_1\mathbf{U}_2^*\big)+\zeta_3\big(\widetilde{C}\eta_2\mathbf{U}_2^*-\widetilde{C}\kappa\mathbf{U}_1^*+\widetilde{A}\sigma_1\mathbf{U}_2^*\big)\\&\quad-\zeta_1(\widetilde{C}\eta_1\mathbf{S}^*\mathbf{U}_1^*)-\frac{\zeta_4}{\zeta_2}\widetilde{B}\sigma_2\mathbf{Q}^*\\&\quad-\frac{\zeta_3}{\zeta_2}\widetilde{B}\eta_2\mathbf{U}_2^*-\frac{\zeta_2}{\zeta_3}\widetilde{C}\kappa\mathbf{U}_1^*-\frac{\zeta_3}{\zeta_4}\widetilde{A}\sigma_1\mathbf{U}_2^*-\frac{1}{\zeta_1}\eta_1\mathbf{S}^*\mathbf{U}_1^*-\mathcal{W}_3\mathbf{S}^*\frac{(1-\zeta_1)^2}{\zeta_1}\\&\quad+\big(\eta_1\mathbf{S}^*\mathbf{U}_1^*+\widetilde{B}\eta_1\mathbf{S}^*\mathbf{U}_1^*+\widetilde{B}\sigma_2\mathbf{Q}^*+\widetilde{B}\eta_2\mathbf{U}_2^*+\widetilde{C}\kappa\mathbf{U}_1^*+\widetilde{A}\sigma_1\mathbf{U}_2^*\big).\end{aligned} \tag{14}$$

Since $\dot{Y}_1>0$ possible only, if all the coefficients of $\zeta_1\zeta_2,\ \zeta_2,\ \zeta_3$ and ζ_4 are zero, then we have

$$\begin{cases}\widetilde{B}\eta_1\mathbf{S}^*\mathbf{U}_1^*-\eta_1\mathbf{S}^*\mathbf{U}_1^*=0,\\ \eta_1\mathbf{S}^*\mathbf{U}_1^*-\widetilde{B}\eta_1\mathbf{S}^*\mathbf{U}_1^*-\widetilde{B}\sigma_2\mathbf{Q}^*-\widetilde{B}\eta_2\mathbf{U}_2^*+\widetilde{C}\sigma_1\mathbf{U}_2^*=0,\\ \widetilde{B}\sigma_2\mathbf{Q}^*-\widetilde{A}\sigma_1\mathbf{U}_2^*=0,\\ \widetilde{C}\eta_2\mathbf{U}_2^*-\widetilde{C}\kappa\mathbf{U}_1^*+\widetilde{A}\sigma_1\mathbf{U}_2^*.\end{cases} \tag{15}$$

Therefore, simple computations yields

$$\widetilde{A}=\sigma_2\mathbf{Q}^*/\sigma_1\mathbf{U}_2^*,\quad \widetilde{B}=1,\quad \widetilde{C}=\sigma_2\mathbf{Q}^*+\eta_2\mathbf{U}_2^*/\kappa\mathbf{U}_1^*. \tag{16}$$

Thus, we conclude that

$$\begin{aligned}\dot{Y}_1&=\sigma_2\mathbf{Q}^*\Big(\frac{3\zeta_2\zeta_3\zeta_4-\zeta_4^2\zeta_3-\zeta_2^2\zeta_4-\zeta_2\zeta_3^2}{\zeta_2\zeta_3\zeta_4}\Big)+\eta_1\mathbf{S}^*\mathbf{U}_1^*\Big(\frac{2\zeta_1-\zeta_1^2-1}{\zeta_1}\Big)\\&\quad+\eta_2\mathbf{U}_2^*\Big(\frac{2\zeta_2\zeta_3-\zeta_3^2-\zeta_2^2}{\zeta_2\zeta_3}\Big)\\&\quad-\mathcal{W}_3\mathbf{S}^*\frac{(\zeta_1-1)^2}{\zeta_1}.\end{aligned} \tag{17}$$

Observe that if $\zeta_1>0$, then $-\mathcal{W}_3\mathbf{S}^*\frac{(\zeta_1-1)^2}{\zeta_1}\le0$, and $-\mathcal{W}_3\mathbf{S}^*\frac{(\zeta_1-1)^2}{\zeta_1}=0$, if and only if $\zeta_1=0$. Clearly, we see that $\dot{Y}_1\le0$ if $\zeta_1,\zeta_2,\zeta_3,\zeta_4>0$ and $\dot{Y}_1=0$ iff $\zeta_1=1$ and $\zeta_2=\zeta_3=\zeta_4$. Setting $\mathbf{S}=\mathbf{S}^*$ and $\mathbf{U}_1/\mathbf{U}_1^*=\mathbf{U}_2/\mathbf{U}_2^*=\mathbf{Q}/\mathbf{Q}^*$ in the first equation of (6), thus we attain $\Theta-\eta_1\mathbf{S}^*\mathbf{U}_1-\mathcal{W}_3\mathbf{S}^*$, which gives that $\mathbf{U}_1=\mathbf{U}_1^*$. Hence, by the maximum invariant set of equations on the set $\{(\zeta_1,\zeta_2,\zeta_3,\zeta_4):Y_1=0\}$ is the singleton $(1,1,1,1)$. Therefore, the largest invariant set where $Y1=0$ is $(\mathbf{S}^*,\mathbf{U}_1^*,\mathbf{U}_2^*,\mathbf{Q}^*)$. In view of Lasalle's invariance principle [42], demonstrate that when $\mathcal{R}_0>1$, then the drug-addiction equilibrium $\mathcal{E}^*$ of system (6) has global asymptotic stability. This ends the proof. □

Mathematical significance of fractional heroin epidemic model

The system of DEs describes the analytical framework that incorporates the hypotheses with relapse and constant immunization, the flow

chart (Fig. 3), and the ABC-fractional derivative.

$$\begin{cases} {}^{ABC}_{0}\mathbf{D}^{\wp}_{\xi}\mathbf{S}(\xi) = \Omega_1(\xi,\mathbf{S}), \\ {}^{ABC}_{0}\mathbf{D}^{\wp}_{\xi}\mathbf{U}_1(\xi) = \Omega_2(\xi,\mathbf{U}_1), \\ {}^{ABC}_{0}\mathbf{D}^{\wp}_{\xi}\mathbf{U}_2(\xi) = \Omega_3(\xi,\mathbf{U}_2), \\ {}^{ABC}_{0}\mathbf{D}^{\wp}_{\xi}\mathbf{Q}(\xi) = \Omega_4(\xi,\mathbf{Q}), \\ {}^{ABC}_{0}\mathbf{D}^{\wp}_{\xi}\mathbf{R}(\xi) = \Omega_5(\xi,\mathbf{R}), \end{cases} \tag{18}$$

where the kernels are defined as

$$\begin{cases} \Omega_1(\xi,\mathbf{S}) = \Theta - \eta_1\mathbf{S}\mathbf{U}_1 - (\nu+\phi_1)\mathbf{S}, \\ \Omega_2(\xi,\mathbf{U}_1) = \eta_1\mathbf{S}\mathbf{U}_1 + \sigma_2\mathbf{Q} + \eta_2\mathbf{U}_2 - (\nu+\gamma_1+\kappa)\mathbf{U}_1, \\ \Omega_3(\xi,\mathbf{U}_2) = \kappa\mathbf{U}_1 - (\nu+\gamma_2+\sigma_1+\eta_2)\mathbf{U}_2, \\ \Omega_4(\xi,\mathbf{Q}) = k_1\mathbf{U}_2 - (\nu+\sigma_2+\phi_2)\mathbf{Q}, \\ \Omega_5(\xi,\mathbf{R}) = \phi_1\mathbf{S} + \phi_2\mathbf{Q} - \nu\mathbf{R}, \end{cases} \tag{19}$$

supplements with ICs $\mathbf{S}(0) = \mathbf{S}_0$, $\mathbf{U}_1(0) = \mathbf{U}_{1_0}$, $\mathbf{U}_2(0) = \mathbf{U}_{2_0}$, $\mathbf{Q}(0) = \mathbf{Q}_0$, $\mathbf{R}(0) = \mathbf{R}_0$. $\mathbf{R}$, as aforementioned, is a comprehensive recuperation compartment. In $\mathbf{R}$, the people are inoculated indefinitely, and one clearly observe that the $\mathbf{R}$ equation is detached from some other formulas (6). As a result, the dispersion of $\mathbf{R}$ is neglected. All components in the system are assumed to be significant model terms, and their interpretations are stated in Table 1.

In what follows, the availability and originality, positivity, and stability of fractional-order system (6) alternatives are discussed. We employ the well-known Banach fixed point theorem to demonstrate the presence of the approach to the problem (6). We recommend that researchers [43] for a comprehensive review of fixed points and contractions.

Now, we will proceed via the steps below to demonstrate the solution's presence and novelty. Formulation (6) is addressed by applying the AB fractional integral:

$$\begin{cases} \mathbf{S}(\xi) - \mathbf{S}(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1(\xi,\mathbf{S}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_1(\tau,\mathbf{S})(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{U}_1(\xi) - \mathbf{U}_1(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1(\xi,\mathbf{U}_1) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_2(\tau,\mathbf{U}_1)(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{U}_2(\xi) - \mathbf{U}_2(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_3(\xi,\mathbf{U}_2) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_3(\tau,\mathbf{U}_2)(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{Q}(\xi) - \mathbf{Q}(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_4(\xi,\mathbf{Q}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_4(\tau,\mathbf{Q})(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{R}(\xi) - \mathbf{R}(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_5(\xi,\mathbf{R}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_5(\tau,\mathbf{R})(\xi-\tau)^{\wp-1}d\tau, \end{cases} \tag{20}$$

Assume that there is an interval $\mathcal{I} = [0,\xi]$ such that $\mathcal{W} = \mathcal{H}(\mathcal{I}) \times \mathcal{H}(\mathcal{I}) \times \mathcal{H}(\mathcal{I}) \times \mathcal{H}(\mathcal{I}) \times \mathcal{H}(\mathcal{I})$ for every Banach space $\mathcal{H}(\mathcal{I}) = \mathbb{C}[0,\xi]$ of real-valued continuous mappings on $\mathcal{I}$ having the respective norm described as follows:

$$\begin{aligned} &\left\|\mathbf{S}\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{S}(\xi)\right|, \ \left\|\mathbf{U}_1\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{U}_1(\xi)\right|, \ \left\|\mathbf{U}_2\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{U}_2(\xi)\right|, \\ &\left\|\mathbf{Q}\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{Q}(\xi)\right|, \ \left\|\mathbf{R}\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{Q}(\xi)\right|. \end{aligned} \tag{21}$$

Our next theorem is based on the contraction and the Lipschitz assumption.

Theorem 6. *If there be the kernels G_j, $j = 1,2,3,4,5$ in (6), then there exists $\mathbb{L}_j$, $j = 1,2,3,4,5$, such that*

$$\begin{cases} \left\|\Omega_1(\xi,\mathbf{S}) - \Omega_1(\xi,\bar{\mathbf{S}})\right\| \le \mathbb{L}_1\left\|\mathbf{S}(\xi) - \mathbf{S}_0(\xi)\right\|, \\ \left\|\Omega_2(\xi,\mathbf{U}_1) - \Omega_1(\xi,\bar{\mathbf{U}_1})\right\| \le \mathbb{L}_2\left\|\mathbf{U}_1(\xi) - \bar{\mathbf{U}_1}(\xi)\right\|, \\ \left\|\Omega_3(\xi,\mathbf{U}_2) - \Omega_3(\xi,\bar{\mathbf{U}_2})\right\| \le \mathbb{L}_3\left\|\mathbf{U}_2(\xi) - \bar{\mathbf{U}_2}(\xi)\right\|, \\ \left\|\Omega_4(\xi,\mathbf{Q}) - \Omega_4(\xi,\bar{\mathbf{Q}})\right\| \le \mathbb{L}_4\left\|\mathbf{Q}(\xi) - \bar{\mathbf{Q}}(\xi)\right\|, \\ \left\|\Omega_5(\xi,\mathbf{R}) - \Omega_5(\xi,\bar{\mathbf{R}})\right\| \le \mathbb{L}_5\left\|\mathbf{R}(\xi) - \bar{\mathbf{R}}(\xi)\right\|, \end{cases} \tag{22}$$

are contraction mappings for $\mathbb{L}_j \in [0,1)$, $j = 1,2,3,4,5$.

Proof. We shall continue from the first compartment $\mathbf{S}$. If $\mathbf{S}$ and $\bar{\mathbf{S}}$ are two mappings, we must consider these factors:

$$\begin{aligned} \left\|\Omega_1(\xi,\mathbf{S}) - \Omega_1(\xi,\bar{\mathbf{S}})\right\| &= \left\| - \eta_1\mathbf{S}\mathbf{U}_1 - (\nu+\phi_1)\mathbf{S} - \left(-\eta_1\bar{\mathbf{S}}\mathbf{U}_1 - (\nu+\phi_1)\bar{\mathbf{S}}\right)\right\| \\ &= \left\| - \eta_1\mathbf{U}_1(\mathbf{S}-\bar{\mathbf{S}}) - (\nu+\phi_1)(\mathbf{S}-\bar{\mathbf{S}})\right\| \\ &\le \left(\eta_1 m_1 + (\nu+\phi_1)\right)\left\|\mathbf{S}-\bar{\mathbf{S}}\right\| \\ &\le \mathbb{L}_1\left\|\mathbf{S}-\bar{\mathbf{S}}\right\|, \end{aligned} \tag{23}$$

where $\mathbb{L}_1 = \left(\eta_1 m_1 + (\nu+\phi_1)\right)$,

$$\begin{aligned} &\left\|\mathbf{S}\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{S}(\xi)\right| = m_1, \ \left\|\mathbf{U}_1\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{U}_1(\xi)\right| = m_2, \ \left\|\mathbf{U}_2\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{U}_2(\xi)\right| = m_3, \\ &\left\|\mathbf{Q}\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{Q}(\xi)\right| = m_4, \ \left\|\mathbf{R}\right\| = \sup_{\xi\in\mathcal{I}}\left|\mathbf{Q}(\xi)\right| = m_5. \end{aligned} \tag{24}$$

So, the Lipschitz condition is satisfied for $\mathbf{S}$, and additionally, if $0 \le \left(\eta_1 m_1 + (\nu+\phi_1)\right) < 1$, it is therefore a contraction. The Lipschitz conditions are presented in the same way for the succeeding contexts.

Nor for $\xi = \xi_{\mathbf{n}}$, $\mathbf{n} = 1,2,\ldots$, we present the subsequent iterative relation of (20):

$$\begin{cases} \mathbf{S}_{\mathbf{n}}(\xi)(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1(\xi,\mathbf{S}_{\mathbf{n}-1}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_1(\tau,\mathbf{S}_{\mathbf{n}-1})(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{U}_{1\mathbf{n}}(\xi)(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_2(\xi,\mathbf{U}_{1\mathbf{n}-1}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_2(\tau,\mathbf{U}_{1\mathbf{n}-1})(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{U}_{2\mathbf{n}}(\xi)(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_3(\xi,\mathbf{U}_{2\mathbf{n}-1}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_3(\tau,\mathbf{U}_{2\mathbf{n}-1})(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{Q}_{\mathbf{n}}(\xi)(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_4(\xi,\mathbf{Q}_{\mathbf{n}-1}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_4(\tau,\mathbf{Q}_{\mathbf{n}-1})(\xi-\tau)^{\wp-1}d\tau, \\ \mathbf{R}_{\mathbf{n}}(\xi)(0) = \frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_5(\xi,\mathbf{R}_{\mathbf{n}-1}) + \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Omega_5(\tau,\mathbf{R}_{\mathbf{n}-1})(\xi-\tau)^{\wp-1}d\tau, \end{cases}$$

along with appropriate starting condition $\mathbf{S}(0) = \mathbf{S}_0$, $\mathbf{U}_1(0) = \mathbf{U}_{10}$, $\mathbf{U}_2(0) = \mathbf{U}_{20}$, $\mathbf{Q}(0) = \mathbf{Q}_0$, $\mathbf{R}(0) = \mathbf{R}_0$.

The following equations is used to compute the difference between the succeeding aspects:

$$\begin{aligned} \mho_{1\mathbf{n}}(\xi) = \mathbf{S}_{\mathbf{n}}(\xi) - \mathbf{S}_{\mathbf{n}-1}(\xi) &= \frac{1-\wp}{\mathbf{AB}(\wp)}\left(\Omega_1(\xi,\mathbf{S}_{\mathbf{n}-1}) - \Omega_1(\xi,\mathbf{S}_{\mathbf{n}-2})\right) \\ &+ \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Big(\Omega_1(\tau,\mathbf{S}_{\mathbf{n}-1}) \\ &- \Omega_1(\tau,\mathbf{S}_{\mathbf{n}-2})\Big)(\xi-\tau)^{\wp-1}d\tau, \\ \mho_{2\mathbf{n}}(\xi) = \mathbf{U}_{1\mathbf{n}}(\xi) - \mathbf{U}_{1\mathbf{n}-1}(\xi) &= \frac{1-\wp}{\mathbf{AB}(\wp)}\left(\Omega_2(\xi,\mathbf{U}_{1\mathbf{n}-1}) - \Omega_2(\xi,\mathbf{U}_{1\mathbf{n}-2})\right) \\ &+ \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Big(\Omega_2(\tau,\mathbf{U}_{1\mathbf{n}-1}) \\ &\Omega_2(\tau,\mathbf{U}_{1\mathbf{n}-2})\Big)(\xi-\tau)^{\wp-1}d\tau, \\ \mho_{3\mathbf{n}}(\xi) = \mathbf{U}_{2\mathbf{n}}(\xi) - \mathbf{U}_{2\mathbf{n}-1}(\xi) &= \frac{1-\wp}{\mathbf{AB}(\wp)}\left(\Omega_3(\xi,\mathbf{U}_{2\mathbf{n}-1}) - \Omega_3(\xi,\mathbf{U}_{2\mathbf{n}-2})\right) \\ &+ \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Big(\Omega_3(\tau,\mathbf{U}_{2\mathbf{n}-1}) \\ &- \Omega_3(\tau,\mathbf{U}_{2\mathbf{n}-2})\Big)(\xi-\tau)^{\wp-1}d\tau, \\ \mho_{4\mathbf{n}}(\xi) = \mathbf{Q}_{\mathbf{n}}(\xi) - \mathbf{Q}_{\mathbf{n}-1}(\xi) &= \frac{1-\wp}{\mathbf{AB}(\wp)}\left(\Omega_4(\xi,\mathbf{Q}_{\mathbf{n}-1}) - \mathbf{Q}(\xi,\mathbf{Q}_{\mathbf{n}-2})\right) \\ &+ \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Big(\Omega_4(\tau,\mathbf{Q}_{\mathbf{n}-1}) \\ &- \Omega_4(\tau,\mathbf{Q}_{\mathbf{n}-2})\Big)(\xi-\tau)^{\wp-1}d\tau, \\ \mho_{5\mathbf{n}}(\xi) = \mathbf{R}_{\mathbf{n}}(\xi) - \mathbf{R}_{\mathbf{n}-1}(\xi) &= \frac{1-\wp}{\mathbf{AB}(\wp)}\left(\Omega_5(\xi,\mathbf{Q}_{\mathbf{n}-1}) - \Omega_5(\xi,\mathbf{Q}_{\mathbf{n}-2})\right) \\ &+ \frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^\xi \Big(\Omega_5(\tau,\mathbf{Q}_{\mathbf{n}-1}) \\ &- \Omega_5(\tau,\mathbf{Q}_{\mathbf{n}-2})\Big)(\xi-\tau)^{\wp-1}d\tau. \end{aligned} \tag{25}$$

Implementing the norm on both sides of (25), we have

$$\left\|\mho_{1\mathbf{n}}(\xi)\right\| = \left\|\mathbf{S}_{\mathbf{n}}(\xi) - \mathbf{S}_{\mathbf{n}-1}(\xi)\right\| = \frac{1-\wp}{\mathbf{AB}(\wp)}\left\|\Omega_1(\xi,\mathbf{S}_{\mathbf{n}-1}) - \Omega_1(\xi,\mathbf{S}_{\mathbf{n}-2})\right\|$$

$$
\begin{aligned}
&+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_1(\tau,\mathbf{S_{n-1}})\\
&-\Omega_1(\tau,\mathbf{S_{n-2}})\Big\|(\xi-\tau)^{\wp-1}d\tau,\\
\Big\|\mho_{2\mathbf{n}}(\xi)\Big\| = \Big\|\mathbf{U_{1n}}(\xi)-\mathbf{U_{1n-1}}(\xi)\Big\| &= \frac{1-\wp}{\mathbf{AB}(\wp)}\Big\|\Omega_2(\xi,\mathbf{U_{1n-1}})-\Omega_2(\xi,\mathbf{U_{1n-2}})\Big\|\\
&+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_2(\tau,\mathbf{U_{1n-1}})\\
&-\Omega_2(\tau,\mathbf{U_{1n-2}})\Big\|(\xi-\tau)^{\wp-1}d\tau,\\
\Big\|\mho_{3\mathbf{n}}(\xi)\Big\| = \Big\|\mathbf{U_{2n}}(\xi)-\mathbf{U_{2n-1}}(\xi)\Big\| &= \frac{1-\wp}{\mathbf{AB}(\wp)}\Big\|\Omega_3(\xi,\mathbf{U_{2n-1}})-\Omega_3(\xi,\mathbf{U_{2n-2}})\Big\|\\
&+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_2(\tau,\mathbf{U_{2n-1}})\\
&-\Omega_3(\tau,\mathbf{U_{2n-2}})\Big\|(\xi-\tau)^{\wp-1}d\tau,\\
\Big\|\mho_{4\mathbf{n}}(\xi)\Big\| = \Big\|\mathbf{Q_n}(\xi)-\mathbf{Q_{n-1}}(\xi)\Big\| &= \frac{1-\wp}{\mathbf{AB}(\wp)}\Big\|\Omega_3(\xi,\mathbf{Q_{n-1}})-\Omega_3(\xi,\mathbf{Q_{n-2}})\Big\|\\
&+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_2(\tau,\mathbf{Q_{n-1}})\\
&-\Omega_3(\tau,\mathbf{Q_{n-2}})\Big\|(\xi-\tau)^{\wp-1}d\tau,\\
\Big\|\mho_{5\mathbf{n}}(\xi)\Big\| = \Big\|\mathbf{R_n}(\xi)-\mathbf{R_{n-1}}(\xi)\Big\| &= \frac{1-\wp}{\mathbf{AB}(\wp)}\Big\|\Omega_5(\xi,\mathbf{R_{n-1}})-\Omega_5(\xi,\mathbf{R_{n-2}})\Big\|\\
&+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_5(\tau,\mathbf{R_{n-1}})\\
&-\Omega_5(\tau,\mathbf{R_{n-2}})\Big\|(\xi-\tau)^{\wp-1}d\tau. \qquad (26)
\end{aligned}
$$

In addition, the first identity in (26) can be simplified to the equations:

$$
\begin{aligned}
\Big\|\mho_{1\mathbf{n}}(\xi)\Big\| &= \Big\|\mathbf{S_n}(\xi)-\mathbf{S_{n-1}}(\xi)\Big\|\\
&\le \frac{1-\wp}{\mathbf{AB}(\wp)}\Big\|\Omega_1(\xi,\mathbf{S_{n-1}})-\Omega_1(\xi,\mathbf{S_{n-2}})\Big\|\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_1(\tau,\mathbf{S_{n-1}})-\Omega_1(\tau,\mathbf{S_{n-2}})\Big\|(\xi-\tau)^{\wp-1}d\tau\\
&\le \frac{1-\wp}{\mathbf{AB}(\wp)}\mathbb{L}_1\Big\|\mathbf{S_{n-1}}-\mathbf{S_{n-2}}\Big\|\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\mathbb{L}_1\Big\|\mathbf{S_{n-1}}-\mathbf{S_{n-2}}\Big\|(\xi-\tau)^{\wp-1}d\tau\\
&\le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_1\Big\|\mho_{1(\mathbf{n}-1)}(\xi)\Big\|. \qquad (27)
\end{aligned}
$$

Consequently, we have

$$
\Big\|\mho_{1\mathbf{n}}(\xi)\Big\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_1\Big\|\mho_{1(\mathbf{n}-1)}(\xi)\Big\|. \qquad (28)
$$

Accordingly, all other representations of (26) can be simplified to the accompanying inequalities:

$$
\begin{aligned}
&\Big\|\mho_{2\mathbf{n}}(\xi)\Big\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_2\Big\|\mho_{2(\mathbf{n}-1)}(\xi)\Big\|,\\
&\Big\|\mho_{3\mathbf{n}}(\xi)\Big\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_3\Big\|\mho_{3(\mathbf{n}-1)}(\xi)\Big\|,\\
&\Big\|\mho_{4\mathbf{n}}(\xi)\Big\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_4\Big\|\mho_{4(\mathbf{n}-1)}(\xi)\Big\|,\\
&\Big\|\mho_{5\mathbf{n}}(\xi)\Big\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_5\Big\|\mho_{5(\mathbf{n}-1)}(\xi)\Big\|. \quad \square \qquad (29)
\end{aligned}
$$

Now we will describe the theorem below.

Theorem 7. *The fractional heroin epidemic model (6) provides exact coupled solutions. If the following assumptions exist, i.e., we can determine M_0 such that*

$$
\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_J<1,\quad J=1,2,3,4,5. \qquad (30)
$$

Proof. By means of (28) and (29), we have

$$
\begin{cases}
\Big\|\mho_{1\mathbf{n}}(\xi)\Big\| \le \Big\|\mathbf{S}(0)\Big\|\Big[\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_1\Big]^{\mathbf{n}},\\
\Big\|\mho_{2\mathbf{n}}(\xi)\Big\| \le \Big\|\mathbf{U}_1(0)\Big\|\Big[\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_2\Big]^{\mathbf{n}},\\
\Big\|\mho_{3\mathbf{n}}(\xi)\Big\| \le \Big\|\mathbf{U}_2(0)\Big\|\Big[\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_3\Big]^{\mathbf{n}},\\
\Big\|\mho_{4\mathbf{n}}(\xi)\Big\| \le \Big\|\mathbf{Q}(0)\Big\|\Big[\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_4\Big]^{\mathbf{n}},\\
\Big\|\mho_{5\mathbf{n}}(\xi)\Big\| \le \Big\|\mathbf{R}(0)\Big\|\Big[\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_5\Big]^{\mathbf{n}}.
\end{cases} \qquad (31)
$$

As a result, the existence and continuity of the aforementioned approaches are established. Additionally, we continue as specified to verify that the aforementioned function is a result of (6):

$$
\begin{cases}
\mathbf{S}(\xi)-\mathbf{S}(0)=\mathbf{S_{n-1}}-\widetilde{a_{1\mathbf{n}}}(\xi),\\
\mathbf{U}_1(\xi)-\mathbf{U}_1(0)=\mathbf{U_{1n-1}}-\widetilde{a_{2\mathbf{n}}}(\xi),\\
\mathbf{U}_2(\xi)-\mathbf{U}_2(0)=\mathbf{U_{2n-1}}-\widetilde{a_{3\mathbf{n}}}(\xi),\\
\mathbf{Q}(\xi)-\mathbf{Q}(0)=\mathbf{Q_{n-1}}-\widetilde{a_{4\mathbf{n}}}(\xi),\\
\mathbf{R}(\xi)-\mathbf{R}(0)=\mathbf{R_{n-1}}-\widetilde{a_{5\mathbf{n}}}(\xi).
\end{cases} \qquad (32)
$$

Consequently, we have

$$
\begin{aligned}
\|\widetilde{a_{1\mathbf{n}}}(\xi)\| &\le \frac{1-\wp}{\mathbf{AB}(\wp)}\Big\|\Omega_1(\xi,\mathbf{S_n})-\Omega_1(\xi,\mathbf{S_{n-1}})\Big\|\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Big\|\Omega_1(\tau,\mathbf{S_n})-\Omega_1(\tau,\mathbf{S_{n-1}})\Big\|(\xi-\tau)^{\wp-1}d\tau\\
&\le \frac{1-\wp}{\mathbf{AB}(\wp)}\mathbb{L}_1\Big\|\mathbf{S_n}-\mathbf{S_{n-1}}\Big\|+\frac{\xi^{\mathbf{n}}}{\mathbf{AB}(\wp)\Gamma(\wp)}\mathbb{L}_1\Big\|\mathbf{S_n}-\mathbf{S_{n-1}}\Big\|. \qquad (33)
\end{aligned}
$$

Continuing the procedures recursively, yields

$$
\|\widetilde{a_{1\mathbf{n}}}(\xi)\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{\xi^{\mathbf{n}}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)^{\mathbf{n}+1}\mathbb{L}_1^{\mathbf{n}}\Big\|\mathbf{S_n}-\mathbf{S_{n-1}}\Big\|^{\mathbf{n}}. \qquad (34)
$$

Setting $\xi=M_0^{\wp}$ gives

$$
\|\widetilde{a_{1\mathbf{n}}}(\xi)\| \le \Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\mathbf{n}}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)^{\mathbf{n}+1}\mathbb{L}_1^{\mathbf{n}}\Big\|\mathbf{S_n}-\mathbf{S_{n-1}}\Big\|^{\mathbf{n}}. \qquad (35)
$$

Since $\|\widetilde{a_{1\mathbf{n}}}(\xi)\|\mapsto 0$. Utilizing limit as $\mathbf{n}\mapsto\infty$ to (35), it is clear that $\|\widetilde{a_{1\mathbf{n}}}(\xi)\|\mapsto 0$ for

$$
\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\mathbf{n}}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_1<1. \qquad (36)
$$

Analogously, we prove that $\|\widetilde{a_{2\mathbf{n}}}(\xi)\|\mapsto 0$, $\|\widetilde{a_{3\mathbf{n}}}(\xi)\|\mapsto 0$, $\|\widetilde{a_{4\mathbf{n}}}(\xi)\|\mapsto 0$, $\|\widetilde{a_{5\mathbf{n}}}(\xi)\|\mapsto 0$, then

$$
\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\mathbf{n}}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_J<1,\ J=1,2,3,4,5. \quad \square \qquad (37)
$$

With the aid of Banach fixed point theorem, Theorems 6 and 7 ensures the existence of the system (6). The uniqueness of the result is provided in our upcoming result.

Theorem 8. *The fractional heroin epidemic model (6) has a unique solution, if*

$$
\Big(\frac{1-\wp}{\mathbf{AB}(\wp)}+\frac{M_0^{\mathbf{n}}}{\mathbf{AB}(\wp)\Gamma(\wp)}\Big)\mathbb{L}_J<1,\ J=1,2,3,4,5. \qquad (38)
$$

Proof. Suppose that $\widehat{\mathbf{S}}(\xi)$, $\widehat{\mathbf{U}_1}(\xi)$, $\widehat{\mathbf{U}_2}(\xi)$, $\widehat{\mathbf{Q}}(\xi)$, $\widehat{\mathbf{R}}(\xi)$ are another solutions to (6). Then

$$
\begin{aligned}
\mathbf{S}(\xi)-\widehat{\mathbf{S}}(\xi) &= \frac{1-\wp}{\mathbf{AB}(\wp)}\big(\Omega_1(\xi,\mathbf{S})-\Omega_1(\xi,\widehat{\mathbf{S}})\big)+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\\
&\quad\times\int_0^{\xi}\big(\Omega_1(\tau,\mathbf{S})-\Omega_1(\tau,\widehat{\mathbf{S}})\big)(\xi-\tau)^{\wp-1}d\tau. \qquad (39)
\end{aligned}
$$

Implementing norm on both sides, we find

$$
\Big\|\mathbf{S}(\xi)-\widehat{\mathbf{S}}(\xi)\Big\| \le \frac{1-\wp}{\mathbf{AB}(\wp)}\mathbb{L}_1\Big\|\mathbf{S}-\widehat{\mathbf{S}}\Big\|+\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\mathbb{L}_1\Big\|\mathbf{S}-\widehat{\mathbf{S}}\Big\|. \qquad (40)
$$

Since $\left(1-\frac{1-\wp}{\mathbf{AB}(\wp)}-\frac{\xi^{\wp}}{\mathbf{AB}(\wp)\Gamma(\wp)}\right)>0$, we find that $\left\|\mathbf{S}-\widehat{\mathbf{S}}\right\|=0$. Hence, we have $\mathbf{S}(\xi)=\widehat{\mathbf{S}}(\xi)$. In the same way, we can prove that $\mathbf{U}_1(\xi)=\widehat{\mathbf{U}_1}(\xi)$, $\mathbf{U}_2(\xi)=\widehat{\mathbf{U}_2}(\xi)$, $\mathbf{Q}(\xi)=\widehat{\mathbf{Q}}(\xi)$, $\mathbf{R}(\xi)=\widehat{\mathbf{R}}(\xi)$. This completes the proof. □

Numerical approaches and simulations

Numerical configurations of the model

In this part, we employ the Toufik–Atangana [38] method to construct a mathematical approach for the framework (6).

In view of the first compartment of (6), we have

$$
\begin{aligned}
&{}_{0}^{ABC}\mathbf{D}_{\xi}^{\wp}\mathbf{S}(\xi)=\Omega_1(\xi,\mathbf{S}(\xi)),\\
&\mathbf{S}(0)=\mathbf{S}_0.
\end{aligned}
\tag{41}
$$

Considering (20), we can solve for (41) in the presented problem (42):

$$
\mathbf{S}(\xi)=\mathbf{S}(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1(\xi,\mathbf{S}(\xi))+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\int_0^{\xi}\Omega_1(\tau,\mathbf{S}(\tau))(\xi-\tau)^{\wp-1}d\tau.
\tag{42}
$$

By means of Lagrange's interpolation polynomial on the interval $[\xi_\rho,\xi_{\rho+1}]$ to the identity $\Omega_1(\mathbf{y},\mathbf{S}(\mathbf{y}))=\frac{\Theta}{\nu+\phi_1}-\eta_1\mathbf{S}(\mathbf{y})\mathbf{U}_1(\mathbf{y})-\mathbf{S}(\mathbf{y})$ yields

$$
S_\rho\approx\frac{1}{\hbar}\left[(\mathbf{y}-\xi_{\rho-1})\Omega_1\left(\xi_\rho,S(\xi_\rho),\mathbf{U}_1(\xi_\rho)\right)-(\mathbf{y}-\xi_\rho)\Omega_1\left(\xi_{\rho-1},S(\xi_{\rho-1}),\mathbf{U}_1(\xi_{\rho-1})\right)\right],
\tag{43}
$$

where $\hbar=\xi_\rho-\xi_{\rho-1}$.

Plugging (43) into (42), we have

$$
\begin{aligned}
\mathbf{S}(\xi_{\mathbf{n}+1}) &= \mathbf{S}(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1\left(\xi_\rho,\mathbf{S}(\xi_\rho),\mathbf{U}_1(\xi_\rho)\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\begin{cases}\frac{\Omega_1\left(\xi_i,\mathbf{S}(\xi_i),\mathbf{U}_1(\xi_i)\right)}{\hbar}\int_{\xi_i}^{\xi_{i+1}}(\mathbf{y}-\xi_{i-1})\\ \times(\xi_{\mathbf{n}+1}-\mathbf{y})^{\wp-1}d\mathbf{y}\\ -\frac{\Omega_1\left(\xi_{i-1},\mathbf{S}(\xi_{i-1}),\mathbf{U}_1(\xi_{i-1})\right)}{\hbar}\int_{\xi_i}^{\xi_{i+1}}(\mathbf{y}-\xi_{i-1})\\ \times(\xi_{\mathbf{n}+1}-\mathbf{y})^{\wp-1}d\mathbf{y}\end{cases}\\
&= \mathbf{S}(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1\left(\xi_{\mathbf{n}},\mathbf{S}(\xi_{\mathbf{n}}),\mathbf{U}_1(\xi_{\mathbf{n}})\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\left(\frac{\Omega_1\left(\xi_i,\mathbf{S}(\xi_i),\mathbf{U}_1(\xi_i)\right)}{\hbar}Y_{i-1}\right.\\
&\quad\left.-\frac{\Omega_1\left(\xi_{i-1},\mathbf{S}(\xi_{i-1}),\mathbf{U}_1(\xi_{i-1})\right)}{\hbar}Y_i\right),
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
Y_{i-1} &= \int_{\xi_i}^{\xi_{i+1}}(\mathbf{y}-\xi_{i-1})(\xi_{\mathbf{n}+1}-\mathbf{y})^{\wp-1}d\mathbf{y}\\
&= -\frac{1}{\wp}\left[(\xi_{i+1}-\xi_{i-1})(\xi_{\mathbf{n}+1}-\xi_{i+1})^{\wp}-(\xi_i-\xi_{i-1})(\xi_{\mathbf{n}+1}-\xi_i)^{\wp}\right]\\
&\quad-\frac{1}{\wp(\wp+1)}\left[(\xi_{\mathbf{n}+1}-\xi_{i+1})^{\wp+1}(\xi_{\mathbf{n}+1}-\xi_{i+1})^{\wp}-(\xi_{\mathbf{n}+1}-\xi_i)^{\wp+1}\right],\\
Y_i &= \int_{\xi_i}^{\xi_{i+1}}(\mathbf{y}-\xi_{i-1})(\xi_{\mathbf{n}+1}-\mathbf{y})^{\wp-1}d\mathbf{y}\\
&= -\frac{1}{\wp}\left[(\xi_{i+1}-\xi_{i-1})(\xi_{\mathbf{n}+1}-\xi_{i+1})^{\wp}\right]\\
&\quad-\frac{1}{\wp(\wp+1)}\left[(\xi_{\mathbf{n}+1}-\xi_{i+1})^{\wp+1}-(\xi_{\mathbf{n}+1}-\xi_i)^{\wp+1}\right].
\end{aligned}
\tag{45}
$$

Moreover, plugging $\xi_i=i\hbar$ into (44) and (45) gives

$$
Y_{i-1}=\frac{\hbar^{\wp+1}}{\wp(\wp+1)}\left[(\mathbf{n}+1-i)^{\wp}(\mathbf{n}-i+2+\wp)-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+2+2\wp)\right],
$$

$$
Y_i=\frac{\hbar^{\wp+1}}{\wp(\wp+1)}\left[(\mathbf{n}+1-i)^{\wp+1}-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+1+\wp)\right].
\tag{46}
$$

Ultimately, we can interpret (44) as simply in contexts of (46):

$$
\begin{aligned}
\mathbf{S}(\xi_{\mathbf{n}+1}) &= \mathbf{S}(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_1\left(\xi_\rho,\mathbf{S}(\xi_\rho),\mathbf{U}_1(\xi_\rho)\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\begin{cases}\frac{\Omega_1\left(\xi_i,\mathbf{S}(\xi_i),\mathbf{U}_1(\xi_i)\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp}(\mathbf{n}-i+2+\wp)\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+2+2\wp)\right]\\ -\frac{\Omega_1\left(\xi_{i-1},\mathbf{S}(\xi_{i-1}),\mathbf{U}_1(\xi_{i-1})\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp+1}\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+1+\wp)\right].\end{cases}
\end{aligned}
\tag{47}
$$

Similarly, for the other system parameters, we get the corresponding formulas:

$$
\begin{aligned}
\mathbf{U}_1(\xi_{\mathbf{n}+1}) &= \mathbf{U}_1(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_2\left(\xi_\rho,\mathbf{S}(\xi_\rho),\mathbf{U}_1(\xi_\rho),\mathbf{U}_2(\xi_\rho),\mathbf{Q}(\xi_\rho)\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\begin{cases}\frac{\Omega_2\left(\xi_i,\mathbf{S}(\xi_i),\mathbf{U}_1(\xi_i),\mathbf{U}_2(\xi_i),\mathbf{Q}(\xi_i)\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp}(\mathbf{n}-i+2+\wp)\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+2+2\wp)\right]\\ -\frac{\Omega_2\left(\xi_{i-1},\mathbf{S}(\xi_{i-1}),\mathbf{U}_1(\xi_{i-1}),\mathbf{U}_2(\xi_{i-1}),\mathbf{Q}(\xi_{i-1})\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp+1}\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+1+\wp)\right],\end{cases}\\
\mathbf{U}_2(\xi_{\mathbf{n}+1}) &= \mathbf{U}_2(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_3\left(\xi_\rho,\mathbf{U}_1(\xi_\rho),\mathbf{U}_2(\xi_\rho)\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\begin{cases}\frac{\Omega_3\left(\xi_i,\mathbf{U}_1(\xi_i),\mathbf{U}_2(\xi_i)\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp}(\mathbf{n}-i+2+\wp)\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+2+2\wp)\right]\\ -\frac{\Omega_3\left(\xi_{i-1},\mathbf{U}_1(\xi_{i-1}),\mathbf{U}_2(\xi_{i-1})\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp+1}\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+1+\wp)\right],\end{cases}\\
\mathbf{Q}(\xi_{\mathbf{n}+1}) &= \mathbf{Q}(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_4\left(\xi_\rho,\mathbf{U}_2(\xi_\rho),\mathbf{Q}(\xi_\rho)\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\begin{cases}\frac{\Omega_4\left(\xi_i,\mathbf{U}_2(\xi_i),\mathbf{Q}(\xi_i)\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp}(\mathbf{n}-i+2+\wp)\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+2+2\wp)\right]\\ -\frac{\Omega_4\left(\xi_{i-1},\mathbf{U}_2(\xi_{i-1}),\mathbf{Q}(\xi_{i-1})\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp+1}\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+1+\wp)\right],\end{cases}\\
\mathbf{R}(\xi_{\mathbf{n}+1}) &= \mathbf{R}(0)+\frac{1-\wp}{\mathbf{AB}(\wp)}\Omega_5\left(\xi_\rho,\mathbf{S}(\xi_\rho),\mathbf{Q}(\xi_\rho),\mathbf{R}(\xi_\rho)\right)\\
&\quad+\frac{\wp}{\mathbf{AB}(\wp)\Gamma(\wp)}\sum_{i=1}^{\mathbf{n}}\begin{cases}\frac{\Omega_5\left(\xi_i,\mathbf{S}(\xi_i),\mathbf{Q}(\xi_i),\mathbf{R}(\xi_i)\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp}(\mathbf{n}-i+2+\wp)\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+2+2\wp)\right]\\ -\frac{\Omega_5\left(\xi_{i-1},\mathbf{S}(\xi_{i-1}),\mathbf{Q}(\xi_{i-1}),\mathbf{R}(\xi_{i-1})\right)}{\Gamma(\wp+2)}\\ \times\hbar^{\wp}\left[(\mathbf{n}+1-i)^{\wp+1}\right.\\ \left.-(\mathbf{n}-i)^{\wp}(\mathbf{n}-i+1+\wp)\right].\end{cases}
\end{aligned}
\tag{48}
$$

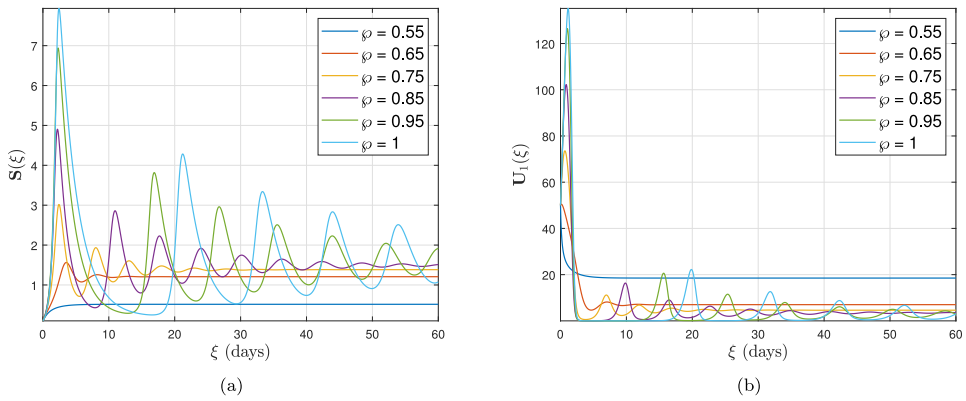

Fig. 4. (a) Susceptible individuals **S** (b) drug users not in treatment $\mathbf{U}_1$ for various fractional-order when $\eta_1 = 0.0002,\ \sigma_1 = 0.008,\ \sigma_2 = 0.00008$ and $\phi_1 = 0.02$ with ICs $(2, 2, 2, 2, 2)$.

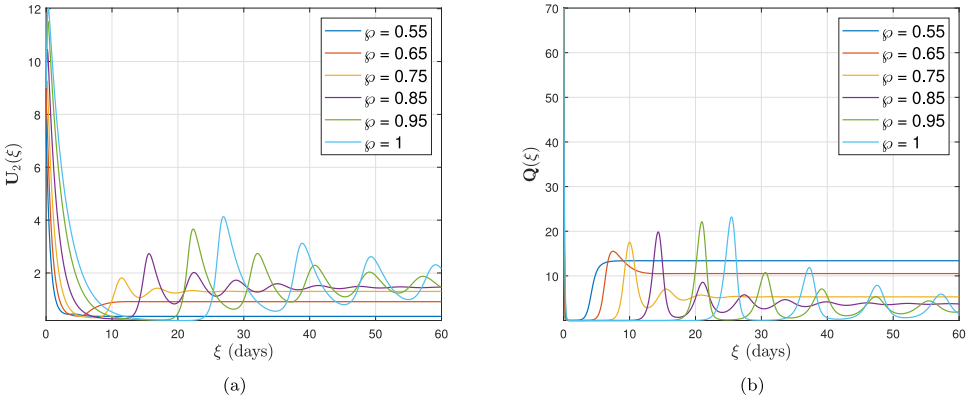

Fig. 5. (a) Drug users in treatment $\mathbf{U}_2$ (b) people who have used drugs in the past but are not doing so now but may do so in the future **Q** for various fractional-order when $\eta_1 = 0.0002,\ \sigma_1 = 0.008,\ \sigma_2 = 0.00008$ and $\phi_1 = 0.02$ with ICs $(2, 2, 2, 2, 2)$.

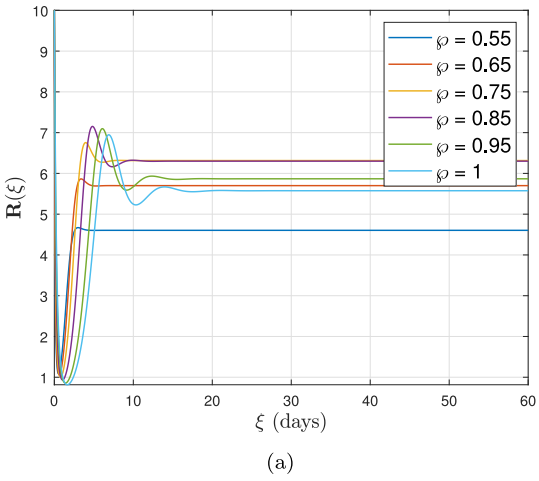

Fig. 6. (a) Individuals who have never consumed drugs **R** for various fractional-order when $\eta_1 = 0.0002,\ \sigma_1 = 0.008,\ \sigma_2 = 0.00008$ and $\phi_1 = 0.02$ with ICs $(2, 2, 2, 2, 2)$.

Numerical results and discussion

Numerous numerical simulations (described in the preface) have subsequently been constructed to characterize the heroin epidemic. The majority of these heroin modelling techniques are ODE simulations that imply the rate of recidivism is unaffected by treatment duration. Following on fractional epidemiology concepts [12–14], we developed a fractional order heroin model approach, including control symptoms, in this research. The approach takes into consideration the risk of relapse, which varies regardless of how long the patient has been treated.

For the fractional order heroin epidemic model (18), σ_1, η_1, σ_2, ϕ_1 and ϕ_2 indicate the decontamination performance level, disinfection malfunction proportion, recidivism incidence from **Q** to perpetrators, persistent removal speeds from **S** to **R**, and the continuous abstinence levels from **Q** to **R**, correspondingly. Figs. 4–6 demonstrates the connection involving $\mathcal{R}_0$ and σ_1, η_1, σ_2, ϕ_1, including other attribute estimates from Table 1. As presented in Figs. 4–6, $\mathcal{R}_0$ rises to σ_2 and η_2 and though declines to σ_1, implying that if we prefer to regulate drug habit, we must lessen σ_2 and boost σ_1, i.e., upsurge the disinfection performance level and compensate consideration to individuals with a collective memory of drug addiction to mitigate their psychological reliance on narcotics. Furthermore, as seen in Plots 4–6, $\mathcal{R}_0$ declines to ϕ_2, implying so if we intend to prevent heroin dependency, we need to enhance community understanding of the adverse effects of heroin and encourage individuals to adopt the responsibility to abstain from heroin. We might conclude that minimizing the fractional-order $\wp$ of derivative substantially reduces the amount of susceptible, exponential contaminated, clinical symptoms compromised, and cured individuals.

The influence of ϕ_1 and ϕ_2 on the fractional heroin epidemic system (18) is depicted in Figs. 7–9. It reveals that the parameters ϕ_1 and ϕ_2 have a considerable impact on the quantity of heroin-substance dynamic systems. Figs. 7–9 depicts that the bigger the ϕ_1 is, the less the individuals that consume narcotics in equilibria are, implying that we should not always focus on illicit substances but also those whom do not abuse substances. The authorities might promote public information about narcotics deterrence initiatives. Attempting to compare Figs. 7–9, we will see that, while the consequence of ϕ_2 on $\mathbf{U}_1$ is far reduced total

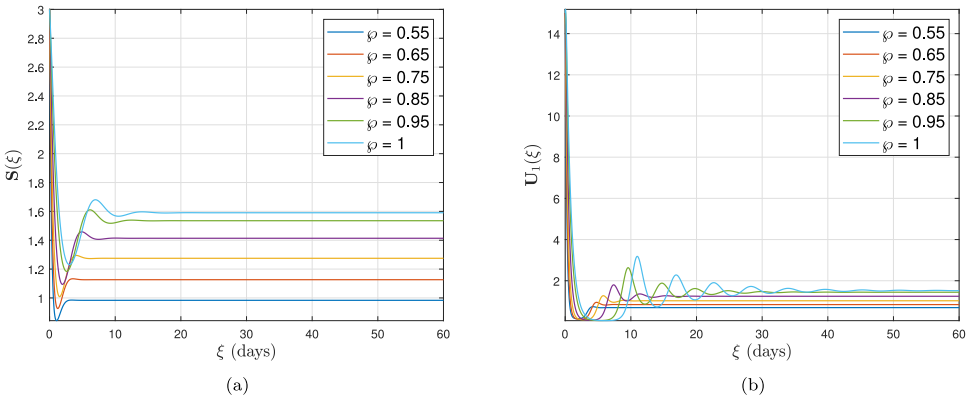

Fig. 7. (a) Susceptible individuals **S** (b) drug users not in treatment $\mathbf{U}_1$ for various fractional-order when $\eta_1 = 0.0002,\ \sigma_1 = 0.008,\ \sigma_2 = 0.00008$ and $\phi_1 = 0.08$ with ICs $(5, 10, 3, 6, 1)$.

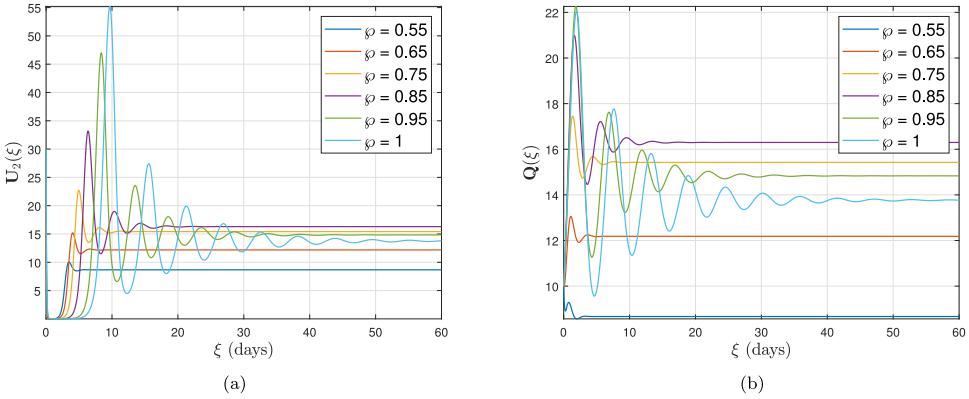

Fig. 8. (a) Drug users in treatment $\mathbf{U}_2$ (b) people who have used drugs in the past but are not doing so now but may do so in the future **Q** for various fractional-order when $\eta_1 = 0.0002,\ \sigma_1 = 0.008,\ \sigma_2 = 0.00008$ and $\phi_1 = 0.08$ with ICs $(5, 10, 3, 6, 1)$.

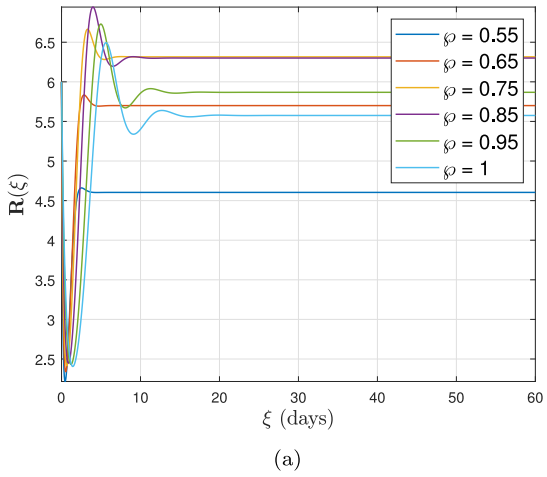

Fig. 9. (a) Individuals who have never consumed drugs **R** for various fractional-order when $\eta_1 = 0.0002,\ \sigma_1 = 0.008,\ \sigma_2 = 0.00008$ and $\phi_1 = 0.08$ with ICs $(5, 10, 3, 6, 1)$.

than influence of ϕ_1, the significantly bigger the ϕ_2 is, the some less individuals who consume substances in equilibration are; thus, we must also incur publicity to persons who have now exited substance misuse, so how they could indeed stay drug free rather than re-purposing substances. Graphs 7–9 exemplifies that lowering the fractional-order $\wp$ from 1 deforms the curves of the suspicious incidents (**S**) for a significant reductions in the amount of instances in the compartments. As the ratio decreases, the majority of instances becomes the constant rate $\mathbf{S}_0$.

Figs. 10–12 reveal that $\mathbf{U}_1$ plummeted dramatically and eventually fluctuated, and moreover, all scheme configurations are endlessly similar to the substance optimum $\mathcal{E}_0$. It proves the presence of $\mathcal{E}_0$ having $\eta_1 = 0.01,\ \sigma_2 = 0.008,\ \sigma_1 = 0.01,\ \phi_1 = 0.01$ and ICs are $(25, 25, 25, 25, 25)$. Figs. 10–12 imply that when the fractional derivative becomes lower, the time required to attain the endemic equilibria becomes higher and higher. That is, when the derivative order is decreased from 1, the static system's reminiscence impact grows. As a result of the spreading phenomenon, drug regulation is becoming progressively more challenging for authorities. Surprisingly, the fractional heroin pandemic model (18) emerges as a narcotic equilibrium, convincing us that heroin proliferation can be avoided.

Conclusion

In this study, we created a distinctive heroin epidemic framework that combines the recidivism component as well as the persistent immunization component via the AB-fractional derivative in the Caputo perspectives. We calculated the fundamental reproductive value, $\mathcal{R}_0$ utilizing the succeeding matrix approach. By creating certain appropriate Lyapunov functions, we were capable of extracting the model's global behaviour. It is demonstrated that when $\mathcal{R}_0 = 1$, the narcotic equilibria is globally asymptotically stable, implying that substance consumption will be exterminated; when $\mathcal{R}_0 > 1$, the pandemic state is asymptotically reliable, implying that drug dependence will remain everlasting. The simulated findings demonstrate that lowering the ordering of the fractional derivative from 1 flattens the curves, and the probability diminishes gradually for suspected ($\mathbb{S}$) situations. This

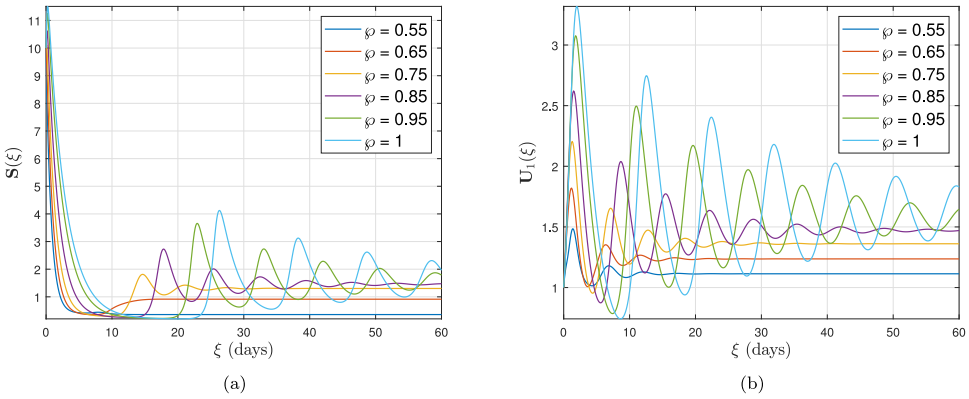

Fig. 10. (a) Susceptible individuals **S** (b) drug users not in treatment $\mathbf{U}_1$ for various fractional-order when $\eta_1 = 0.008,\ \sigma_1 = 0.01,\ \sigma_2 = 0.008$ and $\phi_1 = 0.02$ with ICs $(25, 25, 25, 25, 25)$.

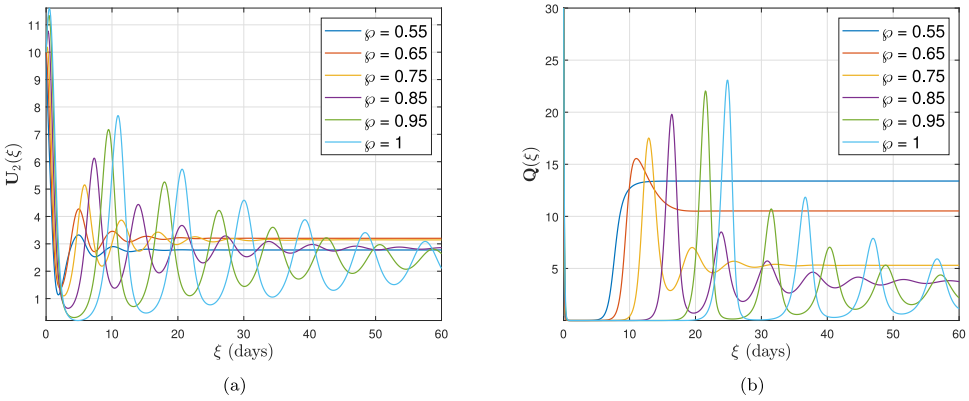

Fig. 11. (a) Drug users in treatment $\mathbf{U}_2$ (b) people who have used drugs in the past but are not doing so now but may do so in the future **Q** for various fractional-order when $\eta_1 = 0.008,\ \sigma_1 = 0.01,\ \sigma_2 = 0.008$ and $\phi_1 = 0.02$ with ICs $(25, 25, 25, 25, 25)$.

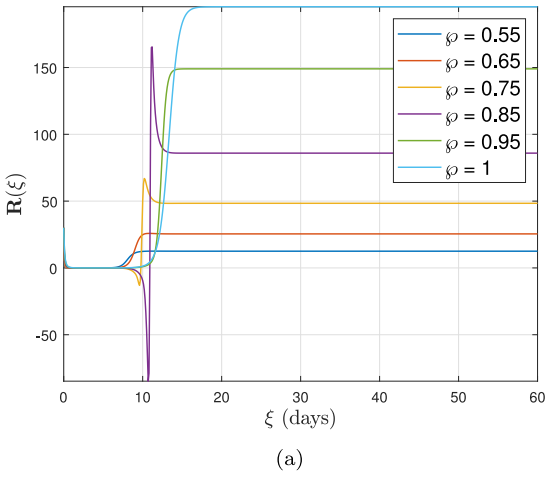

Fig. 12. (a) Individuals who have never consumed drugs **R** for various fractional-order when $\eta_1 = 0.008,\ \sigma_1 = 0.01,\ \sigma_2 = 0.008$ and $\phi_1 = 0.02$ with ICs $(25, 25, 25, 25, 25)$.

significant achievement is attributed to the ABC-fractional operator exhibiting the heredity characteristic. The exponential and power law functions are not as good as the generalized Mittag-Leffler function with robust memory entangled in the Atangana–Baleanu fractional derivative. Furthermore, the Atangana–Baleanu fractional order derivative is at the same time Liouville–Caputo and Caputo–Fabrizio thus possesses Markovian and non-Markovian properties. We, the researchers of this study, contend that numerical methods incorporating the AB-fractional operator can adequately disclose the concealed or underlying features of real-world situations. This hypothesis could be supported by undertaking additional research on the influence of alternative fractional operators, such as fractal–fractional derivatives, and reporting the performance of the ABC-fractional operator findings on analogous systems as well as other relevant infectious systems.

CRediT authorship contribution statement

Saima Rashid: Conceptualization, Methodology, Investigation, Resources, Data curation, Writing – original draft, Supervision, Project administration, Funding acquisition. **Fahd Jarad:** Conceptualization, Investigation, Resources, Data curation, Writing – original draft, Supervision, Project administration, Funding acquisition. **Abdulaziz Garba Ahmad:** Conceptualization, Methodology, Resources, Data curation, Writing – original draft, Supervision, Funding acquisition. **Khadijah M. Abualnaja:** Conceptualization, Methodology, Investigation, Resources, Data curation, Writing – original draft, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of supporting data

No data were used to support this study.

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