AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(3): 4386–4404.
DOI: 10.3934/math.2022244
Received: 09 October 2021
Revised: 05 December 2021
Accepted: 13 December 2021
Published: 21 December 2021

Research article

New results for a coupled system of ABR fractional differential equations with sub-strip boundary conditions

Mohammed A. Almalahi[1,2], **Satish K. Panchal**[2], **Tariq A. Aljaaidi**[2] **and Fahd Jarad**[3,4,*]

[1] Department of Mathematics, Hajjah University, Hajjah, Yemen
[2] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad (M.S), 431001, India
[3] Department of Mathematics, Çankaya University, Etimesgut, Ankara, 06790, Turkey
[4] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

* **Correspondence:** Email: fahd@cankaya.edu.tr.

Abstract: In this article, we investigate sufficient conditions for the existence, uniqueness and Ulam-Hyers (UH) stability of solutions to a new system of nonlinear ABR fractional derivative of order $1 < \varrho \leq 2$ subjected to multi-point sub-strip boundary conditions. We discuss the existence and uniqueness of solutions with the assistance of Leray-Schauder alternative theorem and Banach's contraction principle. In addition, by using some mathematical techniques, we examine the stability results of Ulam-Hyers (UH). Finally, we provide one example in order to show the validity of our results.

Keywords: sub-strip boundary conditions; coupled system; existence; fixed point theorem
Mathematics Subject Classification: 34A08, 34B15, 34A12, 47H10

1. Introduction

For the last three decades, fractional calculus has caught importance and popularity among researchers due to its applicability in modeling many phenomena of the real-world such as propagation in complex mediums, polymers, biological tissues, earth sediments, etc. For more details about applications of fractional calculus, we refer the reader to monographs of Podlubny [1], Samko [2], Kilbas [3], Hilfer [4], and references therein. One of the features of fractional calculus is the fact there are many types of derivatives and thus the researchers can use the most suitable fractional derivative for the model they work on. Some of these researchers realized the need for fractional operators with non-singular kernels in modeling some phenomena. Caputo and Fabrizio in [5] studied a new kind of fractional derivative with an exponential kernel. A new type and interesting fractional derivative with Mittag-Leffler kernels were developed by Atangana and Baleanu in [6]. Abdeljawad in [7] extended this fractional derivative from order between zero and one to higher arbitrary order and formulated their associated integral operators. Atangana [8, 9] introduced some new types of fractional derivatives in the form of power-law and generalized Mittag-Leffler. Many researchers have realized the importance of these new fractional derivatives and applied them to study some properties of solutions for some problems in different fields of science and engineering (see [10–14]). The famous kinds of stability of fractional differential equations are Ulam, Ulam-Hyers, and Ulam-Hyers-Rassias stability. For more details on kinds of stability, we refer the reader to monographs of Ulam [15], Hyers [16] and Rassias [17].

Coupled systems of fractional differential equations appear in modeling many phenomena of real-world problems. Ahmad et al. [18, 19] studied existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Recently, Almalahi et al. [20] studied the existence, uniqueness, and Ulam-Hyers stability results for a coupled system of generalized Hilfer sequential fractional differential equations with two-point boundary conditions by means of Leray-Schauder alternative and Banach fixed point theorem. Almalahi et al. [21] studied stability results of positive solutions for a system of generalized Hilfer fractional differential equations building upper and lower control functions and using some techniques of nonlinear functional analysis. Utilizing the Banach and Krasnoselskii fixed point theorems. Alsaedi et al. [22] studied the existence and uniqueness results for a nonlinear Caputo-Riemann-Liouville type fractional integro-differential boundary value problem with multi-point sub-strip boundary conditions in the form

$$\begin{cases} {}^{C}D^{\varrho}\vartheta(\sigma) + \sum_{i=1}^{k} I^{p_i} g_i(\sigma, \vartheta(\sigma)) = f(\sigma, \vartheta(\sigma)), \\ \vartheta(0) = 0, \vartheta'(0) = 0, \vartheta''(0) = 0, \ldots.., \vartheta^{(m-2)}(0) = 0, \\ \alpha\vartheta(1) + \beta\vartheta'(1) = \varpi_1 \int_0^{\tau} \vartheta(s)ds + \sum_{i=1}^{p} \mu_i \vartheta(\eta_i) + \varpi_2 \int_{\tau}^{1} \vartheta(s)ds, \end{cases}$$

where ${}^{C}D^{\varrho}$ represents the Caputo fractional derivative operator of order $\varrho \in (m-1, m], m \in \mathbb{N}, m \geq 2, p_i > 0, 0 < \tau, \eta_1, \eta_2, ..., \eta_p < 1, \alpha, \beta, \varpi_1, \varpi_2 \in \mathbb{R}, \mu_i \in \mathbb{R}, i = 1, 2, ...p$ and $f, g_i : [0, 1] \times \mathbb{R} \to \mathbb{R}, i = 1, 2, ...k$ are continuous functions.

In [23], Alsaedi et al. discussed the existence and uniqueness of solutions for the following coupled system

$$\begin{cases} {}^{C}D^{\varrho_1}\vartheta_1(\sigma) + \sum_{i=1}^{k} I^{p_i} g_i(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)) = f_1(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)), \\ {}^{C}D^{\varrho_2}\vartheta_1(\sigma) + \sum_{j=1}^{l} I^{\nu_j} g_j(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)) = f_1(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)), \end{cases}$$

subjected to the conditions

$$\begin{cases} \vartheta_1(0) = a_1, \vartheta_2(0) = a_2, \\ \alpha_1\vartheta_1(1) + \beta_1\vartheta_1'(1) = \varpi_1 \int_0^{\tau} \vartheta_2(s)ds + \sum_{i=1}^{m} \mu_i \vartheta_2(\eta_i), \\ \alpha_2\vartheta_2(1) + \beta_2\vartheta_2'(1) = \varpi_2 \int_0^{\tau} \vartheta_1(s)ds + \sum_{i=1}^{m} \xi_i \vartheta_1(\eta_i), \end{cases}$$

where ${}^{C}D^{\varrho_1}, {}^{C}D^{\varrho_2}$ represents the Caputo fractional derivative of order $\varrho_1, \varrho_2 \in (1, 2]$.

Motivated by the novel advancements of Atangana-Baleanu and its applications and the above argumentations, the intent of this work is to investigate the existence, uniqueness, and stability results of a new coupled system under a new fractional derivative so-called ABR fractional derivative of order

$1 < \varrho_1, \varrho_2 \leq 2$ with multi-point sub-strip boundary conditions described by

$$\begin{cases} {}^{ABR}_{0^+}\mathbf{D}^{\varrho_1}\vartheta_1(\sigma) = f_1(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)), \ \sigma \in [0,1], \\ {}^{ABR}_{0^+}\mathbf{D}^{\varrho_2}\vartheta_2(\sigma) = f_2(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)) \ \sigma \in [0,1], \\ \alpha_1\vartheta_1(1) = \varpi_1 \int_0^\tau \vartheta_2(s)ds + \sum_{i=1}^m \mu_i\vartheta_2(\eta_i), \\ \alpha_2\vartheta_2(1) = \varpi_2 \int_0^\tau \vartheta_1(s)ds + \sum_{i=1}^m \xi_i\vartheta_1(\eta_i), \end{cases} \tag{1.1}$$

where

• ${}^{ABR}D^p_{a^+}$ represents the Atangana-Baleanu-Riemann fractional derivative of order $p = \{\varrho_1, \varrho_2\} \subset (1,2]$.

• $\alpha_1, \alpha_2, \varpi_1, \varpi_2, \mu_i, \xi_i \in \mathbb{R}$, and $\eta_i, \tau \in (0,1)$, $i = 1, 2, , ..., m$.

• $f_j : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous function, $j = 1, 2$.

In this work, we consider a new type of coupled system involving new fractional operators which extended lately to higher-order by Abdeljawad [7]. We considered the system (1.1) with multipoint sub strip conditions, which means our results yield some new results related to choosing the parameters, if $\varpi_1 = \varpi_2 = 0$, then the system (1.1) reduce to the system with coupled multi-point boundary conditions and if $\mu_i = \xi_i = 0$, then the system (1.1) reduce to the system with coupled sub-strip boundary conditions.

We investigated the existence and uniqueness of the solution as well as Ulam-Hyers and generalized Ulam-Hyers stability of the proposed coupled system by using minimal conditions.

The main contribution of this work is to find an equivalent fractional integral equation for the suggested system and to prove its existence, uniqueness, and Ulam-Hyers (UH) stability results for a new system under a new fractional derivative. The fixed point theorems of Banach and Leray-Schauder are used in our analysis. Despite the fact that we employ common methods to get our conclusions, the application of it to the suggested system is novel. Furthermore, the results acquired in this study may be extended to an n-tuple fractional system. Our results obtained include the results of Alsaedi et al. in [22, 23]. With regard to the boundary condition at the terminal position $\sigma = 1$ used in this work, the linear combination of the unknown function and its derivative is associated with the contribution due to sub-strip $(0, \tau)$ and finitely many nonlocal positions between them within the domain [0.1]. This boundary condition covers many interesting situations, for example, it corresponds to the two-strip aperture condition for all $\mu_i = \xi_i = 0, i = 1, 2, ..., m$. By choosing $\varpi_1 = \varpi_2 = 0$, this condition reduces to a multi-point nonlocal boundary condition. It's worth noting that integral boundary conditions play a critical role in the research of practical problems like blood flow problems [24] and bacterial self-regularization [25], among others. For more applications about strip conditions in engineering and real-world problems (see [26, 27]). To the best of our knowledge, this is the first work dealing with the ABR fractional derivative of order $\varrho_1, \varrho_2 \in (1,2]$ with multi-point sub-strip boundary conditions. In consequence, the results of this work will be a useful contribution to the existing literature on this topic.

The paper is organized as follows: In Section 2, we present notations and some preliminary facts used throughout the paper. Section 3 discusses the existence and uniqueness results for ABR-System (1.1). The stability analysis in the frame of Ulam-Hyers has been discussed in Section 4. Section 5 provides an example to illustrate the validity of our results. Concluding remarks about our results in the last Section.

2. Preliminaries

To achieve our main objectives, we present here some definitions and basic auxiliary results that are required throughout the paper. Let $\mathcal{J} = [0,1] \subset \mathbb{R}$ and $\mathcal{X} = C(\mathcal{J},\mathbb{R})$ be the space of continuous functions $\vartheta : \mathcal{J} \to \mathbb{R}$ equipped with the norm $\|\vartheta\| = \sup_{\sigma\in\mathcal{J}} |\vartheta(\sigma)|$. Evidently, $(\mathcal{X}, \|\cdot\|)$ is a Banach space and hence the product space $\mathcal{H} := \mathcal{X}\times\mathcal{X}$ is also a Banach space with the following norm $\|(\vartheta_1,\vartheta_2)\| = \|\vartheta_1\| + \|\vartheta_2\|$.

Definition 2.1. *[6] Let $0 < \varrho \leq 1$ and $\vartheta \in H^1(\mathcal{J})$. Then the left-sided ABR fractional derivative of order ϱ for a function ϑ with the lower limit zero is defined by*

$$ {}^{ABR}D^{\varrho}_{0^+}\vartheta(\sigma) = \frac{B(\varrho)}{1-\varrho}\frac{d}{d\sigma}\int_0^{\sigma} E_{\varrho}\left(\frac{\varrho}{\varrho-1}(\sigma-\theta)^{\varrho}\right)\vartheta'(\theta)d\theta, \ \ \sigma > a, $$

where $B(\varrho) = \frac{\varrho}{2-\varrho} > 0$ is the normalization function such that $B(0) = B(1) = 1$ and E_{ϱ} is the Mittag-Leffler function defined by

$$ E_{\varrho}(\vartheta) = \sum_{i=0}^{\infty}\frac{\vartheta^i}{\Gamma(i\varrho+1)}, \ Re(\varrho) > 0, \vartheta \in \mathbb{C}. $$

The associated Atangana-Baleanu (AB) fractional integral is given by

$$ {}^{AB}I^{\varrho}_{0^+}\vartheta(\sigma) = \frac{1-\varrho}{B(\varrho)}\vartheta(\sigma) + \frac{\varrho}{B(\varrho)\Gamma(\varrho)}\int_0^{\sigma}(\sigma-s)^{\varrho-1}\vartheta(s)ds. $$

Definition 2.2. *[7] The relation between the ABR and ABC fractional differential equations is given by*

$$ {}^{ABC}D^{\varrho}_{a^+}\vartheta(\sigma) = {}^{ABR}D^{\varrho}_{a^+}\vartheta(\sigma) + \frac{B(\varrho)}{1-\varrho}\vartheta(a)E_{\varrho}\left(\frac{\varrho}{\varrho-1}(\sigma-a)^{\varrho}\right). $$

Lemma 2.3. *[6] Let $\vartheta > 0$. Then ${}^{AB}I^{\varrho}_{0^+}$ is bounded from $\mathcal{X}$ into $\mathcal{X}$.*

Definition 2.4. *([7] Definition 3.1) Let $n < \varrho \leq n+1$ and $\vartheta^{(n)} \in H^1(0,1)$. Let $\beta = \varrho - n$. Then, $0 < \beta \leq 1$ and the left-sided ABR fractional derivative of order ϱ for a function ϑ with the lower limit zero is defined by*

$$ \left({}^{ABR}\mathbf{D}^{\varrho}_{0^+}\vartheta\right)(\sigma) = \left({}^{ABR}D^{\beta}_{0^+}\vartheta^{(n)}\right)(\sigma). $$

The correspondent fractional integral is given by

$$ \left({}^{AB}\mathbf{I}^{\varrho}_{0^+}\vartheta\right)(\sigma) = \left(I^{n AB}_{0^+}I^{\beta}_{0^+}\vartheta\right)(\sigma). $$

Lemma 2.5. *([7] Proposition 3.1) Let $\vartheta(\sigma)$ be a function defined on $[0,b]$ and $n < \varrho \leq n+1$. Then, for some $n \in \mathbb{N}_0$, we have*

- $\left({}^{ABR}\mathbf{D}^{\varrho AB}_{0^+}\mathbf{I}^{\varrho}_{0^+}\vartheta\right)(\sigma) = \vartheta(\sigma),$.
- $\left({}^{AB}\mathbf{I}^{\varrho ABR}_{0^+}\mathbf{D}^{\varrho}_{0^+}\vartheta\right)(\sigma) = \vartheta(\sigma) - \sum_{i=0}^{n-1}\frac{\vartheta^{(i)}(0)}{i!}\sigma^i.$
- $\left({}^{AB}\mathbf{I}^{\varrho ABC}_{0^+}\mathbf{D}^{\varrho}_{0^+}\vartheta\right)(\sigma) = \vartheta(\sigma) - \sum_{i=0}^{n}\frac{\vartheta^{(i)}(0)}{i!}\sigma^i.$

Lemma 2.6. *([7] Theorem 4.2) Let* $\varrho \in (1,2]$ *and* $\hbar_i \in \mathcal{X}, i = 1,2$*. Then the solution of the following problem*

$$\begin{aligned} {}^{ABR}\mathbf{D}^{\varrho}_{0^+}\vartheta(\sigma) &= \hbar(\sigma), \\ \vartheta(a) &= c, \end{aligned}$$

is given by

$$\vartheta(\sigma) = c + \frac{2-\varrho}{B(\varrho-1)}\int_0^{\sigma}\hbar(s)ds + \frac{\varrho-1}{B(\varrho-1)\Gamma(\varrho)}\int_0^{\sigma}(\sigma-s)^{\varrho-1}\hbar(s)ds.$$

Theorem 2.7. *[28] Let K be closed subset from a Banach space* $\mathcal{X}$*, and* $\mathcal{G} : \mathcal{K} \to \mathcal{K}$*, be a strict contraction i.e.,* $\|\mathcal{G}(x) - \mathcal{G}(y)\| \le L\|x-y\|$ *for some* $0 < L < 1$ *and all* $x, y \in \mathcal{K}$*. Then* $\mathcal{G}$ *has a fixed point in* $\mathcal{K}$*.*

Lemma 2.8. *[29] Let* $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ *be an operator satisfies*

- *The operator* $\mathcal{G}$ *is completely continuous,*
- *The set* $\xi(\mathcal{G}) = \{\vartheta \in \mathcal{X} : \vartheta = \delta\mathcal{G}(\vartheta), \delta \in [0,1]\}$ *is bounded.*

Then, $\mathcal{G}$ *has at least one fixed point.*

Theorem 2.9. *Let* $\varrho_1, \varrho_2 \in (1,2], \Theta = \alpha_1\alpha_2 - \left(\varpi_1\tau + \sum_{i=1}^m \mu_i\eta_i\right)\left(\varpi_2\tau + \sum_{i=1}^m \xi_i\eta_i\right) \neq 0, \alpha_1, \alpha_2, \varpi_1, \varpi_2, \mu_i, \xi_i \in \mathbb{R}$*, and* $\eta_i, \tau \in (0,1)$*,* $i = 1,2,,...,m$ *and* $\hbar_1, \hbar_2 \in \mathcal{X}$*. The unique solution* $(\vartheta_1, \vartheta_2) \in \mathcal{H}$ *of the following problem*

$$\begin{cases} {}^{ABR}\mathbf{D}^{\varrho_1}_{0^+}\vartheta_1(\sigma) = \hbar_1(\sigma),\ \sigma \in [0,1], \\ {}^{ABR}\mathbf{D}^{\varrho_2}_{0^+}\vartheta_2(\sigma) = \hbar_2(\sigma)\ \sigma \in [0,1], \\ \alpha_1\vartheta_1(1) - \varpi_1\int_0^{\tau}\vartheta_2(s)ds = \sum_{i=1}^m \mu_i\vartheta_2(\eta_i), \\ \alpha_2\vartheta_2(1) - \varpi_2\int_0^{\tau}\vartheta_1(s)ds = \sum_{i=1}^m \xi_i\vartheta_1(\eta_i), \end{cases} \tag{2.1}$$

is given by

$$\vartheta_1(\sigma) = \begin{cases} \frac{1}{\Theta}\Big[\frac{\alpha_2\varpi_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^{\tau}\int_0^{s}\hbar_2(u)duds + \frac{\alpha_2\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\tau}\int_0^{s}(s-u)^{\varrho_2-1}\hbar_2(u)duds \\ \quad + \sum_{i=1}^m \mu_i\left(\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)}\int_0^{\eta_i}\hbar_2(s)ds + \frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}\hbar_2(s)ds\right) \\ -\frac{\alpha_2\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^1\hbar_1(s)ds - \frac{\alpha_2\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\hbar_1(s)ds \\ +\pi_1\int_0^{\tau}\int_0^{s}\hbar_1(u)duds + \frac{\pi_2}{\Gamma(\varrho_1)}\int_0^{\tau}\int_0^{s}(s-u)^{\varrho_1-1}\hbar_1(u)duds \\ \quad + \sum_{i=1}^m \xi_i\left(\pi_3\int_0^{\eta_i}\hbar_1(s)ds + \frac{\pi_4}{\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}\hbar_1(s)ds\right) \\ \quad -\pi_5\int_0^1\hbar_2(s)ds - \frac{\pi_6}{\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}\hbar_2(s)ds\Big] \\ +\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^{\sigma}\hbar_1(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\sigma}(\sigma-s)^{\varrho_1-1}\hbar_1(s)ds \end{cases} \tag{2.2}$$

and

$$\vartheta_2(\sigma) = \begin{cases} \frac{1}{\Theta}\Big[\frac{\alpha_1\varpi_2(2-\varrho_1)}{B(\varrho_1-1)}\int_0^{\tau}\int_0^{s}\hbar_1(u)duds + \frac{\alpha_1\varpi_2(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\tau}\int_0^{s}(s-u)^{\varrho_1-1}\hbar_1(u)duds \\ \quad + \sum_{i=1}^m \xi_i\left(\frac{\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^{\eta_i}\hbar_1(s)ds + \frac{\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}\hbar_1(s)ds\right) \\ -\frac{\alpha_2\alpha_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^1\hbar_2(s)ds - \frac{\alpha_2\alpha_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}\hbar_2(s)ds \\ +\psi_1\int_0^{\tau}\int_0^{s}\hbar_2(u)duds + \frac{\psi_2}{\Gamma(\varrho_2)}\int_0^{\tau}\int_0^{s}(s-u)^{\varrho_2-1}\hbar_2(u)duds \\ \quad + \sum_{i=1}^m \mu_i\left(\psi_3\int_0^{\eta_i}\hbar_2(s)ds + \frac{\psi_4}{\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}\hbar_2(s)ds\right) \\ \quad -\psi_5\int_0^1\hbar_1(s)ds - \frac{\psi_6}{\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\hbar_1(s)ds\Big] \\ \quad +\frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^{\sigma}\hbar_2(s)ds + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\sigma}(\sigma-s)^{\varrho_2-1}\hbar_2(s)ds, \end{cases} \tag{2.3}$$

where

$$\begin{cases} \pi_1 = \frac{\left(\varpi_1\tau+\sum_{i=1}^m \mu_i\eta_i\right)\varpi_2(2-\varrho_1)}{B(\varrho_1-1)}, \pi_2 = \frac{\left(\varpi_1\tau+\sum_{i=1}^m \mu_i\eta_i\right)\varpi_2(\varrho_1-1)}{B(\varrho_1-1)}, \\ \pi_3 = \frac{\left(\varpi_1\tau+\sum_{i=1}^m \mu_i\eta_i\right)(2-\varrho_1)}{B(\varrho_1-1)}, \pi_4 = \frac{\left(\varpi_1\tau+\sum_{i=1}^m \mu_i\eta_i\right)(\varrho_1-1)}{B(\varrho_1-1)}, \\ \pi_5 = \frac{\alpha_2\left(\varpi_1\tau+\sum_{i=1}^m \mu_i\eta_i\right)(2-\varrho_2)}{B(\varrho_2-1)}, \pi_6 = \frac{\alpha_2\left(\varpi_1\tau+\sum_{i=1}^m \mu_i\eta_i\right)(\varrho_2-1)}{B(\varrho_2-1)} \end{cases}$$

and

$$\begin{cases} \psi_1 = \frac{\varpi_1\left(\varpi_2\tau+\sum_{i=1}^m \xi_i\eta_i\right)(2-\varrho_2)}{B(\varrho_2-1)}, \psi_2 = \frac{\varpi_1\left(\varpi_2\tau+\sum_{i=1}^m \xi_i\eta_i\right)(\varrho_2-1)}{B(\varrho_2-1)}, \\ \psi_3 = \frac{\left(\varpi_2\tau+\sum_{i=1}^m \xi_i\eta_i\right)(2-\varrho_2)}{B(\varrho_2-1)}, \psi_4 = \frac{\left(\varpi_2\tau+\sum_{i=1}^m \xi_i\eta_i\right)(\varrho_2-1)}{B(\varrho_2-1)}, \\ \psi_5 = \frac{\alpha_1\left(\varpi_2\tau+\sum_{i=1}^m \xi_i\eta_i\right)(2-\varrho_1)}{B(\varrho_1-1)}, \psi_6 = \frac{\alpha_1\left(\varpi_2\tau+\sum_{i=1}^m \xi_i\eta_i\right)(\varrho_1-1)}{B(\varrho_1-1)}. \end{cases}$$

Proof. Assume that $(\vartheta_1, \vartheta_2) \in \mathcal{H}$ is a solution of the following equations

$$\begin{cases} {}^{ABR}\mathbf{D}_{0^+}^{\varrho_1}\vartheta_1(\sigma) = \hbar_1(\sigma), \ \sigma \in [0,1], \\ {}^{ABR}\mathbf{D}_{0^+}^{\varrho_2}\vartheta_2(\sigma) = \hbar_2(\sigma) \ \sigma \in [0,1]. \end{cases}$$

Then, by Lemma 2.6, we get

$$\vartheta_1(\sigma) = c_1 + \frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^\sigma \hbar_1(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\sigma (\sigma-s)^{\varrho_1-1}\hbar_1(s)ds, \tag{2.4}$$

$$\vartheta_2(\sigma) = c_2 + \frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^\sigma \hbar_2(s)ds + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\sigma (\sigma-s)^{\varrho_2-1}\hbar_2(s)ds, \tag{2.5}$$

where, c_1, c_2 are arbitrary constants. Applying the conditions $(\alpha_1\vartheta_1(1) - \varpi_1\int_0^\tau \vartheta_2(s)ds = \sum_{i=1}^m \mu_i\vartheta_2(\eta_i)$ and $\alpha_2\vartheta_2(1) - \varpi_2\int_0^\tau \vartheta_1(s)ds = \sum_{i=1}^m \xi_i\vartheta_1(\eta_i))$, we obtain

$$\begin{aligned} &\alpha_1\left(c_1 + \frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^1 \hbar_1(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1 (1-s)^{\varrho_1-1}\hbar_1(s)ds\right) \\ &-\varpi_1\int_0^\tau\left(c_2 + \frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^s \hbar_2(u)du + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^s (s-u)^{\varrho_2-1}\hbar_2(u)du\right)ds \\ = &\sum_{i=1}^m \mu_i\left(c_2 + \frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^{\eta_i} \hbar_2(s)ds + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\eta_i} (\eta_i-s)^{\varrho_2-1}\hbar_2(s)ds\right) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\alpha_2\left(c_2 + \frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^1 \hbar_2(s)ds + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^1 (1-s)^{\varrho_2-1}\hbar_2(s)ds\right) \\ &-\varpi_2\int_0^\tau\left(c_1 + \frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^s \hbar_1(u)du + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^s (s-u)^{\varrho_1-1}\hbar_1(u)du\right)ds \\ = &\sum_{i=1}^m \xi_i\left(c_1 + \frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^{\eta_i} \hbar_1(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\eta_i} (\eta_i-s)^{\varrho_1-1}\hbar_1(s)ds\right). \end{aligned} \tag{2.7}$$

Equations (2.6) and (2.7) can be written as the following system

$$\begin{cases} \alpha_1 c_1 - \mathcal{Z}_1 c_2 = \mathcal{P}_1, \\ -\mathcal{Z}_2 c_1 + \alpha_2 c_2 = \mathcal{P}_2, \end{cases} \tag{2.8}$$

where

$$\mathcal{Z}_1 = \left(\varpi_1\tau + \sum_{i=1}^{m}\mu_i\right),\ \mathcal{Z}_2 = \left(\varpi_2\tau + \sum_{i=1}^{m}\xi_i\right),$$

$$\mathcal{P}_1 = \begin{cases} \frac{\varpi_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^\tau\int_0^s \hbar_2(u)duds + \frac{\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\tau\int_0^s (s-u)^{\varrho_2-1}\hbar_2(u)duds \\ +\sum_{i=1}^m \mu_i\left(\frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^{\eta_i}\hbar_2(s)ds + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}\hbar_2(s)ds\right) \\ -\frac{\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^1\hbar_1(s)ds - \frac{\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\hbar_1(s)ds \end{cases}$$

and

$$\mathcal{P}_2 = \begin{cases} \frac{\varpi_2(2-\varrho_1)}{B(\varrho_1-1)}\int_0^\tau\int_0^s \hbar_1(u)duds + \frac{\varpi_2(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\tau\int_0^s (s-u)^{\varrho_1-1}\hbar_1(u)duds \\ +\sum_{i=1}^m \xi_i\left(\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^{\eta_i}\hbar_1(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}\hbar_1(s)ds\right) \\ -\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)}\int_0^1\hbar_2(s)ds - \frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}\hbar_2(s)ds. \end{cases}$$

Solving system (2.8) for c_1 and c_2, we obtain

$$c_1 = \frac{\alpha_2\mathcal{P}_1 + \mathcal{Z}_1\mathcal{P}_2}{\alpha_1\alpha_2 - \mathcal{Z}_1\mathcal{Z}_2} \text{ and } c_2 = \frac{\mathcal{Z}_2\mathcal{P}_1 + \alpha_1\mathcal{P}_2}{\alpha_1\alpha_2 - \mathcal{Z}_1\mathcal{Z}_2}.$$

Substituting the values of c_1 and c_2 in (2.4) and (2.5) respectively, we get (2.2) and 2.3. Conversely, apply the operators ${}^{ABR}\mathbf{D}_{0^+}^{\varrho_1}, {}^{ABR}\mathbf{D}_{0^+}^{\varrho_2}$ on (2.2) and (2.3) respectively and making use the Lemma 2.5 and note that ${}^{ABR}\mathbf{D}_{0^+}^{\varrho_i}c = {}^{ABR}D_{0^+}^{\beta_i}\frac{d}{d\sigma}c = 0, (\beta_i = \varrho_i - n), i = 1, 2$, we obtain (2.1). Hence, $(\vartheta_1, \vartheta_2)$ satisfies (2.1) if and only if it satisfies (2.2) and (2.3). The proof is completed. □

3. Existence and uniqueness of solutions for ABR-System (1.1)

In view of Lemma 2.9, we define an operator $\Upsilon : \mathcal{H} \to \mathcal{H}$ by

$$\Upsilon(\vartheta_1, \vartheta_2) = (\Upsilon_1(\vartheta_1, \vartheta_2), \Upsilon_2(\vartheta_1, \vartheta_2)), \tag{3.1}$$

where

$$\Upsilon_1(\vartheta_1, \vartheta_2) = \begin{cases} \frac{1}{\Theta}\Big[\frac{\alpha_2\varpi_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^\tau\int_0^s F_{2,\vartheta}(u)duds + \frac{\alpha_2\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\tau\int_0^s(s-u)^{\varrho_2-1}F_{2,\vartheta}(u)duds \\ +\sum_{i=1}^m\mu_i\left(\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)}\int_0^{\eta_i}F_{2,\vartheta}(s)ds + \frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}F_{2,\vartheta}(s)ds\right) \\ -\frac{\alpha_2\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^1 F_{1,\vartheta}(s)ds - \frac{\alpha_2\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds \\ +\pi_1\int_0^\tau\int_0^s F_{1,\vartheta}(u)duds + \frac{\pi_2}{\Gamma(\varrho_1)}\int_0^\tau\int_0^s(s-u)^{\varrho_1-1}F_{1,\vartheta}(u)duds \\ +\sum_{i=1}^m\xi_i\left(\pi_3\int_0^{\eta_i}F_{1,\vartheta}(s)ds + \frac{\pi_4}{\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds\right) \\ -\pi_5\int_0^1 F_{2,\vartheta}(s)ds - \frac{\pi_6}{\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}F_{2,\vartheta}(s)ds\Big] \\ +\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^\sigma F_{1,\vartheta}(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\sigma(\sigma-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds \end{cases}$$

and

$$\Upsilon_2(\vartheta_1,\vartheta_2)=\begin{cases}\frac{1}{\Theta}\Big[\frac{\alpha_1\varpi_2(2-\varrho_1)}{B(\varrho_1-1)}\int_0^\tau\int_0^s F_{1,\vartheta}(u)duds+\frac{\alpha_1\varpi_2(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\tau\int_0^s(s-u)^{\varrho_1-1}F_{1,\vartheta}(u)duds\\ \quad+\sum_{i=1}^m\xi_i\Big(\frac{\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^{\eta_i}F_{1,\vartheta}(s)ds+\frac{\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds\Big)\\ -\frac{\alpha_2\alpha_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^1F_{2,\vartheta}(s)ds-\frac{\alpha_2\alpha_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}F_{2,\vartheta}(s)ds\\ +\psi_1\int_0^\tau\int_0^s F_{2,\vartheta}(u)duds+\frac{\psi_2}{\Gamma(\varrho_2)}\int_0^\tau\int_0^s(s-u)^{\varrho_2-1}F_{2,\vartheta}(u)duds\\ +\sum_{i=1}^m\mu_i\Big(\psi_3\int_0^{\eta_i}F_{2,\vartheta}(s)ds+\frac{\psi_4}{\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}F_{2,\vartheta}(s)ds\Big)\\ -\psi_5\int_0^1F_{1,\vartheta}(s)ds-\frac{\psi_6}{\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds\Big]\\ +\frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^\sigma F_{2,\vartheta}(s)ds+\frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\sigma(\sigma-s)^{\varrho_2-1}F_{2,\vartheta}(s)ds,\end{cases}$$

such that, $F_{i,\vartheta}(s)=f_i(s,\vartheta_1(s),\vartheta_2(s)), i=1,2$. In the sequel, to simplify our analysis, we take the following notations

$$\begin{aligned}Q_\pi &= \frac{\alpha_2\varpi_1(2-\varrho_2)}{2\Theta B(\varrho_2-1)}+\frac{\alpha_2\varpi_1(\varrho_2-1)}{\Theta B(\varrho_2-1)\Gamma(\varrho_2+2)}+\frac{\alpha_2(2-\varrho_2)\sum_{i=1}^m\mu_i\eta_i}{\Theta B(\varrho_2-1)}\\ &\quad+\frac{\alpha_2(\varrho_2-1)\sum_{i=1}^m\mu_i\eta_i}{\Theta B(\varrho_2-1)\Gamma(\varrho_2+2)}+\frac{\pi_5}{\Theta}+\frac{\pi_6}{\Theta\Gamma(\varrho_2+1)},\end{aligned}$$

$$\begin{aligned}Q_\psi &= \frac{\alpha_1\varpi_2(2-\varrho_1)}{2\Theta B(\varrho_1-1)}+\frac{\alpha_1\varpi_2(\varrho_1-1)}{\Theta B(\varrho_1-1)\Gamma(\varrho_1+2)}+\frac{\alpha_1(2-\varrho_1)\sum_{i=1}^m\mu_i\eta_i}{\Theta B(\varrho_1-1)}\\ &\quad+\frac{\alpha_1(\varrho_1-1)\sum_{i=1}^m\mu_i\eta_i}{\Theta B(\varrho_1-1)\Gamma(\varrho_1+2)}+\frac{\psi_5}{\Theta}+\frac{\psi_6}{\Theta\Gamma(\varrho_1+1)},\end{aligned}$$

$$\begin{aligned}M_\pi &= \frac{\pi_1}{2\Theta}+\frac{\pi_2}{\Theta\Gamma(\varrho_1+2)}+\frac{\alpha_2\alpha_1(2-\varrho_1)}{\Theta B(\varrho_1-1)}+\frac{\alpha_2\alpha_1(\varrho_1-1)}{\Theta B(\varrho_1-1)\Gamma(\varrho_1+1)}\\ &\quad+\frac{\pi_3\sum_{i=1}^m\xi_i\eta_i}{\Theta}+\frac{\pi_4\sum_{i=1}^m\xi_i\eta_i}{\Theta\Gamma(\varrho_1+2)}+\frac{(2-\varrho_1)}{B(\varrho_1-1)}+\frac{(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\end{aligned}$$

and

$$\begin{aligned}M_\psi &= \frac{\psi_1}{2\Theta}+\frac{\psi_2}{\Theta\Gamma(\varrho_2+2)}+\frac{\alpha_2\alpha_1(2-\varrho_2)}{\Theta B(\varrho_2-1)}+\frac{\alpha_2\alpha_1(\varrho_2-1)}{\Theta B(\varrho_2-1)\Gamma(\varrho_2+1)}\\ &\quad+\frac{\psi_3\sum_{i=1}^m\xi_i\eta_i}{\Theta}+\frac{\psi_4\sum_{i=1}^m\xi_i\eta_i}{\Theta\Gamma(\varrho_1+2)}+\frac{(2-\varrho_2)}{B(\varrho_2-1)}+\frac{(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2+1)}.\end{aligned}$$

In the forthcoming theorems, we will prove the existence and uniqueness of solutions for the ABR-System (1.1) utilizing Leray-Schauder alternative and Banach contraction mapping principle.

Theorem 3.1. *Let $f_1, f_2:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be are continuous functions. In addition, we assume that:*
$(H_1): |f_i(\sigma,\vartheta,v)|\le\varepsilon_i+\theta_i|\vartheta|+\lambda_i|v|, \varepsilon_i,\theta_i,\lambda_i>0, i=1,2$.
Then, the ABR-System (1.1) has at least one solution, provided that $\Lambda_1<1$, where

$$\Lambda_1=2\max\{(\theta_1+\theta_2),(\lambda_1+\lambda_2)\}.$$

Proof. Notice that the ABR-System (1.1) has at least one solution $(\vartheta_1, \vartheta_2)$ if the operator Υ defined by (3.1) has a fixed point. For that, we shall divide the proof into the next steps:

Step 1: Υ is continuous. Since the functions f_1 and f_2 are continuous, we conclude that the operator Υ is continuous too.

Step 2: Υ is compact.

Define a closed ball $\mathbb{B}_R = \{(\vartheta_1, \vartheta_2) \in \mathcal{H} : \|(\vartheta_1, \vartheta_2)\| \leq R\}$ with

$$R \geq \frac{\Lambda_2}{1-\Lambda_1} \quad \text{where} \quad \Lambda_2 := \varepsilon_2 \left(Q_\pi + Q_\psi\right) + \varepsilon_1 \left(M_\pi + M_\psi\right). \tag{3.2}$$

First, we show that Υ is uniformly bounded on $\mathbb{B}_R$. For each $(\vartheta_1, \vartheta_2) \in \mathbb{B}_R$, we have

$$\begin{aligned}
& \|\Upsilon_1(\vartheta_1, \vartheta_2)\| \\
\leq\; & \frac{1}{\Theta}\Bigg[\frac{\alpha_2\varpi_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^\tau\int_0^s \left|F_{2,\vartheta}(u)\right| du ds + \frac{\alpha_2\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\tau\int_0^s (s-u)^{\varrho_2-1}\left|F_{2,\vartheta}(u)\right| du ds \\
& + \sum_{i=1}^m \mu_i\left(\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)}\int_0^{\eta_i}\left|F_{2,\vartheta}(s)\right| ds + \frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}\left|F_{2,\vartheta}(s)\right| ds\right) \\
& + \frac{\alpha_2\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^1\left|F_{1,\vartheta}(s)\right| ds + \frac{\alpha_2\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\left|F_{1,\vartheta}(s)\right| ds \\
& + \pi_1\int_0^\tau\int_0^s\left|F_{1,\vartheta}(u)\right| du ds + \frac{\pi_2}{\Gamma(\varrho_1)}\int_0^\tau\int_0^s(s-u)^{\varrho_1-1}\left|F_{1,\vartheta}(u)\right| du ds \\
& + \sum_{i=1}^m \xi_i\left(\pi_3\int_0^{\eta_i}\left|F_{1,\vartheta}(s)\right| ds + \frac{\pi_4}{\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}\left|F_{1,\vartheta}(s)\right| ds\right) \\
& + \pi_5\int_0^1\left|F_{2,\vartheta}(s)\right| ds + \frac{\pi_6}{\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}\left|F_{2,\vartheta}(s)\right| ds\Bigg] \\
& + \frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^\sigma\left|F_{1,\vartheta}(s)\right| ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\sigma(\sigma-s)^{\varrho_1-1}\left|F_{1,\vartheta}(s)\right| ds \\
\leq\; & (\varepsilon_2 + \theta_2\|\vartheta_1\| + \lambda_2\|\vartheta_2\|) \\
& \Bigg(\frac{\alpha_2\varpi_1(2-\varrho_2)\tau^2}{2\Theta B(\varrho_2-1)} + \frac{\alpha_2\varpi_1(\varrho_2-1)\tau^{\varrho_2+1}}{\Theta B(\varrho_2-1)\Gamma(\varrho_2+2)} + \frac{\alpha_2(2-\varrho_2)\sum_{i=1}^m\mu_i\eta_i}{\Theta B(\varrho_2-1)} \\
& + \frac{\alpha_2(\varrho_2-1)\sum_{i=1}^m\mu_i\eta_i}{\Theta B(\varrho_2-1)\Gamma(\varrho_2+2)} + \frac{\pi_5}{\Theta} + \frac{\pi_6}{\Theta\Gamma(\varrho_2+1)}\Bigg) \\
& + (\varepsilon_1 + \theta_1\|\vartheta_1\| + \lambda_1\|\vartheta_2\|) \\
& \Bigg(\frac{\pi_1\tau^2}{2\Theta} + \frac{\pi_2\tau^{\varrho_1+1}}{\Theta\Gamma(\varrho_1+2)} + \frac{\alpha_2\alpha_1(2-\varrho_1)}{\Theta B(\varrho_1-1)} + \frac{\alpha_2\alpha_1(\varrho_1-1)}{\Theta B(\varrho_1-1)\Gamma(\varrho_1+1)} \\
& + \frac{\pi_3\sum_{i=1}^m\xi_i\eta_i}{\Theta} + \frac{\pi_4\sum_{i=1}^m\xi_i\eta_i}{\Theta\Gamma(\varrho_1+2)} + \frac{(2-\varrho_1)\sigma}{B(\varrho_1-1)} + \frac{(\varrho_1-1)\sigma^{\varrho_1}}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\Bigg) \\
=\; & (\varepsilon_2 + \theta_2\|\vartheta_1\| + \lambda_2\|\vartheta_2\|) Q_\pi + (\varepsilon_1 + \theta_1\|\vartheta_1\| + \lambda_1\|\vartheta_2\|) M_\pi.
\end{aligned}$$

Similarly, we can find that

$$\|\Upsilon_2(\vartheta_1, \vartheta_2)\| \leq (\varepsilon_2 + \theta_2\|\vartheta_1\| + \lambda_2\|\vartheta_2\|) Q_\psi + (\varepsilon_1 + \theta_1\|\vartheta_1\| + \lambda_1\|\vartheta_2\|) M_\psi.$$

Consequently, we get

$$
\begin{aligned}
\|\Upsilon(\vartheta_1,\vartheta_2)\| &\leq \|\Upsilon_1(\vartheta_1,\vartheta_2)\| + \|\Upsilon_2(\vartheta_1,\vartheta_2)\| \\
&\leq \varepsilon_2\left(Q_\pi + Q_\psi\right) + \varepsilon_1\left(M_\pi + M_\psi\right) + 2(\theta_1+\theta_2)\|\vartheta_1\| \\
&\quad +2(\lambda_1+\lambda_2)\|\vartheta_2\| \\
&\leq \varepsilon_2\left(Q_\pi + Q_\psi\right) + \varepsilon_1\left(M_\pi + M_\psi\right) \\
&\quad +2\max\{(\theta_1+\theta_2),(\lambda_1+\lambda_2)\}(\|\vartheta_1\|+\|\vartheta_2\|) \\
&= \varepsilon_2\left(Q_\pi + Q_\psi\right) + \varepsilon_1\left(M_\pi + M_\psi\right) \\
&\quad +2\max\{(\theta_1+\theta_2),(\lambda_1+\lambda_2)\}(\|(\vartheta_1,\vartheta_2)\|) \\
&= \Lambda_2 + \Lambda_1 R \leq R.
\end{aligned}
$$

Hence Υ is uniformly bounded.

Next, we show that Υ is equicontinuous. Since $\mathbb{B}_R \subset \mathcal{H}$ is bounded. Then, for all $(\vartheta_1,\vartheta_2) \in \mathbb{B}_R$, there exist constants $\varphi_1,\varphi_2 > 0$ such that $\left|F_{1,\vartheta}(\sigma)\right| \leq \varphi_1$ and $\left|F_{2,\vartheta}(\sigma)\right| \leq \varphi_2$. Let $\sigma_1,\sigma_2 \in \mathcal{J}$ such that $\sigma_1 < \sigma_2$. Then, we have

$$
\begin{aligned}
&\|\Upsilon_1(\vartheta_1(\sigma_2),\vartheta_2(\sigma_2)) - \Upsilon_1(\vartheta_1(\sigma_1),\vartheta_2(\sigma_1))\| \\
= &\left|\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^{\sigma_2} F_{1,\vartheta}(s)ds - \frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^{\sigma_1} F_{1,\vartheta}(s)ds\right. \\
&+\frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\sigma_2}(\sigma_2-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds \\
&\left.-\frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\sigma_1}(\sigma_1-s)^{\varrho_1-1}F_{1,\vartheta}(s)ds\right| \\
\leq &\frac{(2-\varrho_1)\varphi_1}{B(\varrho_1-1)}(\sigma_2-\sigma_2) + \frac{(\varrho_1-1)\varphi_1}{B(\varrho_1-1)\Gamma(\varrho_1)} \\
&\int_0^{\sigma_1}\left((\sigma_2-s)^{\varrho_1-1}-(\sigma_1-s)^{\varrho_1-1}\right)ds \\
&+\frac{(\varrho_1-1)\varphi_1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_{\sigma_1}^{\sigma_2}(\sigma_2-s)^{\varrho_1-1}ds \\
\leq &\frac{(2-\varrho_1)\varphi_1}{B(\varrho_1-1)}(\sigma_2-\sigma_2) + \frac{(\varrho_1-1)\varphi_1}{B(\varrho_1-1)\Gamma(\varrho_1+1)} \\
&\left[-(\sigma_2-\sigma_1)^{\varrho_1} + \sigma_2^{\varrho_1} - \sigma_1^{\varrho_1}\right].
\end{aligned}
$$

Take $\sigma_2 \to \sigma_1$, we get

$$\|\Upsilon_1(\vartheta_1(\sigma_2),\vartheta_2(\sigma_2)) - \Upsilon_1(\vartheta_1(\sigma_1),\vartheta_2(\sigma_1))\| \to 0 \text{ as } \sigma_2 \to \sigma_1.$$

In the same technique, we get

$$\|\Upsilon_2(\vartheta_1(\sigma_2),\vartheta_2(\sigma_2)) - \Upsilon_2(\vartheta_1(\sigma_1),\vartheta_2(\sigma_1))\| \to 0 \text{ as } \sigma_2 \to \sigma_1.$$

It follows that

$$\|\Upsilon(\vartheta_1(\sigma_2),\vartheta_2(\sigma_2)) - \Upsilon(\vartheta_1(\sigma_1),\vartheta_2(\sigma_1))\| \to 0 \text{ as } \sigma_2 \to \sigma_1.$$

Hence Υ is equicontinuous. Due to the Arzelá-Ascoli theorem, we conclude that the operator Υ is compact in $\mathcal{H}$. Therefore, from the above steps, we deduce that Υ is completely continuous.

Step 3: In the last step, we show tht the set $\xi(\Upsilon) = \{(\vartheta_1, \vartheta_2) \in \mathcal{H} : (\vartheta_1, \vartheta_2) = \beta\Upsilon(\vartheta_1, \vartheta_2), \beta \in (0, 1)\}$ is bounded.

Let $(\vartheta_1, \vartheta_2) \in \xi(\Upsilon)$. Then $(\vartheta_1, \vartheta_2) = \beta\Upsilon(\vartheta_1, \vartheta_2)$. Now, for $\sigma \in \mathcal{J}$, we have $\vartheta_1(\sigma) = \beta\Upsilon_1(\vartheta_1, \vartheta_2)$ and $\vartheta_2(\sigma) = \beta\Upsilon_2(\vartheta_1, \vartheta_2)$. According to (H_1), we obtain

$$\begin{aligned}
\|\vartheta_1\| &= \sup_{\sigma\in[0,1]} |\beta\Upsilon_1(\vartheta_1, \vartheta_2)(\sigma)| \\
&\leq \|\Upsilon_1(\vartheta_1, \vartheta_2)\| \\
&\leq \varepsilon_2 Q_\pi + \varepsilon_1 M_\pi + (\theta_1 + \theta_2)\|\vartheta_1\| \\
&\quad + (\lambda_1 + \lambda_2)\|\vartheta_2\|.
\end{aligned}$$

By the same technique, we get

$$\begin{aligned}
\|\vartheta_2\| &\leq \varepsilon_2 Q_\psi + \varepsilon_1 M_\psi + (\theta_1 + \theta_2)\|\vartheta_1\| \\
&\quad + (\lambda_1 + \lambda_2)\|\vartheta_2\|.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\|(\vartheta_1, \vartheta_2)\| &= \|\vartheta_1\| + \|\vartheta_2\| \\
&\leq \varepsilon_2\left(Q_\pi + Q_\psi\right) + \varepsilon_1\left(M_\pi + M_\psi\right) + 2(\theta_1 + \theta_2)\|\vartheta_1\| \\
&\quad +2(\lambda_1 + \lambda_2)\|\vartheta_2\| \\
&= \varepsilon_2\left(Q_\pi + Q_\psi\right) + \varepsilon_1\left(M_\pi + M_\psi\right) \\
&\quad +2\max\{(\theta_1 + \theta_2), (\lambda_1 + \lambda_2)\}(\|(\vartheta_1, \vartheta_2)\|) \\
&= \Lambda_2 + \Lambda_1\|(\vartheta_1, \vartheta_2)\|.
\end{aligned}$$

Since $\Lambda_1 < 1$, therefore

$$\|(\vartheta_1, \vartheta_2)\| \leq \frac{\Lambda_2}{1 - \Lambda_1} \leq R.$$

Hence, the set $\xi(\Upsilon)$ is bounded. Due to the above steps with Theorem 2.8, we deduce that Υ has at least one fixed point. Consequently, the ABR-System (1.1) has at least one solution. □

Theorem 3.2. *Let $f_1, f_2 : [0, 1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be are continuous functions. In addition, we assume that:*
$(H_2) : |f_i(\sigma, \vartheta_1, v_1) - f_i(\sigma, \vartheta_2, v_2)| \leq L_i(|\vartheta_1 - \vartheta_2| + |v_1 - v_2|), L_i > 0, i = 1, 2.$
Then, the system (2.1) has a unique solution, provided that $\sigma < 1$, where

$$\sigma = L_2\left(Q_\pi + Q_\psi\right) + L_1\left(M_\pi + M_\psi\right).$$

Proof. Let us consider a closed ball set $\mathbb{B}_R$ defined in Theorem 3.1. In order to apply Theorem 2.7, we will divide the proof into the following steps:

Step (1): We show that $\Upsilon(\mathbb{B}_R) \subset \mathbb{B}_R$. By the second step in Theorem 3.1, we have $\Upsilon(\mathbb{B}_R) \subset \mathbb{B}_R$.

Step (1): We need to prove that Υ is a contraction map. Let $(\vartheta_1, \vartheta_2), (x_1, x_2) \in \mathcal{H}$ and $\sigma \in \mathcal{J}$. Then, we obtain

$$\|\Upsilon_1(\vartheta_1, \vartheta_2) - \Upsilon_1(x_1, x_2)\|$$

$$
\begin{aligned}
\leq\ & \sup_{\sigma\in[0,1]} \frac{1}{|\Theta|}\left[\frac{\alpha_2\varpi_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^\tau\int_0^s \left|F_{2,\vartheta}(u)-F_{2,x}(u)\right| duds\right.\\
& +\frac{\alpha_2\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\tau\int_0^s (s-u)^{\varrho_2-1}\left|F_{2,\vartheta}(u)-F_{2,x}(u)\right| duds\\
& +\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)}\sum_{i=1}^m \mu_i\int_0^{\eta_i}\left|F_{2,\vartheta}(s)-F_{2,x}(s)\right| ds\\
& +\frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\sum_{i=1}^m \mu_i\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}\left|F_{2,\vartheta}(s)-F_{2,x}(s)\right| ds\\
& +\frac{\alpha_2\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^1\left|F_{1,\vartheta}(s)-F_{1,x}(s)\right| ds\\
& +\frac{\alpha_2\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\left|F_{1,\vartheta}(s)-F_{1,x}(s)\right| ds+\pi_1\int_0^\tau\int_0^s\left|F_{1,\vartheta}(u)-F_{1,x}(u)\right| duds\\
& +\frac{\pi_2}{\Gamma(\varrho_1)}\int_0^\tau\int_0^s(s-u)^{\varrho_1-1}\left|F_{1,\vartheta}(u)-F_{1,x}(u)\right| duds+|\pi_3|\sum_{i=1}^m\xi_i\int_0^{\eta_i}\left|F_{1,\vartheta}(s)-F_{1,x}(s)\right| ds\\
& +\frac{\pi_4}{\Gamma(\varrho_1)}\sum_{i=1}^m\xi_i\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}\left|F_{1,\vartheta}(s)-F_{1,x}(s)\right| ds\\
& \left.+\pi_5\int_0^1\left|F_{2,\vartheta}(s)-F_{2,x}(s)\right| ds+\frac{\pi_6}{\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}\left|F_{2,\vartheta}(s)-F_{2,x}(s)\right| ds\right]\\
& +\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^\sigma\left|F_{1,\vartheta}(s)-F_{1,x}(s)\right| ds\\
& +\frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\sigma(\sigma-s)^{\varrho_1-1}\left|F_{1,\vartheta}(s)-F_{1,x}(s)\right| ds\\
\leq\ & (L_2Q_\pi+L_1M_\pi)\left(\|\vartheta_1-x_1\|+\|\vartheta_2-x_2\|\right),
\end{aligned}
$$

and consequently, we obtain

$$
\begin{aligned}
& \|\Upsilon_1(\vartheta_1,\vartheta_2)-\Upsilon_1(x_1,x_2)\|\\
\leq\ & (L_2Q_\pi+L_1M_\pi)\left(\|\vartheta_1-x_1\|+\|\vartheta_2-x_2\|\right).
\end{aligned}
\tag{3.3}
$$

By the same way, one can obtain

$$
\begin{aligned}
& \|\Upsilon_1(\vartheta_1,\vartheta_2)-\Upsilon_1(x_1,x_2)\|\\
\leq\ & \left(L_2Q_\psi+L_1M_\psi\right)\left(\|\vartheta_1-x_1\|+\|\vartheta_2-x_2\|\right).
\end{aligned}
\tag{3.4}
$$

It follows from (3.3) and (3.4) that

$$
\begin{aligned}
& \|\Upsilon(\vartheta_1,\vartheta_2)-\Upsilon(x_1,x_2)\|\\
& =\|\Upsilon_1(\vartheta_1,\vartheta_2)-\Upsilon_1(x_1,x_2)\|+\|\Upsilon_1(\vartheta_1,\vartheta_2)-\Upsilon_1(x_1,x_2)\|\\
& \leq L_2\left(Q_\pi+Q_\psi\right)+L_1\left(M_\pi+M_\psi\right)\left(\|\vartheta_1-x_1\|+\|\vartheta_2-x_2\|\right)\\
& \leq \sigma\left(\|\vartheta_1-x_1\|+\|\vartheta_2-x_2\|\right)
\end{aligned}
$$

Due to $\sigma<1$, we conclude that the operator Υ is a contraction. Hence, by Theorem 2.7, the ABR-System (1.1) has a unique solution. □

4. Ulam-Hyers stability

In this section, we shall discuss the Ulam-Hyers (UH) stability of the ABR-System (1.1).

Remark 4.1. *[20] A function $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) \in \mathcal{H}$ satisfies the following inequalities*

$$\begin{cases} \left| {}^{ABR}\mathbf{D}_{0^+}^{\varrho_1} \widehat{\vartheta}_1(\sigma) - F_{1,\widehat{\vartheta}}(\sigma) \right| \le \varepsilon_1, \\ \left| {}^{ABR}\mathbf{D}_{0^+}^{\varrho_2} \widehat{\vartheta}_2(\sigma) - F_{2,\widehat{\vartheta}}(\sigma) \right| \le \varepsilon_2, \end{cases} \tag{4.1}$$

if and only if there exists a functions $\kappa_1, \kappa_2 \in D$ such that

(i) $\begin{cases} |\kappa_1(\sigma)| \le \varepsilon_1, \\ |\kappa_2(\sigma)| \le \varepsilon_2 \,. \end{cases}$

(ii) $\begin{cases} {}^{ABR}\mathbf{D}_{0^+}^{\varrho_1} \widehat{\vartheta}_1(\sigma) = F_{1,\widehat{\vartheta}}(\sigma) + \kappa_1(\sigma), \\ {}^{ABR}\mathbf{D}_{0^+}^{\varrho_2} \widehat{\vartheta}_2(\sigma) = F_{2,\widehat{\vartheta}}(\sigma) + \kappa_2(\sigma). \end{cases}$.

Definition 4.2. *[20] The ABR-System (1.1) is UH stable if there exists $\mathcal{M} > 0$ such that, for each $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and each solution $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) \in \mathcal{H}$ of the inequalities (4.1), there exists a solution $(\vartheta_1, \vartheta_2) \in \mathcal{H}$ of the ABR-System (1.1) with*

$$\left\| \left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) - (\vartheta_1, \vartheta_2) \right\| \le \mathcal{M}\varepsilon, \;\; \sigma \in \mathcal{J}$$

Lemma 4.3. *Let $\varrho_1, \varrho_2 \in (1, 2)$. If a function $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) \in \mathcal{H}$ satisfies the inequalities (4.1), then $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right)$ satisfies the following integral inequalities*

$$\begin{cases} \left| \widehat{\vartheta}_1(\sigma) - \Re_{\widehat{\vartheta}_1} - \frac{2-\varrho_1}{B(\varrho_1-1)} \int_0^\sigma F_{1,\widehat{\vartheta}}(s)ds - \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)} \int_0^\sigma (\sigma - s)^{\varrho_1-1} F_{1,\widehat{\vartheta}}(s)ds \right| \\ \qquad \le \varepsilon_2 Q_\pi + \varepsilon_1 M_\pi, \\ \left| \widehat{\vartheta}_2(\sigma) - \Re_{\widehat{\vartheta}_2} - \frac{2-\varrho_2}{B(\varrho_2-1)} \int_0^\sigma F_{2,\widehat{\vartheta}}(s)ds - \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)} \int_0^\sigma (\sigma - s)^{\varrho_2-1} F_{2,\widehat{\vartheta}}(s)ds \right| \\ \qquad \le \varepsilon_2 Q_\psi + \varepsilon_1 M_\psi, \end{cases}$$

where

$$\Re_{\widehat{\vartheta}_1} = \begin{cases} \frac{1}{\Theta} \Big[\frac{\alpha_2\varpi_1(2-\varrho_2)}{B(\varrho_2-1)} \int_0^\tau \int_0^s F_{2,\widehat{\vartheta}}(u)duds + \frac{\alpha_2\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)} \int_0^\tau \int_0^s (s-u)^{\varrho_2-1} F_{2,\widehat{\vartheta}}(u)duds \\ \quad + \sum_{i=1}^m \mu_i \left(\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)} \int_0^{\eta_i} F_{2,\widehat{\vartheta}}(s)ds + \frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)} \int_0^{\eta_i} (\eta_i - s)^{\varrho_2-1} F_{2,\widehat{\vartheta}}(s)ds \right) \\ -\frac{\alpha_2\alpha_1(2-\varrho_1)}{B(\varrho_1-1)} \int_0^1 F_{1,\widehat{\vartheta}}(s)ds - \frac{\alpha_2\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)} \int_0^1 (1-s)^{\varrho_1-1} F_{1,\widehat{\vartheta}}(s)ds \\ +\pi_1 \int_0^\tau \int_0^s F_{1,\widehat{\vartheta}}(u)duds + \frac{\pi_2}{\Gamma(\varrho_1)} \int_0^\tau \int_0^s (s-u)^{\varrho_1-1} F_{1,\widehat{\vartheta}}(u)duds \\ \quad + \sum_{i=1}^m \xi_i \left(\pi_3 \int_0^{\eta_i} F_{1,\widehat{\vartheta}}(s)ds + \frac{\pi_4}{\Gamma(\varrho_1)} \int_0^{\eta_i} (\eta_i - s)^{\varrho_1-1} F_{1,\widehat{\vartheta}}(s)ds \right) \\ -\pi_5 \int_0^1 F_{2,\widehat{\vartheta}}(s)ds - \frac{\pi_6}{\Gamma(\varrho_2)} \int_0^1 (1-s)^{\varrho_2-1} F_{2,\widehat{\vartheta}}(s)ds \Big], \end{cases}$$

and

$$\Re_{\widehat{\vartheta}_2}=\begin{cases}\frac{1}{\Theta}\Big[\frac{\alpha_1\varpi_2(2-\varrho_1)}{B(\varrho_1-1)}\int_0^\tau\int_0^s F_{1,\widehat{\vartheta}}(u)duds+\frac{\alpha_1\varpi_2(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\tau\int_0^s(s-u)^{\varrho_1-1}F_{1,\widehat{\vartheta}}(u)duds\\ \quad+\sum_{i=1}^m\xi_i\Big(\frac{\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^{\eta_i}F_{1,\widehat{\vartheta}}(s)ds+\frac{\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}F_{1,\widehat{\vartheta}}(s)ds\Big)\\ -\frac{\alpha_2\alpha_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^1F_{2,\widehat{\vartheta}}(s)ds-\frac{\alpha_2\alpha_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}F_{2,\widehat{\vartheta}}(s)ds\\ +\psi_1\int_0^\tau\int_0^sF_{2,\widehat{\vartheta}}(u)duds+\frac{\psi_2}{\Gamma(\varrho_2)}\int_0^\tau\int_0^s(s-u)^{\varrho_2-1}F_{2,\widehat{\vartheta}}(u)duds\\ \quad+\sum_{i=1}^m\mu_i\Big(\psi_3\int_0^{\eta_i}F_{2,\widehat{\vartheta}}(s)ds+\frac{\psi_4}{\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}F_{2,\widehat{\vartheta}}(s)ds\Big)\\ -\psi_5\int_0^1F_{1,\widehat{\vartheta}}((s)ds-\frac{\psi_6}{\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}F_{1,\widehat{\vartheta}}((s)ds\Big].\end{cases}$$

Proof. By Remark 4.1, we have

$$^{ABR}\mathbf{D}_{0^+}^{\varrho_1}\widehat{\vartheta}_1(\sigma)=F_{1,\widehat{\vartheta}}(\sigma)+\kappa_1(\sigma).$$

Then, in view of Theorem 2.9 and Lemma 4.3, we get

$$\begin{aligned}&\left|\widehat{\vartheta}_1(\sigma)-\Re_{\widehat{\vartheta}_1}-\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^\sigma F_{1,\widehat{\vartheta}}(s)ds-\frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\sigma(\sigma-s)^{\varrho_1-1}F_{1,\widehat{\vartheta}}(s)ds\right|\\ \le\ &\frac{1}{\Theta}\left[\frac{\alpha_2\varpi_1(2-\varrho_2)}{B(\varrho_2-1)}\int_0^\tau\int_0^s\kappa_2(u)duds+\frac{\alpha_2\varpi_1(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\tau\int_0^s(s-u)^{\varrho_2-1}\kappa_2(u)duds\right.\\ &+\sum_{i=1}^m\mu_i\left(\frac{\alpha_2(2-\varrho_2)}{B(\varrho_2-1)}\int_0^{\eta_i}\kappa_2(s)ds+\frac{\alpha_2(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_2-1}\kappa_2(s)ds\right)\\ &-\frac{\alpha_2\alpha_1(2-\varrho_1)}{B(\varrho_1-1)}\int_0^1\kappa_1(s)ds-\frac{\alpha_2\alpha_1(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\kappa_1(s)ds\\ &+\pi_1\int_0^\tau\int_0^s\kappa_1(u)duds+\frac{\pi_2}{\Gamma(\varrho_1)}\int_0^\tau\int_0^s(s-u)^{\varrho_1-1}\kappa_1(u)duds\\ &+\sum_{i=1}^m\xi_i\left(\pi_3\int_0^{\eta_i}\kappa_1(s)ds+\frac{\pi_4}{\Gamma(\varrho_1)}\int_0^{\eta_i}(\eta_i-s)^{\varrho_1-1}\kappa_1(s)ds\right)\\ &\left.-\pi_5\int_0^1\kappa_2(s)ds-\frac{\pi_6}{\Gamma(\varrho_2)}\int_0^1(1-s)^{\varrho_2-1}\kappa_2(s)ds\right]\\ &+\frac{2-\varrho_1}{B(\varrho_1-1)}\int_0^\sigma\kappa_1(s)ds+\frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)}\int_0^\sigma(\sigma-s)^{\varrho_1-1}\kappa_1(s)ds\\ &\le\varepsilon_2Q_\pi+\varepsilon_1M_\pi\,.\end{aligned}$$

In the same way, one can obtain

$$\begin{aligned}&\left|\widehat{\vartheta}_2(\sigma)-\Re_{\widehat{\vartheta}_2}-\frac{2-\varrho_2}{B(\varrho_2-1)}\int_0^\sigma F_2(s)ds-\frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)}\int_0^\sigma(\sigma-s)^{\varrho_2-1}F_2(s)ds\right|\\ \le\ &\varepsilon_2Q_\psi+\varepsilon_1M_\psi.\end{aligned}$$

□

Theorem 4.4. *Assume that* (H_2) *hold. If*

$$\Omega=\max\left\{L_1\left(\frac{2-\varrho_1}{B(\varrho_1-1)}+\frac{(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\right),L_2\left(\frac{2-\varrho_2}{B(\varrho_2-1)}+\frac{(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2+1)}\right)\right\}<1.$$

Then

$$ {}^{ABR}\mathbf{D}_{0^+}^{\varrho_1}\widehat{\vartheta}_2(\sigma) = F_{1,\widehat{\vartheta}}(\sigma), \qquad {}^{ABR}\mathbf{D}_{0^+}^{\varrho_2}\widehat{\vartheta}_2(\sigma) = F_{2,\widehat{\vartheta}}(\sigma), \tag{4.2} $$

are Ulam-Hyers stable.

Proof. Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ and $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) \in \mathcal{H}$ be a function that satisfying the inequalities 4.1 and let $(\vartheta_1, \vartheta_2) \in \mathcal{H}$ be the unique solution of the following system

$$ \begin{cases} {}_{0^+}^{ABR}\mathbf{D}^{\varrho_1}\vartheta_1(\sigma) = F_{1,\vartheta}(\sigma), \ \sigma \in [0,1], \ \varrho \in (1,2], \\ {}_{0^+}^{ABR}\mathbf{D}^{\varrho_2}\vartheta_2(\sigma) = F_{2,\vartheta}(\sigma) \ \sigma \in [0,1], \varrho \in (1,2], \\ \alpha_1\vartheta_1(1) = \alpha_1\widehat{\vartheta}_1(1) = \varpi_1 \int_0^\tau \widehat{\vartheta}_2(s)ds + \sum_{i=1}^m \mu_i \widehat{\vartheta}_2(\eta_i), \\ \alpha_2\vartheta_2(1) = \alpha_2\widehat{\vartheta}_2(1) = \varpi_2 \int_0^\tau \widehat{\vartheta}_1(s)ds + \sum_{i=1}^m \xi_i \widehat{\vartheta}_1(\eta_i) \end{cases}. $$

Now, by Theorem 2.9, we have

$$ \begin{cases} \vartheta_1(\sigma) = \Re_{\vartheta_1} + \frac{2-\varrho_1}{B(\varrho_1-1)} \int_0^\sigma F_{1,\vartheta}(s)ds + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)} \int_0^\sigma (\sigma-s)^{\varrho_1-1} F_{1,\vartheta}(s)ds, \\ \vartheta_2(\sigma) = \Re_{\vartheta_2} + \frac{2-\varrho_2}{B(\varrho_2-1)} \int_0^\sigma F_{2,\vartheta}(s)ds + \frac{\varrho_2-1}{B(\varrho_2-1)\Gamma(\varrho_2)} \int_0^\sigma (\sigma-s)^{\varrho_2-1} F_{2,\vartheta}(s)ds. \end{cases} $$

Since $\alpha_1\vartheta_1(1) = \alpha_1\widehat{\vartheta}_1(1)$ and $\alpha_2\vartheta_2(1) = \alpha_2\widehat{\vartheta}_2(1)$, we can proof that $\Re_{\vartheta_1} = \Re_{\widehat{\vartheta}_1}$ and $\Re_{\vartheta_2} = \Re_{\widehat{\vartheta}_2}$. Hence, from ($H_2$) with Lemma 4.3, and for each $\sigma \in [0,1]$, we have

$$ \begin{aligned} \left|\widehat{\vartheta}_1(\sigma) - \vartheta_1(\sigma)\right| &\leq \left|\widehat{\vartheta}_1(\sigma) - \Re_{\widehat{\vartheta}_1} - \frac{2-\varrho_1}{B(\varrho_1-1)} \int_0^\sigma F_{1,\widehat{\vartheta}}(s)ds - \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)} \int_0^\sigma (\sigma-s)^{\varrho_1-1} F_{1,\widehat{\vartheta}}(s)ds\right| \\ &\quad + \frac{2-\varrho_1}{B(\varrho_1-1)} \int_0^\sigma \left|F_{1,\widehat{\vartheta}}(s) - F_{1,\vartheta}(s)\right| ds \\ &\quad + \frac{\varrho_1-1}{B(\varrho_1-1)\Gamma(\varrho_1)} \int_0^\sigma (\sigma-s)^{\varrho_1-1} \left|F_{1,\widehat{\vartheta}}(s) - F_{1,\vartheta}(s)\right| ds \\ &\leq \varepsilon_2 Q_\pi + \varepsilon_1 M_\pi + L_1 \left(\left\|\widehat{\vartheta}_1 - \vartheta_1\right\| + \left\|\widehat{\vartheta}_2 - \vartheta_2\right\|\right)\left(\frac{2-\varrho_1}{B(\varrho_1-1)} + \frac{(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\right). \end{aligned} \tag{4.3} $$

Hence

$$ \left\|\widehat{\vartheta}_1 - \vartheta_1\right\| \leq \varepsilon_2 Q_\pi + \varepsilon_1 M_\pi + L_1 \left(\left\|\widehat{\vartheta}_1 - \vartheta_1\right\| + \left\|\widehat{\vartheta}_2 - \vartheta_2\right\|\right)\left(\frac{2-\varrho_1}{B(\varrho_1-1)} + \frac{(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\right). \tag{4.4} $$

By the same technique, we get

$$ \left\|\widehat{\vartheta}_2 - \vartheta_2\right\| \leq \varepsilon_2 Q_\psi + \varepsilon_1 M_\psi + L_2 \left(\left\|\widehat{\vartheta}_1 - \vartheta_1\right\| + \left\|\widehat{\vartheta}_2 - \vartheta_2\right\|\right)\left(\frac{2-\varrho_2}{B(\varrho_2-1)} + \frac{(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2+1)}\right). \tag{4.5} $$

Thus

$$ \begin{aligned} &\left\|\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) - (\vartheta_1, \vartheta_2)\right\| \\ &\leq \left\|\widehat{\vartheta}_1 - \vartheta_1\right\| + \left\|\widehat{\vartheta}_2 - \vartheta_2\right\| \\ &\leq \varepsilon_2 Q_\pi + \varepsilon_1 M_\pi + L_1 \left(\left\|\widehat{\vartheta}_1 - \vartheta_1\right\| + \left\|\widehat{\vartheta}_2 - \vartheta_2\right\|\right)\left(\frac{2-\varrho_1}{B(\varrho_1-1)} + \frac{(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\right) \end{aligned} $$

$$
\begin{aligned}
&+\varepsilon_2 Q_\psi + \varepsilon_1 M_\psi + L_2\left(\left\|\widehat{\vartheta}_1 - \vartheta_1\right\| + \left\|\widehat{\vartheta}_2 - \vartheta_2\right\|\right)\left(\frac{2-\varrho_2}{B(\varrho_2-1)} + \frac{(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2+1)}\right)\\
\leq\ & \left(\left(M_\pi + M_\psi\right)\varepsilon_1 + \left(Q_\pi + Q_\psi\right)\varepsilon_2\right)\\
&+\max\left\{L_1\left(\frac{2-\varrho_1}{B(\varrho_1-1)} + \frac{(\varrho_1-1)}{B(\varrho_1-1)\Gamma(\varrho_1+1)}\right), L_2\left(\frac{2-\varrho_2}{B(\varrho_2-1)} + \frac{(\varrho_2-1)}{B(\varrho_2-1)\Gamma(\varrho_2+1)}\right)\right\}\\
&\left\|\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) - (\vartheta_1, \vartheta_2)\right\|\\
&\leq \left(\left(M_\pi + M_\psi\right)\varepsilon_1 + \left(Q_\pi + Q_\psi\right)\varepsilon_2\right) + \Omega\left\|\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) - (\vartheta_1, \vartheta_2)\right\|\\
\leq\ & \varepsilon\mathcal{K},
\end{aligned}
\tag{4.6}
$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and

$$\mathcal{K} = \frac{M_\pi + M_\psi + Q_\pi + Q_\psi}{1-\Omega}.$$

Hence, from (4.6) and Definition 4.2, we deduce that the coupled system (4.2) is Ulam-Hyers (UH) stable. □

5. An example

In this section, we will demonstrate the applicability of our main results through the following example.

Example 5.1. *Consider the following system*

$$
\begin{cases}
{}^{ABR}_{0^+}\mathbf{D}^{\frac{3}{2}}\vartheta_1(\sigma) = f_1(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)),\ \sigma \in [0,1],\\
{}^{ABR}_{0^+}\mathbf{D}^{\frac{5}{4}}\vartheta_2(\sigma) = f_2(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma))\ \sigma \in [0,1],\\
\frac{1}{4}\vartheta_1(1) = \int_0^{\frac{1}{7}} \vartheta_2(s)ds + \frac{1}{2}\vartheta_2(\frac{1}{5}) + \frac{3}{4}\vartheta_2(\frac{2}{5}) + \vartheta_2(\frac{3}{5}),\\
\frac{1}{2}\vartheta_2(1) = \int_0^{\frac{1}{7}} \vartheta_1(s)ds + \frac{1}{3}\vartheta_2(\frac{1}{5}) + \frac{2}{3}\vartheta_2(\frac{2}{5}) + \vartheta_2(\frac{3}{5}).
\end{cases}
\tag{5.1}
$$

Here $\varrho_1 = \frac{3}{2}, \varrho_2 = \frac{5}{4}, \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{2}, \tau = \frac{1}{7}, \varpi_1 = \varpi_2 = 1, m = 3, \mu_1 = \frac{1}{2}, \mu_2 = \frac{3}{4}, \mu_3 = \xi_3 = 1,$ $\xi_i = \frac{i}{3}, (i = 1,2), \eta_i = \frac{i}{5}, (i = 1,2,3)$ *and*

$$f_1(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)) = \frac{\sigma}{18(169+\sigma^4)^{\frac{1}{2}}}\left(\frac{|\vartheta_1(\sigma)|}{1+|\vartheta_1(\sigma)|} + \frac{|\vartheta_2(\sigma)|}{1+|\vartheta_2(\sigma)|} + \cos\sigma\right),$$

$$f_2(\sigma, \vartheta_1(\sigma), \vartheta_2(\sigma)) = \frac{\sigma}{8(30+\sigma^2)}\left(\frac{|\vartheta_1(\sigma)|}{1+|\vartheta_1(\sigma)|} + \tan^{-1}\vartheta_2(\sigma) + \frac{1}{1+\sigma^2}\right).$$

Clearly, for each $\vartheta_i, v_i \in \mathbb{R}, i = 1,2,$ *we have*

$$|f_1(t, \vartheta_1, v_1) - f_1(t, \vartheta_2, v_2)| \leq \frac{1}{234}\left(|\vartheta_1 - \vartheta_2| + |v_1 - v_2|\right)$$

and

$$|f_2(t, \vartheta_1, v_1) - f_2(t, \mathfrak{v}_2, v_2)| \leq \frac{1}{240}\left(|\vartheta_1 - \vartheta_2| + |v_1 - v_2|\right),$$

with $L_1 = \frac{1}{234}$ *and* $L_2 = \frac{1}{240}$. *By the given data, we get* $Q_\psi \simeq 5.7, Q_\pi \simeq 1.8, M_\pi \simeq 1.7, M_\psi \simeq 0.3$ *and* $\sigma \simeq 0.04 < 1$. *Then, all conditions in Theorem 3.2 are hold. Consequently, the coupled system (5.1) has a unique solution. On the other hand, by simple calculation, we get* $\Lambda_1 \simeq 0.02 < 1$ *and hence all hypothesis in Theorem 3.1 are satisfied. Thus, the coupled system (5.1) has at least one solution in* $[0,1]$. *For every* $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ *and each* $\left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) \in \mathcal{H}$ *satisfies*

$$\begin{cases} \left| {}_{0^+}^{ABR}\mathbf{D}_1^{\varrho_1}\widehat{\vartheta}(\sigma) - F_{1,\widehat{\vartheta}}(\sigma)\right| \leq \varepsilon_1 \\ \left| {}_{0^+}^{ABR}\mathbf{D}_2^{\varrho_2}\widehat{\vartheta}(\sigma) - F_{2,\widehat{\vartheta}}(\sigma)\right| \leq \varepsilon_2 \end{cases},$$

there exists a solution $(\vartheta_1, \vartheta_2) \in \mathcal{H}$ *of the coupled system (5.1) with*

$$\left\| \left(\widehat{\vartheta}_1, \widehat{\vartheta}_2\right) - (\vartheta_1, \vartheta_2) \right\| \leq \mathcal{K}\varepsilon, \;\; \sigma \in \mathcal{J}$$

where

$$\mathcal{K} = \frac{M_\pi + M_\psi + Q_\pi + Q_\psi}{1 - \Omega} \simeq 9 > 0.$$

and

$$\Omega = \max\left\{ \frac{1}{234}\left(1.6 + \frac{1.6}{\Gamma(\frac{3}{2}+1)}\right), \frac{1}{240}\left(5 + \frac{1.7}{\Gamma(\frac{5}{4}+1)}\right)\right\} \simeq 0.02 < 1.$$

Therefore, all conditions in Theorem 4.4 are satisfied and hence the coupled system 4.2 is UH stable and GUH.

6. Conclusions

In recent years, the subject of fractional operators involving nonsingular kernels is novel and has very important significance in modeling many phenomena in the real world, thus there is interest from some researchers to study some qualitative properties of FDEs. In this paper, we have discussed a new system of the nonlinear fractional operators with nonsingular Mittag-Leffler function kernels from order $1 < \varrho_1, \varrho_2 \leq 2$ with multipoint sub-strip boundary conditions. The results obtained in this work are new and cover some new results by choices of the parameters. We proved the existence, uniqueness, and UH, GUH results by means of the Banach fixed point theorem for initial value problems in the frame of ABR derivatives.

Conflict of interest

The authors declare no conflict of interest.

References

1. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.
2. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, Gordon & Breach, Yverdon, 1993.

3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

4. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, **35** (2000).

5. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.,* **1** (2015), 73–85.

6. A. Atangana, D. Baleanu, New fractional derivative with non-local and non-singular kernel, *Therm. Sci.,* **20** (2016), 757–763.

7. T. Abdeljawad, A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, *J. Inequal. Appl.,* **1** (2017), 130. https://doi.org/10.1186/s13660-017-1400-5

8. A. Atangana, Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system, *Chaos Soliton. Fract.,* **102** (2017), 396–406. https://doi.org/10.1016/j.chaos.2017.04.027

9. A. Atangana, J. F. Gómez-Aguilar, Fractional derivatives with no-index law property: Application to chaos and statistics, *Chaos Soliton. Fract.,* **114** (2018), 516–535. https://doi.org/10.1016/j.chaos.2018.07.033

10. F. Jarad, T. Abdeljawad, Z. Hammouch, On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative, *Chaos Soliton. Fract.,* **117** (2018), 16–20. https://doi.org/10.1016/j.chaos.2018.10.006

11. M. A. Almalahi, S. K. Panchal, On the theory of ψ-Hilfer nonlocal Cauchy problem, *J. Sib. Fed. Univ.-Math.,* **14** (2021), 159–175. https://doi.org/10.17516/1997-1397-2021-14-2-161-177

12. M.S. Abdo, T. Abdeljawad, K. D. Kucche, M. A. Alqudah, S. M. Ali, M. B. Jeelani, On nonlinear pantograph fractional differential equations with Atangana-Baleanu-Caputo derivative, *Adv. Differ. Equ.,* **1** (2021), 1–17. https://doi.org/10.1186/s13662-021-03229-8

13. M. S. Abdo, T. Abdeljawad, S. M. Ali, K. Shah, On fractional boundary value problems involving fractional derivatives with Mittag-Leffler kernel and nonlinear integral conditions, *Adv. Differ. Equ.,* **37** (2021). https://doi.org/10.1186/s13662-020-03196-6

14. M. A. Almalahi, S. K. Panchal, W. Shatanawi, M. S. Abdo, K. Shah, K. Abodayeh, Analytical study of transmission dynamics of 2019-nCoV pandemic via fractal fractional operator, *Results Phys.,* **24** (2021), 104045. https://doi.org/10.1016/j.rinp.2021.104045

15. S. M. Ulam, *A collection of mathematical problems,* Interscience Tracts in Pure and Applied Mathematics, Interscience, New York, **8** (1960).

16. D. H. Hyers, G. Isac, T. Rassias, *Stability of functional equations in several variables,* Spring, Boston,1998.

17. T. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.,* **72** (1978), 297–300.

18. B. Ahmad, S. K. Ntouyas, Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, *Appl. Math. Comput.,* **266** (2015), 615–622. https://doi.org/10.1016/j.amc.2015.05.116

19. B. Ahmad, S. K. Ntouyas, A. Alsaedi, On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions, *Chaos Soliton. Fract.,* **83** (2016), 234–241. https://doi.org/10.1016/j.chaos.2015.12.014
20. M. A. Almalahi, M. S. Abdo, S. K. Panchal, Existence and Ulam-Hyers stability results of a coupled system of ψ -Hilfer sequential fractional differential equations, *Results Appl. Math.,* **10** (2021), 100142. https://doi.org/10.1016/j.rinam.2021.100142
21. M. A. Almalahi, S. K. Panchal, F. Jarad, Stability results of positive solutions for a system of ψ-Hilfer fractional differential equations, *Chaos Soliton. Fract.,* **147** (2021), 110931. https://doi.org/10.1016/j.chaos.2021.110931
22. A. Alsaedi, A. F. Albideewi, S. K. Ntouyas, B. Ahmad, On Caputo-Riemann-Liouville type fractional integro-differential equations with multi-point sub-strip boundary conditions, *Mathematics,* **8** (2020), 1899. https://doi.org/10.3390/math8111899
23. A. Alsaedi, A. F. Albideewi, S. K. Ntouyas, B. Ahmad, Existence results for a coupled system of Caputo type fractional integro-differential equations with multi-point and sub-strip boundary conditions, *Adv. Differ. Equ.,* **19** (2021). https://doi.org/10.1186/s13662-020-03174-y
24. B. Ahmad, A. Alsaedi, B. S. Alghamdi, Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions, *Nonlinear Anal.-Real,* **9** (2008), 1727–1740. https://doi.org/10.1016/j.nonrwa.2007.05.005
25. R. Ciegis, A. Bugajev, Numerical approximation of one model of the bacterial self-organization, *Nonlinear Anal.-Model.,* **17** (2012), 253–270. https://doi.org/10.15388/NA.17.3.14054
26. B. Ahmad, S. Asghar, T. Hayat, Diffraction of a plane wave by an elastic knife-edge adjacent to a rigid strip, *Canad. Appl. Math. Quart.,* **9** (2001), 303–316.
27. E. Yusufoglu, I. Turhan, A mixed boundary value problem in orthotropic strip containing a crack, *J. Franklin Inst.,* **349** (2012), 2750–2769. https://doi.org/10.1016/j.jfranklin.2012.09.001
28. K. Deimling, *Nonlinear functional analysis*, Springer,New York, 1985.
29. A. Granas, J. Dugundji, *Fixed point theory*, Springer Science & Business Media, 2013.