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Research article

Non-instantaneous impulsive fractional-order delay differential systems with Mittag-Leffler kernel

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Abstract: The existence of fractional-order functional differential equations with non-instantaneous impulses within the Mittag-Leffler kernel is examined in this manuscript. Non-instantaneous impulses are involved in such equations and the solution semigroup is not compact in Banach spaces. We suppose that the nonlinear term fulfills a non-compactness measure criterion and a local growth constraint. We further assume that non-instantaneous impulsive functions satisfy specific Lipschitz criteria. Finally, an example is given to justify the theoretical results.

Keywords: Atangana-Baleanu fractional derivative; non-instantaneous impulses; Mittag-Leffler kernel; fixed point theorem
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1. Introduction

Newton developed fractional calculus in 1695, although it has only lately caught the interest of many scholars. Over the last two decades, the most intriguing developments in scientific and engineering applications have been discovered within the framework of fractional calculus. The notion of the fractional derivative has been established because to the challenges associated with a heterogeneity issue [36–38]. The important point to remember is that fractional derivatives and integrals have different applications and consequences depending on the definitions utilized, such as Riemann-Liouville, Hadamard, Grunwald Letnikov, Caputo, Riesz-Caputo, Chen, Weyl, Erd Iyi-Kober and so on. Recently, Caputo and Fabrizio [12] introduced a novel non-local and non-singular kernel idea in the non-typical Banach space H^1 in 2015. This concept was first difficult to implement, but it soon gained traction in a number of fields, including thermal science, mechanical engineering, and groundwater research; for additional details, see [4, 6, 11], and the references therein. Later, Atangana and Baleanu [1] presented a new concept of non-local derivatives with non-singular kernel depending on the Mittag-Leffler function, which supported the Caputo-Fabrizio's one based on exponential function. The Atangana and Baleanu interpretations have improved the understanding of the relationship between fractional calculus and the Mittag-Leffler function, as well as the important applications that they achieve together. For further information, see [1, 3, 7].

The theory of instantaneous impulsive differential equations covers processes that undergo a sudden change in state at certain times. Such processes occur often and spontaneously, especially in phenomena studied in science, control systems, engineering, and biological sciences ([10, 26]). The concept of instantaneous impulsive differential equations has emerged as an important research area in recent decades in Banach spaces, see the references [2, 3, 14, 15, 17, 18, 35] for more information on this idea and its applications, which contain comprehensive bibliographies as well as a wide variety of features of their solutions. In short, differential systems with instantaneous impulses looks at situations with abrupt and instantaneous impulses. Models with instantaneous impulses, on the other hand, are clearly unable of explaining several elements of pharmacological evolution processes. As Hernandez and O'Regan pointed out in [20], when we look at a simplified scenario of a person's hemodynamic equilibrium, the absorption of medications into the circulation and the body's subsequent absorption are slow and continuous processes. As a consequence, this situation may be understood as an impulsive behaviour that starts rapidly and ends after a certain amount of time has passed. We call this phenomenon that occurs while creating mathematical models "non-instantaneous impulses". According to the study, non-instantaneous impulses may describe a variety of models drawn from real-world models as partial differential systems.

Several authors have investigated non-instantaneous impulses in recent years and come up with some intriguing conclusions, see for instance [5, 9, 13, 19, 20, 25, 27, 33, 34]. In [25], authors studied the trajectory approximately controllability and optimal control for non-instantaneous impulsive inclusions without compactness in Banach spaces and finally as an application, the controllability for a differential inclusion system governed by a heat equation is considered. Recently, Qiu et al. [33] discussed the consistent tracking problem of non-instantaneous impulsive multi-agent systems and further authors shows that all agents of linear systems are driven to achieve a given asymptotical consensus as the number of iteration increases by using the standard urn:x-wiley:rnc:media:rnc5627:rnc5627-math-0003-type learning law with the initial state learning rule. Furthermore, very few authors studied the existence and controllability of fractional-order differential system through Atangana-Baleanu derivative, see for instance [3, 27–29]. In particular, in [27], authors investigated the existence results for fractional-order differential systems having non-instantaneous impulses utilizing the Atangana-Baleanu derivative in Banach spaces through measures of non-compactness. Recently, Mallika Arjunan et al. [28–30] studied the existence results of various fractional-order differential systems through Atangana-Baleanu derivative under suitable fixed point theorems.

In light of the preceding, in this manuscript, we investigate the existence results for a class of fractional-order functional differential equations with non-instantaneous impulses of the form

$$\mathscr{D}^{\vartheta}_{ABC}p(\varsigma) = Ap(\varsigma) + \mathscr{F}(\varsigma, p_\varsigma), \quad \cup_{\ell=0}^{m}(s_\ell, \varsigma_{\ell+1}], \tag{1.1}$$

$$p(\varsigma) = \kappa_\ell(\varsigma, p_\varsigma), \quad \varsigma \in \cup_{\ell=1}^{m}(\varsigma_\ell, s_\ell], \tag{1.2}$$

$$p(\varsigma) = \varphi(\varsigma) \in \mathscr{B}, \tag{1.3}$$

where $J = [0, \xi], \xi > 0$ is the operational interval, $\vartheta \in (0, 1)$, $A : D(A) \subset E \to E$ is the infinitesimal generator of an ϱ-resolvent family $\widehat{\mathscr{B}_\vartheta}(\varsigma)_{\varsigma \geq 0}$, the solution operator $\mathscr{B}_\vartheta(\varsigma)_{\varsigma\geq 0}$ is described on a complex Banach space E, $\mathscr{D}^{\vartheta}_{ABC}$ is the Atangana-Baleanu-Caputo derivative, $0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_m < \varsigma_{m+1} = \xi$, $s_0 = 0$ and $s_\ell \in (\varsigma_\ell, \varsigma_{\ell+1})$ for each $\ell = 1, 2, \ldots, m$; $\mathscr{F} : \cup_{\ell=0}^{m}(s_\ell, \varsigma_{\ell+1}] \times \mathscr{B} \to E$ is a given function which satisfies certain assumptions to be specified later on. We consider the non-instantaneous impulsive functions $\kappa_\ell : (\varsigma_\ell, s_\ell] \times \mathscr{B} \to E$, $\ell = 1, 2, \ldots, m$; we assume that $p_\varsigma : (-\infty, 0] \to E, p_\varsigma(x) = p(\varsigma + x)$, $x \leq 0$, and $\varphi \in \mathscr{B}$, where $\mathscr{B}$ is an abstract phase space defined in Section 2.2.

The rest of the manuscript is organized as follows. We give some fundamental concepts on Atangana-Baleanu fractional derivatives, phase space axioms ($\mathscr{B}$), sectorial operator and mild solution of the systems (1.1)–(1.3) in Section 2. The proof of our main results are given in Section 3. In the final section, an example is shown.

2. Preliminaries

The essential definitions and results of the sectorial operator, piece-wise continuous functions, measures of non-compactness, phase space axioms, and Atangana-Baleanu fractional derivative are covered in this part, which will assist us in proving our primary points.

Let $(E, \|\cdot\|_E)$ be a complex Banach space. $L(E)$ is the Banach space of all bounded linear operators from X into X with $\|\cdot\|_{L(E)}$ as the corresponding norm.

$\mathscr{C}([0, \xi], E)$ is the Banach space of all continuous functions from $[0, \xi]$ into E with the norm

$$\|p\|_{\mathscr{C}([0,\xi],E)} = \sup\{\|p(\varsigma)\| : \varsigma \in [0, \xi]\}.$$

The functions $p : [0, \xi] \to E$ that are integrable in the Bochner notion with regard to the Lebesgue measure, equipped with

$$\|p\|_{L^1} = \int_0^{\xi} \|p(x)\| dx$$

is denoted by $L^1([0, \xi], E)$.

Here, we recall some fundamental definitions of Atangnan-Baleanu fractional derivative.

Definition 2.1. *[1] The Atangnan-Baleanu fractional integral of order* $\vartheta \in (0, 1)$ *of a function* $r : (d, \xi) \to \mathbb{R}$ *is described as*

$$^{AB}I^{\vartheta}_{d^+}r(\varsigma) = \frac{1-\vartheta}{B(\vartheta)}r(\varsigma) + \frac{\vartheta}{B(\vartheta)\Gamma(\vartheta)}\int_d^{\varsigma}(\varsigma - x)^{\vartheta-1}r(x)dx,$$

where $B(\vartheta) = (1-\vartheta) + \frac{\vartheta}{\Gamma(\vartheta)}$ *is the normalising function that fulfills the condition* $B(0) = 1$ *and* $B(1) = 1$.

Definition 2.2. *[1] As $r \in H^1(d,\xi), d < \xi$, the Atangnan-Baleanu fractional derivative of order $\vartheta \in (0,1)$ of a function r in Caputo sense is described by*

$$^{ABC}\mathscr{D}^{\vartheta}_{d^+} r(\varsigma) = \frac{B(\vartheta)}{1-\vartheta}\int_d^{\varsigma} r'(s)E_{\vartheta}\left(-\frac{\vartheta}{1-\vartheta}(\varsigma - x)^{\vartheta}\right)dx$$

for each $\varsigma \in (d,\xi)$. Here E_{ϑ} is the Mittag-Leffler function.

We recommend readers to refer the following papers to prevent repeats of several definitions used in this manuscript: sectorial operator [23] and solution operator (see Definition 2.7 in [29]). For more information on this topic and its applications, we recommend reading [1, 3, 27, 28, 32, 35].

2.1. Piece-wise Continuous Functions

When incorporating impulsive constraints, we must first construct the piece-wise continuous functions.

We discuss it in detail here.

$$\begin{aligned}\mathscr{PC}([0,\xi],E) = \{p : [0,\xi] \to E : p \text{ is continuous for } \varsigma \neq \varsigma_\ell,\\ \text{left continuous at } \varsigma = \varsigma_\ell \text{ and } p(\varsigma_\ell^+) \text{ serves for } \ell = 1,2,\ldots,m\}.\end{aligned}$$

The fact that $\mathscr{PC}([0,\xi],E)$ is a Banach space equipped with the $\mathscr{PC}$-norm

$$\|p\|_{\mathscr{PC}} = \max\Big\{\sup_{\varsigma\in[0,\xi]}\|p(\varsigma^+)\|, \sup_{\varsigma\in[0,\xi]}\|p(\varsigma^-)\|\Big\}, \quad p \in \mathscr{PC}([0,\xi],E),$$

where $p(\varsigma^+)$ and $p(\varsigma^-)$ are the right and left limits of $p(\varsigma)$ at $\varsigma \in [0,\xi]$, accordingly.

2.2. Phase space axioms

To employ delay criteria, we must first establish the phase space axioms $\mathscr{B}$ introduced by Hale and Kato in [21] and utilize the terminology used in [24]. As a result, $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is a semi-normed linear space of functions mapping $(-\infty, 0]$ into E and satisfying the axioms below.

If $p :]-\infty,\xi] \to E, \xi > 0$, is such that $p_0 \in \mathscr{B}$, then for all $\varsigma \in [0,\xi]$, the subsequent assumptions hold:

(C1) $p_\varsigma \in \mathscr{B}$,
(C2) $\left\|p_\varsigma\right\|_{\mathscr{B}} \leq Q_1(\varsigma)\sup_{0\leq x\leq\varsigma}\|p(x)\| + Q_2(\varsigma)\|p_0\|_{\mathscr{B}}$,
(C3) $\|p(\varsigma)\| \leq \overline{W}\left\|p_\varsigma\right\|_{\mathscr{B}}$, where $\overline{W} > 0$ is a constant and $Q_1 : [0,\infty) \to [0,\infty)$ is continuous, $Q_2 : [0,\infty) \to [0,\infty)$ is locally bounded, and Q_1, Q_2 are independent of $p(\cdot)$. Furthermore, $\|\varphi(0)\| \leq \overline{W}\|\varphi\|_{\mathscr{B}}$ for every $\varphi \in \mathscr{B}$.
(C4) p_ς is a $\mathscr{B}$-valued continuous function on $[0,\xi]$ and $\mathscr{B}$ is complete. For more details, see [22].

Now, we define the space

$$Y_\xi = \{p : (-\infty,\xi] \to E \quad \text{such that} \quad p_0 \in \mathscr{B} \text{ and the assumption } \quad p|_{[0,\xi]} \in \mathscr{PC}\}.$$

In Y_ξ, the function $\|\cdot\|_{Y_\xi}$ is defined as a seminorm,

$$\|p\|_{Y_\xi} = \|\varphi\|_{\mathscr{B}} + \sup_{x\in[0,\xi]}\|p(x)\|, \quad x \in Y_\xi.$$

We recommend that the reader go to [22, 24] for further information on phase space axioms and examples.

Definition 2.3. *A function $\mathscr{F} : [0, \xi] \times \mathscr{B} \to E$ is said to be Caratheodory if it satisfies the following criteria:*

(i) the function $\mathscr{F}(\varsigma, \cdot) : \mathscr{B} \to E$ is continuous for almost every $\varsigma \in [0, \xi]$;
(ii) the function $\mathscr{F}(\cdot, p) : [0, \xi] \to E$ is measurable for every $p \in \mathscr{B}$.

2.3. Measures of Noncompactness

The idea of measure of non-compactness supports a number of our results. With this in mind, we shall now recall some of this concept's properties. The reader will be directed to [8, 16] for fundamental information. Throughout this manuscript, we only employ the Kuratowski measure idea of non-compactness.

Definition 2.4. *[8, 16] (Kuratowski measure of non-compactness) Let $B(E)$ be a family of bounded subset of E, where E is a Banach space. Then $\beta : B(E) \to \mathbb{R}_+$ is set to*

$$\beta(\mathcal{U}) := \inf\{\delta > 0 : \mathcal{U} = \cup_{i=1}^{k} \mathcal{U}_i \text{ with } \operatorname{diam}(\mathcal{U}_i) \leq \delta \text{ for } i = 1, 2, \ldots, k\},$$

where $\mathcal{U} \in B(E)$ is known as Kuratowski measure of non-compactness.

Lemma 2.1 ([8, 16]). *For any bounded sets $\mathcal{U}, \mathcal{U}_1$ and $\mathcal{U}_2$ of a Banach space E, we obtain*

(i) $\beta(\mathcal{U}) = 0$ iff $\mathcal{U}$ is totally bounded;
(ii) $\beta(\mathcal{U}) = \beta(\overline{\mathcal{U}})$, where $\overline{\mathcal{U}}$ means the closure of $\mathcal{U}$;
(iii) For each $\mathcal{U}_1 \subset \mathcal{U}_2$ implies $\beta(\mathcal{U}_1) \leq \beta(\mathcal{U}_2)$;
(iv) $\beta(\mathcal{U}_1 + \mathcal{U}_2) \leq \beta(\mathcal{U}_1) + \beta(\mathcal{U}_2)$;
(v) $\beta(\mathcal{U}_1 \cup \mathcal{U}_2) = \max\{\beta(\mathcal{U}_1), \beta(\mathcal{U}_2)\}$;
(vi) $\beta(\lambda\mathcal{U}) = |\lambda|\beta(\mathcal{U})$ for any $\lambda \in \mathbb{R}$.
(vii) $\beta(\mathcal{U}) = \beta(\overline{co}(\mathcal{U}))$.

Lemma 2.2. *[8] If $\mathcal{U} \subset \mathscr{C}([\tau_1, \tau_2], E)$ is bounded and equi-continuous on $[\tau_1, \tau_2]$, then $\beta(\mathcal{U}(\varsigma))$ is continuous for $\varsigma \in [\tau_1, \tau_2]$ and $\beta_{\mathscr{C}}(\mathcal{U}) = \sup\{\beta(\mathcal{U}(\varsigma)), \varsigma \in [\tau_1, \tau_2]\}$, where $\mathcal{U}(\varsigma) = \{p(\varsigma) : p \in \mathcal{U}\} \subset E$.*

Lemma 2.3. *[8] If $\mathcal{U}$ is a bounded set in $\mathscr{C}([\tau_1, \tau_2], E)$, then $\mathcal{U}(\varsigma)$ is bounded in E and $\beta(\mathcal{U}(\varsigma)) \leq \beta_{\mathscr{C}}(\mathcal{U})$.*

Lemma 2.4. *[13] If $\mathcal{U} \subset E$ is bounded for a Banach space E, then a countable subset $\mathcal{U}_0 \subset \mathcal{U}$ exists, for which $\beta(\mathcal{U}) \leq 2\beta(\mathcal{U}_0)$ exists.*

Lemma 2.5 ([13]). *Let E be a Banach space, and let $\mathcal{U} = \{v_n\} \subset \mathscr{PC}([\tau_1, \tau_2], E)$ be a bounded and countable set for constants $-\infty < \tau_1 < \tau_2 < +\infty$. Then $\beta(\mathcal{U}(\varsigma))$ is Lebesgue integral on $[\tau_1, \tau_2]$, and*

$$\beta\Big(\Big\{\int_{\tau_1}^{\tau_2} v_n(\varsigma) d\varsigma : n \in \mathbb{N}\Big\}\Big) \leq 2 \int_{\tau_1}^{\tau_2} \beta(\mathcal{U}(\varsigma)) d\varsigma.$$

The Kuratowski measure of non-compactness on the bounded set of E, $\mathscr{C}([0, \xi], E)$, and $\mathscr{PC}([0, \xi], E)$ is denoted by $\beta(\cdot)$, $\beta_{\mathscr{C}}(\cdot)$, and $\beta_{\mathscr{PC}}(\cdot)$, respectively, in this manuscript.

We can now define the mild solution for the systems (1.1)–(1.3).

Definition 2.5. *[28] A function $p \in Y_\xi$ is called a mild solution of the systems* (1.1)–(1.3) *if $p_0 = \varphi \in \mathscr{B}$ and $p(\varsigma) = \kappa_\ell(\varsigma, p_\varsigma)$ for $\varsigma \in (\varsigma_\ell, s_\ell]$, and each $\ell = 1, 2, \ldots, m$, satisfies the following integral equation*

$$p(\varsigma) = \begin{cases} \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) + \dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \displaystyle\int_0^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, p_s) ds \\ + \dfrac{\vartheta \mathbb{S}^2}{B(\vartheta)} \displaystyle\int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) \mathscr{F}(s, p_s) ds, \quad \varsigma \in (0, \varsigma_1], \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\kappa_\ell(s_\ell, p_{s_\ell}) + \dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \displaystyle\int_{s_\ell}^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, p_s) ds \\ + \dfrac{\vartheta \mathbb{S}^2}{B(\vartheta)} \displaystyle\int_{s_\ell}^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) \mathscr{F}(s, p_s) ds, \quad \varsigma \in \cup_{\ell=1}^m (s_\ell, \varsigma_{\ell+1}], \end{cases} \tag{2.1}$$

$\mathbb{S} = \zeta(\zeta I - A)^{-1}$ *and* $\mathbb{T} = -\widetilde{\gamma} A(\zeta I - A)^{-1}$ *with* $\zeta = \frac{B(\vartheta)}{1-\vartheta}$, $\widetilde{\gamma} = \frac{\vartheta}{1-\vartheta}$ *and*

$$\mathscr{B}_\vartheta(\varsigma) = E_\vartheta(-\mathbb{T}\varsigma^\vartheta) = \frac{1}{2\pi i} \int_\Gamma e^{x\varsigma} x^{\vartheta-1} (x^\vartheta I - \mathbb{T})^{-1} dx, \tag{2.2}$$

$$\widehat{\mathscr{B}}_\vartheta(\varsigma) = \varsigma^{\vartheta-1} E_{\vartheta,\vartheta}(-\mathbb{T}\varsigma^\vartheta) = \frac{1}{2\pi i} \int_\Gamma e^{x\varsigma} (x^\vartheta I - \mathbb{T})^{-1} dx, \tag{2.3}$$

where Γ denotes the Bromwich path [7].

Remark 2.1. *Before we can present and verify the primary result of the following section, we must first construct the operator estimates specified in* (2.2) *and* (2.3).

If $\vartheta \in (0, 1)$ and $A \in \mathscr{A}^\vartheta(\beta_0, \omega_0)$, then for any $p \in E$ and $\varsigma > 0$, we have $\|\mathscr{B}_\vartheta(\varsigma)\| \leq \widehat{\Lambda} e^{\omega\varsigma}$ and $\|\widehat{\mathscr{B}}_\vartheta(\varsigma)\| \leq C e^{\omega\varsigma}(1 + \varsigma^{\vartheta-1})$, for every $\varsigma > 0, \omega > \omega_0$. Hence, we get $\|\mathscr{B}_\vartheta(\varsigma)\| \leq \widehat{M}_{\mathscr{B}}$ and $\|\widehat{\mathscr{B}}_\vartheta(\varsigma)\| \leq \varsigma^{\vartheta-1} \widehat{M}_{\widehat{\mathscr{B}}}$. Since $\widehat{M}_{\mathscr{B}} = \sup\limits_{0 \leq \varsigma \leq \xi} \|\mathscr{B}_\vartheta(\varsigma)\|$ and $\widehat{M}_{\widehat{\mathscr{B}}} = \sup\limits_{0 \leq \varsigma \leq \xi} C e^{\omega\varsigma}(1 + \varsigma^{1-\vartheta})$. For additional details, see [23, 27, 35].

At the end of this section, we mention the crucial Monch fixed point theorem [MFPT] and it is very helpful in proving our results [9, 31].

Theorem 2.1. *For any bounded, closed and convex subset of a Banach space E. Let $\Upsilon : B \to B$ and if the implication*

$$W = \overline{conv}\, \Upsilon(W) \quad or \quad W = \Upsilon(W) \cup 0 \implies \beta(W) = 0$$

holds for every subset W of B, then Υ has a fixed point.

3. Existence results

This section presents and proves the existence findings for the systems (1.1)–(1.3) under the MFPT [31].

The following are the conditions for applying the fixed point theorem [31].

(A1) The function $\mathscr{F} : [0, \xi] \times \mathscr{B} \to E$ is Caratheodory. There exists a non-decreasing continuous function $\Omega : [0, \infty) \to (0, \infty)$ and a function $\gamma \in \mathscr{L}^1([0, \xi], \mathbb{R}_+)$ is such that

$$\|\mathscr{F}(\varsigma, v)\|_E \leq \gamma(\varsigma)\Omega(\|v\|_{\mathscr{B}}) \quad \text{for a.e.} \quad \varsigma \in [0, \xi] \text{ and } v \in \mathscr{B}.$$

(A2) For each $\ell = 1, 2, \ldots, m$ the functions $\kappa_\ell : (\varsigma_\ell, s_\ell] \times \mathscr{B} \to E$ are continuous and fulfills the subsequent assumptions:

(i) There are constants $L_{\kappa_\ell}, \overline{L}_{\kappa_\ell}$, $\ell = 1, 2, \ldots, m$ in ways that

$$\|\kappa_\ell(\varsigma, v)\|_E \leq L_{\kappa_\ell}\|v\|_{\mathscr{B}} + \overline{L}_{\kappa_\ell}, \quad \text{for a.e.} \quad \varsigma \in (\varsigma_\ell, s_\ell],\ v \in \mathscr{B}.$$

(ii) The constants $\gamma_\ell > 0$ in a way that, for each bounded $\mathcal{U}_1 \subset \mathscr{B}$,

$$\beta(\kappa_\ell(\varsigma, \mathcal{U}_1)) \leq \gamma_\ell \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_1(\theta)), \quad \text{for a.e.} \quad \varsigma \in (\varsigma_\ell, s_\ell],\ \ell = 1, 2, \ldots, m.$$

(A3) The sets $\{\varsigma \to \kappa_\ell(\varsigma, p_\varsigma) : p_\varsigma \in \mathcal{U}\}$, $\ell = 1, 2, \ldots, m$ are equi-continuous in $\mathscr{B}$ for any bounded set $\mathcal{U} \subset \mathscr{B}$.

(A4) For each $\ell = 0, 1, 2, \ldots, m$, there exist constants $L_\ell > 0$ in ways that for any countable set $\mathcal{U}_2 \subset \mathscr{B}$

$$\beta(\mathscr{F}(\varsigma, \mathcal{U}_2)) \leq L_\ell \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_2(\theta)) \quad \forall \quad \varsigma \in (s_\ell, \varsigma_{\ell+1}].$$

(A5) $\mathbb{S}$ and $\mathbb{T}$ are bounded linear operators and there exist constants $\mu, \overline{\mu}$ such that $\|\mathbb{S}\| \leq \mu$ and $\|\mathbb{T}\| \leq \overline{\mu}$.

Theorem 3.1. *Assume that (A1)–(A5) hold. If*

$$\widehat{\Theta} = \left[2\mu\widehat{M}_{\mathscr{B}}\widehat{\gamma} + 4\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\widetilde{L}\right] < 1, \tag{3.1}$$

where $\widehat{\gamma} = \max_{\ell=1,2,\ldots,m} \gamma_\ell$ *and* $\widetilde{L} = \max_{\ell=0,1,2,\ldots,m} \{(\varsigma_{\ell+1} - s_\ell)^\vartheta L_\ell\}$. *Then the systems* (1.1)–(1.3) *has at least one mild solution on* $[0, \xi]$.

Proof. Now the operator $\Upsilon : Y_\xi \to Y_\xi$ defined by

$$(\Upsilon p)(\varsigma) = \begin{cases} \varphi(\varsigma), & \varsigma \leq 0, \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0) + \dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\displaystyle\int_0^\varsigma (\varsigma - s)^{\vartheta-1}\mathscr{F}(s, p_s)ds & \\ +\dfrac{\vartheta\mathbb{S}^2}{B(\vartheta)}\displaystyle\int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s)\mathscr{F}(s, p_s)ds, & \varsigma \in (0, \varsigma_1], \\ \kappa_\ell(\varsigma, p_\varsigma), & \varsigma \in \cup_{\ell=1}^m (\varsigma_\ell, s_\ell], \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\kappa_\ell(s_\ell, p_{s_\ell}) + \dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\displaystyle\int_{s_\ell}^\varsigma (\varsigma - s)^{\vartheta-1}\mathscr{F}(s, p_s)ds & \\ +\dfrac{\vartheta\mathbb{S}^2}{B(\vartheta)}\displaystyle\int_{s_\ell}^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s)\mathscr{F}(s, p_s)ds, & \varsigma \in \cup_{\ell=1}^m (s_\ell, \varsigma_{\ell+1}]. \end{cases}$$

Let $u(\cdot) : (-\infty, \xi] \to E$ be the function described by

$$u(\varsigma) = \begin{cases} \varphi(\varsigma), & \varsigma \in (-\infty, 0], \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0), & \varsigma \in [0, \xi], \end{cases}$$

then $u_0 = \varphi$. For every $v \in \mathscr{C}([0,\xi],\mathbb{R})$ with $v(0) = 0$, we denote by $\overline{v}$ the function defined by

$$\overline{v}(\varsigma) = \begin{cases} 0, & \varsigma \in (-\infty, 0]; \\ v(\varsigma), & \varsigma \in [0,\xi]. \end{cases}$$

Let $p(\cdot)$ satisfies (2.1), then we decompose $p(\cdot)$ as $p(\varsigma) = v(\varsigma) + u(\varsigma)$ for $\varsigma \in [0,\xi]$, which suggests $p_\varsigma = v_\varsigma + u_\varsigma$, and the function $v(\cdot)$ fulfills

$$v(\varsigma) = \begin{cases} \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, v_s + u_s) ds \\ + \frac{\vartheta \mathbb{S}^2}{B(\vartheta)} \int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) \mathscr{F}(s, v_s + u_s) ds, & \varsigma \in (0, \varsigma_1], \\ \kappa_\ell(\varsigma, v_\varsigma + u_\varsigma), & \varsigma \in \cup_{\ell=1}^m (\varsigma_\ell, s_\ell], \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\kappa_\ell(s_\ell, v_{s_\ell} + u_{s_\ell}) + \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_{s_\ell}^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, v_s + u_s) ds \\ + \frac{\vartheta \mathbb{S}^2}{B(\vartheta)} \int_{s_\ell}^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) \mathscr{F}(s, v_s + u_s) ds, & \varsigma \in \cup_{\ell=1}^m (s_\ell, \varsigma_{\ell+1}]. \end{cases}$$

Set $Y_\xi^0 = \{v \in Y_\xi : v_0 = 0 \in \mathscr{B}\}$ and for any $v \in Y_\xi^0$, we have

$$\|v\|_{Y_\xi^0} = \|v_0\|_{\mathscr{B}} + \sup\{\|v(x)\|_E : 0 \le s < +\infty\} = \sup\{\|v(x)\|_E : 0 \le x < +\infty\}.$$

As a result, the Banach space $(Y_\xi^0, \|\cdot\|_{Y_\xi^0})$ exists. Consider $\overline{\Upsilon} : Y_\xi^0 \to Y_\xi^0$, which is defined as:

$$(\overline{\Upsilon} v)(\varsigma) = \begin{cases} \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, v_s + u_s) ds \\ + \frac{\vartheta \mathbb{S}^2}{B(\vartheta)} \int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) \mathscr{F}(s, v_s + u_s) ds, & \varsigma \in (0, \varsigma_1], \\ \kappa_\ell(\varsigma, v_\varsigma + u_\varsigma), & \varsigma \in \cup_{\ell=1}^m (\varsigma_\ell, s_\ell], \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\kappa_\ell(s_\ell, v_{s_\ell} + u_{s_\ell}) + \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_{s_\ell}^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, v_s + u_s) ds \\ + \frac{\vartheta \mathbb{S}^2}{B(\vartheta)} \int_{s_\ell}^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s) \mathscr{F}(s, v_s + u_s) ds, & \varsigma \in \cup_{\ell=1}^m (s_\ell, \varsigma_{\ell+1}]. \end{cases}$$

It is obvious that operator Υ has a fixed-point if and only if $\overline{\Upsilon}$ has a fixed-point.

To prove the result, we first determine the estimate of the phase space axioms. For every $\varsigma \in [0,\xi]$, we have from Section 2.2,

$$\begin{aligned} \|v_\varsigma + u_\varsigma\|_{\mathscr{B}} &\le \|v_\varsigma\|_{\mathscr{B}} + \|u_\varsigma\|_{\mathscr{B}} \\ &\le Q_1(\varsigma)\|v(\varsigma)\|_E + Q_2(\varsigma)\|v_0\|_{\mathscr{B}} + Q_1(\varsigma)\|u(\varsigma)\|_E + Q_2(\varsigma)\|u_0\|_{\mathscr{B}} \\ &\le Q_1(\varsigma)\|v(\varsigma)\|_E + Q_1(\varsigma)[\|\mathbb{S}\mathscr{B}_\vartheta(\varsigma)\varphi(0)\|_E] + Q_2(\varsigma)\|\varphi\|_{\mathscr{B}} \\ &\le Q_1^* \sup_{0\le x\le\varsigma} \|v(x)\|_E + Q_1^*\mu\widehat{M}_{\mathscr{B}}\overline{W}\|\varphi\|_{\mathscr{B}} + Q_2^*\|\varphi\|_{\mathscr{B}} \\ &= Q_1^* \sup_{0\le x\le\varsigma} \|v(x)\|_E + (Q_1^*\mu\widehat{M}_{\mathscr{B}}\overline{W} + Q_2^*)\|\varphi\|_{\mathscr{B}}, \end{aligned}$$

where $Q_1^* = \sup_{x\in[0,\xi]} Q_1(x)$ and $Q_2^* = \sup_{x\in[0,\xi]} Q_2(x)$.

Then, we get

$$\|v_\varsigma + u_\varsigma\|_{\mathscr{B}} \le e + Q_1^* \sup_{0\le x\le\varsigma} \|v(x)\|_E, \tag{3.2}$$

where $e = (Q_1^*\mu\widehat{M}_{\mathscr{B}}\overline{W} + Q_2^*)\|\varphi\|_{\mathscr{B}}$.

To make the proof more understandable, we have separated it into phases.

Step 1: $\overline{\Upsilon}$ is continuous.

Let v^n be a sequence such that $v^n \to v$ in Y_ξ^0.

$$(\overline{\Upsilon}v^n)(\varsigma) = \begin{cases} \dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \displaystyle\int_0^\varsigma (\varsigma - s)^{\vartheta-1} \mathscr{F}(s, v_s^n + u_s) ds \\ + \dfrac{\vartheta\mathbb{S}^2}{B(\vartheta)} \displaystyle\int_0^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s)\mathscr{F}(s, v_s^n + u_s) ds, \quad \varsigma \in (0, \varsigma_1], \\ \kappa_\ell(\varsigma, v_\varsigma^n + u_\varsigma),\ \varsigma \in \cup_{\ell=1}^m (\varsigma_\ell, s_\ell], \\ \mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\kappa_\ell(s_\ell, v_{s_\ell}^n + u_{s_\ell}) + \dfrac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \displaystyle\int_{s_\ell}^\varsigma (\varsigma - s)^{\vartheta-1}\mathscr{F}(s, v_s^n + u_s) ds \\ + \dfrac{\vartheta\mathbb{S}^2}{B(\vartheta)} \displaystyle\int_{s_\ell}^\varsigma \widehat{\mathscr{B}}_\vartheta(\varsigma - s)\mathscr{F}(s, v_s^n + u_s) ds, \quad \varsigma \in \cup_{\ell=1}^m (s_\ell, \varsigma_{\ell+1}]. \end{cases}$$

By using the dominated convergence theorem along with (A5), for $\varsigma \in [0, \varsigma_1]$, we have

$$\begin{aligned} \|(\overline{\Upsilon}v^n)(\varsigma) - (\overline{\Upsilon}v)(\varsigma)\|_E &\le \Bigg\| \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1}[\mathscr{F}(s, v_s^n + u_s) - \mathscr{F}(s, v_s + u_s)] ds \\ &\quad + \frac{\vartheta\mathbb{S}^2}{B(\vartheta)} \int_0^\varsigma \mathscr{B}_\vartheta(\varsigma - s)[\mathscr{F}(s, v_s^n + u_s) - \mathscr{F}(s, v_s + u_s)] ds \Bigg\|_E \\ &\le \frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1} \|\mathscr{F}(s, v_s^n + u_s) - \mathscr{F}(s, v_s + u_s)\|_E ds \\ &\quad + \frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)} \int_0^\varsigma (\varsigma - s)^{\vartheta-1} \|\mathscr{F}(s, v_s^n + u_s) - \mathscr{F}(s, v_s + u_s)\|_E ds \\ &\le \left[\frac{\mu\overline{\mu}(1-\vartheta)\varsigma_1^\vartheta}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}\varsigma_1^\vartheta}{B(\vartheta)}\right] \sup_{x\in[0,\xi]} \|\mathscr{F}(s, v_s^n + u_s) - \mathscr{F}(s, v_s + u_s)\|_E \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

For any $\varsigma \in (\varsigma_\ell, s_\ell]$, $\quad \ell = 1, 2, \ldots, m$, we obtain

$$\begin{aligned} \|(\overline{\Upsilon}v^n)(\varsigma) - (\overline{\Upsilon}v)(\varsigma)\|_E &\le \|\kappa_\ell(\varsigma, v_\varsigma^n + u_\varsigma) - \kappa_\ell(\varsigma, v_\varsigma + u_\varsigma)\|_E \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

For $\varsigma \in \cup_{\ell=0}^m (s_\ell, \varsigma_{\ell+1}]$, we get

$$\|(\overline{\Upsilon}v^n)(\varsigma) - (\overline{\Upsilon}v)(\varsigma)\|_E \le \|\mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\|_E[\|\kappa_\ell(\varsigma, v_\varsigma^n + u_\varsigma) - \kappa_\ell(\varsigma, v_\varsigma + u_\varsigma)\|_E]$$

$$+\left[\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}+\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right]\widehat{L}\sup_{x\in[0,\xi]}\|\mathscr{F}(s,v_s^n+u_s)-\mathscr{F}(s,v_s+u_s)\|_E$$
$$\to 0 \quad \text{as} \quad n\to\infty,$$

where $\widehat{L}=\max_{\ell=0,1,2,\dots,m}(\varsigma_{\ell+1}-s_\ell)^{\vartheta}$.

It is easy to see that

$$\lim_{n\to\infty}\|(\overline{\Upsilon}v^n)-(\overline{\Upsilon}v)\|_{Y_\xi^0}=0.$$

As a result, in Y_ξ^0, the operator $\overline{\Upsilon}$ is continuous.

Step 2: Any closed ball B_R of Y_ξ^0 is mapped into bounded sets in Y_ξ^0 by $\overline{\Upsilon}$.

Indeed, it is enough to show that there exists $N>0$ in a way that for every $v\in B_R=\{v\in Y_\xi^0:\|v\|_{Y_\xi^0}\le R\}$ one has $\|\overline{\Upsilon}(v)\|_{Y_\xi^0}\le N$.

Let $v\in Y_\xi^0$ and denote $\gamma^*=\sup_{0\le x\le\xi}\gamma(x)$. If $\varsigma\in[0,\varsigma_1]$, then by (A1) and (3.2), we have

$$\begin{aligned}
\|(\overline{\Upsilon}v)(\varsigma)\|_E &\le \frac{\|\mathbb{S}\|\|\mathbb{T}\|(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\|\mathscr{F}(s,v_s+u_s)\|_E ds\\
&\quad+\frac{\vartheta\|\mathbb{S}\|^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\|\mathscr{F}(s,v_s+u_s)\|_E ds\\
&\le \frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\gamma(s)\Omega(\|v_s+u_s\|_{\mathscr{B}})ds\\
&\quad+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\gamma(s)\Omega(\|v_s+u_s\|_{\mathscr{B}})ds\\
&\le\left[\frac{\mu\overline{\mu}(1-\vartheta)\varsigma_1^{\vartheta}}{B(\vartheta)\Gamma(\vartheta+1)}+\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}\varsigma_1^{\vartheta}}{B(\vartheta)}\right]\gamma^*\Omega(e+Q_1^*R).
\end{aligned}$$

For $\varsigma\in(\varsigma_\ell,s_\ell]$, $\ell=1,2,\dots,m$, then by (A2) and (3.2), we get

$$\begin{aligned}
\|(\overline{\Upsilon}v)(\varsigma)\|_E &= \|\kappa_\ell(\varsigma,v_\varsigma+u_\varsigma)\|_E\\
&\le L_{\kappa_\ell}\|v_\varsigma+u_\varsigma\|_{\mathscr{B}}+\overline{L}_{\kappa_\ell}\\
&\le L_{\kappa_\ell}(e+Q_1^*R)+\overline{L}_{\kappa_\ell}\\
&\le L(1+e+Q_1^*R),
\end{aligned}$$

where $L=\max_{\ell=1,2,\dots,m}\{L_{\kappa_\ell},\overline{L}_{\kappa_\ell}\}$.

For $\varsigma\in(s_\ell,\varsigma_{\ell+1}]$, $\ell=1,2,\dots,m$, then by conditions (A1) and (A2) along with (3.2), we obtain

$$\begin{aligned}
\|(\overline{\Upsilon}v)(\varsigma)\|_E &\le \|\mathbb{S}\|\|\mathscr{B}_\vartheta(\varsigma-s_\ell)\|\|\kappa_\ell(s_\ell,v_{s_\ell}+u_{s_\ell})\|_E\\
&\quad+\frac{\|\mathbb{S}\|\|\mathbb{T}\|(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_{s_\ell}^{\varsigma}(\varsigma-s)^{\vartheta-1}\|\mathscr{F}(s,v_s+u_s)\|_E ds\\
&\quad+\frac{\vartheta\|\mathbb{S}\|^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\int_{s_\ell}^{\varsigma}(\varsigma-s)^{\vartheta-1}\|\mathscr{F}(s,v_s+u_s)\|_E ds
\end{aligned}$$

$$\leq \mu\widehat{M}_{\mathscr{B}}L(1+e+Q_1^*R)+\left[\frac{\mu\bar{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}+\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right](\varsigma_{\ell+1}-s_\ell^\vartheta)\gamma^*\Omega(e+Q_1^*R)$$
$$\leq \mu\widehat{M}_{\mathscr{B}}L(1+e+Q_1^*R)+\left[\frac{\mu\bar{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}+\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right]\gamma^*\Omega(e+Q_1^*R)\widehat{L}$$
$$\leq N,$$

where $\widehat{L}=\max\limits_{\ell=0,1,2,\dots,m}(\varsigma_{\ell+1}-s_\ell)^\vartheta$.

Step 3: $\overline{\Upsilon}$ maps bounded sets of Y_ξ^0 into equi-continuous sets on Y_ξ^0.

Let B_R be the same as defined in Step 2. Let $0\leq\nu_1\leq\nu_2\leq\varsigma_1$ for each $\nu\in B_R$, we sustain

$$\|(\overline{\Upsilon}\nu)(\nu_2)-(\overline{\Upsilon}\nu)(\nu_1)\|_E\leq\sum_{i=1}^4\|I_i\|,$$

where

$$I_1=\frac{\mathbb{ST}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_0^{\nu_1}[(\nu_2-s)^{\vartheta-1}-(\nu_1-s)^{\vartheta-1}]\mathscr{F}(s,\nu_s+u_s)ds;$$
$$I_2=\frac{\mathbb{ST}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_{\nu_1}^{\nu_2}(\nu_2-s)^{\vartheta-1}\mathscr{F}(s,\nu_s+u_s)ds;$$
$$I_3=\frac{\vartheta\mathbb{S}^2}{B(\vartheta)}\int_0^{\nu_1}[\widehat{\mathscr{B}}_\vartheta(\nu_2-s)-\widehat{\mathscr{B}}_\vartheta(\nu_1-s)]\mathscr{F}(s,\nu_s+u_s)ds;$$
$$I_4=\frac{\vartheta\mathbb{S}^2}{B(\vartheta)}\int_{\nu_1}^{\nu_2}\widehat{\mathscr{B}}_\vartheta(\nu_2-s)\mathscr{F}(s,\nu_s+u_s)ds.$$

Therefore

$$\|I_1\|\leq\frac{\mu\bar{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}\gamma^*\Omega(e+Q_1^*R)[(\nu_2^\vartheta-\nu_1^\vartheta)-(\nu_2-\nu_1)^\vartheta]$$
$$\to 0\quad\text{as}\quad\nu_2\to\nu_1.$$
$$\|I_2\|\leq\frac{\mu\bar{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}\gamma^*\Omega(e+Q_1^*R)[(\nu_2-\nu_1)^\vartheta]$$
$$\to 0\quad\text{as}\quad\nu_2\to\nu_1.$$
$$\|I_3\|\leq\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\gamma^*\Omega(e+Q_1^*R)[(\nu_2^\vartheta-\nu_1^\vartheta)-(\nu_2-\nu_1)^\vartheta]$$
$$\to 0\quad\text{as}\quad\nu_2\to\nu_1.$$
$$\|I_4\|\leq\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\gamma^*\Omega(e+Q_1^*R)[(\nu_2-\nu_1)^\vartheta]$$
$$\to 0\quad\text{as}\quad\nu_2\to\nu_1.$$

Hence $\|(\overline{\Upsilon}\nu)(\nu_2)-(\overline{\Upsilon}\nu)(\nu_1)\|_E\to 0$ as $\nu_2\to\nu_1$ by using the continuity of $\widehat{\mathscr{B}}_\vartheta$.

For any $\nu_1,\nu_2\in\cup_{\ell=1}^m(\varsigma_\ell,s_\ell]$, we get

$$\|(\overline{\Upsilon}\nu)(\nu_2)-(\overline{\Upsilon}\nu)(\nu_1)\|_E=\|\kappa_\ell(\nu_2,\nu_{\nu_2}+u_{\nu_2})-\kappa_\ell(\nu_1,\nu_{\nu_1}+u_{\nu_1})\|_E$$

$$\to 0 \quad \text{as} \quad \nu_2 \to \nu_1.$$

In the similar manner, for any $\nu_1, \nu_2 \in \cup_{\ell=0}^{m}(s_\ell, \varsigma_{\ell+1}]$, $s_\ell \le \nu_1 < \nu_2 \le \varsigma_{\ell+1}$, we obtain

$$\|(\overline{\Upsilon}v)(\nu_2) - (\overline{\Upsilon}v)(\nu_1)\|_E \le \sum_{i=5}^{9} \|I_i\|,$$

where

$$\begin{aligned}
I_5 &= \mathbb{S}[\mathscr{B}_\vartheta(\nu_2 - s_\ell) - \mathscr{B}_\vartheta(\nu_1 - s_\ell)]\kappa_\ell(s_\ell, v_{s_\ell} + u_{s_\ell})\| \\
I_6 &= \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_{s_\ell}^{\nu_1} [(\nu_2 - s)^{\vartheta-1} - (\nu_1 - s)^{\vartheta-1}]\mathscr{F}(s, v_s + u_s)ds; \\
I_7 &= \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} \int_{\nu_1}^{\nu_2} (\nu_2 - s)^{\vartheta-1}\mathscr{F}(s, v_s + u_s)ds; \\
I_8 &= \frac{\vartheta\mathbb{S}^2}{B(\vartheta)} \int_{s_\ell}^{\nu_1} [\widehat{\mathscr{B}}_\vartheta(\nu_2 - s) - \widehat{\mathscr{B}}_\vartheta(\nu_1 - s)]\mathscr{F}(s, v_s + u_s)ds; \\
I_9 &= \frac{\vartheta\mathbb{S}^2}{B(\vartheta)} \int_{\nu_1}^{\nu_2} \widehat{\mathscr{B}}_\vartheta(\nu_2 - s)\mathscr{F}(s, v_s + u_s)ds.
\end{aligned}$$

Now

$$\begin{aligned}
\|I_5\| &\le \mu\|[\mathscr{B}_\vartheta(\nu_2 - s_\ell) - \mathscr{B}_\vartheta(\nu_1 - s_\ell)]\kappa_\ell(s_\ell, v_{s_\ell} + u_{s_\ell})\|_E \\
&\to 0 \quad \text{as} \quad \nu_2 \to \nu_1. \\
\|I_6\| &\le \frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}\gamma^*\Omega(e + Q_1^*R)[(\nu_2 - s_\ell)^\vartheta - (\nu_2 - \nu_1)^\vartheta - (\nu_1 - s_\ell)^\vartheta] \\
&\to 0 \quad \text{as} \quad \nu_2 \to \nu_1. \\
\|I_7\| &\le \frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}\gamma^*\Omega(e + Q_1^*R)[(\nu_2 - \nu_1)^\vartheta] \\
&\to 0 \quad \text{as} \quad \nu_2 \to \nu_1. \\
\|I_8\| &\le \frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\gamma^*\Omega(e + Q_1^*R)[(\nu_2 - s_\ell)^\vartheta - (\nu_2 - \nu_1)^\vartheta - (\nu_1 - s_\ell)^\vartheta] \\
&\to 0 \quad \text{as} \quad \nu_2 \to \nu_1. \\
\|I_9\| &\le \frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\gamma^*\Omega(e + Q_1^*R)[(\nu_2 - \nu_1)^\vartheta] \\
&\to 0 \quad \text{as} \quad \nu_2 \to \nu_1.
\end{aligned}$$

From the above discussion, we conclude that $\|(\overline{\Upsilon}v)(\nu_2) - (\overline{\Upsilon}v)(\nu_1)\|_E \to 0$ as $\nu_2 \to \nu_1$. On Y_ξ^0, the operator $\overline{\Upsilon}$ is hence equi-continuous.

Now, let W be a subset of B_R such that $W \subset \overline{conv}\ (\Upsilon(W) \cup \{0\})$. Furthermore, for every bounded set $\mathcal{U}$, by Lemma 2.4, we note that there exists a countable set $\mathcal{U}_0 = \{w_n\} \subset \mathcal{U}$, such that $\beta(\overline{\Upsilon}(\mathcal{U})) \le 2\beta(\overline{\Upsilon}(\mathcal{U}_0))$. Thus for $\{w_n\} \subset \mathcal{U}$, noting that the choice of W. For every $\varsigma \in [0, \varsigma_1]$, by utilizing Lemma 2.5, condition (A4) and properties of the measure β, we obtain

$$\begin{aligned}
\beta(\overline{\Upsilon}(w_n)) &= \beta\left(\left\{\frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\mathscr{F}(s,w_{ns}+u_s)ds\right\}\right)\\
&\quad+\beta\left(\left\{\frac{\vartheta\mathbb{S}^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\mathscr{F}(s,w_{ns}+u_s)ds\right\}\right)\\
&=\beta\left(\left\{\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right\}\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}\mathscr{F}(s,w_{ns}+u_s)ds\right)\\
&\le 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}L_\ell\left[\sup_{-\infty<\theta\le 0}\beta(w_n(\theta+s)+u(\theta+s))\right]ds\\
&\le 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}L_\ell\sup_{0<\mu\le\xi}\beta(w_n(\mu))ds\\
&\le 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}L_\ell\sup_{0<s\le\xi}\beta(w_n(s))ds\\
&\le 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)L_\ell\beta(\{w_n\})\int_0^{\varsigma}(\varsigma-s)^{\vartheta-1}ds\\
&\le 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}+\frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)L_\ell\cdot\frac{\varsigma_1^{\vartheta}}{\vartheta}\beta(\{w_n\}),
\end{aligned}$$

which ensures that

$$\beta(\overline{\Upsilon}(W))\le 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)}+\frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\widehat{L_1}\beta_{\mathscr{P}\mathscr{C}}(W),$$

where $\widehat{L_1}=\max\limits_{\ell=0,1,2,\dots,m}\{L_\ell\varsigma_1^{\vartheta}\}$.

For any $\varsigma\in\cup_{\ell=1}^{m}(\varsigma_\ell,s_\ell]$, we obtain

$$\begin{aligned}
\beta(\overline{\Upsilon}(w_n)) &= \beta(\{\kappa_\ell(\kappa,w_{n\varsigma}+u_\varsigma)\})\\
&\le 2\gamma_\ell\sup_{-\infty<\theta\le 0}\beta(w_n(\theta+\varsigma)+u(\theta+\varsigma))\\
&\le 2\gamma_\ell\sup_{0<\mu\le\xi}\beta(w_n(\mu))\\
&\le 2\gamma_\ell\sup_{0<s\le\xi}\beta(w_n(s))\\
&\le 2\gamma_\ell\beta(\{w_n\}),
\end{aligned}$$

which ensures that

$$\beta(\overline{\Upsilon}(W))\le 2\widehat{\gamma}\beta_{\mathscr{P}\mathscr{C}}(W),$$

where $\widehat{\gamma}=\max\limits_{\ell=1,2,\dots,m}\gamma_\ell$.

Likewise, for any $\varsigma \in (s_\ell, \varsigma_{\ell+1}]$, $\ell = 0, 1, 2, \dots, m$, we get

$$\begin{aligned}
\beta(\overline{\Upsilon}(w_n)) &= \beta\Bigg(\Bigg\{\mathbb{S}\mathscr{B}_\vartheta(\varsigma - s_\ell)\kappa_\ell(s_\ell, w_{ns_\ell} + u_{s_\ell}) + \frac{\mathbb{S}\mathbb{T}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)}\int_{s_\ell}^{\vartheta}(\varsigma - s)^{\vartheta-1}\mathscr{F}(s, w_{ns} + u_s)ds \\
&\quad + \frac{\vartheta\mathbb{S}^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\int_{s_\ell}^{\varsigma}(\varsigma - s)^{\vartheta-1}\mathscr{F}(s, w_{ns} + u_s)ds\Bigg\}\Bigg) \\
&\le 2\mu\widehat{M}_{\mathscr{B}}\beta(\{\kappa_\ell(s_\ell, w_{ns_\ell} + u_{s_\ell})\}) \\
&\quad + 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\int_{s_\ell}^{\varsigma}(\varsigma - s)^{\vartheta-1}L_\ell\left[\sup_{-\infty<\theta\le 0}\beta(w_n(\theta + s) + u(\theta + s))\right]ds \\
&\le 2\mu\widehat{M}_{\mathscr{B}}\gamma_\ell \sup_{-\infty<\theta\le 0}\beta(w_n(\theta + s_\ell) + u(\theta + s_\ell)) \\
&\quad + 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\int_{s_\ell}^{\varsigma}(\varsigma - s)^{\vartheta-1}L_\ell \sup_{s_\ell<\mu\le\xi}\beta(w_n(\mu)ds \\
&\le 2\mu\widehat{M}_{\mathscr{B}}\gamma_\ell \sup_{0<s\le\xi}\beta(w_n(s)) \\
&\quad + 2\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta)} + \frac{\vartheta\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\int_{s_\ell}^{\varsigma}(\varsigma - s)^{\vartheta-1}L_\ell \sup_{0<s\le\xi}\beta(w_n(s))ds,
\end{aligned}$$

which ensures that

$$\beta(\overline{\Upsilon}(W)) \le \left(2\mu\widehat{M}_{\mathscr{B}}\widehat{\gamma} + 4\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\widetilde{L}\right)\beta_{\mathscr{PC}}(W),$$

where $\widehat{\gamma} = \max_{\ell=1,2,\dots,m}\gamma_\ell$ and $\widetilde{L} = \max_{\ell=0,1,2,\dots,m}\{(\varsigma_{\ell+1} - s_\ell)^{\vartheta}L_\ell\}$.

Then

$$\beta_{\mathscr{PC}}(W) \le \beta(\overline{\Upsilon}(W))_{\mathscr{PC}} \le \widehat{\Theta}\beta_{\mathscr{PC}}(W).$$

That is to say

$$\beta_{\mathscr{PC}}(W)(1 - \widehat{\Theta}) \le 0.$$

Hence, we get $\beta(W) = 0$. The theorem of Arzela-Ascoli shows that W in Y_ξ^0 is relatively compact. The Monch fixed point theorem 2.1 concludes that $\overline{\Upsilon}$ has a fixed point $v \in Y_\xi^0$.

□

4. Applications

Consider the following fractional differential equation with non-instantaneous impulsive condition of the form

$$\begin{aligned}
\mathscr{D}^{\vartheta}_{ABC}z(\varsigma, w) &= \frac{\partial^2}{\partial w^2}z(\varsigma, w) \\
&\quad + \int_{-\infty}^{0}\frac{e^{\sigma\tau}}{49}\frac{|z(\varsigma + \tau, w)|}{\sqrt{1 + |z(\varsigma + \tau, w)|}(1 + |z(\varsigma + \tau, w)|)}d\tau, \ (\varsigma, w) \in \cup_{\ell=1}^{m}(s_\ell, \varsigma_{\ell+1}] \times [0, \pi], \quad (4.1)
\end{aligned}$$

$$z(\varsigma, 0) = z(\varsigma, \pi) = 0,\ \varsigma \in [0, 1], \tag{4.2}$$

$$z(\tau, w) = z_0(\tau, w),\ -\infty < \tau \leq 0,\ w \in [0, \pi], \tag{4.3}$$

$$z(\varsigma, w) = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{36} \frac{|z(\varsigma+\tau, w)|}{(1 + |z(\varsigma+\tau, w)|)} d\tau,\ (\varsigma, w) \in (\varsigma_\ell, s_\ell] \times [0, \pi],\ \ell = 1, 2, \ldots, m, \tag{4.4}$$

where $\sigma > 0$, $\mathscr{D}^{\vartheta}_{ABC}$ is the Atangana-Baleanu-Caputo derivative of order $\vartheta \in (0, 1)$, $0 = \varsigma_0 = s_0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_m < s_m < \varsigma_{m+1} = 1$ are prefixed numbers and $z_0 \in \mathscr{B}$.

Let $E = L^2([0, \pi], E)$ and $A : D(A) \subseteq E \to E$ be an operator described by $A\Xi = \Xi'', \Xi \in D(A)$ with domain $D(A) = \{\Xi \in E; \Xi$ and Ξ' are absolutely continuous, $\Xi'' \in E, \Xi(0) = \Xi(\pi) = 0\}$. Then

$$A\Xi = \sum_{n=1}^{\infty} n^2 \langle \Xi, \Xi_n \rangle \Xi_n,\ \Xi \in D(A),$$

where $\Xi_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns)$, $n \in \mathbb{N}$ is the orthonormal set of eigenvectors of A and in E, A generates a C_0 semigroup $\{\mathscr{B}(\varsigma)\}_{\varsigma \geq 0}$ which is described by

$$\mathscr{B}(\varsigma)\Xi = \sum_{n=1}^{\infty} e^{-n^2\varsigma} \langle \Xi, \Xi_n \rangle \Xi_n,\ \Xi \in E,\ \varsigma > 0$$

which is uniformly bounded and also a compact semigroup and hence the operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is compact for each $\lambda \in \rho(A)$, i.e., $A \in \mathscr{A}^{\rho}(\beta_0, \Xi_0)$. From [27], we have $\|\mathscr{B}_\vartheta(\varsigma)\| \leq \widehat{M}_{\mathscr{B}}$ for each $\varsigma \in [0, 1]$.

For the phase space $\mathscr{B}$, we choose the well-known space $BUC(\mathbb{R}_-, E)$, the space of bounded uniformly continuous functions and satisfies the phase space axioms $(C1)$ and $(C2)$. Further, it is defined from $(-\infty, 0]$ to E endowed with the following norm:

$$\|\varphi\| = \sup_{\tau \leq 0} |\varphi(\tau)|, \quad \text{for each} \quad \varphi \in \mathscr{B}.$$

If $\varphi \in BUC(\mathbb{R}_-, E)$ and $w \in [0, \pi]$,

$$z(\varsigma)(w) = z(\varsigma, w), \quad \varsigma \in [0, \xi],\ w \in [0, \pi],$$

$$\varphi(\tau)(w) = z_0(\tau, w),\ -\infty < \tau \leq 0,\ w \in [0, \pi],$$

$$\mathscr{F}(\varsigma, \varphi)(w) = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{49} \frac{|\varphi(\tau)(w)|}{\sqrt{1 + |\varphi(\tau)(w)|}(1 + |\varphi(\tau)(w)|)} d\tau, \quad -\infty < \tau \leq 0,\ w \in [0, \pi],\ \sigma > 0,$$

$$\kappa_\ell(\varsigma, \varphi)(w) = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{36} \frac{|\varphi(\tau)(w)|}{(1 + |\varphi(\tau)(w)|)} d\tau.$$

The systems (4.1)–(4.4) can then be written as (1.1)–(1.3) in an abstract form.

Verification of the hypotheses:

We now check that the presumptions (A1)–(A5) for the problems (4.1)–(4.4) are correct.

The function $\mathscr{F} : \cup_{\ell=0}^{m} (s_\ell, \varsigma_{\ell+1}] \times \mathscr{B} \to E$ defined by

$$\|\mathscr{F}(\varsigma, \varphi)\|(w) = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{49} \frac{\|\varphi(\tau)(w)\|}{\sqrt{1 + \|\varphi(\tau)(w)\|}(1 + \|\varphi(\tau)(w)\|)} d\tau$$

$$\leq \frac{1}{\sqrt{1+\|\varphi(\varsigma)(w)\|}}\Omega(\|\varphi(\varsigma)(w)\|)$$
$$\leq \gamma(\varsigma)\Omega(\|\varphi(\varsigma)(w)\|),$$

where $\gamma(\varsigma) = \frac{1}{\sqrt{1+\|\varphi(\varsigma)(w)\|}}$ and $\int_{-\infty}^{0} \frac{e^{\sigma\tau}}{49} d\tau < \infty$.

Furthermore

$$\beta(\mathscr{F}(\varsigma, \mathcal{U}_2)) \leq \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{49} d\tau \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_2(\theta))$$
$$\leq L_\ell \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_2(\theta))$$
$$\leq \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_2(\theta)),$$

where $L_\ell = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{49} d\tau < \infty$, $\ell = 1, 2, \ldots, m$. Take $\max\{L_\ell, \ell = 1, 2, \ldots, m\} = 1$.

From this, we can conclude that $\mathscr{F}$ satisfies the conditions (A1) and (A4).

Consider the non-instantaneous impulsive functions $\kappa_\ell : (\varsigma_\ell, s_\ell] \times \mathscr{B} \to E,\ \ell = 1, 2, \ldots, m$, we have

$$\|\kappa_\ell(\varsigma, \varphi)(w)\| = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{36} \frac{\|\varphi(\tau)(w)\|}{(1+\|\varphi(\tau)(w)\|)} d\tau$$
$$\leq L_{\kappa_\ell}\|\varphi(\tau)(w)\| + \overline{L}_{\kappa_\ell},$$

where $L_{\kappa_\ell} = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{36}$ and $\overline{L}_{\kappa_\ell} = 0$.

Moreover,

$$\beta(\kappa(\varsigma, \mathcal{U}_1)) \leq \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{36} d\tau \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_1(\theta)), \quad \varsigma \in (\varsigma_\ell, s_\ell],\ \ell = 1, 2, \ldots, m$$
$$\leq \gamma_\ell \sup_{-\infty<\theta\leq 0} \beta(\mathcal{U}_1(\theta)),$$

where $\gamma_\ell = \int_{-\infty}^{0} \frac{e^{\sigma\tau}}{36} d\tau < \infty$. Thus, conditions (A2) and (A3) are fulfilled.

By thinking of Definition 2.5, we obtain $\mathbb{S} = \zeta(\zeta I - A)^{-1}$ and $\mathbb{T} = -\widetilde{\gamma}A(\zeta I - A)^{-1}$ with $\zeta = \frac{B(\vartheta)}{1-\vartheta}$, $\widetilde{\gamma} = \frac{\vartheta}{1-\vartheta}$. We assume that $\vartheta = \frac{3}{4}$, then $B\left(\frac{3}{4}\right) = \left(1 - \frac{3}{4}\right) + \frac{\frac{3}{4}}{\Gamma(\frac{3}{4})} = 0.86$, since $\Gamma(0.75) = 1.2254$. Thus, we have $\zeta = 3.44$ and $\widetilde{\gamma} = 3$.

From the above discussion, we have $\|\mathbb{S}\| \leq \mu$ and $\|\mathbb{T}\| \leq \overline{\mu}$ for the bounded linear operators $\mathbb{S}$ and $\mathbb{T}$. Hence, the assumption (A5) is verified.

Finally, to verify the inequality (3.1), we take $\mu = \overline{\mu} = \frac{1}{20}$, $\widehat{M}_{\mathscr{B}} = \widehat{M}_{\widehat{\mathscr{B}}} = \widetilde{L} = \widehat{\gamma} = 1$. Then, we have

$$\widehat{\Theta} = \left[2\mu\widehat{M}_{\mathscr{B}}\widehat{\gamma} + 4\left(\frac{\mu\overline{\mu}(1-\vartheta)}{B(\vartheta)\Gamma(\vartheta+1)} + \frac{\mu^2\widehat{M}_{\widehat{\mathscr{B}}}}{B(\vartheta)}\right)\widetilde{L}\right]$$

$$= 0.1 + 4\left[\frac{(0.05)(0.05)(0.25)}{(0.8620)(0.75)(1.2254)} + \frac{0.0025}{0.8620}\right]$$
$$= 0.11475 < 1.$$

From the above discussion, we can confirm that the Theorem 3.1 assumptions hold. As a consequence, problems (4.1)–(4.4) has a mild solution from the Theorem 3.1.

5. Conclusions

There are various types of fractional derivative definitions, with the RiemannLiouville fractional derivative (RLFD) and the Caputo fractional derivative (CFD) being two of the most prominent in applications [32]. Under suitable regularity assumptions, the RLFD can be transformed to the Caputo fractional derivative. The CFD are often used to determine the time-fractional derivatives in fractional partial differential equations. The fundamental difficulty is because the RL technique requires initial conditions including the RLFD limit values at the origin of time $t = 0$, which have unclear physical interpretations. The initial conditions for time-fractional Caputo derivatives, on the other hand, are the same as for integer-order differential equations, that is, the initial values of integer-order derivatives of functions at the origin of time $t = 0$ [32]. The benefit of using the Caputo definition is that it not only allows for the consideration of easily interpreted initial conditions, but it is also bounded, meaning that the derivative of a constant is equal to 0.

Caputo and Fabrizio [12] have proposed a novel definition of fractional derivative without singular kernel by substituting the function $\exp(-\frac{\vartheta}{1-\vartheta}(\varsigma - s))$ for the kernel $(\varsigma - s)^{-\vartheta}$. The extended Mittag-Leffler function was employed as a nonlocal and nonsingular kernel by both Atangana and Baleanu [1] a year later. The kernel's nonlocality allows for a more accurate representation of memory within structures of varying scales. For these reasons, we are using Atangana-Baleanu-Caputo fractional derivative in this manuscript. In this study, we used their [1] new result to our differential systems (1.1)–(1.3). Theorem 3.1 is proved to investigate the existence of the addressing models (1.1)–(1.3) by means of Monch fixed point theorem. Then, in Example 4, we check that the hypotheses (A1)–(A5) for the problems (4.1)–(4.4) are correct individually. The effectiveness of present research to approximate controllability with non-instantaneous impulses for diverse models may be developed using an appropriate fixed point theorem.

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A
2. A. Anguraj, M. Mallika Arjunan, E. Hernandez, Existence results for an impulsive neutral functional differential equation with state-dependent delay, *Appl. Anal.*, **86** (2007), 861–872. https://doi.org/10.1080/00036810701354995

3. D. Aimene, D. Baleanu, D. Seba, Controllability of semilinear impulsive Atangana-Baleanu fractional differential equations with delay, *Chaos Soliton. Fract.*, **128** (2019), 51–57. https://doi.org/10.1016/j.chaos.2019.07.027

4. N. Al-Salti, E. Karimov, K. Sadarangani, On a differential equation with Caputo- Fabrizio fractional derivative of order $1 < \beta > 2$ and application to mass-spring-damper system, *Progr. Fract. Differ. Appl.*, **2** (2016), 257–263. https://dx.doi.org/10.18576/pfda/020403

5. S. Abbas, M. Benchohra, Uniqueness and Ulam stabilities results for partial fractional differential equations with not instantaneous impulses, *Appl. Math. Comput.*, **257** (2015), 190–198. https://doi.org/10.1016/j.amc.2014.06.073

6. E. Bas, R. Ozarslan, Real world applications of fractional models by Atangana-Baleanu fractional derivative, *Chaos Soliton. Fract.*, **116** (2018), 121–125. https://doi.org/10.1016/j.chaos.2018.09.019

7. G. Bahaa, A. Hamiaz, Optimality conditions for fractional differential inclusions with nonsingular Mittag-Leffler kernel, *Adv. Differ. Equ.*, **2018** (2018), 257. https://doi.org/10.1186/s13662-018-1706-8

8. J. Banaŝ, K. Goebel, Measures of noncompactness in Banach spaces, In: *Lecture notes in pure and applied mathematics*, New York: Marcel Dekker, 1980.

9. M. Benchohra, S. Litimein, Juan J. Nieto, Semilinear fractional differential equations with infinite delay and non-instantaneous impulses, *J. Fix. Point Theory A.*, **21** (2019), 21. https://doi.org/10.1007/s11784-019-0660-8

10. M. Benchohra, J. Henderson, S. Ntouyas, *Impulsive differential equations and inclusions*, New York: Hindawi Publishing Corporation, 2006.

11. N. Bastos, Calculus of variations involving Caputo-Fabrizio fractional differentiation, *Stat. Optim. Inform. Comput.*, **6** (2018), 12–21. https://doi.org/10.19139/soic.v6i1.466

12. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 73–85. https://dx.doi.org/10.12785/pfda/010201

13. P. Chen, X. Zhang, Y. Li, Existence of mild solutions to partial differential equations with non-instantaneous impulses, *Electron. J. Differ. Equ.*, **2016** (2016), 1–11.

14. P. Chen, Y. Li, H. Yang, Perturbation method for nonlocal impulsive evolution equations, *Nonlinear Anal. Hybri.*, **8** (2013), 22–30. https://doi.org/10.1016/j.nahs.2012.08.002

15. Y. K. Chang, A. Anguraj, M. Mallika Arjunan, Existence results for impulsive neutral functional differential equations with infinite delay, *Nonlinear Anal. Hybr.*, **2** (2008), 209–218. https://doi.org/10.1016/j.nahs.2007.10.001

16. K. Deimling, *Nonlinear functional analysis*, New York: Springer-Verlag, 1985.

17. Z. Fan, G. Li, Existence results for semilinear differential equations with nonlocal and impulsive conditions, *J. Funct. Anal.*, **258** (2010), 1709–1727. https://doi.org/10.1016/j.jfa.2009.10.023

18. D. Guo, Existence of positive solutions for *n*th-order nonlinear impulsive singular integro-differential equations in Banach spaces, *Nonlinear Anal. Theor.*, **68** (2008), 2727–2740. https://doi.org/10.1016/j.na.2007.02.019

19. G. R. Gautam, J. Dabas, Mild solutions for a class of neutral fractional functional differential equations with not instantaneous impulses, *Appl. Math. Comput.*, **259** (2015), 480–489. https://doi.org/10.1016/j.amc.2015.02.069

20. E. Hernández, D. O'Regan, On a new class of abstract impulsive differential equations, *Proc. Amer. Math. Soc.*, **141** (2013), 1641–1649. https://doi.org/10.1090/S0002-9939-2012-11613-2

21. J. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.*, **21** (1978), 11–41.

22. J. K. Hale, *Theory of functional differential equations*, New York: Springer-Verlag, 1977.

23. M. Haase, *The functional calculus for sectorial operators*, Springer Science & Business Media, 2006.

24. Y. Hino, S. Murakami, T. Naito, *Functional differential equations with unbounded delay*, Springer, 1991.

25. S. Liu, J. Wang, D. O'Regan, Trajectory approximately controllability and optimal control for noninstantaneous impulsive inclusions without compactness, *Topol. Method. Nonl. An.*, **58** (2021), 19–49. https://doi.org/10.12775/TMNA.2020.069

26. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations*, Singapore: World Scientific, 1989.

27. A. Kumar, Dwijendra N. Pandey, Existence of mild solution of Atangana-Baleanu fractional differential equations with non-instantaneous impulses and with non-local conditions, *Chaos Soliton. Fract.*, **132** (2020), 109551. https://doi.org/10.1016/j.chaos.2019.109551

28. M. Mallika Arjunan, T. Abdeljawad, V. Kavitha, A. Yousef, On a new class of Atangana-Baleanu fractional Volterra-Fredholm integro-differential inclusions with non-instantaneous impulses, *Chaos Soliton. Fract.*, **148** (2021), 111075. https://doi.org/10.1016/j.chaos.2021.111075

29. M. Mallika Arjunan, A. Hamiaz, V. Kavitha, Existence results for Atangana-Baleanu fractional neutral integro-differential systems with infinite delay through sectorial operators, *Chaos Soliton. Fract.*, **149** (2021), 111042. https://doi.org/10.1016/j.chaos.2021.111042

30. M. Mallika Arjunan, V. Kavitha, Existence results for Atangana-Baleanu fractional integro-differential systems with non-instantaneous impulses, *Nonlinear Stud.*, **28** (2021), 865–877.

31. H. Monch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal. Theor.*, **4** (1980), 985–999. https://doi.org/10.1016/0362-546X(80)90010-3

32. I. Podlubny, *Fractional differential equations*, San Diego California: Academic Press, 1999.

33. W. Qiu, J. Wang, Iterative learning control for multi-agent systems with non-instantaneous impulsive consensus tracking, *Int. J. Robust Nonlin.*, **31** (2021), 6507–6524. https://doi.org/10.1002/rnc.5627

34. S. Suganya, D. Baleanu, P. Kalamani, M. Mallika Arjunan, On fractional neutral integro-differential systems with state-dependent delay and non-instantaneous impulses, *Adv. Differ. Equ.*, **2015** (2015), 372. https://doi.org/10.1186/s13662-015-0709-y

35. X. B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal. Theor.*, **74** (2011), 2003–2011. https://doi.org/10.1016/j.na.2010.11.007

36. H. Zhang, Y. Miaolin, Y. Renyu, Jinde Cao, Synchronization stability of Riemann-Liouville fractional delay-coupled complex neural networks, *Physica. A*, **508** (2018), 155–165. https://doi.org/10.1016/j.physa.2018.05.060

37. H. Zhang, Y. Renyu, Jinde Cao, A. Alsaedi, Delay-independent stability of Riemann-Liouville fractional neutral-type delayed neural networks, *Neural Process Lett.*, **47** (2018), 427–442. https://doi.org/10.1007/s11063-017-9658-7

38. W. Zhang, H. Zhang, Jinde Cao, Fuad E. Alsaadi, D. Chen, Synchronization in uncertain fractional-order memristive complex-valued neural networks with multiple time delays, *Neural Netw.*, **110** (2019), 186–198. https://doi.org/10.1016/j.neunet.2018.12.004