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On a Fractional Parabolic Equation with Regularized Hyper-Bessel Operator and Exponential Nonlinearities

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Abstract: Recent decades have witnessed the emergence of interesting models of fractional partial differential equations. In the current work, a class of parabolic equations with regularized Hyper-Bessel derivative and the exponential source is investigated. More specifically, we examine the existence and uniqueness of mild solutions in Hilbert scale-spaces which are constructed by a uniformly elliptic symmetry operator on a smooth bounded domain. Our main argument is based on the Banach principle argument. In order to achieve the necessary and sufficient requirements of this argument, we have smoothly combined the application of the Fourier series supportively represented by Mittag-Leffler functions, with Hilbert spaces and Sobolev embeddings. Because of the presence of the fractional operator, we face many challenges in handling proper integrals which appear in the representation of mild solutions. Besides, the source term of an exponential type also causes trouble for us when deriving the desired results. Therefore, powerful embeddings are used to limit the growth of nonlinearity.

Keywords: exponential nonlinearity; fractional diffusion equation; Hyper-Bessel operators; symmetric elliptic operator

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1. Introduction

In this paper, we modify the classical parabolic equation $\partial_t u - \Delta u = J(u)$ by changing the usual time-derivative by the following fractional Caputo-type Hyper Bessel derivative

$$ {}_C(t^\sigma \partial_t)^\alpha u(t) := (t^\sigma \partial_t)^\alpha u(t) - \frac{(1-\sigma)^\alpha}{\Gamma(1-\alpha)} u(0) t^{\alpha(\sigma-1)}, \tag{1} $$

where $\sigma \in (-\infty, 1)$, $\alpha \in (0,1)$, Γ is the Gamma function and $(t^\sigma \partial_t)^\alpha$ is the Hyper-Bessel operator (see [1] and also the interesting work [2] for more extensive discussion about the properties of the fractional Caputo-type Hyper Bessel derivative). According to this modification, for a bounded domain $\mathcal{D} \subset \mathbb{R}^n$ $(n \geqslant 1)$ with suitably smooth boundary $\partial\mathcal{D}$, we study the following initial-value boundary problem

$$\begin{cases} {}_C(t^{\sigma}\partial_t)^{\alpha} u(t,x) - \Delta u(t,x) = J(u(t,x)), & \text{in } [0,T]\times\mathcal{D}, \\ u(t,x) = 0 & \text{on } [0,T]\times\partial\mathcal{D}, \\ u(0,x) = u_0(x) & \text{in } \mathcal{D}, \end{cases} \tag{2}$$

where H is the source function that satisfies the following exponential growth

$$\begin{aligned} &\left|J(u)-J(v)\right| \leqslant L_0\left(|u|^q e^{u^2} + |v|^q e^{v^2}\right)|u-v|, && u,v\in\mathbb{R},\ q>1,\ L_0>0 \\ &J(u)=0, && u=0. \end{aligned} \tag{3}$$

In recent decades, many models of PDEs have been proposed as an alternative to classical models in many situations. For example, based on the law of the classical heat equation, the heat can be transfered with infinite speed. However in real modeling, the speed of the heat flow can be finite because of disruption of the response of the material. Many authors have proved that it is reasonable to investigate heat model with memory term and the most common way is replacing the classical derivative by the fractional one (see [3] for more details). This alternative leads us to fractional partial differential equations which have been proven to be applicable to many fields of applied science such as physics, hydrology, engineering, finance, see e.g., [4–8] and references given there. One of the most common famous counterparts of the first Equation in (2) is the time-fractional parabolic equation given by

$$D_t^{\alpha} u(t,x) - \Delta u(t,x) = J(u(t,x)), \quad \text{in } [0,T]\times\mathcal{D}, \tag{4}$$

where D_t^{α} is defined in the sense of Riemann-Liouville or Caputo. Derived from many practical application problems, many similar versions of (4) were produced by replacing D_t^{α} with other types of non-integer derivatives. For sake of clarity, we refer the reader to [9–29] and references therein, for engaging studies about (4), other models and relative problems.

The main object of this work, Problem (2), is studied with the fractional Caputo-type Hyper Bessel derivative instead of Riemann-Liouville or Caputo operators. It turns out that, compared with results in [30–32], there are many differences in approach and method for dealing with the well-posedness of mild solutions. In fact, in the mild formula of solutions to Problem (2), the singular integral is given by $\int_0^t (t-\tau)^{\alpha-1} d\tau$ while in (9), the integral term is more complicated. In view of this variation, it seems that (9) causes more trouble for us in deriving desired results. We note that, until the time we carry out this work, there are not many studies about the initial-value boundary problems similar to (2). However, there are still high-quality papers about mild solutions of parabolic equations with regularized Hyper-Bessel operators. Among them, we would like to make an overview of beautiful works which are our great motivation to carry out this paper. In [33], Tuan et al. investigated an initial data recovering problem associated with the first Equation of (2). They have showed the mild solution uniquely exists. However, this solution is not stable. Therefore, they applied a Tikhonov method to construct a approxiamting solution which converges to the unstable one. Au et al. [34] studied a fractional parabolic equation with ${}_C(t^{\sigma}\partial_t)^{\alpha}$. In this work, they provided results about the local existence, uniqueness ans regularity for mild solutions for three cases: linear source, global Lipschitz source and semi-linear source. Moreover, for the case of locally Lipschitz source term, they showed that the solution exists globally or blows up in finite time. Furthermore, another fascinating point of our work is the presence of the source function J which satisfies the exponential growth. It is obvious for us whether it is a classical or fractional model, linear version is often easier to handle than nonlinear one, in term of existence and uniqueness of solutions. One of the most famous and most frequently surveyed nonlinear source term is the polynomial given by u^{q+1} or $|u|^q u$ $(q>1)$. It should be noticed that parabolic equations with these polynomial source have been studied almost completely, to our awareness. Indeed, we desire to mention great works [14,15,23,24] as proof of the great interest among mathematicians around the world on this subject. However, as derived by many authors [35–38] that approximation to infinity behavior of q is more appriciate in some specific cases. In these cases, they proposed nonlinearities of exponential type as alternatives. Some typical examples for J are $e^{u^2}-1$ and $(e^{u^2}-1-u^2)u$.

In order to help the reader has a more complete view of our work, we clearly explain the difficulties in studying Problem (2) and shortly sketch our methods for dealing with these troubles.

- The first drawback is the integral term of the form $\int_0^t (t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}\mathrm{d}\tau^{\widetilde{\sigma}}$, where $\widetilde{\sigma}=1-\sigma$. Because of the complicated definition of ${}_C(t^{\sigma}\partial_t)^{\alpha}$, we use Fourier series of functions in $L^2(\mathcal{D})$ as our basis for defining mild solutions to Problem (2). Furthermore, besides the singularity of the kernel in the integral symbol, the upper limit does not possesses the same power as the integrating variable. Hence, we can not easily apply the Beta function to derive wished results. In our proof, we recall the bounded property of Mittag-Leffler functions and basic inequalities to handle the singular kernel and obtain sharp upper bound for mild solutions.
- The second and also the most difficult problem for us as mentioned above, the fast growth of the nonlinearity J. In order to overcome this issue, previous work [23,24] made smallness assumption on the initial data function. It seems to be a efficient method. In this study, instead of following this method, we apply powerful embeddings to get L^∞-bounds for the exponential term. Then, by making the relationships between Hilbert scale spaces and well-known Sobolev spaces, we can apply the Picard ilteration to derive the local existence and uniqueness of mild solutions to Problem (2).

The rest of this study is outlined as follows. Section 2 provides basic settings about function spaces, useful lemma and mild formula. The main result is stated and proved in Section 3. Section 4 is the summary of our work and proposes potential developing results of this study in the future.

2. Preliminaries

Throughout this paper, the symbols $\mathbb{N}, \mathbb{B}(0,K)$ respectively stand for the set of nonzero natural numbers and an open ball with center at zero and radius $K>0$. We begin this section by recalling the Lebesgue space

$$L^2(\mathcal{D}) := \left\{ u : \mathcal{D} \to \mathbb{R} \;\middle|\; \int_{\mathcal{D}} |u(x)|^2 \mathrm{d}x < \infty \right\}.$$

Also, for a Banach space $(X, \|\|_X)$, we define

$$C\Big(0,T;\mathcal{D}\Big) := \left\{ u : [0,T] \to X \;\middle|\; u \text{ is continuous on } [0,T] \right\}.$$

Next, for a bounded domain $\mathcal{D}$ with smooth boundary $\partial\mathcal{D}$, the Laplace operator (a uniform elliptic symmetry operator) subject to Dirichlet conditions possesses a set of eigenvalues $\{\kappa_j\}_{j\geqslant 1}$ which satisfies

$$0 < \kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_j \nearrow \infty,$$

and a corresponding eigenvectors $\{\phi_j\}_{j\geqslant 1}$ which is also an orthonormal basis of $L^2(\mathcal{D})$ such that

$$\begin{cases} -\Delta\phi_j = \kappa_j\phi_j, & x \in \mathcal{D},\ j \geqslant 1, \\ \phi_j = 0, & x \in \partial\mathcal{D},\ j \geqslant 1. \end{cases} \tag{5}$$

Based on these settings, we define Hilbert scale spaces by which we provide main results more efficiently. For $\eta \geqslant 0$, we define the Hilbert scale space $\mathbf{H}^\eta(\mathcal{D})$ by

$$\mathbf{H}^\eta(\mathcal{D}) := \left\{ u \in L^2(\mathcal{D}) \,\middle|\, \sum_{j\geqslant 1} \kappa_j^\eta (u, \vartheta_j)_{L^2}^2 < \infty \right\}.$$

The space $\mathbf{H}^\eta(\mathcal{D})$ is equipped with the following norm

$$\|u\|_{\mathbf{H}^\eta(\mathcal{D})} := \left[\sum_{j\geqslant 1} \kappa_j^\eta (u, \vartheta_j)_{L^2}^2 \right]^{\frac{1}{2}}, \quad u \in \mathbf{H}^\eta(\mathcal{D}).$$

Throughout this paper, we use the convention that $X \hookrightarrow Y$, where X, Y are Banach spaces, implies $X \subset Y$ and the identity operator from X into Y is continuous (it is equivalent that a constant $C_0 > 0$ exists such that $\|\cdot\|_Y \leqslant C_0 \|\cdot\|_X$).

We now provide the representation of mild solutions of Problem (2). First, the definition of Mittag-Leffler functions are given. For any $\alpha \in (0,1)$, two-parameters Mittag-Leffler functions $E_{\alpha,1}$ and $E_{\alpha,2\alpha}$ are defined as follows

$$E_{\alpha,1}(w) := \sum_{j\geqslant 1} \frac{w^j}{\Gamma(\alpha j + 1)}, \quad w \in \mathbb{C}$$

and

$$E_{\alpha,2\alpha}(w) := \sum_{j\geqslant 1} \frac{w^j}{\Gamma(\alpha j + \alpha)}, \quad w \in \mathbb{C}.$$

We also provide the following lemmas for upper bounds of Mittag-Leffler functions and solution formula of a fractional ordinary differential equation which is a counterpart of (2).

Lemma 1 ([39], Theorem 1.6)**.** *Let $(\alpha_1, \alpha_2) \in (0,1) \times \mathbb{R}$ and $\phi \in (\frac{\pi\alpha_1}{2}; \pi)$. Then, there exists a positive constant $C_1 > 0$ such that*

$$\left| E_{\alpha_1,\alpha_2}(w) \right| \leqslant \frac{C_1}{1 + |w|},$$

for any $w \in \mathbb{C}$ which satisfies $\phi \leqslant |\arg(w)| \leqslant \pi$.

Lemma 2 ([1], Theorem 2.4)**.** *The solution of the following non-homogeneous fractional differential equation*

$$\begin{cases} {}_C(t^\sigma \partial_t)^\alpha u(t) + \gamma u(t) = f(t), & t > 0, \\ u(0) = u_0, \end{cases}$$

is given by the integral equality

$$u(t) = E_{\alpha,1}\left(-\gamma \widetilde{\sigma}^{-\alpha} t^{\widetilde{\sigma}\alpha}\right) u_0 + \int_0^t \frac{(t^{\widetilde{\sigma}} - \tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha} E_{\alpha,2\alpha}\left[-\frac{\gamma (t^{\widetilde{\sigma}} - \tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha} \right] f(t) \mathrm{d}(\tau^{\widetilde{\sigma}}),$$

In view of (5), we take the inner product $(\cdot,\cdot)_{L^2}$ to the first Equation of (2) with respect to ϑ_j $(j \geqslant 1)$ to obtain the following ordinary differential equation

$$\big((t^\sigma\partial_t)^\alpha u(t),\vartheta_j\big)_{L^2}+\kappa_j\big(u(t),\vartheta_j\big)_{L^2}=\big(J(u(t)),\vartheta_j\big)_{L^2},\quad t>0, \tag{6}$$

associated to the initial condition

$$\big(u(0),\vartheta_j\big)_{L^2}=\big(u_0,\vartheta_j\big)_{L^2}. \tag{7}$$

According to Lemma 2, the solution of (6) and (7) is given as follows

$$\begin{aligned}\big(u(t),\vartheta_j\big)_{L^2}&=E_{\alpha,1}\Big(-\kappa_j\widetilde{\sigma}^{-\alpha}t^{\widetilde{\sigma}\alpha}\Big)(u_0,\vartheta_j)_{L^2}\\&\quad+\int_0^t\frac{(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha}E_{\alpha,2\alpha}\Bigg[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\Bigg]\big(J(u(\tau)),\vartheta_j\big)_{L^2}\mathrm{d}(\tau^{\widetilde{\sigma}}),\end{aligned} \tag{8}$$

here we denote $\widetilde{\sigma}=1-\sigma$. Suppose that $u\in L^2(\mathcal{D})$, from (8), u can be defined via the following Fourier series

$$\begin{aligned}u(t,x)&=\sum_{j\geqslant1}\big(u(t),\vartheta_j\big)_{L^2}\vartheta_j(x)\\&=\sum_{j\geqslant1}E_{\alpha,1}\Big(-\kappa_j\widetilde{\sigma}^{-\alpha}t^{\widetilde{\sigma}\alpha}\Big)(u_0,\vartheta_j)_{L^2}\vartheta_j(x)\\&\quad+\sum_{j\geqslant1}\int_0^t\frac{(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha}E_{\alpha,2\alpha}\Bigg[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\Bigg]\big(J(u(\tau)),\vartheta_j\big)_{L^2}\vartheta_j(x)\mathrm{d}(\tau^{\widetilde{\sigma}}).\end{aligned} \tag{9}$$

3. Existence and Uniqueness

Before providing main results of the paper, we first introduce two different ways to estimate the source function J.

Lemma 3. *Let $n\in\{1,2,3\}$ and $s\in(n/2;2)$. A positve constant L exists such that for any u_1 and u_2 in $\mathbf{H}^s(\mathcal{D})$, the following estimate holds*

$$\Big\|J(u_1)-J(u_2)\Big\|_{L^2(\mathcal{D})}\leqslant L\Big(e^{\lambda\|u_1^2\|_{\boldsymbol{H}^\eta(\mathcal{D})}}\|u_1\|^q_{\boldsymbol{H}^\eta(\mathcal{D})}+e^{\lambda\|u_2^2\|_{\boldsymbol{H}^\eta(\mathcal{D})}}\|u_2\|^q_{H^\eta(\mathcal{D})}\Big)\|u_1-u_2\|_{\boldsymbol{H}^\eta(\mathcal{D})}.$$

Proof. Using Hölder's inequality and the triangle inequality, for any $u_1,u_2\in$ we obtain

$$\begin{aligned}\Big\|J(u_1)-J(u_2)\Big\|_{L^2(\mathcal{D})}&\leqslant L_0\Big(\Big\||u_1|^qe^{\lambda u_1^2}\Big\|_{L^4(\mathcal{D})}+\Big\||u_2|^qe^{\lambda u_2^2}\Big\|_{L^4(\mathcal{D})}\Big)\|u_1-u_2\|_{L^4(\mathcal{D})}\\&\leqslant L_0\Big(e^{\lambda\|u_1^2\|_{L^\infty(\mathcal{D})}}\|u_1\|^q_{L^{4q}(\mathcal{D})}+e^{\lambda\|u_2^2\|_{L^\infty(\mathcal{D})}}\|u_2\|^q_{L^{4q}(\mathcal{D})}\Big)\|u_1-u_2\|_{L^4(\mathcal{D})}.\end{aligned} \tag{10}$$

Then, by applying the following embeddings

$$L^\infty(\mathcal{D})\hookrightarrow L^p(\mathcal{D}),\quad \text{for } p\geqslant1$$

and

$$H^\eta(\mathcal{D})\hookrightarrow C(\bar{\mathcal{D}}),\quad \text{for } \frac{d}{2}<\eta\leqslant2,$$

we can derive

$$\left\|J(u_1)-J(u_2)\right\|_{L^2(\mathcal{D})} \leqslant C_0L_0\Big(e^{\lambda\|u_1^2\|_{\mathbf{H}^\eta(\mathcal{D})}}\|u_1\|^q_{\mathbf{H}^\eta(\mathcal{D})}+e^{\lambda\|u_2^2\|_{\mathbf{H}^\eta(\mathcal{D})}}\|u_2\|^q_{\mathbf{H}^\eta(\mathcal{D})}\Big)\|u_1-u_2\|_{\mathbf{H}^\eta(\mathcal{D})}.$$

The proof is completed. □

Theorem 1. *Let $\alpha \in (0,1), \sigma \in (-\infty,1), n \in \{1,2,3\}$ and $\eta \in (n/2;2)$. Suppose that $u_0 \in \mathbf{H}^\eta(\mathcal{D})$ and positive real constants K and T exist such that*

$$K = 2C_1\|u_0\|_{\mathbf{H}^\eta(\mathcal{D})}$$

and

$$T \leqslant \frac{1K}{4}\left[\frac{C_1L\kappa_1^{\frac{\eta-2\theta}{2}}}{\widetilde{\sigma}^{\alpha-\theta\alpha}(\alpha-\theta\alpha)}e^{\lambda K}K^{q+1}\right]^{\frac{-1}{\widetilde{\sigma}(\alpha-\theta\alpha)}}. \tag{11}$$

Then, Problem (2) possesses a unique mild solution $u \in C([0,T];\mathbf{H}^\eta(\mathcal{D}))$.

Proof. Our main aim is to apply the Banach principle argument. To this end, we define the sequence of approximating solutions $\{u_k\}_{k\in\mathbb{N}}$ as follows

$$\begin{cases} u_1(t,x) := \displaystyle\sum_{j\geqslant 1} E_{\alpha,1}\Big(-\kappa_j\widetilde{\sigma}^{-\alpha}t^{\widetilde{\sigma}\alpha}\Big)(u_0,\vartheta_j)_{L^2}\vartheta_j(x), \\ u_k(t,x) := u_1(t,x) + \displaystyle\sum_{j\geqslant 1}\int_0^t \frac{(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha}E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\right]\big(J(u(\tau)),\vartheta_j\big)_{L^2}\vartheta_j(x)\mathrm{d}(\tau^{\widetilde{\sigma}}). \end{cases}$$

We prove that $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{B}(0,K) \subset C([0,T];\mathbf{H}^\eta(\mathcal{D}))$. Our proof includes two main parts
Part 1: We prove that $\{u_k\}_{k\in\mathbb{N}}$ is a subset of $\mathbb{B}(0,K)$. For a clear presentation, we devide the proof into 2 steps.
Step 1: By Parseval's identity, for $u_0 \in \mathbf{H}^\eta(\mathcal{D})$, we have

$$\begin{aligned}\left\|\sum_{j\geqslant 1} E_{\alpha,1}\Big(-\kappa_j\widetilde{\sigma}^{-\alpha}t^{\widetilde{\sigma}\alpha}\Big)(u_0,\vartheta_j)_{L^2}\vartheta_j(x)\right\|_{\mathbf{H}^\eta(\mathcal{D})} &\leqslant C_1\left[\sum_{j\geqslant 1}\kappa_j^\eta(u_0,\vartheta_j)^2_{L^2}\right]^{\frac{1}{2}} \\ &\leqslant C_1\|u_0\|_{\mathbf{H}^\eta(\mathcal{D})}.\end{aligned}$$

This result implies that

$$\|u_1\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))} \leqslant C_1\|u_0\|_{\mathbf{H}^\eta(\mathcal{D})} \tag{12}$$

Step 2: Suppose that $w_k \in \mathbb{B}(0,K)$ for $k \geqslant 1$, we show that $w_{k+1} \in \mathbb{B}(0,K)$. To this end, we first observe that

$$\begin{aligned}&\|w_{k+1}(t)-u_1(t)\|_{\mathbf{H}^\eta(\mathcal{D})} \\ &\quad\leqslant \int_0^t \frac{(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha}\left\|\sum_{j\geqslant 1}E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\right]\big(J(u(\tau)),\vartheta_j\big)_{L^2}\vartheta_j(x)\right\|_{\mathbf{H}^\eta(\mathcal{D})}\mathrm{d}(\tau^{\widetilde{\sigma}}).\end{aligned} \tag{13}$$

Similar to Step 1, we apply Parseval's formula to derive

$$\left\|\sum_{j\geqslant 1}E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\right]\big(J(u(\tau)),\vartheta_j\big)_{L^2}\vartheta_j(x)\right\|^2_{\mathbf{H}^\eta(\mathcal{D})}$$

$$= \sum_{j\geqslant 1}\kappa_j^{\eta}\left|E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha}}{\widetilde{\sigma}^{\alpha}}\right]\right|^2\big(J(u(\tau)),\vartheta_j\big)_{L^2}^2.$$

Then, a repeated application of Lemma 1 implies

$$\left|E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha}}{\widetilde{\sigma}^{\alpha}}\right]\right|^2 \leqslant \frac{C_1^2}{\left[1+\kappa_j\widetilde{\sigma}^{-\alpha}(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha}\right]^2}.$$

For $\theta\in(\eta/2,1)$, we can derive from basic inequalities that

$$\left|E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha}}{\widetilde{\sigma}^{\alpha}}\right]\right|^2 \leqslant \frac{C_1^2}{\kappa_j^{2\theta}\widetilde{\sigma}^{-2\theta\alpha}(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{2\theta\alpha}}.$$

Then, it follows that

$$\begin{aligned}&\left\|\sum_{j\geqslant 1}E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha}}{\widetilde{\sigma}^{\alpha}}\right]\big(J(u(\tau)),\vartheta_j\big)_{L^2}\vartheta_j(x)\right\|^2_{\mathbf{H}^{\eta}(\mathcal{D})}\\ &\qquad\leqslant \frac{C_1^2}{\widetilde{\sigma}^{2\theta\alpha}}(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{-2\theta\alpha}\sum_{j\geqslant 1}\kappa_j^{\eta-2\theta}\big(J(u(\tau)),\vartheta_j\big)_{L^2}^2.\end{aligned}$$

Based on this result, (13) is equivalent to the following estimate

$$\|w_{k+1}(t)-u_1(t)\|_{\mathbf{H}^{\eta}(\mathcal{D})}\leqslant \frac{C_1}{\widetilde{\sigma}^{\alpha-\theta\alpha}}\kappa_1^{\frac{\eta-2\theta}{2}}\int_0^t(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-\theta\alpha-1}\Big\|J(w_k(\tau))\Big\|_{L^2(\mathcal{D})}\mathrm{d}(\tau^{\widetilde{\sigma}}). \tag{14}$$

In view of Lemma 3, we can find that

$$\begin{aligned}\Big\|J(w_k(\tau))\Big\|_{L^2(\mathcal{D})} &\leqslant Le^{\lambda\|w_k(\tau)\|_{\mathbf{H}^{\eta}(\mathcal{D})}}\|w_k(\tau)\|^{q+1}_{\mathbf{H}^{\eta}(\mathcal{D})}\\ &\leqslant Le^{\lambda\|w_k\|_{C([0,T];\mathbf{H}^{\eta}(\mathcal{D}))}}\|w_k\|^{q+1}_{C([0,T];\mathbf{H}^{\eta}(\mathcal{D}))}\\ &\leqslant Le^{\lambda K}K^{q+1}.\end{aligned}$$

for every $\tau\in[0,T]$, provided that $w_k\in\mathbb{B}(0,K)$. This result together with (14) ensure

$$\begin{aligned}&\|w_{k+1}(t)-u_1(t)\|_{\mathbf{H}^{\eta}(\mathcal{D})}\\ &\qquad\leqslant L\frac{C_1}{\widetilde{\sigma}^{\alpha-\theta\alpha}}\kappa_1^{\frac{\eta-2\theta}{2}}\left[\int_0^t(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-\theta\alpha-1}\mathrm{d}(\tau^{\widetilde{\sigma}})\right]e^{\lambda K}K^{q+1}.\end{aligned} \tag{15}$$

Since $\theta<1$, we can easily obtain

$$\int_0^t(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-\theta\alpha-1}\mathrm{d}(\tau^{\widetilde{\sigma}})=\frac{T^{\widetilde{\sigma}(\alpha-\theta\alpha)}}{\alpha-\theta\alpha}.$$

combining the above equality and (15) enables us to derive the following estimate

$$\begin{aligned}\|w_{k+1}-u_1\|_{C([0,T];\mathbf{H}^{\eta}(\mathcal{D}))} &\leqslant \frac{C_1L\kappa_1^{\frac{\eta-2\theta}{2}}}{\widetilde{\sigma}^{\alpha-\theta\alpha}(\alpha-\theta\alpha)}e^{\lambda K}K^{q+1}T^{\widetilde{\sigma}(\alpha-\theta\alpha)}\\ &\leqslant \frac{3K}{4}.\end{aligned} \tag{16}$$

As a result of (12) and (16), we use the triangle inequality to find that

$$\|u_{k+1}\|_{C([0,T];\mathbf{H}^{\eta}(\mathcal{D}))}\leqslant\|u_1\|_{C([0,T];\mathbf{H}^{\eta}(\mathcal{D}))}+\|u_{k+1}-u_1\|_{C([0,T];\mathbf{H}^{\eta}(\mathcal{D}))}$$

$$\leqslant K,$$

provided that $u_k \in \mathbb{B}(0,K)$.

As a consequence of Step 1 and Step 2, we can apply a conductive argument to conclude that $\{u_k\}_{k\geqslant 1}$ is a subset of $\mathbb{B}(0,K)$. We end Part 1 and move on to the next part. Part 2: We show that $\{u_k\}_{k\geqslant 1}$ is a Cauchy sequence in $\mathbb{B}(0,K)$. First, for $k \geqslant 2$, w e suppose that $u_{k-1}, u_k \in \mathbb{B}(0,R)$. Then, by Parseval's identity we have

$$\begin{aligned}&\left\|u_{k+1}(t)-u_k(t)\right\|_{\mathbf{H}^\eta(\mathcal{D})}\\&\leqslant \int_0^t \frac{(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha}\left[\sum_{j\geqslant 1}\kappa_j^\eta\left|E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\right]\right|^2\left(J(u_k(\tau)-J(u_{k-1}(\tau)),\vartheta_j\right)_{L^2}^2\right]^{\frac{1}{2}}\mathrm{d}(\tau^{\widetilde{\sigma}}).\end{aligned}$$

Then, we can now proceed analogously to arguments in Step 1, there holds

$$\begin{aligned}&\left\|u_{k+1}(t)-u_k(t)\right\|_{\mathbf{H}^\eta(\mathcal{D})}\\&\qquad\leqslant \frac{C_1}{\widetilde{\sigma}^{\alpha-\theta\alpha}}\kappa_1^{\frac{\eta-2\theta}{2}}\int_0^t (t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-\theta\alpha-1}\left\|J(w_k(\tau))-J(w_{k-1}(\tau))\right\|_{L^2(\mathcal{D})}\mathrm{d}(\tau^{\widetilde{\sigma}}).\end{aligned}\tag{17}$$

Repeated application of Lemma 3 yields

$$\begin{aligned}\left\|J(u_k(t))-J(u_{k-1}(t))\right\|_{L^2(\mathcal{D})} &\leqslant Le^{\lambda\|u_k^2(t)\|_{\mathbf{H}^\eta(\mathcal{D})}}\|u_k(t)\|_{\mathbf{H}^\eta(\mathcal{D})}\|u_k(t)-u_{k-1}(t)\|_{\mathbf{H}^\eta(\mathcal{D})}\\&\quad+Le^{\lambda\|u_k^2(t)\|_{\mathbf{H}^\eta(\mathcal{D})}}\|u_k(t)\|_{\mathbf{H}^\eta(\mathcal{D})}\|u_k(t)-u_{k-1}(t)\|_{\mathbf{H}^\eta(\mathcal{D})}.\end{aligned}$$

It follows from the assumption $w_k, w_{k-1} \in \mathbb{B}(0,K)$ that

$$\left\|J(u_k(t))-J(u_{k-1}(t))\right\|_{L^2(\mathcal{D})} \leqslant 2Le^{\lambda K}K^q\|u_k-u_{k-1}\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))},\quad \text{for any } t\in[0,T].\tag{18}$$

Combining (17) and (18), we can assert that

$$\begin{aligned}\left\|u_{k+1}(t)-u_k(t)\right\|_{\mathbf{H}^\eta(\mathcal{D})} &\leqslant \frac{2C_1L\kappa_1^{\frac{\eta-2\theta}{2}}}{\widetilde{\sigma}^{\alpha-\theta\alpha}}e^{\lambda K}K^q\left[\int_0^t (t^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-\theta\alpha-1}\mathrm{d}(\tau^{\widetilde{\sigma}})\right]\|u_k-u_{k-1}\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))}\\&\leqslant \frac{2C_1L\kappa_1^{\frac{\eta-2\theta}{2}}}{\widetilde{\sigma}^{\alpha-\theta\alpha}(\alpha-\theta\alpha)}e^{\lambda K}K^qT^{\widetilde{\sigma}(\alpha-\theta\alpha)}\|u_k-u_{k-1}\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))},\end{aligned}$$

for any $t \in [0,T]$. Consequently, we have

$$\|u_{k+1}-u_k\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))} \leqslant \frac{2C_1L\kappa_1^{\frac{\eta-2\theta}{2}}}{\widetilde{\sigma}^{\alpha-\theta\alpha}(\alpha-\theta\alpha)}e^{\lambda K}K^qT^{\widetilde{\sigma}(\alpha-\theta\alpha)}\|u_k-u_{k-1}\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))}.$$

$$\|u_{k+1}-u_k\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))} \leqslant \frac{1}{2}\|u_k-u_{k-1}\|_{C([0,T];\mathbf{H}^\eta(\mathcal{D}))}.$$

Hence, by some basic arguments, one can conclude that $\{u_k\}_{k\geqslant 1}$ is a Cauchy sequence in $\mathbb{B}(0,K)$.

Having disposed of these two parts, we can now use the completeness of $C([0,T];\mathbf{H}^\eta(\mathcal{D}))$ to deduce that $\{u_k\}_{k\geqslant 1}$ possesses a unique limit $u \in \mathbb{B}(0,K)$ which is the unique mild solution of Problem (2). The theorem is thus proved. □

Comment 1. *In this paper, since we focus only on the mild solution of Problem* (2) *which is represented by* (9)*, the solution is showed to be in* $C([0,T];\boldsymbol{H}^\eta(\mathcal{D}))$*. Then, Lemma* 3 *implies that*

$J(u) \in C\big([0,T]; L^2(\mathcal{D})\big)$. *More results about the regularity of the solution will be investigated in the future.*

4. Numerical Example

The aim of this section is considering an example to show the asymptotic behavior of the mild solutions in the non-homogeneous source function. Firstly, we choose the operator $-\Delta$ on the domain $D = (0,\pi)$ with the homogeneous Dirichlet boundary condition, then the eigenvectors and eigenvalues of $-\Delta$ are given by $\phi_j(x) = \sqrt{2/\pi}\sin(jx)$ and j^2, $(j = 1,2,3,\dots)$, respectively.

We consider the problem to find a function $u\colon [0,1] \to L^2(0,\pi)$ satisfying

$$\begin{cases} {}_C(t^\sigma\partial_t)^\alpha u(t,x) - \Delta u(t,x) = J(u(t,x)), & \text{in } [0,1]\times[0,\pi], \\ u(t,x) \quad = \quad 0 & \text{on } [0,1]\times\{0,\pi\}, \\ u(0,x) \quad = \quad u_0(x) & \text{in } [0,\pi], \end{cases} \tag{19}$$

By using the following finite difference method and partitions of spatial and temporal variables $\mho_t \times \mho_x$

$$\mho_t := \left\{ t_n = (n-1)\frac{1}{N_t+1}, \text{ for } n = 1,2,\dots,N_t+1 \right\},$$
$$\mho_x := \left\{ x_m = (m-1)\frac{\pi}{M_x+1}, \text{ for } m = 1,2,\dots,M_x+1 \right\}.$$

Additionally, thanks to the code in Python software `mlf(a,b,z)` to calculate the Mittag-Leffler function $E_{\alpha,\beta}(z)$ as follows

$$E_{\alpha,\beta}(z) = \texttt{mlf(a,b,z)}.$$

Next, using the Simpson rule of the approximation of numerical integration, we have

$$\frac{t_{i+1}-t_i}{3k}\left[f(z_1) + 2\sum_{k=1}^{(k+1)/2-1} f(z_{2k}) + 4\sum_{k=1}^{(k+1)/2} f(z_{2k-1}) + f(z_{k+1}) \right] \approx \int_{t_i}^{t_{i+1}} f(z)\mathrm{d}z.$$

In this example, we apply the truncation Fourier series by parameters F_j, the mild solution of Problem (19) is giving by a matrix form by fixing the temporal variable t

$$\begin{aligned} u(t_n,x_m) &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{F_j} U_j(t_n)\sin(jx_m) \\ &= \sqrt{\frac{2}{\pi}}\begin{bmatrix} U_1(t_n) & U_2(t_n) & U_3(t_n) & \cdots & U_{F_j}(t_n) \end{bmatrix} \times \begin{bmatrix} \sin(x_m) \\ \sin(2x_m) \\ \sin(3x_m) \\ \vdots \\ \sin(F_j x_m) \end{bmatrix}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} U_j(t_n) &= E_{\alpha,1}\left(-\kappa_j\widetilde{\sigma}^{-\alpha}t_n^{\widetilde{\sigma}\alpha}\right)\int_0^\pi u_0(\star)\vartheta_j(\star)\mathrm{d}\star \\ &+ \int_0^{t_n}\frac{(t_n^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^{\alpha-1}}{\widetilde{\sigma}^\alpha}E_{\alpha,2\alpha}\left[-\frac{\kappa_j(t_n^{\widetilde{\sigma}}-\tau^{\widetilde{\sigma}})^\alpha}{\widetilde{\sigma}^\alpha}\right]\int_0^\pi J(t_n,\star)\vartheta_j(\star)\mathrm{d}\star\ \mathrm{d}(\tau^{\widetilde{\sigma}}). \end{aligned} \tag{21}$$

Then, a matrix form of the solution (20) can be presented in as follows

$$\begin{bmatrix} u(t_1,x_1) & u(t_1,x_2) & \cdots & u(t_1,x_{M_x}) & u(t_1,x_{M_x+1}) \\ u(t_2,x_1) & u(t_2,x_2) & \cdots & u(t_2,x_{M_x}) & u(t_2,x_{M_x+1}) \\ u(t_3,x_1) & u(t_3,x_2) & \cdots & u(t_3,x_{M_x}) & u(t_3,x_{M_x+1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u(t_{N_t},x_1) & u(t_{N_t},x_2) & \cdots & u(t_{N_t},x_{M_x}) & u(t_{N_t},x_{M_x+1}) \\ u(t_{N_t+1},x_1) & u(t_{N_t+1},x_2) & \cdots & u(t_{N_t+1},x_{M_x}) & u(t_{N_t+1},x_{M_x+1}) \end{bmatrix}_{(N_t+1)\times(M_x+1)}.$$

For $(t,x) \in [0,1] \times [0,\pi]$, we choose $\sigma = \alpha = 0.5$, $F_j = 10$, $N_t = M_x = 100$ and the giving functions as follows

$$J(t,x) = 16(t^\alpha + 1)\sin(4x) - \left[\left(\frac{1}{2}\right)^\alpha t^{-\frac{1}{4}} + \frac{t^{-\frac{\alpha}{2}}}{2^\alpha \Gamma(\frac{1}{2})} \right] \sin(4x), (t,x) \in [0,1] \times [0,\pi] \quad (22)$$

$$u_0(x) = \sin(4x), x \in [0,\pi]. \quad (23)$$

The graphs of the solution u in some cases of $\alpha \in \{0.1, 0.5, 0.7, 0.9\}$ are mentioned, which shown in Figures 1–4. For detail, case 1: $t = 0.2$ is presented in Figure 1; case 2: $t = 0.4$ is presented in Figure 2; case 3: $t = 0.6$ is presented in Figure 3. And this final case with $t = 0.8$ is shown in Figure 4.

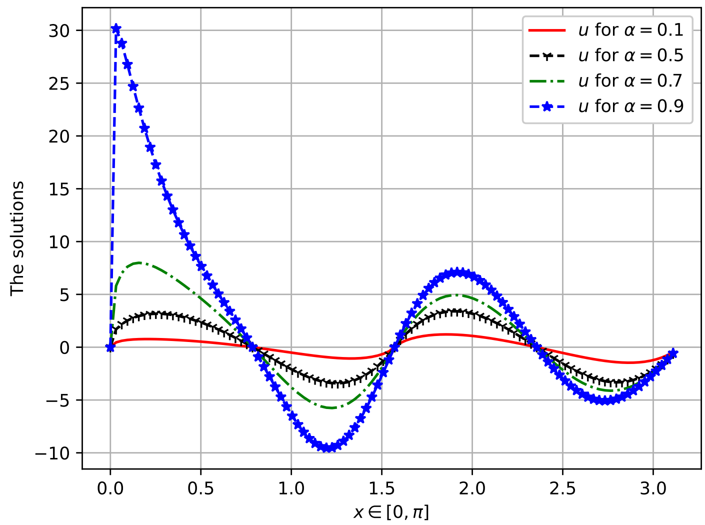

Figure 1. The solution u at $t = 0.2$.

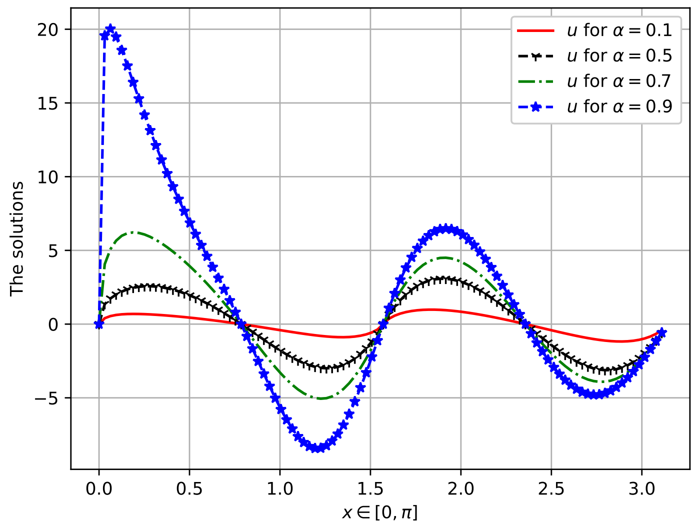

Figure 2. The solution u at $t = 0.4$.

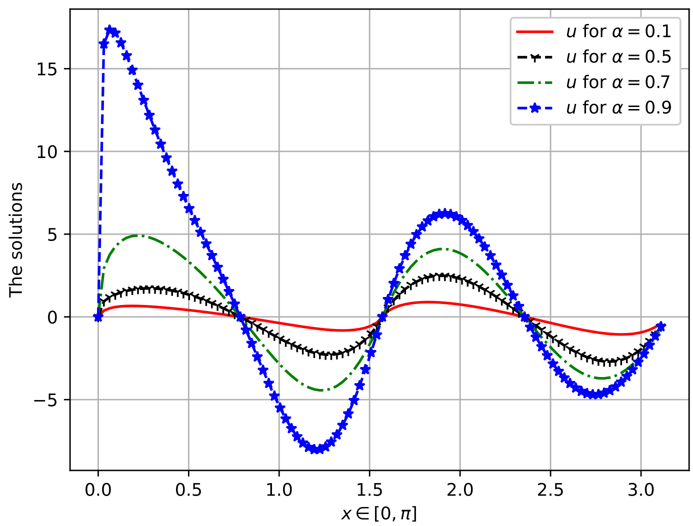

Figure 3. The solution u at $t = 0.6$.

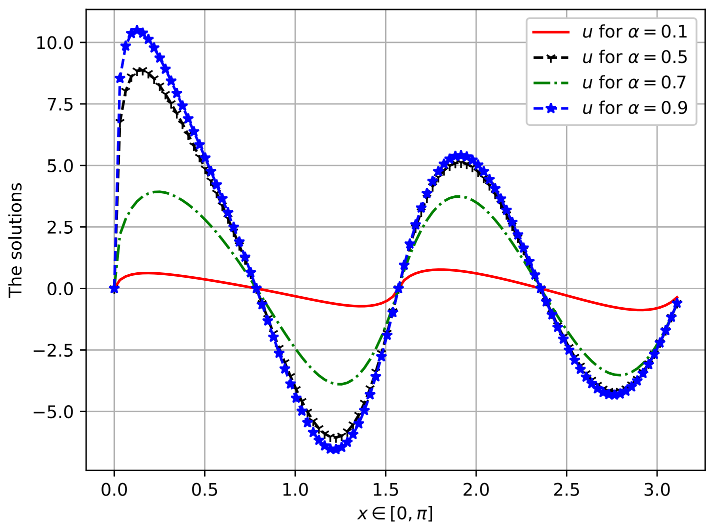

Figure 4. The solution u at $t = 0.8$.

5. Conclusions

In this study, we apply the Banach principle argument to derive the well-posedness of an initial-value boundary problem associated with fractional parabolic equation with regularized Hyper-Bessel operator and exponential nonlinearity. Thanks to properties of Mittag-Leffler functions, powerful Sobolev embeddings and the usefulness of Hilbert scale spaces, we have proved the local existence and uniqueness of a mild solution. In the future, we aim to improve our result to global one and consider more regularity results for solutions. It seems to be very difficult, but also very engaging. In addition, the case $\sigma \geqslant 1$ will also be studied in upcomming works.

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References

1. Al-Musalhi, F.; Al-Salti, N.; Karimov, E. Initial boundary value problems for a fractional differential equation with hyper-Bessel operator. *Fract. Calc. Appl. Anal.* **2018**, *21*, 200–219. [CrossRef]
2. Shishkina, E.; Sitnik, S. A fractional equation with left-sided fractional Bessel derivatives of Gerasimov–Caputo type. *Mathematics* **2019**, *7*, 1216. [CrossRef]
3. Chen, Z.Q.; Kim, K.H.; Kim, P. Fractional time stochastic partial differential equations. *Stoch. Process. Their Appl.* **2015**, *125*, 1470–1499. [CrossRef]
4. Akdemir, O.A.; Dutta, H.; Atangana, A. *Fractional Order Analysis: Theory, Methods and Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2020. [CrossRef]
5. Herrmann, R. *Fractional Calculus: An Introduction for Physicists*; World Scientific: Singapore, 2011.
6. Meerschaert, M.M.; Sikorskii, A. *Stochastic Models for Fractional Calculus*; De Gruyter Studies in Mathematics; Walter de Gruyter: Berlin/Heidelber, Germany; Boston, MA, USA, 2012; Volume 43.
7. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77.
8. Saichev, A.I.; Zaslavsky, G.M. Fractional kinetic equations: Solutions and applications. *Chaos Interdiscip. J. Nonlinear Sci.* **1997**, *7*, 753–764. [CrossRef]
9. Adiguzel, R.S.; Aksoy, U.; Karapinar, E.; Erhan, I.M. On the solution of a boundary value problem associated with a fractional differential equation. *Math. Methods Appl. Sci.* **2020**, 1–12. [CrossRef]
10. Adiguzel, R.S.; Aksoy, U.; Karapinar, E.; Erhan, I.M. Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. *RACSAM* **2021**, *115*, 1–16. [CrossRef]
11. Adiguzel, R.S.; Aksoy, U.; Karapinar, E.; Erhan, I.M. On The Solutions of Fractional Differential Equations Via Geraghty Type Hybrid Contractions. *Appl. Comput. Math.* **2021**, *20*, 313–333. [CrossRef]
12. Akdemir, A.O.; Karaoğlan, A.; Ragusa, M.A.; Set, E. Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions. *J. Funct. Spaces.* **2021**, *2021*, 10. [CrossRef]
13. Butt, S.I.; Akdemir, A.O.; Nadeem, M.; Raza, M.A. Grüss type inequalities via generalized fractional operators. *Math. Methods Appl. Sci.* **2021**, *44*, 12559–12574. [CrossRef]
14. Bao, T.N.; Caraballo, T.; Tuan, N.H. Existence and regularity results for terminal value problem for nonlinear fractional wave equations. *Nonlinearity* **2021**, *34*, 1448–1503. [CrossRef]
15. de Carvalho-Neto, P.M.; Planas, G. Mild solutions to the time fractional Navier-Stokes equations in $\mathbb{R}^N$. *J. Differ. Equ.* **2015**, *259*, 2948–2980. [CrossRef]

16. Caraballo, T.; Guo, B.; Tuan, N.H.; Wang, R. Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains. *Proc. R. Soc. Edinb. Sect. Math.* **2021**, *151*, 1700–1730. [CrossRef]
17. Chen, P.; Wang, B.; Wang, R.; Zhang, X. Multivalued random dynamics of Benjamin-Bona-Mahony equations driven by nonlinear colored noise on unbounded domains. *Math. Ann.* **2022**, 1–31. [CrossRef]
18. Caraballo, T.; Ngoc, T.B.; Tuan, N.H.; Wang, R. On a nonlinear Volterra integrodifferential equation involving fractional derivative with Mittag-Leffler kernel. *Proc. Amer. Math. Soc.* **2021**, *149*, 3317–3334. [CrossRef]
19. Karapinar, E.; Binh, H.D.; Luc, N.H.; Can, N.H. On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems. *Adv. Differ. Equ.* **2021**, *2021*, 1–24. [CrossRef]
20. Önalan, H.K.; Akdemir, A.O.; Ardıç, M.A.; Baleanu, D. On new general versions of Hermite–Hadamard type integral inequalities via fractional integral operators with Mittag-Leffler kernel. *J. Inequalities Appl.* **2021**, *2021*, 1–16. [CrossRef]
21. Nghia, B.; Luc, N.; Binh, H.; Long, L. Regularization method for the problem of determining the source function using integral conditions. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 351–361.
22. Nass, A.; Mpungu, K. Symmetry Analysis of Time Fractional Convection-reaction-diffusion Equation with a Delay. *Results Nonlinear Anal.* **2019**, *2*, 113–124. [CrossRef]
23. Nguyen, A.T.; Caraballo, T.; Tuan, N.H. On the initial value problem for a class of nonlinear biharmonic equation with time-fractional derivative. *Proc. Roy. Soc. Edinburgh Sect. A* **2021**, *26*, 1–43.
24. Nguyen, H.T.; Tuan, N.A.; Yang, C. Global well-posedness for fractional Sobolev-Galpern type equations. *Discrete Contin. Dyn. Syst.* **2022**, *42*, 2637–2665. [CrossRef]
25. Wang, R.; Guo, B.; Wang, B. Well-posedness and dynamics of fractional FitzHugh-Nagumo systems on $\mathbb{R}^N$ driven by nonlinear noise. *Sci. China Math.* **2021**, *64*, 2395–2436. [CrossRef]
26. Wang, R.; Li, Y.; Wang, B. Random dynamics of fractional nonclassical diffusion equations driven by colored noise. *Discret. Contin. Dyn. Syst.* **2019**, *39*, 4091. [CrossRef]
27. Wang, R.; Shi, L.; Wang, B. Asymptotic behavior of fractional nonclassical diffusion equations driven by nonlinear colored noise on $\mathbb{R}^N$. *Nonlinearity* **2019**, *32*, 4524. [CrossRef]
28. Wang, R.; Wang, B. Random dynamics of p-Laplacian lattice systems driven by infinite-dimensional nonlinear noise. *Stoch. Process. Their Appl.* **2020**, *130*, 7431–7462. [CrossRef]
29. Wang, R. Long-time dynamics of stochastic lattice plate equations with nonlinear noise and damping. *J. Dyn. Differ. Equ.* **2021**, *33*, 767–803. [CrossRef]
30. Tuan, N.H.; Debbouche, A.; Ngoc, T.B. Existence and regularity of final value problems for time fractional wave equations. *Comput. Math. Appl.* **2019**, *78*, 1396–1414. [CrossRef]
31. Tuan, N.H.; Zhou, Y.; Thach, T.N.; Can, N.H. Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *78*, 104873. [CrossRef]
32. Tuan, N.H.; Nane, E. Inverse source problem for time-fractional diffusion with discrete random noise. *Stat. Probab. Lett.* **2017**, *120*, 126–134. [CrossRef]
33. Tuan, N.H.; Huynh, L.N.; Baleanu, D.; Can, N.H. On a terminal value problem for a generalization of the fractional diffusion equation with hyper-Bessel operator. *Math. Methods Appl. Sci.* **2020**, *43*, 2858–2882. [CrossRef]
34. Au, V.V.; Singh, J.; Nguyen, A.T. Well-posedness results and blow-up for a semi-linear time fractional diffusion equation with variable coefficients. *Electron. Res. Arch.* **2021**, *29*, 581. [CrossRef]
35. Furioli, G.; Kawakami, T.; Ruf, B.; Terraneo, E. Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity. *J. Differ. Equ.* **2017**, *262*, 145–180. [CrossRef]
36. Ioku, N. The Cauchy problem for heat equations with exponential nonlinearity. *J. Differ.* **2011**, *251*, 1172–1194. [CrossRef]
37. Nakamura, M.; Ozawa, T. Nonlinear Schrödinger equations in the Sobolev space of critical order. *J. Funct. Anal.* **1998**, *155*, 364–380. [CrossRef]
38. Nakamura, M.; Ozawa, T. Global solutions in the critical Sobolev space for the wave equations with nonlinearity of exponential growth. *Math. Z* **1999**, *231*, 79–487. [CrossRef]
39. Podlubny, I. Fractional Differential Equations. In *Mathematics in Sciene and Engineering*; Academic Press: San Diego, CA, USA, 1999; Volume 198. [CrossRef]