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ON RITZ APPROXIMATION FOR A CLASS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

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Abstract

We apply the Ritz method to approximate the solution of optimal control problems through the use of polynomials. The constraints of the problem take the form of differential equations of

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fractional order accompanied by the boundary and initial conditions. The ultimate goal of the algorithm is to set up a system of equations whose number matches the unknowns. Computing the unknowns enables us to approximate the solution of the objective function in the form of polynomials.

Keywords: Fractional Optimal Control Problems; Optimal Control Problems; Polynomial Basis Functions; Caputo Fractional Derivative.

1. INTRODUCTION

Fractional-order dynamics emerge in various problems in engineering and science such as biomathematics,[1] bioengineering,[2,3] viscoelasticity,[4,5] dynamics of interfaces between substrates and nanoparticles.[6] It is also shown that the materials with hereditary and memory effects and dynamical processes including heat conduction and gas diffusion in fractal porous media can be modeled by fractional-order models better than integer models.[7] Although the optimal control theory is an area in mathematics which has been under development for years, the fractional optimal control problem (FOCP) theory is a very new area in mathematics. An FOCP can be defined with respect to different definitions of fractional derivatives. But the most important types of fractional derivatives are the Caputo and the Riemann–Liouville derivatives.[8,9] General necessary conditions of optimality have been developed for FOCPs. For instance, in Refs. 10 and 11 the authors have achieved the necessary conditions of optimization for FOCPs with the Riemann–Liouville derivative and also have solved the problem numerically by solving the necessary conditions. There also exist other numerical simulations for FOCPs with Riemann–Liouville fractional derivatives such as Oustaloup's approximation into a state-space realization form.[12] In Ref. 13, the necessary conditions of optimization are achieved for FOCPs with the Caputo fractional derivative. There exist numerical technique for such problems such as in Refs. 13 and 14, where the author has solved the problem by solving the necessary conditions approximately.

In principle, it is difficult to find an analytical solution of optimal control problems, to circumvent this, researchers have suggested techniques that are able to simulate the solutions. Generally, the approaches adopted in the solution of optimal control problems are classified into two categories, indirect and direct. In the former approach, the original optimal control problem is transformed into a different state that can be easily solved. In the latter approach, the solution of the optimal control problem is approximated using numerical techniques.

A deeper insight into the solution of the optimal control problems for indirect methods is found in Refs. 11, 15, 16 and for direct methods in Refs. 17–24. Since we intend to approximate the solutions of FOCPs in this research, it implies our methodology falls under direct methods. Our research solely focuses on the methods of solution of FOCPs, therefore we will not be engaged in the formulation of optimal control problems, instead we will make use of examples to demonstrate the suggested scheme.

We separate the rest of our work into different sections, we start by detailing the methodology for approximating the FOCPs, we then explore the convergence of the technique in Sec. 3, we demonstrate the use of the methodology in Sec. 4 and then we finally give our conclusion in the last section.

2. SOLUTION OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

We give a concise description of our methodology in this section by explaining how to go about solving a typical FOCP.

Let $0 < \alpha_i < n_i$ and L, f_i: $[a, +\infty[\times \mathbb{R}^{2n+1} \to \mathbb{R}$ be differentiable functions and also f_i is a linear and invertible function for $1 \leq i \leq n$.

Consider the following FOCPs:

$$\text{Min} \quad \aleph(r_1, \ldots, r_n, u_1, \ldots, u_n, T) = \int_a^T L(t, r_1, \ldots, r_n, u_1, \ldots, u_n)dt, \qquad (1)$$

subject to the dynamic system

$$\begin{cases} D_t^{\alpha_1} r_1(t) = f_1(t, r_1, \ldots, r_n, u_1, \ldots, u_n), \\ D_t^{\alpha_2} r_2(t) = f_2(t, r_1, \ldots, r_n, u_1, \ldots, u_n), \\ \quad \vdots \\ D_t^{\alpha_n} r_n(t) = f_n(t, r_1, \ldots, r_n, u_1, \ldots, u_n), \\ \quad 0 \leq t \leq 1, \end{cases} \qquad (2)$$

where the boundary conditions are as follows:

$$\begin{cases} r_1(0) = r_{10}, & r_1(1) = r_{11}, \\ r_2(0) = r_{20}, & r_2(1) = r_{21}, \\ \vdots & \vdots \\ r_n(0) = r_{n0}, & r_n(1) = r_{n1}, \end{cases} \tag{3}$$

where T, r_{i0} and $r_{i1}, i = 0, 2, \ldots, n$ are fixed real numbers, $D_t^{\alpha_j}, j = 1, 2, \ldots, n$ represents the Caputo fractional derivative with respect to t.[8,9]

We consider the functions $r_i(t)$, $i = 1, 2, \ldots, n$

$$r_i(t) \cong r_{ik}(t) = \sum_{j=0}^{k} a_{ij} t(t-1)\psi_j(t) + w_i(t). \tag{4}$$

In the above equation, a_{ij} are unknown coefficients, $\psi_j(t)$ and $w_i(t)$ are basis polynomial functions.

The functions $w_i(t)$ are constructed in a manner so as to satisfy the boundary conditions

$$w_i(t) = r_{i0} + (r_{i1} - r_{i0})t.$$

Then we consider

$$\begin{cases} u_1(t) = f_1^{-1}(D_t^{\alpha} r_1(t)) - g_1^{-1}(t, \boldsymbol{r}, u_2, \ldots, u_n), \\ u_2(t) = f_2^{-1}(D_t^{\alpha} r_1(t)) - g_2^{-1}(t, \boldsymbol{r}, u_1, u_3, \ldots, u_n), \\ \quad\vdots \\ u_n(t) = f_n^{-1}(D_t^{\alpha} r_1(t)) - g_n^{-1}(t, \boldsymbol{r}, u_1, \ldots, u_{n-1}), \end{cases} \tag{5}$$

where g_i are linear functions and $\boldsymbol{r} = (r_1, \ldots, r_n)$. We substitute (4) and (5) in (1), and define

$$\aleph[a_{10}, \ldots, a_{1k}, \ldots, a_{n0}, \ldots, a_{nk}] = \int_a^T L(t, r_{1k}, \ldots, r_{nk})dt. \tag{6}$$

We substitute $r_i(t)$ terms represented by Eq. (4) in the above equation. Minimizing the function given by Eq. (6) demands that we impose the necessary condition as

$$\frac{\partial \aleph}{\partial a_{ij}} = 0, \quad j = 0, \ldots, k, \;\; i = 1, \ldots, n. \tag{7}$$

The format of the above equation lends itself to a system of equations with a_{ij} as unknowns. Our final task is to solve for the unknowns and substitute them in Eq. (4). In most cases, the system of equations that results from Eq. (7) is so complex that they require the assistance of computer algebraic systems such as Mathematica.

3. CONVERGENCE

Our major objective in this section is to show that increasing the degree of the polynomial, that is, the value of k in Eq. (4), is accompanied by an increase in the accuracy of the method.

Definition 1. We define the Banach space as

$$C^n(K) = \{r(t) | r^{(n)}(t) \in C(K)\},$$
$$\|r\|_n = \sum_{i=0}^{n} \|r^{(i)}\|_\infty \qquad \text{that } K = [0, 1]$$

and

$$H_i(K) = \{r_i(t) \in C^n(K) | r_i(0) = r_{i0}, r_i(1) = r_{i1}\}.$$

Lemma 2. *Assume $r_i(t) \in H_i(K)$, under the norm $\|\cdot\|_n$, there exists a polynomial $\{l_{ij}(t)\}_{j\in N} \subset H_i(K)$ such that $l_{ij} \to r_i$.*

The proof of Lemma 2 is found in Ref. 25.

Definition 3. We define $M_{ik}(K)$ and $M_k(K)$ as

$$\begin{aligned} M_{ik}(K) &= H_i(K) \cap \left\langle \{\psi_j(t)\}_{j=0}^{j=k} \right\rangle, \\ M_k(K) &= \bigcup_{i=1}^{n} M_{ik}, \end{aligned} \tag{8}$$

where $\left\langle \{\psi_j(t)\}_{j=0}^{j=k} \right\rangle$ is the Banach subspace of $C^n(K)$. In Eq. (8), $M_{ik}(K)$ is a metric subspace of $H_i(K)$.

Theorem 4. *Assume $r_i \in C^n(K)$ and $\mathcal{D}^{\alpha_i} r(t) \in C(K)$ then*

$$\left\| \frac{\partial^{\alpha_i} r_i(t)}{\partial t^{\alpha_i}} \right\|_\infty \le \frac{\left\| r_i^{(n_i)} \right\|_\infty}{\Gamma(n_i - \alpha_i + 1)}. \tag{9}$$

Proof. It is similar to given proof in Refs. 8 and 26. ◻

Theorem 5 (Ref. 27). *A map p that operates from a metric space into another metric space is regarded as uniformly continuous.*

Lemma 6. *On the Banach space $((C^n(K), \ldots, C^n(K)), \|\cdot\|_n)$, $\aleph$ is assumed to be continuous.*

Proof. Presume $r_i^* \in C^n(K)$, $s > 0$ and

$$I = K \times \prod_{i=0}^{n}[-L_i - s, L_i + s]$$

$$\times \prod_{i=0}^{n}[-H_i - s, H_i + s],$$

where

$$L_i = \|r_i^*\|_\infty, \quad H_i = \|\mathcal{D}^{\alpha_i} r_i^*\|_\infty.$$

If $t \in K$,

$$R^* = (t, r_1^*, \ldots, r_n^*, \mathcal{D}^{\alpha_1} r_1^*, \ldots, \mathcal{D}^{\alpha_n} r_n^*) \in I.$$

Suppose $\mu > 0$ and $\|r_i - r_i^*\| < \mu$, then in view of Eq. (9),

$$\|\mathcal{D}^{\alpha_i} r_i - \mathcal{D}^{\alpha_i} r_i^*\|_\infty \le \frac{1}{\Gamma(n_i - \alpha_i + 1)}\|r_i - r_i^*\|_\infty < \frac{\mu}{\Gamma(n_i - \alpha_i + 1)}.$$

We have the following result for μ that is small enough:

$$R = (t, r_1, \ldots, r_n, \mathcal{D}^{\alpha_1} r_1, \ldots, \mathcal{D}^{\alpha_n} r_n) \in I, \quad t \in T.$$

If we consider

$$L[(r_1(t), \ldots, r_n(t)] = L(t, r_1, \ldots, r_n, u_1, \ldots, u_n),$$

L is a continuous mapping on the compact set I and according to Theorem 5, L will be uniformly continuous on set I. Accordingly, if $\mu > 0$ is very small then

$$|R - R^*| < \mu \quad \Rightarrow \quad |L(R) - L(R^*)| < \xi,$$

and therefore we have

$$|\aleph[(r_1(t), \ldots, r_n(t))] - \aleph[(r_1^*(t), \ldots, r_n^*(t))]| < \xi,$$

as required. □

Theorem 7. *We assume the $\omega_k \in M_k(K)$ to be the minimum value of $\aleph$, then we have*

$$\lim_{k\to\infty} \omega_k = 0.$$

Proof. For arbitrary value $\xi > 0$, assume $r_i^* \in M_{ik}(K)$ such that $\aleph[(r_1^*, \ldots, r_n^*)] < \xi$. According to Lemma 6, we know that on space

$$((C^n(K), \ldots, C^n(K)), \|\cdot\|_n),$$

the functional $\aleph$ is continuous. Thus,

$$|\aleph[(r_1, \ldots, r_n)] - \aleph[(r_1^*, \ldots, r_n^*)]| < \xi \qquad (10)$$

provided that $\|(r_1, \ldots, r_n) - (r_1^*, \ldots, r_n^*)\|$. According to Lemma 2, there exists $\eta_k \in M_k(K)$, for large enough values of k, we have $\|\eta_k - (r_1^*, \ldots, r_n^*)\| < \xi$. Moreover, suppose r_{ik} are the elements of $M_{ik}(K)$ such that $\aleph[(r_1, \ldots, x_n)] = \omega_k$, then using (10) we have

$$0 \le \aleph[(r_1, \ldots, r_n)] \le \aleph[\eta_k] < 2\xi.$$

As $\xi > 0$ is arbitrary, we obtain

$$\lim_{karrow\infty} \omega_k = \lim_{karrow\infty} \aleph[(r_{1k}, \ldots, r_{nk})] = 0. \qquad \square$$

4. ILLUSTRATIVE EXAMPLES

In this section, we use the suggested technique to solve two nonlinear FOCPs.

Example 8. Consider the nonlinear FOCP[20]

$$\min \aleph[r_1, r_2] = \int_0^1 \left[(r_1 - t^{\frac{5}{2}} - t^2 - 1)^2 + (r_2 - t^{\frac{9}{2}})^2 + \left(D^{\frac{1}{2}} r_2 - \frac{315\sqrt{\pi}t^4}{3\sqrt{\pi}}\right)^2 + \left(D^{\frac{1}{2}} r_1 - \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{15\pi t^2}{16\sqrt{\pi}}\right)^2\right] dt$$

accompanied by the conditions

$$r_1(0) = 1, \quad r_1(1) = 3, \quad r_2(0) = 0, \quad r_2(1) = 1.$$

The analytic solution of the above problem is

$$r_1(t) = t^{\frac{5}{2}} + t^2 + 1,$$
$$r_2(t) = t^{\frac{9}{2}}.$$

We formulate the approximate solution in the form

$$r_{1k}(t) = \sum_{i=0}^{k} a_{1i}(t-1)t^{i+1} + 1 + 2t, \qquad (11)$$

$$r_{2k}(t) = \sum_{i=0}^{k} a_{2i}(t-1)t^{i+1} + t. \qquad (12)$$

We set up a system of equations that enables us to solve for a_{1i} and a_{2i} as discussed in the previous section. Thereafter, we choose the value of k to use in (11) and (12), for example if $k = 6$,

$$r_{16(t)} = 0.211t^7 + 0.457t^6 + 0.217t^5 - 1.602t^4 + 1.503t^3 - 0.786t^2 + 4.003t - 1.003,$$

$$r_{26(t)} = 0.029t^8 + 0.094t^7 - 0.384t^6 + 0.902t^5 - 1.668t^4 + 2.025t^3 + 0.002t^2.$$

Figures 1 and 2 display the absolute error between the above system of equations and the analytic solution. The errors are insignificant, implying a good approximate solution.

In Table 1, the minimum of the functional $\aleph$ can be seen for different values of k.

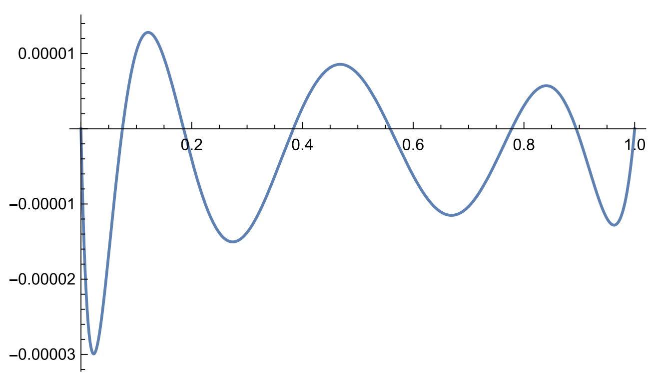

Fig. 1 Absolute error between approximate and exact solution for $r_1(t)$.

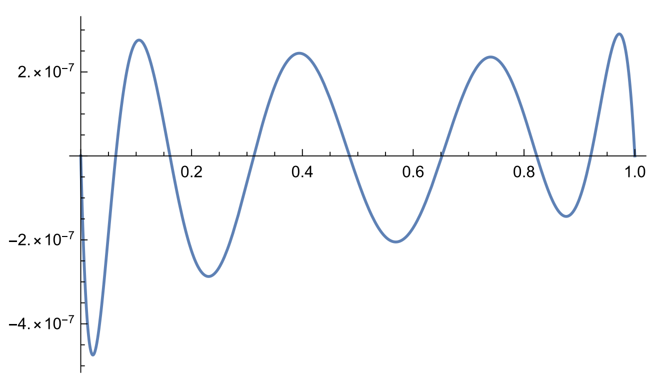

Fig. 2 Absolute error between approximate and exact solution for $r_2(t)$.

Table 1 Minimum of the Functional $\aleph$.

	$k = 2$	$k = 4$	$k = 6$
$\aleph$	0.0000171	6.64259×10^{-8}	2.44447×10^{-9}

Example 9. We intend to approximate the solution of Ref. 20

$$\min \aleph[r_1, r_2] = \int_0^1 \frac{1}{2}\left[(r_1 - t^{\frac{3}{2}} - 1)^2 + (r_2 - t^{\frac{5}{2}})^2 + \left(u(t) - \frac{3\sqrt{\pi}}{4}t + t^{\frac{5}{2}}\right)^2\right] dt, \quad (13)$$

subject to the dynamical system

$$\mathcal{D}_t^{0.5} r_1(t) = r_2(t) + u(t), \quad (14)$$

$$\mathcal{D}_t^{0.5} r_2(t) = r_1(t) + \frac{15\sqrt{\pi}}{16}t^2 - t^{\frac{3}{2}} - 1, \quad (15)$$

accompanied by the conditions

$$r_1(0) = 1, \quad r_2(0) = 0.$$

The exact solution of this problem is

$$r_1(t) = t^{\frac{3}{2}} + 1, \quad r_2(t) = t^{\frac{5}{2}}, \quad u(t) = \frac{3\sqrt{\pi}}{4}t - t^{\frac{5}{2}}. \quad (16)$$

We approximate the solution of the problem in the form of polynomials:

$$r_{1k}(t) = \sum_{i=0}^{m} a_{1i}t^{i+1} + 1, \quad (17)$$

$$r_{2k}(t) = \sum_{i=0}^{m} a_{2i}t^{i+1}. \quad (18)$$

We employ the technique that we described in the previous section to solve it for a_{1i} and a_{2i}, and then substituting these obtained values in Eqs. (17) and (18) yields

$$r_{16(t)} = 1 + 0.132t + 2.135t^2 - 3.687t^3 + 4.946t^4 - 3.551t^5 + 1.024t^6,$$

$$r_{26(t)} = -0.016t + 0.291t^2 + 1.752t^3 - 3.887t^4 + 6.8002t^5 - 5.866t6 + 1.028t^7.$$

In Figs. 3 and 4, we compare the approximate solution with (16) through the use of the absolute error. The magnitude of the errors is small, indicating a good approximate solution. In Table 2, the minimum of the functional $\aleph$ can be seen for different values of k.

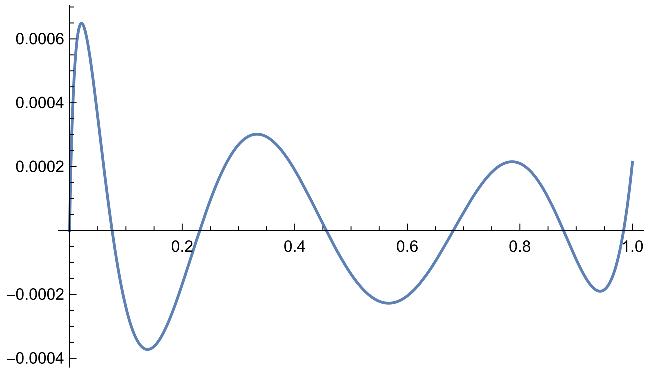

Fig. 3 Absolute error between approximate and exact solution for $r_1(t)$.

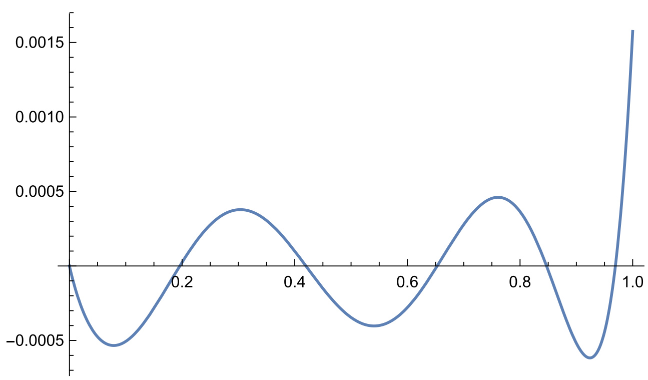

Fig. 4 Absolute error between approximate and exact solution for $r_2(t)$.

Table 2 Minimum of the Functional ℵ.

k	$k = 1$	$k = 3$	$k = 5$
ℵ	0.000419	9.64694×10^{-6}	1.00695×10^{-6}

5. CONCLUSION

A numerical method that possesses the capability of handling nonlinear optimal control problems is profoundly explained. Two applications serve as examples to support the theoretical aspects of the methodology. To prove the accuracy of the numerical technique, the obtained results are compared with those from the known analytic solutions. The computed errors prove that the suggested technique is both reliable and accurate. In terms of implementation, this technique is not cumbersome as it requires a few and easy steps to get to the final answer. As with other numerical schemes, the accuracy of the suggested approach improves with increasing number of unknowns in the equations, however, solving a few equations is generally adequate to yield acceptable levels of accuracy. Finally, it should be noted that this method, although given for linear functions in conditions, has worked very well, but more discussions of nonlinear functions should be given.

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